Vibration characteristics of piezoelectric functionally graded carbon nanotube-reinforced composite doubly-curved shells*

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Abstract This paper presents an analytical solution for the free vibration behavior of functionally graded carbon nanotube-reinforced composite (FG-CNTRC) doubly curved shallow shells with integrated piezoelectric layers. Here, the linear distribution of electric potential across the thickness of the piezoelectric layer and five different types of carbon nanotube (CNT) distributions through the thickness direction are considered. Based on the four-variable shear deformation refined shell theory, governing equations are obtained by applying Hamilton's principle. Navier's solution for the shell panels with the simply supported boundary condition at all four edges is derived. Several numerical examples validate the accuracy of the presented solution. New parametric studies regarding the effects of different material properties, shell geometric parameters, and electrical boundary conditions on the free vibration responses of the hybrid panels are investigated and discussed in detail.

Key words free vibration, four-variable shear deformation refined theory, functionally graded carbon nanotube-reinforced composite (FG-CNTRC), piezoelectric material

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1 Introduction

Functionally graded carbon nanotube-reinforced composite (FG-CNTRC) materials are a new generation of composite materials in which carbon nanotubes (CNTs) are designed purposefully to grade with specific rules along with desired directions within an isotropic matrix. Thanks to the outstanding properties of FG-CNTRC and electromechanical properties of piezoelectric materials, the complicated FG-CNTRC structures with integrated piezoelectric layers (PFG-CNTRC) offer great potential for use in engineering, advanced aerospace, medicine, military, and automotive structural applications. The PFG-CNTRC structures are applied as smart structures to control the free vibration, suppress the forced vibration, decrease the deflection and stresses, delay the buckling, decrease the post-buckling deflection, flutter control,

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and suppress the snap-through phenomenon. Subsequently, researchers started to investigate the behavior of the PFG-CNTRC structures in the last few years.

Based on the three-dimensional (3D) theory of elasticity, Alibeigloo^[1–5] performed the bending and free vibration analysis of PFG-CNTRC plates and cylindrical panels (CYLs) subjected to mechanical uniform pressure, thermal load, and applied voltage field. Using a unified formulation of a finite layer model based on Reissner's mixed variational theorem, the 3D buckling responses of the PFG-CNTRC plates under bi-axial compressive loads were investigated by Wu and Chang^[6]. In the classical lamination theory based on the Kirchhoff plate theory, the interlaminar shear deformation is neglected. Nasihatgozar et al.^[7] studied the stability of the PFG-CNTRC cylindrical panels under the axial and biaxial loadings. Applying the Ritz energy approach, the post-buckling behavior of PFG-CNTRC cylindrical shells subjected to axial compression and lateral loads in the thermal environment was investigated by Ansari et al.^[8]. Using the first-order shear deformation plate theory (FSDT) with the assumption of constant transverse shear deformation, Rafiee et al.^[9] analyzed the nonlinear dynamic stability of PFG-CNTRC plates with initial geometric imperfection subjected to a combined thermal and electrical loadings and interaction of parametric and external resonance. Rafiee et al.^[10] also investigated the large amplitude free vibration of the immovable simply supported PFG-CNTRC plates. Based on the FSDT and the Ritz method, Kiani^[11] developed a model for the free vibration analysis of FG-CNTRC plates with various mechanical and electrical boundary conditions. Sharma et al.^[12] implemented a finite element model with the nonlinear fuzzy logic controller to perform active vibration control of the FG-CNTRC plates. Some research papers on the analysis of PFG-CNTRC plates and shells based on the higher-order shear deformation plate theory are also available. Kolahchi et al.^[13] presented a model based on the refined piezoelectricity zig-zag theory for general wave propagation analysis of the piezoelectric sandwich plate. The free vibration behavior of the PFG-CNTRC quadrilateral spherical panel was examined by Setoodeh et al.^[14]. Nguyen-Quang et al.^[15] introduced an extension of the isogeometric approach based on the non-uniform rational B-spline (NURBS) basis functions for the dynamic response of the laminated PFG-CNTRC plates. Based on Reddy's higher-order shear deformation theory, Selim et al.^[16] studied the impact analysis of the PFG-CNTRC plates using the element-free IMLS-Ritz model. Song et al.^[17] presented the active vibration control of FG-CNTRC plates with surface-bonded piezoelectric actuator and sensor. Selim et al.^[18] presented a novel element-free IMLS-Ritz model for the active vibration control of the FG-CNTRC plate with different positions of piezoelectric sensor layer and actuator layer. Zhang et al.^[19] presented an optimal shape control of PFG-CNTRC plates with the open-loop control and the displacement feedback control gain.

Free vibration characteristics of shell structures were reported by many authors such as Tran et al.^[20], Baghlani et al.^[21] for FGM cylindrical shells; Song et al.^[22] for FG-CNTRC cylindrical shells; Zhu et al. for cylindrical nano-shell^[23], and doubly curved nano-shell^[24].

In the present work, the previous works of Tran et al.^[25] and HUU et al.^[26] for PFG-CNTRC plates are utilized and developed wherein multilayer FG-CNTRC doubly curved panels (DCPs) with surface-bonded piezoelectric layers are modeled using four-variable shear deformation refined theory for the first time. By applying Navier's solution, the governing equations can be solved to obtain the vibrational responses of hybrid shell panels. Several numerical examples validate the accuracy of the present model. The effects of material properties, geometric parameters, and electrical boundary conditions on the vibrational responses of the hybrid panels are also investigated and discussed in detail.

2 Mathematical formulation

2.1 Geometry and material properties

Consider a laminated FG-CNTRC doubly curved panel (FG-CNTRC-DCP) with two perfect bonded piezoelectric layers at the top and bottom surfaces, as shown in Fig. 1.

Fig. 1 Schematic of laminated PFG-CNTRC doubly curved shell panel (color online)

The panel with constant principal curvatures is referred to as an orthogonal curvilinear coordinate system (x, y, z) . The length, width, and two radii of principal curvatures of the middle surface of the panel are denoted by a, b , and R_x and R_y , respectively. The thickness of the core and each piezoelectric layer are also marked by h_c and h_p , respectively.

As shown in Fig. 2, for each layer, five types of distributions of CNT are considered. UD represents the uniform distributions, and the other four types of functionally graded distributions of CNT are denoted by FG-A, FG-V, FG-X, and FG-O.

Fig. 2 Distribution types of FG-CNT reinforced composite layers: (a) UD layer; (b) FG-A layer; (c) FG-V layer; (d) FG-X layer; (e) FG-O layer (color online)

According to the distributions of CNTs, the CNT volume fractions $V_{\text{CNT}}(z)$ for each FG-CNTRC layer are given as^[27]

$$V_{\text{CNT}}(z) = \begin{cases} V_{\text{CNT}}^* & \text{for UD CNTRC,} \\ 2V_{\text{CNT}}^* \frac{z - z_k}{z_{k+1} - z_k} & \text{for FG-V CNTRC,} \\ 2V_{\text{CNT}}^* \frac{z_{k+1} - z}{z_{k+1} - z_k} & \text{for FG-A CNTRC,} \\ 2V_{\text{CNT}}^* \left(1 - \frac{|2z - z_k - z_{k+1}|}{z_{k+1} - z_k} \right) & \text{for FG-O CNTRC,} \\ 2V_{\text{CNT}}^* \left(\frac{|2z - z_k - z_{k+1}|}{z_{k+1} - z_k} \right) & \text{for FG-X CNTRC,} \end{cases} \quad (1)$$

where

$$V_{\text{CNT}}^* = \frac{w_{\text{CNT}}}{w_{\text{CNT}} + (\rho_{\text{CNT}}/\rho_{\text{m}}) - (\rho_{\text{CNT}}/\rho_{\text{m}})w_{\text{CNT}}}, \quad (2)$$

in which, w_{CNT} is the mass fraction of the CNT, ρ_{m} and ρ_{CNT} are mass densities of the matrix and CNT, respectively. By using the extended rule of the mixture, the effective material properties of each FG-CNT layer can be expressed as follows^[28]:

$$\begin{cases} E_{11}(z) = \eta_1 V_{\text{CNT}}(z) E_{11}^{\text{CNT}} + V_{\text{m}}(z) E^{\text{m}}, & \frac{\eta_2}{E_{22}(z)} = \frac{V_{\text{CNT}}(z)}{E_{22}^{\text{CNT}}} + \frac{V_{\text{m}}(z)}{E^{\text{m}}}, \\ \frac{\eta_3}{G_{12}(z)} = \frac{V_{\text{CNT}}(z)}{G_{12}^{\text{CNT}}} + \frac{V_{\text{m}}(z)}{G^{\text{m}}}, & \rho(z) = V_{\text{CNT}}(z) \rho^{\text{CNT}} + V_{\text{m}}(z) \rho^{\text{m}}, \\ \nu_{12} = V_{\text{CNT}}^* \nu_{12}^{\text{CNT}} + V_{\text{m}}(z) \nu^{\text{m}}, \end{cases} \quad (3)$$

where E_{11}^{CNT} , E_{22}^{CNT} , and E^{m} are Young's moduli of CNT and matrix; G_{12}^{CNT} and G^{m} are the shear modulus of CNT and matrix; η_1, η_2 , and η_3 are CNT efficiency parameters; $V_{\text{m}}(z)$ and $V_{\text{CNT}}(z)$ are volume fractions of matrix and CNT with the relation of $V_{\text{CNT}}(z) + V_{\text{m}}(z) = 1$; ν_{12}^{CNT} and ν^{m} are Poisson's ratios of CNT and matrix.

