Surface effects on a mode-III reinforced nano-elliptical hole embedded in one-dimensional hexagonal piezoelectric quasicrystals*

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Abstract To effectively reduce the field concentration around a hole or crack, an anti-plane shear problem of a nano-elliptical hole or a nano-crack pasting a reinforcement layer in a one-dimensional (1D) hexagonal piezoelectric quasicrystal (PQC) is investigated subject to remotely mechanical and electrical loadings. The surface effect and dielectric characteristics inside the hole are considered for actuality. By utilizing the technique of conformal mapping and the complex variable method, the phonon stresses, phason stresses, and electric displacements in the matrix and reinforcement layer are exactly derived under both electrically permeable and impermeable boundary conditions. Three size-dependent field intensity factors near the nano-crack tip are further obtained when the nano-elliptical hole is reduced to the nano-crack. Numerical examples are illustrated to show the effects of material properties of the surface layer and reinforced layer, the aspect ratio of the hole, and the thickness of the reinforcing layer on the field concentration of the nano-elliptical hole and the field intensity factors near the nano-crack tip. The results indicate that the properties of the surface layer and reinforcement layer and the electrical boundary conditions have great effects on the field concentration of the nano-hole and nano-crack, which are useful for optimizing and designing the microdevices by PQC nanocomposites in engineering practice.

Key words surface effect, reinforcement layer, exact solution, piezoelectric quasicrystal (PQC), nano-hole/crack

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1 Introduction

Different from traditional forms of solid state, quasicrystals (QCs) exhibit a conventionally forbidden rotational symmetry, incompatible with translational periodicity. The discovery of QCs since 1984 has changed people's inherent understanding of solid structures^[1]. A lot of work has been done on the structures, optics, and mechanical properties of QCs^[2-5]. The research shows that QCs have the characteristics of low friction coefficient, low thermal conductivity, wear resistance, corrosion resistance, and plasticity at high temperature^[6-8]. Based on these properties, QCs are widely used in surface coatings, thermal insulation materials, solar energy, industrial thin film materials, and reinforced phase of structural materials^[9-10]. It is found experimentally that at room temperature, QCs are so brittle that cracks or holes may appear during the manufacturing process^[11]. The presence of defects can affect the physical and mechanical properties of QCs. Therefore, it is of significance to investigate the fracture behavior of QCs with defects.

Based on the classical elasticity theory, a lot of research has been done on the defects of QCs. Wang and Pan^[12] studied some typical defect problems in one-dimensional (1D) hexagonal and two-dimensional (2D) octagonal QCs. Li et al.^[13] derived a set of three-dimensional (3D) general solutions of 1D hexagonal piezoelectric quasicrystals (PQCs) by introducing two displacement functions and utilizing the rigorous operator theory. Yang and Li^[14] investigated the anti-plane shear problem of a circular hole with a straight crack in 1D hexagonal PQCs by utilizing the complex variable function method and the technique of conformal mapping. Furthermore, Bai and Ding^[15] presented the analytical solutions of stress distributions and field intensity factors at the crack tip near the regular hexagonal hole in 1D hexagonal PQCs under the electrically impermeable boundary condition. Jiang and Liu^[16] considered the interaction between a screw dislocation and a wedge-shaped crack in 1D hexagonal PQCs by the conformal mapping and perturbation techniques. Based on the local Petrov-Galerkin approach, Sladek et al.^[17] proposed a meshless method to solve the initial-boundary-value crack problems in decagonal QCs. Wang and Ricoeur^[18] developed numerical tools in a finite element environment to compute the fracture quantities of 1D QCs and analyzed the phonon-phason coupling effect on crack paths. Based on the operator theory and the Fourier transform, Fan et al.^[19] proposed the extended displacement discontinuity method to analyze cracks in the periodic plane of 1D hexagonal QCs with the heat effect. Zhao et al.^[20] used this method to consider a 3D interface crack of arbitrary shapes in a 1D hexagonal thermo-electro-elastic QC bi-material. Tupholme^[21] derived the closed-form solutions of a moving non-constantly loaded anti-plane Griffith-type strip crack embedded in 1D PQCs. Zhou and Li^[22] considered a Yoffe-type moving crack in 1D hexagonal PQCs by using the Fourier transform technique. By using the integral transform technique, Hu et al.^[23] studied an interface crack between dissimilar 1D hexagonal PQCs under anti-plane shear and in-plane electrical loadings. Li et al.^[24] analyzed the interaction between a screw dislocation and an elliptical hole with two asymmetrical cracks in 1D hexagonal PQCs. Loboda et al.^[25] presented an analytical approach to an electrically permeable interface crack in 1D PQCs. Also, the elastic field near the tip of an anticrack^[26] and the arbitrarily shaped planar crack^[27] in 2D QCs were analyzed.