2.2 Approximation on mechanical displacement

The displacements field for doubly curved shell based on the four-variable shear deformation refined theory is given by the following equations^[29]:

$$\begin{cases} u(x, y, z, t) = \left(1 + \frac{z}{R_x}\right) u_0(x, y, t) - z \frac{\partial w_b(x, y, t)}{\partial x} - f(z) \frac{\partial w_s(x, y, t)}{\partial x}, \\ v(x, y, z, t) = \left(1 + \frac{z}{R_y}\right) v_0(x, y, t) - z \frac{\partial w_b(x, y, t)}{\partial y} - f(z) \frac{\partial w_s(x, y, t)}{\partial y}, \\ w(x, y, z, t) = w_b(x, y, t) + w_s(x, y, t), \end{cases} \quad (4)$$

where u_0 and v_0 are the in-plane displacements in the directions of x and y ; w_b and w_s represent the bending and shear components of the transverse displacement, respectively. The polynomial shape function is used as $f(z) = z\left(-\frac{1}{8} + \frac{3}{2}\left(\frac{z}{h}\right)^2\right)$ ^[26]. The strains at any point in the shell space associated with the displacement field in Eq. (4) can be obtained by applying the fundamental kinematic relations of a 3D body in curvilinear coordinates as follows^[30]:

$$\begin{cases} \varepsilon_x = \frac{1}{1 + z/R_x} (\varepsilon_x^0 + z\kappa_x^b + f(z)\kappa_x^s), & \varepsilon_y = \frac{1}{1 + z/R_y} (\varepsilon_y^0 + z\kappa_y^b + f(z)\kappa_y^s), \\ \gamma_{xy} = \frac{1}{1 + z/R_x} (\gamma_{xy}^0 + z\kappa_{xy}^b + f(z)\kappa_{xy}^s), & \gamma_{yx} = \frac{1}{1 + z/R_y} (\gamma_{yx}^0 + z\kappa_{yx}^b + f(z)\kappa_{yx}^s), \\ \gamma_{xz} = \frac{1}{1 + z/R_x} g(z)\gamma_{xz}^s, & \gamma_{yz} = \frac{1}{1 + z/R_y} g(z)\gamma_{yz}^s, \end{cases} \quad (5)$$

where

$$\begin{cases} \varepsilon_x^0 = \left(\frac{\partial u_0}{\partial x} + \frac{w_b}{R_x} + \frac{w_s}{R_x}\right), & \kappa_x^b = \left(\frac{1}{R_x} \frac{\partial u_0}{\partial x} - \frac{\partial w_b^2}{\partial x^2}\right), & \kappa_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \\ \varepsilon_y^0 = \left(\frac{\partial v_0}{\partial y} + \frac{w_b}{R_y} + \frac{w_s}{R_y}\right), & \kappa_y^b = \left(\frac{1}{R_y} \frac{\partial v_0}{\partial y} - \frac{\partial w_b^2}{\partial y^2}\right), & \kappa_y^s = -\frac{\partial^2 w_s}{\partial y^2}, & \gamma_{xy}^0 = \frac{\partial v_0}{\partial x}, \\ \kappa_{xy}^b = \left(\frac{1}{R_y} \frac{\partial v_0}{\partial x} - \frac{\partial w_b^2}{\partial x \partial y}\right), & \kappa_{xy}^s = -\frac{\partial^2 w_s}{\partial x \partial y}, & \gamma_{yx}^0 = \frac{\partial u_0}{\partial y}, \\ \kappa_{yx}^b = \left(\frac{1}{R_x} \frac{\partial u_0}{\partial y} - \frac{\partial w_b^2}{\partial x \partial y}\right), & \kappa_{yx}^s = -\frac{\partial^2 w_s}{\partial x \partial y}, & \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \\ \gamma_{yz}^s = \frac{\partial w_s}{\partial y}, & g(z) = (1 - f'(z)). \end{cases} \quad (6)$$

2.3 Approximation on electric potential

In this work, the electric potential distribution in the transverse direction of each piezoelectric layer is approximated by the linear function as follows^[31–33]:

$$\begin{cases} \Phi^t(x, y, z, t) = \left(z - \frac{h}{2}\right) \frac{1}{h_p} \phi^t(x, y, t) = Z_p^t \phi^t(x, y, t), & \frac{h_c}{2} \leq z \leq \frac{h_c}{2} + h_p, \\ \Phi^b(x, y, z, t) = -\left(z + \frac{h}{2}\right) \frac{1}{h_p} \phi^b(x, y, t) = Z_p^b \phi^b(x, y, t), & -\frac{h_c}{2} - h_p \leq z \leq -\frac{h_c}{2}. \end{cases} \quad (7)$$

The electric field is related to electric potential by the following relation:

$$E_i = -\nabla \Phi_i, \quad i = x, y, z, \quad (8)$$

where ∇ denotes the gradient operator. Thus, the components of the electric field are obtained as

$$\begin{cases} (E_x^t, E_y^t, E_z^t) = -\left(\frac{1}{1+z/R_x} Z_p^t \phi_x^t, \frac{1}{1+z/R_y} Z_p^t \phi_y^t, \frac{1}{h_p} \phi^t\right), \\ (E_x^b, E_y^b, E_z^b) = -\left(\frac{1}{1+z/R_x} Z_p^b \phi_x^b, \frac{1}{1+z/R_y} Z_p^b \phi_y^b, \frac{1}{h_p} \phi^b\right). \end{cases} \quad (9)$$

2.4 Constitutive equations

The stress-strain relation for the k th CNT layer is expressed as^[34]

$$\begin{pmatrix} \sigma_x^k \\ \sigma_y^k \\ \tau_{yz}^k \\ \tau_{xz}^k \\ \tau_{xy}^k \\ \tau_{yx}^k \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11}^k(z) & \bar{Q}_{12}^k(z) & 0 & 0 & \bar{Q}_{16}^k(z) & \bar{Q}_{16}^k(z) \\ \bar{Q}_{12}^k(z) & \bar{Q}_{22}^k(z) & 0 & 0 & \bar{Q}_{26}^k(z) & \bar{Q}_{26}^k(z) \\ 0 & 0 & \bar{Q}_{44}^k(z) & \bar{Q}_{45}^k(z) & 0 & 0 \\ 0 & 0 & \bar{Q}_{45}^k(z) & \bar{Q}_{55}^k(z) & 0 & 0 \\ \bar{Q}_{16}^k(z) & \bar{Q}_{26}^k(z) & 0 & 0 & \bar{Q}_{66}^k(z) & \bar{Q}_{66}^k(z) \\ \bar{Q}_{16}^k(z) & \bar{Q}_{26}^k(z) & 0 & 0 & \bar{Q}_{66}^k(z) & \bar{Q}_{66}^k(z) \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ \gamma_{yx} \end{pmatrix}, \quad (10)$$

in which $\bar{Q}_{ij}^k(z)$ are transformed material constants expressed in terms of material constants^[35],

$$\begin{cases} \bar{Q}_{11}^k(z) = Q_{11}^c(z) \cos^4 \theta^k + 2(Q_{12}^c(z) + 2Q_{66}^c(z)) \sin^2 \theta^k \cos^2 \theta^k + Q_{22}^c(z) \sin^4 \theta^k, \\ \bar{Q}_{12}^k(z) = (Q_{11}^c(z) + Q_{22}^c(z) - 4Q_{66}^c(z)) \sin^2 \theta^k \cos^2 \theta^k + Q_{12}^c(z) (\sin^4 \theta^k + \cos^4 \theta^k), \\ \bar{Q}_{22}^k(z) = Q_{11}^c(z) \sin^4 \theta^k + 2(Q_{12}^c(z) + 2Q_{66}^c(z)) \sin^2 \theta^k \cos^2 \theta^k + Q_{22}^c(z) \cos^4 \theta^k, \\ \bar{Q}_{16}^k(z) = (Q_{11}^c(z) - Q_{12}^c(z) - 2Q_{66}^c(z)) \sin \theta^k \cos^3 \theta^k \\ \quad + (Q_{12}^c(z) - Q_{22}^c(z) + 2Q_{66}^c(z)) \sin^3 \theta^k \cos \theta^k, \\ \bar{Q}_{26}^k(z) = (Q_{11}^c(z) - Q_{12}^c(z) - 2Q_{66}^c(z)) \sin^3 \theta^k \cos \theta^k \\ \quad + (Q_{12}^c(z) - Q_{22}^c(z) + 2Q_{66}^c(z)) \sin \theta^k \cos^3 \theta^k, \\ \bar{Q}_{66}^k(z) = (Q_{11}^c(z) + Q_{22}^c(z) - 2Q_{12}^c(z) - 2Q_{66}^c(z)) \sin^2 \theta^k \cos^2 \theta^k \\ \quad + Q_{66}^c(z) (\sin^4 \theta^k + \cos^4 \theta^k), \\ \bar{Q}_{44}^k(z) = Q_{44}^c(z) \cos^2 \theta^k + Q_{55}^c(z) \sin^2 \theta^k, \\ \bar{Q}_{45}^k(z) = (Q_{55}^c(z) - Q_{44}^c(z)) \cos \theta^k \sin \theta^k, \\ \bar{Q}_{55}^k(z) = Q_{55}^c(z) \cos^2 \theta^k + Q_{44}^c(z) \sin^2 \theta^k, \end{cases} \quad (11)$$

where $Q_{ij}^c(z)$ are the plane stress-reduced stiffnesses defined in terms of the engineering constants in the material axes of the layer,

$$\begin{cases} Q_{11}^c(z) = \frac{E_{11}(z)}{1 - \nu_{12}\nu_{21}}, & Q_{12}^c(z) = \frac{\nu_{12}E_{22}(z)}{1 - \nu_{12}\nu_{21}}, & Q_{22}^c(z) = \frac{E_{22}(z)}{1 - \nu_{12}\nu_{21}}, \\ Q_{44}^c = G_{23}(z), & Q_{55}^c = G_{13}(z), & Q_{66}^c = G_{12}(z). \end{cases} \quad (12)$$

For the kp th piezoelectric layer ($kp = \text{top, bottom}$), the stress components and electric displacement relations are given as^[30,34,36–37]

$$\begin{pmatrix} \sigma_x^{kp} \\ \sigma_y^{kp} \\ \tau_{yz}^{kp} \\ \tau_{xz}^{kp} \\ \tau_{xy}^{kp} \\ \tau_{yx}^{kp} \end{pmatrix} = \begin{pmatrix} \bar{C}_{11}^{kp} & \bar{C}_{12}^{kp} & 0 & 0 & 0 & 0 \\ \bar{C}_{12}^{kp} & \bar{C}_{11}^{kp} & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{C}_{55}^{kp} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{55}^{kp} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{C}_{1122}^{kp} & \tilde{C}_{1122}^{kp} \\ 0 & 0 & 0 & 0 & \tilde{C}_{1122}^{kp} & \tilde{C}_{1122}^{kp} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ \gamma_{yx} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \bar{e}_{31}^{kp} \\ 0 & 0 & \bar{e}_{31}^{kp} \\ 0 & e_{24}^{kp} & 0 \\ e_{15}^{kp} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x^{kp} \\ E_y^{kp} \\ E_z^{kp} \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} D_x^{kp} \\ D_y^{kp} \\ D_z^{kp} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & e_{15}^{kp} & 0 & 0 \\ 0 & 0 & e_{24}^{kp} & 0 & 0 & 0 \\ \bar{e}_{31}^{kp} & \bar{e}_{31}^{kp} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \\ \gamma_{yx} \end{pmatrix} + \begin{pmatrix} p_{11}^{kp} & 0 & 0 \\ 0 & p_{11}^{kp} & 0 \\ 0 & 0 & \bar{p}_{33}^{kp} \end{pmatrix} \begin{pmatrix} E_x^{kp} \\ E_y^{kp} \\ E_z^{kp} \end{pmatrix}, \quad (14)$$

in which, the elastic constants for the piezoelectric layer^[36–37] are defined as

$$\begin{cases} \bar{C}_{11}^{kp} = C_{11}^{kp} - \frac{(C_{13}^{kp})^2}{C_{33}^{kp}}, & \bar{C}_{12}^{kp} = C_{12}^{kp} - \frac{(C_{13}^{kp})^2}{C_{33}^{kp}}, & \bar{C}_{55}^{kp} = C_{55}^{kp}, \\ \tilde{C}_{1122}^{kp} = \frac{(\bar{C}_{11} - \bar{C}_{12})}{2}, & \bar{e}_{31}^{kp} = e_{31}^{kp} - \frac{C_{13}^{kp}}{C_{33}^{kp}} e_{33}^{kp}, & \bar{p}_{33}^{kp} = p_{33}^{kp} + \frac{(e_{33}^{kp})^2}{C_{33}^{kp}}, \end{cases} \quad (15)$$

where C_{ij}^{kp} are stiffness coefficients for the piezoelectric layers; e_{ij} is the electromechanical coupling matrix; p_{ij} is the dielectric permittivity matrix; E is the electric field, and D is the electrical displacement in the piezoelectric layer.

2.5 Governing equation

Hamilton's principle is applied herein to obtain the equations of motion of laminated PFG-CNTRC doubly curved shell panels,

$$\int_0^t (\delta U - \delta T) dt = 0, \quad (16)$$

where δU is the variation of strain energy, and δK is the variation of kinetic energy. The expression of variation of strain energy for the PFG-CNTRC panels is

$$\begin{aligned} \delta U = & \int_0^b \int_0^a \int_{-h/2}^{h/2} (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{yx} \delta \gamma_{yx} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz} - D_x^t \delta E_x^t \\ & - D_y^t \delta E_y^t - D_z^t \delta E_z^t - D_x^b \delta E_x^b - D_y^b \delta E_y^b - D_z^b \delta E_z^b) \left(1 + \frac{z}{R_x}\right) \left(1 + \frac{z}{R_y}\right) dz dx dy. \end{aligned} \quad (17)$$

Substituting Eq. (5), Eq. (9), Eq. (13), and Eq. (14) into Eq. (17), we can get the variation strain energy δU as

$$\begin{aligned} \delta U = & \int_0^b \int_0^a \left(N_x \delta \varepsilon_x^0 + N_y \delta \varepsilon_y^0 + N_{xy} \delta \gamma_{xy}^0 + N_{yx} \delta \gamma_{yx}^0 + M_x^b \delta \kappa_x^b + M_y^b \delta \kappa_y^b + M_{xy}^b \delta \kappa_{xy}^b \right. \\ & + M_{yx}^b \delta \kappa_{yx}^b + M_x^s \delta \kappa_x^s + M_y^s \delta \kappa_y^s + M_{xy}^s \delta \kappa_{xy}^s + M_{yx}^s \delta \kappa_{yx}^s + Q_{ys} \delta \gamma_{yz}^s + Q_{xs} \delta \gamma_{xz}^s \\ & \left. - \bar{D}_x^t \frac{\partial \delta \Phi^t}{\partial x} - \bar{D}_y^t \frac{\partial \delta \Phi^t}{\partial y} - \bar{D}_z^t \frac{\partial \delta \Phi^t}{\partial z} - \bar{D}_x^b \frac{\partial \delta \Phi^b}{\partial x} - \bar{D}_y^b \frac{\partial \delta \Phi^b}{\partial y} - \bar{D}_z^b \frac{\partial \delta \Phi^b}{\partial z} \right) dx dy, \end{aligned} \quad (18)$$

where $(N_x, N_y, N_{xy}, N_{yx})$, $(M_x^b, M_y^b, M_{xy}^b, M_{yx}^b)$, $(M_x^s, M_y^s, M_{xy}^s, M_{yx}^s)$, and (Q_{xs}, Q_{ys}) are stress resultants given in detail in Appendix A. The variation of kinetic energy for the PFG-CNTRC panels is given as

$$\begin{aligned}\delta T &= \int_0^b \int_0^a \int_{-h/2}^{h/2} \rho(z) (\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) \left(1 + \frac{z}{R_x}\right) \left(1 + \frac{z}{R_y}\right) dz dx dy \\ &= \delta T_1 + \delta T_2 + \delta T_3 + \delta T_4,\end{aligned}\quad (19)$$

where

$$\begin{aligned}\delta T_1 &= \int_0^b \int_0^a \left(\frac{1}{R_x} \left(- \left(\bar{I}_1 + \frac{\bar{I}_2}{R_x} \right) \ddot{u}_0 + \bar{I}_2 \frac{\partial \ddot{w}_b}{\partial x} + \bar{J}_2 \frac{\partial \ddot{w}_s}{\partial x} \right) \right. \\ &\quad \left. - \bar{I}_0 \ddot{u}_0 - \bar{I}_1 \left(\frac{\ddot{u}_0}{R_x} - \frac{\partial \ddot{w}_b}{\partial x} \right) + \left(\bar{J}_1 \frac{\partial \ddot{w}_s}{\partial x} \right) \right) \delta u_0 dx dy,\end{aligned}\quad (20)$$

$$\begin{aligned}\delta T_2 &= \int_0^b \int_0^a \left(\frac{1}{R_y} \left(- \left(\bar{I}_1 + \frac{\bar{I}_2}{R_y} \right) \ddot{v}_0 + \bar{I}_2 \frac{\partial \ddot{w}_b}{\partial y} + \bar{J}_2 \frac{\partial \ddot{w}_s}{\partial y} \right) \right. \\ &\quad \left. - \bar{I}_0 \ddot{v}_0 - \bar{I}_1 \left(\frac{\ddot{v}_0}{R_y} - \frac{\partial \ddot{w}_b}{\partial y} \right) + \left(\bar{J}_1 \frac{\partial \ddot{w}_s}{\partial y} \right) \right) \delta v_0 dx dy,\end{aligned}\quad (21)$$

$$\begin{aligned}\delta T_3 &= \int_0^b \int_0^a \left(- \left(\bar{I}_1 + \frac{\bar{I}_2}{R_x} \right) \frac{\partial \ddot{u}_0}{\partial x} + \bar{I}_2 \frac{\partial^2 \ddot{w}_b}{\partial x^2} + \bar{J}_2 \frac{\partial^2 \ddot{w}_s}{\partial x^2} - \left(\bar{I}_1 + \frac{\bar{I}_2}{R_y} \right) \frac{\partial \ddot{v}_0}{\partial y} \right. \\ &\quad \left. + \bar{I}_2 \frac{\partial^2 \ddot{w}_b}{\partial y^2} + \bar{J}_2 \frac{\partial^2 \ddot{w}_s}{\partial y^2} - \bar{I}_0 (\ddot{w}_b + \ddot{w}_s) \right) \delta w_b dx dy,\end{aligned}\quad (22)$$

$$\begin{aligned}\delta T_4 &= \int_0^b \int_0^a \left(- \left(\bar{J}_1 + \frac{\bar{J}_2}{R_x} \right) \ddot{u}_0 + \bar{J}_2 \frac{\partial \ddot{w}_b}{\partial x} + \bar{K}_1 \frac{\partial \ddot{w}_s}{\partial x} - \left(\bar{J}_1 + \frac{\bar{J}_2}{R_y} \right) \ddot{v}_0 \right. \\ &\quad \left. + \bar{J}_2 \frac{\partial \ddot{w}_b}{\partial y} + \bar{K}_1 \frac{\partial \ddot{w}_s}{\partial y} - \bar{I}_0 (\ddot{w}_b + \ddot{w}_s) \right) \delta w_s dx dy,\end{aligned}\quad (23)$$

in which $\rho(z)$ is the mass density, and $\bar{I}_i$ ($i = 0, 1, 2$) and $\bar{J}_i$ ($i = 1, 2$) are inertias defined by

$$\begin{cases} \bar{I}_i = I_i + I_{i+1} \left(\frac{1}{R_x} + \frac{1}{R_y} \right) + \frac{I_{i+2}}{R_x R_y}, \\ (I_0, I_1, I_2, I_3, I_4) = \sum_{k=1}^N \int_{z_k}^{z_{k+1}} \rho(z) (1, z, z^2, z^3, z^4) dz, \\ \bar{J}_i = f(z) \bar{I}_{i-1}, \quad \bar{K}_1 = f^2(z) \bar{I}_0. \end{cases}\quad (24)$$

The governing equations can be obtained by substituting the variation of strain energy and kinetic energy from Eq. (18) and Eq. (19) into Eq. (16) and then set the coefficients of $\delta u_0, \delta v_0, \delta w_b, \delta w_s, \delta \varphi^t$, and $\delta \varphi^b$ to be zero,

$$\begin{cases} \delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + \frac{Q_{xb}}{R_x} = \bar{I}_0 \ddot{u}_0 + \bar{I}_1 \left(\frac{\ddot{u}_0}{R_x} - \frac{\partial \ddot{w}_b}{\partial x} \right) - \bar{J}_1 \frac{\partial \ddot{w}_s}{\partial x}, \\ \delta v_0 : \frac{\partial N_y}{\partial y} + \frac{\partial N_{yx}}{\partial x} + \frac{Q_{yb}}{R_y} = \bar{I}_0 \ddot{v}_0 + \bar{I}_1 \left(\frac{\ddot{v}_0}{R_y} - \frac{\partial \ddot{w}_b}{\partial y} \right) - \bar{J}_1 \frac{\partial \ddot{w}_s}{\partial y}, \\ \delta w_b : \frac{N_x}{R_x} + \frac{N_y}{R_y} - \frac{\partial Q_{xb}}{\partial x} - \frac{\partial Q_{yb}}{\partial y} = -\bar{I}_0 (\ddot{w}_b + \ddot{w}_s), \\ \delta w_s : \frac{N_x}{R_x} + \frac{N_y}{R_y} - \frac{\partial \bar{Q}_{xs}}{\partial x} - \frac{\partial \bar{Q}_{ys}}{\partial y} = -\bar{I}_0 (\ddot{w}_b + \ddot{w}_s), \\ \delta \varphi^t : \frac{\partial \tilde{D}_x^t}{\partial x} + \frac{\partial \tilde{D}_y^t}{\partial y} + \tilde{D}_z^t = 0, \\ \delta \varphi^b : \frac{\partial \tilde{D}_x^b}{\partial x} + \frac{\partial \tilde{D}_y^b}{\partial y} + \tilde{D}_z^b = 0. \end{cases}\quad (25)$$

where

$$\left\{ \begin{array}{l} Q_{xb} = \frac{\partial M_{xx}^b}{\partial x} + \frac{\partial M_{yx}^b}{\partial y} - \left(\bar{I}_1 + \frac{\bar{I}_2}{R_x} \right) \ddot{u}_0 + \bar{I}_2 \frac{\partial \ddot{w}_b}{\partial x} + \bar{J}_2 \frac{\partial \ddot{w}_s}{\partial x}, \\ Q_{yb} = \frac{\partial M_{yy}^b}{\partial y} + \frac{\partial M_{xy}^b}{\partial x} - \left(\bar{I}_1 + \frac{\bar{I}_2}{R_y} \right) \ddot{v}_0 + \bar{I}_2 \frac{\partial \ddot{w}_b}{\partial y} + \bar{J}_2 \frac{\partial \ddot{w}_s}{\partial y}, \\ \bar{Q}_{xs} = \frac{\partial M_{xx}^s}{\partial x} + \frac{\partial M_{yx}^s}{\partial y} + Q_{xs} - \left(\bar{J}_1 + \frac{\bar{J}_2}{R_x} \right) \ddot{u}_0 + \bar{J}_2 \frac{\partial \ddot{w}_b}{\partial x} + \bar{K}_1 \frac{\partial \ddot{w}_s}{\partial x}, \\ \bar{Q}_{ys} = \frac{\partial M_{yy}^s}{\partial y} + \frac{\partial M_{xy}^s}{\partial x} + Q_{ys} - \left(\bar{J}_1 + \frac{\bar{J}_2}{R_y} \right) \ddot{v}_0 + \bar{J}_2 \frac{\partial \ddot{w}_b}{\partial y} + \bar{K}_1 \frac{\partial \ddot{w}_s}{\partial y}, \\ \tilde{D}_x^t = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_y} \right) Z_p^t D_x^t dz, \quad \tilde{D}_y^t = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_x} \right) Z_p^t D_y^t dz, \\ \tilde{D}_z^t = \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_x} \right) \left(1 + \frac{z}{R_y} \right) \frac{\partial Z_p^t}{\partial z} D_z^t dz, \\ \tilde{D}_x^b = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_y} \right) Z_p^b D_x^b dz, \quad \tilde{D}_y^b = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_x} \right) Z_p^b D_y^b dz, \\ \tilde{D}_z^b = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_x} \right) \left(1 + \frac{z}{R_y} \right) \frac{\partial Z_p^b}{\partial z} D_z^b dz. \end{array} \right. \quad (26)$$

3 Solution procedure

In this study, two sets of simply supported boundary conditions named cross-ply (SS-1) and angle-ply (SS-2) laminates shown in Table 1 are considered.

Table 1 Two sets of simply supported boundary conditions

Edge	Boundary condition	
	SS-1	SS-2
$x = 0$ and $x = a$	$v_0 = w_b = w_s = w_{b,y} = w_{s,y} = 0$ $N_x = M_x^b = M_x^s = \phi^t = \phi^b = 0$	$u_0 = w_b = w_s = w_{b,y} = w_{s,y} = 0$ $N_{xy} = M_x^b = M_x^s = M_{xy}^b = \phi^t = \phi^b = 0$
$y = 0$ and $y = b$	$u_0 = w_b = w_s = w_{b,x} = w_{s,x} = 0$ $N_y = M_y^b = M_y^s = \phi^t = \phi^b = 0$	$v_0 = w_b = w_s = w_{b,x} = w_{s,x} = 0$ $N_{yx} = M_y^b = M_y^s = M_{yx}^b = \phi^t = \phi^b = 0$

Following the Navier solution procedure, the following expansion displacements for $u_0, v_0, w_s, w_b, \phi^t$, and ϕ^b are chosen to satisfy the boundary conditions given in Table 1. Here, $u_{mn}, v_{mn}, w_{mn}^b, w_{mn}^s, \psi_{mn}^t$, and ψ_{mn}^b are arbitrary parameters to be determined, $\alpha = \frac{m\pi}{a}$, $\beta = \frac{n\pi}{b}$, and m, n denote the numbers of half-waves in the x - and y -directions, respectively.

By substituting the expansion displacement functions in Table 2 into Eq. (25), the following matrix form can be obtained:

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{12} & m_{22} & m_{23} & m_{24} & 0 & 0 \\ m_{13} & m_{23} & m_{33} & m_{34} & 0 & 0 \\ m_{14} & m_{24} & m_{34} & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{u}_{mn} \\ \ddot{v}_{mn} \\ \ddot{w}_{mn}^b \\ \ddot{w}_{mn}^s \\ \ddot{\psi}_{mn}^t \\ \ddot{\psi}_{mn}^b \end{pmatrix}$$

$$+ \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{12} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{13} & k_{23} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{45} & k_{46} \\ k_{15} & k_{25} & k_{35} & k_{45} & k_{55} & k_{56} \\ k_{16} & k_{26} & k_{36} & k_{46} & k_{56} & k_{66} \end{pmatrix} \begin{pmatrix} u_{mn} \\ v_{mn} \\ w_{mn}^b \\ w_{mn}^s \\ \psi_{mn}^t \\ \psi_{mn}^b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (27)$$

where m_{ij} and k_{ij} are given in Appendices B1 and B2 for SS-1 and SS-2 boundary conditions, respectively. Equation (27) can be rewritten in the short form as

$$\begin{pmatrix} M_{uu} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{u} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} K_{uu} & K_{u\phi} \\ K_{\phi u} & K_{\phi\phi} \end{pmatrix} \begin{pmatrix} u \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (28)$$

Table 2 Expansion displacements ($u_0, v_0, w_s, w_b, \phi^t, \phi^b$)

Displacement	Boundary condition	
	SS-1	SS-2
$u_0(x, y, t)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \cos(\alpha x) \sin(\beta y)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \sin(\alpha x) \cos(\beta y)$
$v_0(x, y, t)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} \sin(\alpha x) \cos(\beta y)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} \cos(\alpha x) \sin(\beta y)$
$w_b(x, y, t)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^b \sin(\alpha x) \sin \beta y$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^b \sin(\alpha x) \sin(\beta y)$
$w_s(x, y, t)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^s \sin(\alpha x) \sin \beta y$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}^s \sin(\alpha x) \sin(\beta y)$
$\phi^t(x, y, t)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^t \sin(\alpha x) \sin \beta y$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^t \sin(\alpha x) \sin(\beta y)$
$\phi^b(x, y, t)$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^b \sin(\alpha x) \sin \beta y$	$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn}^b \sin(\alpha x) \sin(\beta y)$

where M_{uu} is the mass matrix, K_{uu} is the elastic matrix, $K_{u\phi}$ is the piezoelectric matrix, and $K_{\phi\phi}$ is the permittivity matrix. Substituting the solution of the second equation into the first equation in Eq. (24), we obtain a short form of the equation for vibration characteristics of FG-CNTRC shell panels integrated with piezoelectric layers,

$$(M_{uu})(\ddot{u}) + ((K_{uu}) - (K_{u\phi})(K_{\phi\phi})^{-1}(K_{\phi u}))(u) = 0. \quad (29)$$

For free vibration, Eq. (29) reduces to an eigenvalue problem by setting $(u) = (\hat{u})e^{i\omega t}$

$$((K_{uu}) - (K_{u\phi})(K_{\phi\phi})^{-1}(K_{\phi u}) - \omega^2(M_{uu}))(\hat{u}) = 0. \quad (30)$$

Equation (30) is associated with the natural frequencies of an open-circuit (Opc) PFG-CNTRC doubly curved shell panels (see Fig. 3(a)). For the closed-circuit (Clc) (see Fig. 3(b)) condition, both the upper and lower piezoelectric layers are grounded cause the electric displacement disappears from Eq. (28). Hence, we obtain the following eigenvalue problem for the Clc boundary condition:

$$((K_{uu}) - \omega^2(M_{uu}))(\hat{u}) = 0. \quad (31)$$

The eigensolution of free vibration for PFG-CNTRC-DCP can be obtained using a general eigenvalue approach.

Fig. 3 Electrical boundary conditions: (a) closed-circuit and (b) open-circuit

4 Numerical results and discussion

In this section, several comparison studies are conducted to verify the accuracy of instant solution in predicting natural frequencies of a simply supported isotropic doubly curved shell panel and a rectangular FG-CNTRC; both of these structures are integrated with piezoelectric layers at the top and bottom surfaces. Furthermore, the effects of involved parameters such as CNT volume fractions, distribution types of CNT, and different geometric parameters on free vibration response of PFG-CNTRC doubly curved shell panel are also investigated in detail. Unless mentioned otherwise, the material properties of PFG-CNTRC doubly curved shell panels are given in Table 3.

Table 3 Material properties of piezoelectric and FG-CNT materials

Property	Substrate layer			Piezoelectric layer	
	PmPV ^[15]	CNT ^[15]	Al ₂ O ₃ ^[32]	PZT-4 ^[32]	PZT-5A ^[15]
$\rho/(\text{kg} \cdot \text{m}^{-3})$	1 150	1 400	3 800	7 500	7 750
E_{11}/GPa	$E^m=2.5$	5.64×10^3	380	—	$E = 63$
E_{22}/GPa	$E^m=2.5$	7.08×10^3	380	—	—
G_{12}/GPa	—	1.945 5	—	—	$G = 23.3$
ν_{12}	0.34	0.175	0.3	—	0.35
C_{11}/GPa	—	—	—	139	—
C_{12}/GPa	—	—	—	77.8	—
C_{13}/GPa	—	—	—	74.3	—
C_{33}/GPa	—	—	—	115	—
$C_{44,55}/\text{GPa}$	—	—	—	25.6	—
C_{66}/GPa	—	—	—	30.6	—
e_{31}/cm^{-2}	—	—	—	−5.2	−7.209
e_{33}/cm^{-2}	—	—	—	15.1	15.12
e_{15}/cm^{-2}	—	—	—	12.7	12.322
$p_{11}/(\text{nFm}^{-1})$	—	—	—	6.75	1.53
$p_{33}/(\text{nFm}^{-1})$	—	—	—	5.9	1.50

Also, the CNT efficiency parameters η_j ($j = 1, 2, 3$) associated with a given volume fraction V_{CNT}^* are^[15]: $\eta_1 = 0.137$, $\eta_2 = 1.022$ for the case of $V_{\text{CNT}}^* = 0.12$, $\eta_1 = 0.142$, $\eta_2 = 1.626$ for the case of $V_{\text{CNT}}^* = 0.17$, and $\eta_1 = 0.141$, $\eta_2 = 1.585$ for the case of $V_{\text{CNT}}^* = 0.28$, and in all cases $\eta_3 = 0.7\eta_2$.

4.1 Comparison studies

Since there are no published results for the PFG-CNTRC doubly curved shell in the open literature, two numerical examples predict the free vibration behaviors of the piezoelectric isotropic doubly curved shell panels, and the PFG-CNTRC plates are performed to validate the present approach.

4.1.1 Example 1

Firstly, the fundamental frequencies of the simply supported isotropic doubly curved shell panel with surface-bonded piezoelectric layers for different geometric parameters are presented in Table 4. The Opc and Clc of electric boundary conditions are also considered. The shell

Table 4 Fundamental frequencies ω of piezoelectric isotropic DCP ($a/b = 1$, $R_y/b = 5$)

a/R_x	h/a	h_p/h	ω_{Clc}/Hz			ω_{Opc}/Hz		
			Present	Ref. [38]	Ref. [32]	Present	Ref. [38]	Ref. [32]
-0.2	0.1	0.1	823.126	824.049	823.997	840.081	841.189	843.029
		0.2	787.613	788.430	788.091	819.929	820.477	826.775
0	0.2	0.1	1484.893	1483.998	1483.888	1514.076	1512.112	1515.550
		0.2	1400.806	1398.792	1398.093	1457.032	1449.614	1461.337
	0.1	0.1	839.079	839.368	839.316	855.961	856.455	858.249
		0.2	801.488	801.794	801.457	833.706	833.781	840.162
0.1	0.2	0.1	1498.803	1498.407	1498.297	1528.068	1526.689	1530.183
		0.2	1412.678	1411.727	1411.025	1469.096	1462.898	1474.891
	0.1	0.1	853.032	853.147	853.095	869.733	870.057	871.706
		0.2	813.257	813.413	813.081	845.172	845.108	851.309
0.2	0.2	0.1	1506.389	1506.211	1506.101	1535.551	1534.422	1537.884
		0.2	1418.902	1418.331	1417.631	1475.145	1469.403	1481.404
	0.1	0.1	870.670	870.705	870.655	887.107	887.350	888.783
		0.2	828.026	828.101	827.775	859.486	859.347	865.231
	0.2	0.1	1514.343	1514.349	1514.240	1543.307	1542.389	1545.778
		0.2	1425.281	1424.974	1424.279	1481.171	1475.765	1487.688

panel substrate is made of alumina (Al_2O_3), and two piezoelectric layers are made of PZT-4. It should be noted that the results reported by Sayyaadi et al.^[32,38] were based on the HSDT. The excellent agreement between the results shows the accuracy of the current approach.

4.1.2 Example 2

Secondly, the free vibration of the laminated cross-ply and angle-ply PFG-CNTRC plates with simply supported boundary conditions is investigated. The fundamental frequencies are given in Table 5 for different CNT volume fractions, CNT distribution types, and electrical boundary conditions (EBC). Properties of the plate are set equal to $a = b = 0.4$ m, $h = a/20$, and $h_p = h/10$. The substrate is made of a multi-laminate of armchair SWCNT, and piezoelectric layers are PZT-5A. It can be seen that the present results agree well with those by Nguyen-Quang et al.^[15] based on the isogeometric approach and the HSDT. The maximum difference is only 1.41% for the case that $V_{\text{CNT}}^* = 0.28$, FG-X, and $[\text{p}/-45^\circ/45^\circ/-45^\circ/-45^\circ/45^\circ/-45^\circ/\text{p}]$ configuration.

4.2 Parametric studies

4.2.1 Free vibration of PFG-CNTRC doubly curved shell panels

Parametric studies are carried out to understand the effects of material properties, geometric parameters, laminate configurations, and electrical boundaries on the free vibration responses of PFG-CNTRC doubly curved shell panels. The substrate of the panel is made of armchair SWCNT, and two piezoelectric layers are PZT-5A. Tables 6–8 list the fundamental frequencies of the CYL, spherical panel (SPH), and hyperbolic paraboloid (HPR) with various inlet parameters. It is observed from these tables that the distribution types of CNT have a significant effect on the stiffness of the panel. In detail, the FG-O panel has the lowest value of frequencies, while the FG-X panels have the highest ones. This conclusion is compatible with the conclusion of other researchers in the related studies in the literature. In three forms of doubly-curved shell panels, the HPR shell panels have the lowest frequencies, and the SPH shell panels have the highest frequencies. These results may become from the fact that SHP has a curvature effect, while the HPR has both positive and negative curvature that neutralize the effect of each other. These tables also reveal that the Opc of electrical boundary conditions always have higher frequencies than the Clc with all other parameters.

4.2.2 Effect of CNT distribution types

Figure 4 depicts the effect of distribution types of CNT on the fundamental frequencies of cross-ply PFG-CNTRC doubly curved shell panels for different R_x/R_y ratio in case of the Opc condition. It is observed that the previous conclusions regarding the CNT distribution types are

Table 5 Fundamental frequencies ω of square FG-CNTRC plates ($a = b$, $a/h = 20$, $h_p = h/10$, and $R_x = R_y = \infty$)

V_{CNT}^*	FG type	EBC	Laminate configuration					
			[p/0°/90°/0°/p]			[p/−45°/45°/−45°/−45°/45°/−45°/p]		
			Present	Ref. [15]	Diff/%	Present	Ref. [15]	Diff/%
0.12	UD	Clc	587.093	583.510	0.61	662.572	656.538	0.92
		Opc	622.030	627.716	−0.91	692.902	695.085	−0.31
	FG-X	Clc	593.270	588.372	0.83	667.328	658.696	1.31
		Opc	627.814	632.184	−0.69	697.410	697.103	0.04
	FG-V	Clc	585.306	581.714	0.62	661.675	655.606	0.93
		Opc	620.461	626.205	−0.92	692.094	694.272	−0.31
	FG-O	Clc	580.947	578.737	0.38	657.878	654.510	0.51
		Opc	616.286	623.343	−1.13	688.462	693.196	−0.68
0.17	UD	Clc	628.442	624.543	0.62	727.596	720.800	0.94
		Opc	660.887	665.615	−0.71	754.830	755.388	−0.07
	FG-X	Clc	636.977	631.317	0.90	733.961	723.781	1.41
		Opc	668.952	671.913	−0.44	760.925	758.217	0.36
	FG-V	Clc	625.826	621.914	0.63	726.448	719.594	0.95
		Opc	658.552	663.359	−0.72	753.785	754.324	−0.07
	FG-O	Clc	620.132	618.126	0.32	721.527	718.247	0.46
		Opc	653.061	659.687	−1.00	749.042	752.995	−0.52
0.28	UD	Clc	692.016	686.852	0.75	828.983	821.713	0.88
		Opc	720.749	723.150	−0.33	851.843	850.524	0.16
	FG-X	Clc	704.853	697.260	1.09	838.282	826.415	1.44
		Opc	733.029	732.991	0.01	860.872	855.093	0.68
	FG-V	Clc	688.165	682.974	0.76	827.912	820.463	0.91
		Opc	717.276	719.788	−0.35	850.901	849.465	0.17
	FG-O	Clc	680.340	677.986	0.35	821.193	818.750	0.30
		Opc	709.665	714.904	−0.73	844.364	847.767	−0.40

confirmed. The FG-X and FG-O have the highest and lowest frequencies, respectively. Figure 4 also shows that the value of the fundamental frequencies is minimum at the ratio $R_x/R_y = 1$. This observation, maybe because of the curvature effect, is suppressed when the panels have the same value of positive and negative curvatures.

Moreover, the percentage change of frequency (f_{PC}) of the SPH panel is shown in Fig. 5, where f_{PC} is defined as^[39]

$$f_{\text{PC}} = \left(\frac{\omega_{\text{FG}} - \omega_{\text{UD}}}{\omega_{\text{UD}}} \right) \times 100\%. \quad (32)$$

It can be seen from Fig. 5 that among four CNT distribution types, only FG-X shows the positive of f_{PC} , while others show the negative of f_{PC} . These results indicate that only the FG-X panel has higher stiffness than the UD panel, while UD is the simplest distribution type.

4.2.3 Effect of CNT volume fractions

Figure 6 indicates that the fundamental frequencies of the PFG-CNTRC doubly curved shell panels strongly increase with the increase of CNT volume fractions. The panels are set by $a/h = 20$, $a/b = 1$, $R_x/a = 5$, FG-X in Opc condition and [p/0°/90°/0°/p] of configuration.

4.2.4 Effect of electrical boundary conditions

The natural frequency of laminated cross-ply FG-CNTRC doubly curved shell panels coupled with closed and open piezoelectric layers are shown in Fig. 7. The parameters of the panels in this example are set by $a/h = 20$; $a/b = 1$; $R_x/a = 5$; FG-X; $V_{\text{CNT}}^* = 28\%$. Figure 7 shows that the FG-CNTRC panels coupled with the Opc always vibrate with the higher value of frequencies compared to the FG-CNTRC panels coupled with the Clc because the Opc converts electric potential to mechanical energy while the Clc does not.

Table 6 Fundamental frequencies ω of PFG-CNTRC-DCP ($a = b$, $R_x/a = 5$, and $V_{\text{CNT}}^* = 0.12$)

Shell panel	FG type	EBC	Laminate configuration					
			[p/0°/p]		[p/0°/90°/0°/p]		[p/(-45°/45°/-45°) _{as} /p]	
			$a/h = 20$	$a/h = 100$	$a/h = 20$	$a/h = 100$	$a/h = 20$	$a/h = 100$
Cylindrical (CYL, $R_y = \infty$)	UD	Clc	592.304	154.618	593.607	159.006	687.170	235.130
		Opc	626.968	161.568	628.015	165.064	717.108	241.601
	FG-A	Clc	567.175	147.982	589.313	156.416	687.365	235.617
		Opc	604.219	155.591	624.305	162.746	717.696	242.306
	FG-V	Clc	571.471	153.409	594.398	161.032	685.268	234.439
		Opc	607.158	160.109	628.652	166.879	714.979	240.706
	FG-X	Clc	630.839	161.458	599.683	160.097	692.114	236.894
		Opc	663.076	168.121	633.711	166.108	721.811	243.326
	FG-O	Clc	550.429	147.529	587.575	157.968	682.292	233.388
		Opc	588.048	154.803	622.373	164.072	712.477	239.897
Spherical (SPH, $R_y/b = 5$)	UD	Clc	614.685	227.995	618.604	239.508	761.658	405.509
		Opc	648.704	234.979	651.724	244.267	791.004	413.788
	FG-A	Cloc	586.781	221.745	611.821	236.324	762.835	406.178
		Opc	623.529	229.384	645.821	241.370	792.821	414.711
	FG-V	Clc	598.356	229.060	622.037	242.409	758.969	404.781
		Opc	632.817	235.592	654.682	246.917	787.815	412.812
	FG-X	Clc	651.893	232.777	624.478	240.418	767.182	408.539
		Opc	683.647	239.612	657.247	245.140	796.300	416.780
	FG-O	Clc	574.487	223.264	612.812	238.726	756.211	402.522
		Opc	611.225	230.398	646.288	243.522	785.789	410.839
Hyperbolic paraboloid (HPR, $R_y/b = -5$)	UD	Clc	582.399	120.108	582.436	120.108	656.344	136.697
		Opc	617.059	127.493	617.099	127.493	686.410	143.221
	FG-A	Clc	560.869	115.105	580.953	119.736	655.459	136.496
		Opc	597.416	122.836	615.839	127.165	685.614	143.038
	FG-V	Clc	557.348	114.953	580.372	119.711	655.459	136.496
		Opc	593.608	122.670	615.246	127.140	685.614	143.038
	FG-X	Clc	621.407	128.706	588.566	121.415	661.040	137.732
		Opc	653.585	135.620	622.840	128.725	690.861	144.209
	FG-O	Clc	539.908	110.864	576.336	118.804	651.711	135.669
		Opc	577.610	118.829	611.398	126.266	682.029	142.240

Fig. 4 Effects of distribution types of CNT on ω of PFG-CNTRC-DCP for different R_x/R_y ratio ([p/0°/90°/0°/p], $a/h = 20$, $a/b = 1$, $R_x/a = 5$, and $V_{\text{CNT}}^* = 0.28$) (color online) **Fig. 5** Variation of PCF of PFG-CNTRC spherical shell panel with R_x/R_y ratio (color online)

Table 7 Fundamental frequencies ω of PFG-CNTRC-DCP ($a = b$, $R_x/a = 5$, and $V_{\text{CNT}}^* = 0.17$)

Shell panel	FG type	EBC	Laminate configuration					
			[p/0°/p]		[p/0°/90°/0°/p]		[p/(-45°/45°/-45°) _{as} /p]	
			$a/h = 20$	$a/h = 100$	$a/h = 20$	$a/h = 100$	$a/h = 20$	$a/h = 100$
Cylindrical (CYL, $R_y = \infty$)	UD	Clc	633.572	164.396	635.074	169.546	758.173	269.051
		Opc	665.777	170.885	667.020	175.122	784.980	274.812
	FG-A	Clc	598.652	155.938	629.723	166.595	758.225	269.554
		Opc	633.715	163.155	662.292	172.434	785.445	275.530
	FG-V	Clc	603.036	161.691	635.332	171.781	755.967	268.333
		Opc	636.713	167.967	667.202	177.165	782.551	273.892
	FG-X	Clc	685.601	173.875	643.487	171.108	764.763	271.334
		Opc	714.948	180.017	674.977	176.629	791.309	277.054
	FG-O	Clc	575.813	154.583	626.931	168.188	751.889	266.853
		Opc	611.696	161.471	659.343	173.815	778.967	272.653
Spherical (SPH, $R_y/b = 5$)	UD	Clc	656.589	240.614	661.155	254.209	849.365	469.563
		Opc	688.210	247.117	691.855	258.431	875.501	476.964
	FG-A	Clc	619.120	233.156	653.114	250.730	850.514	470.304
		Opc	653.929	240.360	684.716	255.223	877.308	477.950
	FG-V	Clc	631.141	240.924	664.426	257.588	846.377	468.865
		Opc	663.602	246.972	694.739	261.581	872.015	476.024
	FG-X	Clc	706.953	247.435	669.331	255.599	856.660	473.443
		Opc	735.893	253.754	699.614	259.786	882.553	480.800
	FG-O	Clc	601.151	234.135	653.377	253.302	842.404	465.845
		Opc	636.156	240.819	684.501	257.557	868.792	473.287
Hyperbolic paraboloid (HPR $R_y/b = -5$)	UD	Clc	623.421	128.847	623.462	128.848	720.586	150.857
		Opc	655.611	135.726	655.656	135.726	747.584	156.763
	FG-A	Clc	592.150	121.654	621.209	128.289	719.455	150.584
		Opc	626.700	128.968	653.684	135.224	746.555	156.509
	FG-V	Clc	588.329	121.488	620.521	128.260	719.454	150.584
		Opc	622.596	128.789	652.991	135.194	746.555	156.509
	FG-X	Clc	676.025	140.546	631.932	130.654	726.872	152.244
		Opc	705.298	146.873	663.660	137.441	753.602	158.098
	FG-O	Clc	564.824	116.046	615.213	127.073	714.596	149.505
		Opc	600.798	123.644	647.887	134.043	741.873	155.462

Fig. 6 Effect of V_{CNT}^* on fundamental frequencies ω of PFG-CNTRC shell panel (color online) **Fig. 7** Effect of electrical boundary conditions on ω of PFG-CNTRC-DCP (color online)

4.2.5 Effect of curvature of shell panels

The effect of radii of curvature on the natural frequencies of the SPH, CYL, and HPR shell panels are shown in Figs. 8 to 10, respectively. It can be seen from these figures that with the

Table 8 Fundamental frequencies ω of PFG-CNTRC-DCP ($a = b$, $R_x/a = 5$, and $V_{\text{CNT}}^* = 0.28$)

Shell panel	FG type	EBC	Laminate configuration					
			[p/0°/p]		[p/0°/90°/0°/p]		[p/(-45°/45°/-45°) _{as} /p]	
			$a/h = 20$	$a/h = 100$	$a/h = 20$	$a/h = 100$	$a/h = 20$	$a/h = 100$
Cylindrical (CYL, $R_y = \infty$)	UD	Clc	696.079	176.288	697.771	182.254	869.286	323.219
		Opc	724.661	182.264	726.093	187.293	891.715	328.106
	FG-A	Clc	645.376	165.016	691.385	179.245	869.397	323.703
		Opc	677.457	171.831	720.359	184.516	892.263	328.795
	FG-V	Clc	648.885	170.533	696.794	184.631	867.253	322.606
		Opc	679.588	176.355	725.199	189.519	889.487	327.300
	FG-X	Clc	770.002	190.798	710.510	184.890	878.859	326.396
		Opc	795.215	196.326	738.290	189.858	901.030	331.240
	FG-O	Clc	611.394	161.438	686.367	180.466	861.225	320.333
		Opc	644.696	167.950	715.260	185.559	883.954	325.259
Spherical (SPH, $R_y/b = 5$)	UD	Clc	717.463	250.215	722.558	266.395	987.409	570.737
		Opc	745.644	256.334	749.826	270.100	1009.173	577.071
	FG-A	Clc	665.080	241.344	713.761	263.385	988.655	571.585
		Opc	697.073	248.260	741.898	267.320	1011.083	578.149
	FG-V	Clc	675.734	248.955	724.873	270.667	984.732	570.292
		Opc	705.374	254.601	751.949	274.180	1006.032	576.397
	FG-X	Clc	789.565	261.511	735.169	269.205	997.876	576.086
		Opc	814.537	267.361	761.933	272.869	1019.404	582.372
	FG-O	Clc	635.961	240.559	711.788	265.641	978.484	565.926
		Opc	668.532	246.925	739.573	269.371	1000.521	572.303
Hyperbolic paraboloid (HPR $R_y/b = -5$)	UD	Clc	686.512	142.888	686.568	142.888	820.668	174.254
		Opc	715.023	149.054	715.082	149.054	843.322	179.332
	FG-A	Clc	638.958	131.868	683.163	141.999	819.621	173.881
		Opc	670.478	138.592	712.050	148.236	842.404	178.979
	FG-V	Clc	634.839	131.687	682.322	141.962	819.621	173.881
		Opc	666.077	138.397	711.212	148.199	842.404	178.979
	FG-X	Clc	761.021	159.902	699.305	145.597	829.855	176.247
		Opc	786.116	165.428	727.267	151.652	852.241	181.268
	FG-O	Clc	600.719	123.820	674.977	140.327	812.987	172.392
		Opc	634.061	130.896	704.078	146.602	835.950	177.524

increase of R_x/a ratio, the frequencies of SPH and CYL decrease while the frequencies of HPR increase. The frequencies of all three types of shell panels are approximately equal to those of the plate with the corresponding input parameters when the value of R_x/a ratio reaches 20. This observation once again confirms that the opposite curvature will reduce the stiffness of the shells.

4.2.6 Effect of piezoelectric layer thickness

In order to investigate the effect of piezoelectric layer thickness on the free vibration response of the composite shell panel, the variation of percentage difference in natural frequency β is defined as^[40]

$$\beta = \frac{\omega_{\text{with piezoelectric layer}} - \omega_{\text{without piezoelectric layer}}}{\omega_{\text{without piezoelectric layer}}} \times 100\%. \quad (33)$$

Figures 11, 12, and 13 present the percentage difference in natural frequency β versus the h_p/h ratios for different forms of the panel, distribution types of CNT, and CNT volume fractions. It is observed from these figures that the HPR panel has a higher value of β than CYL and SPH panel, the panel reinforced with lower CNT volume fractions has a higher value of β , among five CNT distribution types, the FG-O panel has the highest value of β while the FG-X has lowest one. It can be concluded that the piezoelectric effects are more effective in the case of smaller stiffness panels rather than the case of greater stiffness ones. Furthermore,

Fig. 8 Effect of radius of curvature on ω of PFG-CNTRC spherical shell panels ($a/h = 20$, $a/b = 1$, $V_{\text{CNT}}^* = 0.28$, $R_y = R_x$, $[p/0^\circ/90^\circ/0^\circ/p]$) (color on-line)

Fig. 9 Effect of radius of curvature on ω of PFG-CNTRC cylindrical shell panels ($a/h = 20$, $a/b = 1$, $V_{\text{CNT}}^* = 0.28$, $R_y = \infty$, $[p/0^\circ/90^\circ/0^\circ/p]$) (color on-line)

Fig. 10 Effect of radius of curvature on ω of PFG-CNTRC HPR shell panels ($a/h = 20$, $a/b = 1$, $V_{\text{CNT}}^* = 0.28$, $R_y = -R_x$, $[p/0^\circ/90^\circ/0^\circ/p]$) (color online)

Fig. 11 Variation of β of the PFG-CNTRC-DCP versus h_p/h ratios for electrical boundary conditions ($a/h = 20$, $a/b = 1$, $R_x/a = 5$, FG-X, $V_{\text{CNT}}^* = 0.28$, $[0^\circ/90^\circ/0^\circ]$) (color online)

Fig. 12 Variation of β of spherical panel versus h_p/h ratios for different CNT distribution types ($a/h = 20$, $a/b = 1$, $R_x/a = R_y/b = 5$, $V_{\text{CNT}}^* = 0.28$, $[p/0^\circ/90^\circ/0^\circ/p]$) (color online)

Fig. 13 Variation of β of spherical panel versus h_p/h ratios for different CNT volume fractions ($a/h = 20$, $a/b = 1$, $R_x/a = R_y/b = 5$, FG-X, $[p/0^\circ/90^\circ/0^\circ/p]$) (color online)

these figures also indicate that increasing the thickness of piezoelectric layers from zero to a specific value leads to a decrement of β . It can be explained by the fact that the piezoelectric material has a higher mass density and lower elastic moduli compared with the core material. After that specific value of h_p/h , the percentage difference in natural frequency β increases by the increment of piezoelectric layers thickness because the electromechanical coupling effect increases with an increase in h_p/h and should increase the value of β . The electromechanical coupling effect is lower than the combined effects of the increase in the mass density, and the decrease in the stiffness results in negative values of β for values of h_p/h .

5 Conclusions

In this paper, an analytical solution based on the four-variable shear deformation refined theory is developed for carrying out the free vibration of the laminated functionally graded nanotube-reinforced composite doubly curved shell panels with surface-bonded piezoelectric layers. Comparison studies validate the accuracy of the model. Numerical results are provided to explore the effects of the CNT volume fraction, the CNT distribution type, the thickness of the piezoelectric layers, laminate configurations, and mechanical and electrical boundary conditions on the natural frequencies of the hybrid panels. Through the present formulation and numerical results, some conclusions can be drawn. (i) The volume fraction of CNT has substantial effects on dynamic responses of PFG-CNTRC panels. (ii) The natural frequencies of PFG-CNTRC panel with Opc electrical boundary condition are always higher than those with the Clc case when all other inlet parameters are the same. (iii) CNT reinforcements distributed close to top and bottom are more efficient than those distributed near the mid-plane for increasing the stiffness of the panels.

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Appendix A

$$\begin{aligned} \begin{pmatrix} N_{xx} \\ N_{xy} \end{pmatrix} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^k \\ \tau_{xy}^k \end{pmatrix} dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^{\text{pie}} \\ \tau_{xy}^{\text{pie}} \end{pmatrix}^B dz \\ &\quad - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^{\text{pie}} \\ \tau_{xy}^{\text{pie}} \end{pmatrix}^T dz, \end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} N_{yy} \\ N_{yx} \end{pmatrix} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^k \\ \tau_{yx}^k \end{pmatrix} dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^{\text{pie}} \\ \tau_{yx}^{\text{pie}} \end{pmatrix}^B dz \\
&\quad - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^{\text{pie}} \\ \tau_{yx}^{\text{pie}} \end{pmatrix}^T dz, \\
\begin{pmatrix} M_{xx}^b \\ M_{xy}^b \end{pmatrix} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^k \\ \sigma_{xy}^k \end{pmatrix} z dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^{\text{pie}} \\ \tau_{xy}^{\text{pie}} \end{pmatrix}^B z dz \\
&\quad - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^{\text{pie}} \\ \tau_{xy}^{\text{pie}} \end{pmatrix}^T z dz, \\
\begin{pmatrix} M_{yy}^b \\ M_{yx}^b \end{pmatrix} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^k \\ \sigma_{yx}^k \end{pmatrix} z dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^{\text{pie}} \\ \tau_{yx}^{\text{pie}} \end{pmatrix}^B z dz \\
&\quad - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^{\text{pie}} \\ \tau_{yx}^{\text{pie}} \end{pmatrix}^T z dz, \\
\begin{pmatrix} M_{xx}^s \\ M_{xy}^s \end{pmatrix} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^k \\ \sigma_{xy}^k \end{pmatrix} f(z) dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^{\text{pie}} \\ \tau_{xy}^{\text{pie}} \end{pmatrix}^B f(z) dz \\
&\quad - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_y}\right) \begin{pmatrix} \sigma_{xx}^{\text{pie}} \\ \tau_{xy}^{\text{pie}} \end{pmatrix}^T f(z) dz, \\
\begin{pmatrix} M_{yy}^s \\ M_{yx}^s \end{pmatrix} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^k \\ \sigma_{yx}^k \end{pmatrix} f(z) dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^{\text{pie}} \\ \tau_{yx}^{\text{pie}} \end{pmatrix}^B f(z) dz \\
&\quad - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_x}\right) \begin{pmatrix} \sigma_{yy}^{\text{pie}} \\ \tau_{yx}^{\text{pie}} \end{pmatrix}^T f(z) dz, \\
Q_{xs} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_y}\right) \tau_{xz}^k dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_y}\right) \tau_{xz}^{\text{pieB}} dz - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_y}\right) \tau_{xz}^{\text{pieT}} dz, \\
Q_{ys} &= \sum_{k=1}^n \int_{z_k}^{z_{k+1}} \left(1 + \frac{z}{R_x}\right) \tau_{yz}^k dz - \int_{-h/2-h_p}^{-h/2} \left(1 + \frac{z}{R_x}\right) \tau_{yz}^{\text{pieB}} dz - \int_{h/2}^{h/2+h_p} \left(1 + \frac{z}{R_x}\right) \tau_{yz}^{\text{pieT}} dz.
\end{aligned}$$

Appendix B1

$$\begin{aligned}
m_{11} &= -\left(\bar{I}_0 + 2\frac{\bar{I}_1}{R_x} + \frac{\bar{I}_2}{R_x^2}\right), \quad m_{12} = 0, \quad m_{13} = \left(\bar{I}_1 + \frac{\bar{I}_2}{R_x}\right)\alpha_m, \quad m_{14} = \left(\bar{J}_1 + \frac{\bar{J}_2}{R_x}\right)\alpha_m, \\
m_{22} &= -\left(\bar{I}_0 + \frac{\bar{I}_1}{R_y} + \frac{\bar{I}_2}{R_y^2}\right), \quad m_{23} = \left(\bar{I}_1 + \frac{\bar{I}_2}{R_y}\right)\beta_n, \quad m_{24} = \left(\bar{J}_1 + \frac{\bar{J}_2}{R_y}\right)\beta_n, \\
m_{33} &= -\bar{I}_0 - \bar{I}_2(\alpha_m^2 + \beta_n^2), \quad m_{34} = -\bar{I}_0 - \bar{J}_2(\alpha_m^2 + \beta_n^2), \quad m_{44} = -\bar{I}_0 - \bar{K}_1(\alpha_m^2 + \beta_n^2), \\
k_{11} &= -\left(\bar{A}_{11} + 2\frac{\bar{B}_{11}}{R_x} + \frac{\bar{D}_{11}}{R_x^2}\right)\alpha_m^2 - \left(\hat{A}_{66} + 2\frac{\hat{B}_{66}}{R_x} + \frac{\hat{D}_{66}}{R_x^2}\right)\beta_n^2, \\
k_{12} &= -(A_{12} + A_{66} + (B_{12} + B_{66})\left(\frac{1}{R_x} + \frac{1}{R_y}\right) + \frac{1}{R_x R_y}(D_{12} + D_{66}))\beta_n, \\
k_{13} &= \left(\frac{\bar{A}_{11}}{R_x} + \frac{A_{12}}{R_y} + \frac{1}{R_x}\left(\frac{\bar{B}_{11}}{R_x} + \frac{B_{12}}{R_y}\right) + (B_{12} + B_{66} + \hat{B}_{66} + \frac{1}{R_x}(D_{12} + D_{66} + \hat{D}_{66}))\beta_n^2\right)\alpha_m \\
&\quad + \left(\bar{B}_{11} + \frac{\bar{D}_{11}}{R_x}\right)\alpha_m^3, \\
k_{14} &= \left(\bar{B}_{11} + \frac{\bar{D}_{11}}{R_x}\right)\alpha_m^3 + \left((B_{12}^s + B_{66}^s + \hat{B}_{66}^s + \frac{D_{66}^s}{R_x} + \frac{\hat{D}_{66}^s}{R_x} + \frac{D_{12}^s}{R_x})\beta_n^2\right. \\
&\quad \left.+ \frac{\bar{A}_{11}}{R_x} + \frac{A_{12}}{R_y} + \frac{\bar{B}_{11}}{R_x^2} + \frac{B_{12}}{R_x R_y}\right)\alpha_m,
\end{aligned}$$

$$\begin{aligned}
k_{15} &= -\left(A_{11}^p + \frac{B_{11}^p}{R_x}\right)\alpha_m, \quad k_{16} = -\left(A_{12}^p + \frac{B_{12}^p}{R_x}\right)\alpha_m, \\
k_{22} &= -\left(\bar{A}_{66} + 2\frac{\bar{B}_{66}}{R_y} + \frac{\bar{D}_{66}}{R_y^2}\right)\alpha_m^2 - \left(\hat{A}_{22} + 2\frac{\hat{B}_{22}}{R_y} + \frac{\hat{D}_{22}}{R_y^2}\right)\beta_n^2, \\
k_{23} &= \left(\frac{A_{12}}{R_x} + \frac{\hat{A}_{22}}{R_y} + \frac{1}{R_y}\left(\frac{B_{12}}{R_x} + \frac{\hat{B}_{22}}{R_y}\right)\right)\beta_n + \left(\hat{B}_{22} + \frac{\hat{D}_{22}}{R_y}\right)\beta_n^3 \\
&\quad + \left(B_{12} + B_{66} + \bar{B}_{66} + \frac{1}{R_y}(D_{12} + D_{66} + \bar{D}_{66})\right)\beta_n\alpha_m^2, \\
k_{24} &= \left(\frac{A_{12}}{R_x} + \frac{\hat{A}_{22}}{R_y} + \frac{1}{R_y}\left(\frac{B_{12}}{R_x} + \frac{\hat{B}_{22}}{R_y}\right)\right)\beta_n + \left(\hat{B}_{22}^s + \frac{\hat{D}_{22}^s}{R_y}\right)\beta_n^3 \\
&\quad + \left(B_{12}^s + B_{66}^s + \bar{B}_{66}^s + \frac{1}{R_y}(D_{12}^s + D_{66}^s + \bar{D}_{66}^s)\right)\beta_n\alpha_m^2, \\
k_{25} &= -\left(A_{21}^p + \frac{B_{21}^p}{R_y}\right)\beta_n, \quad k_{26} = -\left(A_{22}^p + \frac{B_{22}^p}{R_y}\right)\beta_n, \\
k_{33} &= -\frac{\bar{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\hat{A}_{22}}{R_y^2} - 2\left(\frac{B_{12}}{R_x} + \frac{\hat{B}_{22}}{R_y}\right)\beta_n^2 - 2\left(\frac{\bar{B}_{11}}{R_x} + \frac{B_{12}}{R_y}\right)\alpha_m^2 \\
&\quad - \bar{D}_{11}\alpha_m^4 - (2D_{12} + 2D_{66} + \bar{D}_{66} + \hat{D}_{66})\alpha_m^2\beta_n^2 - \hat{D}_{22}\beta_n^4, \\
k_{34} &= -\frac{\bar{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\hat{A}_{22}}{R_y^2} - \left(\frac{\bar{B}_{11}}{R_x} + \frac{\bar{B}_{11}^s}{R_x} + \frac{B_{12}}{R_y} + \frac{B_{12}^s}{R_y}\right)\alpha_m^2 - \left(\frac{B_{12}}{R_x} + \frac{B_{12}^s}{R_x} + \frac{\hat{B}_{22}}{R_y} + \frac{\hat{B}_{22}^s}{R_y}\right)\beta_n^2 \\
&\quad - (2D_{12}^s + 2D_{66}^s + \bar{D}_{66}^s + \hat{D}_{66}^s)\alpha_m^2\beta_n^2 - \bar{D}_{11}^s\alpha_m^4 - \hat{D}_{22}^s\beta_n^4, \\
k_{35} &= B_{21}^p\beta_n^2 + B_{11}^p\alpha_m^2 + \frac{A_{21}^p}{R_y} + \frac{A_{11}^p}{R_x}, \quad k_{36} = \frac{A_{12}^p}{R_x} + \frac{A_{22}^p}{R_y} + B_{12}^p\alpha_m^2 + B_{22}^p\beta_n^2, \\
k_{44} &= -\frac{\bar{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\hat{A}_{22}}{R_y^2} - \left(\hat{A}_{44} + 2\frac{B_{12}^s}{R_x} + 2\frac{\hat{B}_{22}^s}{R_y}\right)\beta_n^2 - \left(\bar{A}_{55} + 2\frac{\bar{B}_{11}^s}{R_x} + 2\frac{B_{12}^s}{R_y}\right)\alpha_m^2 \\
&\quad - (2E_{12}^s + 2E_{66}^s + \bar{E}_{66}^s + \hat{E}_{66}^s)\alpha_m^2\beta_n^2 - \bar{E}_{11}^s\alpha_m^4 - \hat{E}_{22}^s\beta_n^4, \\
k_{45} &= \frac{A_{21}^p}{R_y} + \frac{A_{11}^p}{R_x} + (D_{21}^p + C_{11}^p)\alpha_m^2 + (C_{21}^p + D_{11}^p)\beta_n^2, \\
k_{46} &= \frac{A_{12}^p}{R_x} + \frac{A_{22}^p}{R_y} + (C_{12}^p + D_{22}^p)\alpha_m^2 + (D_{12}^p + C_{22}^p)\beta_n^2, \\
k_{55} &= -D_{x12}^p\alpha_m^2 - D_{y12}^p\beta_n^2 + Q^{pT}, \quad k_{56} = 0, \quad k_{66} = -D_{x22}^p\alpha_m^2 - D_{y22}^p\beta_n^2 + Q^{pB}.
\end{aligned}$$

Appendix B2

$$\begin{aligned}
m_{11} &= -\left(\bar{I}_0 + 2\frac{\bar{I}_1}{R_x} + \frac{\bar{I}_2}{R_x^2}\right), \quad m_{12} = m_{13} = m_{14} = 0, \quad m_{22} = -\left(\bar{I}_0 + \frac{\bar{I}_1}{R_y} + \frac{\bar{I}_2}{R_y^2}\right), \\
m_{23} &= m_{24} = 0, \quad m_{33} = -\bar{I}_0 - \bar{I}_2(\alpha_m^2 + \beta_n^2), \quad m_{34} = -\bar{I}_0 - \bar{I}_2(\alpha_m^2 + \beta_n^2), \quad m_{44} = -\bar{I}_0 - \bar{K}_1(\alpha_m^2 + \beta_n^2), \\
k_{11} &= -\left(\bar{A}_{11} + 2\frac{\bar{B}_{11}}{R_x} + \frac{\bar{D}_{11}}{R_x^2}\right)\alpha_m^2 + \left(-2\frac{\bar{B}_{66}}{R_x} - \frac{\hat{D}_{66}}{R_x^2} - \hat{A}_{66}\right)\beta_n^2, \\
k_{12} &= -\left(A_{12} + A_{66} + (B_{12} + B_{66})\left(\frac{1}{R_x} + \frac{1}{R_y}\right) + \frac{1}{R_x R_y}(D_{12} + D_{66})\right)\beta_n\alpha_m, \\
k_{13} &= \left(\frac{A_{16}}{R_x} + \frac{\hat{A}_{16}}{R_y} + \frac{1}{R_x}\left(\frac{B_{16}}{R_x} + \frac{\hat{B}_{26}}{R_y}\right) + \left(2B_{16} + \bar{B}_{16} + 2\frac{D_{16}}{R_x} + \frac{\bar{D}_{16}}{R_x}\right)\beta_n\alpha_m^2\right)\beta_n + \left(\hat{B}_{26} + \frac{\hat{D}_{26}}{R_x}\right)\beta_n^3, \\
k_{14} &= \left(\frac{\hat{D}_{26}^s}{R_x} + \hat{B}_{26}^s\right)\beta_n^3 + \left(\left(2B_{16}^s + \bar{B}_{16}^s + \frac{\bar{D}_{16}^s}{R_x} + 2\frac{D_{16}^s}{R_x}\right)\alpha_m^2 + \frac{A_{16}}{R_x} + \frac{\hat{A}_{26}}{R_y} + \frac{B_{16}}{R_x^2} + \frac{\hat{B}_{26}}{R_x R_y}\right)\beta_n, \\
k_{15} &= 0, \quad k_{16} = 0, \quad k_{22} = -\left(\bar{A}_{66} + 2\frac{\bar{B}_{66}}{R_y} + \frac{\bar{D}_{66}}{R_y^2}\right)\alpha_m^2 - \left(\hat{A}_{22} + 2\frac{\hat{B}_{22}}{R_y} + \frac{\hat{D}_{22}}{R_y^2}\right)\beta_n^2,
\end{aligned}$$

$$\begin{aligned}
k_{23} &= \left(\frac{\bar{A}_{16}}{R_x} + \frac{A_{26}}{R_y} + \frac{1}{R_y} \left(\frac{B_{26}}{R_y} + \frac{\bar{B}_{16}}{R_x} \right) \right) \alpha_m + \left(\frac{\bar{D}_{16}}{R_y} + \bar{B}_{16} \right) \alpha_m^3 + \left(2B_{26} + \hat{B}_{26} + \frac{\hat{D}_{26}}{R_y} + 2\frac{D_{26}}{R_y} \right) \beta_n^2 \alpha_m, \\
k_{24} &= \left(\frac{\bar{A}_{16}}{R_x} + \frac{A_{26}}{R_y} + \frac{1}{R_y} \left(\frac{B_{26}}{R_y} + \frac{\bar{B}_{16}}{R_x} \right) \right) \alpha_m + \left(\frac{\bar{D}_{16}^s}{R_y} + \bar{B}_{16}^s \right) \alpha_m^3 + \left(2B_{26}^s + \hat{B}_{26}^s + \frac{1}{R_y} (D_{26}^s + \hat{D}_{26}^s) \right) \alpha_m \beta_n^2, \\
k_{25} &= 0, \quad k_{26} = 0, \\
k_{33} &= -\frac{\bar{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\hat{A}_{22}}{R_y^2} - 2\left(\frac{B_{12}}{R_x} + \frac{\hat{B}_{22}}{R_y} \right) \beta_n^2 - 2\left(\frac{\bar{B}_{11}}{R_x} + \frac{B_{12}}{R_y} \right) \alpha_m^2 \\
&\quad - \bar{D}_{11} \alpha_m^4 - (2D_{12} + 2D_{66} + \bar{D}_{66} + \hat{D}_{66}) \alpha_m^2 \beta_n^2 - \hat{D}_{22} \beta_n^4, \\
k_{34} &= -\frac{\bar{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\hat{A}_{22}}{R_y^2} - \left(\frac{\bar{B}_{11}}{R_x} + \frac{\bar{B}_{11}^s}{R_x} + \frac{B_{12}}{R_y} + \frac{B_{12}^s}{R_y} \right) \alpha_m^2 \\
&\quad - \left(\frac{B_{12}}{R_x} + \frac{B_{12}^s}{R_x} + \frac{\hat{B}_{22}}{R_y} + \frac{\hat{B}_{22}^s}{R_y} \right) \beta_n^2 - \left(2D_{12}^s + 2D_{66}^s + \bar{D}_{66}^s + \hat{D}_{66}^s \right) \alpha_m^2 \beta_n^2 - \bar{D}_{11}^s \alpha_m^4 - \hat{D}_{22}^s \beta_n^4, \\
k_{35} &= B_{21}^p \beta_n^2 + B_{11}^p \alpha_m^2 + \frac{A_{21}^p}{R_y} + \frac{A_{11}^p}{R_x}, \quad k_{36} = \frac{A_{12}^p}{R_x} + \frac{A_{22}^p}{R_y} + B_{12}^p \alpha_m^2 + B_{22}^p \beta_n^2, \\
k_{44} &= -\frac{\bar{A}_{11}}{R_x^2} - 2\frac{A_{12}}{R_x R_y} - \frac{\hat{A}_{22}}{R_y^2} - \left(\hat{A}_{44} + 2\frac{B_{12}^s}{R_x} + 2\frac{\hat{B}_{22}^s}{R_y} \right) \beta_n^2 - \left(\bar{A}_{55} + 2\frac{\bar{B}_{11}^s}{R_x} + 2\frac{B_{12}^s}{R_y} \right) \alpha_m^2 \\
&\quad - (2E_{12}^s + 2E_{66}^s + \bar{E}_{66}^s + \hat{E}_{66}^s) \alpha_m^2 \beta_n^2 - \bar{E}_{11}^s \alpha_m^4 - \hat{E}_{22}^s \beta_n^4, \\
k_{45} &= \frac{A_{21}^p}{R_y} + \frac{A_{11}^p}{R_x} + (D_{21}^p + C_{11}^p) \alpha_m^2 + (C_{21}^p + D_{11}^p) \beta_n^2, \\
k_{46} &= \frac{A_{12}^p}{R_x} + \frac{A_{22}^p}{R_y} + (C_{12}^p + D_{22}^p) \alpha_m^2 + (D_{12}^{hp} + C_{22}^p) \beta_n^2, \\
k_{55} &= -D_{x12}^p \alpha_m^2 - D_{y12}^p \beta_n^2 + Q^{pT}, \quad k_{56} = 0, \quad k_{66} = -D_{x22}^p \alpha_m^2 - D_{y22}^p \beta_n^2 + Q^{pB},
\end{aligned}$$

where

$$\begin{aligned}
A_{11}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_y} \right) \bar{e}_{31} \frac{-\partial Z_p^t}{\partial z} \phi^t dz, \quad A_{12}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_y} \right) \bar{e}_{31} \frac{-\partial Z_p^d}{\partial z} \phi^d dz, \\
A_{21}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_x} \right) \bar{e}_{32} \frac{-\partial Z_p^t}{\partial z} \phi^t dz, \quad A_{22}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_x} \right) \bar{e}_{32} \frac{-\partial Z_p^d}{\partial z} \phi^d dz, \\
B_{11}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_y} \right) \bar{e}_{31} \frac{-\partial Z_p^t}{\partial z} \phi^t z dz, \quad B_{12}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_y} \right) \bar{e}_{31} \frac{-\partial Z_p^d}{\partial z} \phi^d z dz, \\
B_{21}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_x} \right) \bar{e}_{32} \frac{-\partial Z_p^t}{\partial z} \phi^t z dz, \quad B_{22}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_x} \right) \bar{e}_{32} \frac{-\partial Z_p^d}{\partial z} \phi^d z dz, \\
C_{11}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_y} \right) \bar{e}_{31} \frac{-\partial Z_p^t}{\partial z} \phi^t z^2 dz, \quad C_{12}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_y} \right) \bar{e}_{31} \frac{-\partial Z_p^d}{\partial z} \phi^d z^2 dz, \\
C_{21}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_x} \right) \bar{e}_{32} \frac{-\partial Z_p^t}{\partial z} \phi^t z^2 dz, \quad C_{22}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_x} \right) \bar{e}_{32} \frac{-\partial Z_p^d}{\partial z} \phi^d z^2 dz, \\
D_{11}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_x} \right) g(z) e_{24} \frac{-1}{\left(1 + \frac{z}{R_y} \right)} Z_p^t dz, \quad D_{12}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_x} \right) g(z) e_{24} \frac{-1}{\left(1 + \frac{z}{R_y} \right)} Z_p^d dz, \\
D_{21}^p &= \int_{\frac{h_c}{2}}^{\frac{h_c}{2} + h_p} \left(1 + \frac{z}{R_y} \right) g(z) e_{15} \frac{-1}{\left(1 + \frac{z}{R_x} \right)} Z_p^t dz, \quad D_{22}^p = \int_{-\frac{h_c}{2} - h_p}^{-\frac{h_c}{2}} \left(1 + \frac{z}{R_y} \right) g(z) e_{15} \frac{-1}{\left(1 + \frac{z}{R_x} \right)} Z_p^d dz.
\end{aligned}$$