Although a sufficiently large number of solutions associated with defects have been obtained for QCs based on the classical elasticity theory, very few investigations have been known for nano-QCs. Nano-QC composites normally produced by rapid solidification (melt-spinning or gas atomization) containing icosahedral QC particles with typically size under 500 nm^[28] are attractive potential structural materials in the automotive and aeronautical industries due to their high strength at elevated temperatures^[29]. Experimental observation showed that decreasing the reinforcement particle size to nanoscale could improve both the mechanical strength and the ductility of metal matrix composites^[30]. Defect problems in nano-QCs are more complicated than those in macro-scale due to the introduction of the non-classical boundary

conditions. The surface/interface effect should be considered in the analysis of nano-structures due to the increasing ratio of surface to bulk volume. Based on the surface/interface model, some defects in piezoelectric materials were analyzed^[31–32]. However, very few studies on defects in nano-QCs and nano-PQCs have been done up to now.

The stress concentration around the hole or crack can be effectively reduced by pasting the reinforcement layer with proper shapes and material characteristics^[33–35]. Therefore, this paper focuses on an anti-plane shear problem of a reinforced nano-elliptical hole or nano-crack with surface effects in a 1D hexagonal PQC composite to adjust the phonon stress concentration, phason stress concentration, and electric displacement concentration around the hole or crack, which may provide a theoretical reference for designing new composites in engineering practice. This paper is organized as follows. In Section 2, we present the problem description and basic equations for an anti-plane shear problem of 1D hexagonal PQC composites with surface effects. In Section 3, the size-dependent phonon stress, phason stress, and electric displacement in the reinforced layer and matrix are derived exactly. Numerical examples are illustrated in Section 4, and conclusions are drawn in Section 5.

2 Problem description and basic equations

An anti-plane shear problem of an electrically permeable reinforced nano-elliptical hole in 1D hexagonal PQCs is considered (see Fig. 1(a)). The regions Ω_E , Ω_L , and Ω_M represent the nano-elliptical hole, the reinforcement layer, and the matrix, respectively. It is assumed that the reinforcement layer and matrix have different material properties but they are poled along the z -direction. According to the Gurtin-Murdoch surface/interface model^[36–37], the material properties of the holed surface without thickness are different from those of the reinforcement layer and matrix. For simplicity, we assume that the two ellipses surrounding the hole and the reinforcement layer are confocal with the common foci $(-c, c)$. (a_1, b_1) and (a_2, b_2) are the major semi-axes and minor semi-axes of the ellipse around the hole and the ellipse around the reinforcement layer, respectively. The interface along the ellipse between the reinforcement layer and the matrix is perfectly bonded.

To solve this problem, we introduce a conformal mapping^[31–32] $z = \omega(\zeta) = \zeta + n/\zeta$, with $n = (a_1^2 + b_1^2)/4$, which maps the region of matrix Ω_M in the z -plane into the exterior region Ω'_M of the circle with radius $\rho_2 = (a_2 + b_2)/2$ in the ζ -plane, the region of the reinforced layer Ω_L in the z -plane into the annular region $\rho_1 \leq |\zeta| < \rho_2$ in the ζ -plane, where $\rho_1 = (a_1 + b_1)/2$, and the internal region Ω_E of the nano-elliptical hole in the z -plane into the circular ring region Ω'_E ($\rho_0 \leq |\zeta| < \rho_1$) in the ζ -plane, where $\rho_0 = \sqrt{n}$, as shown in Fig. 1(b).

For simplicity, the generalized stresses (phonon stress, phason stress, and electric displacement) and the generalized strains (phonon strain, phason strain, and electric field) are denoted by the vectors $\Sigma_j = (\sigma_{zj}, H_{zj}, D_j)^T$ and $Z_j = (\varepsilon_{zj}, \omega_{zj}, -E_j)^T$, respectively, where $j = x, y$, and the superscript T denotes the transpose of the vector or matrix. Using the complex variable function method, an analytic function vector $\mathbf{f}(z)$ is introduced. Then, the phonon displacement u_z , the phason displacement w_z , and the electric potential ϕ for the current anti-plane shear problem are only functions of the variables x and y , and $\mathbf{u} = (u_z, w_z, \phi)^T = \text{Im}(\mathbf{f}(z))$.

The basic equations for the anti-plane shear problem of 1D hexagonal PQC materials are

$$\Sigma_y + i\Sigma_x = \mathbf{B}(Z_y + iZ_x) = \mathbf{B}\mathbf{f}'(z), \quad \mathbf{B}\nabla^2 \mathbf{u} = \mathbf{0}, \quad (1)$$

where $i^2 = -1$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a 2D Laplace operator, and $\mathbf{B}$ is the material coefficient matrix, i.e.,

$$\mathbf{B} = \begin{pmatrix} c_{44} & R_3 & e_{15} \\ R_3 & K_2 & d_{15} \\ e_{15} & d_{15} & -\lambda_{11} \end{pmatrix}, \quad (2)$$

Fig. 1 A reinforced nano-elliptical hole in an infinite 1D hexagonal PQC matrix in (a) the z -plane and (b) the ζ -plane (color online)

in which c_{44} , K_2 , and R_3 are the phonon elastic constant, the phason elastic constant, and the coupling elastic constant between the phonon and phason fields, respectively, e_{15} and d_{15} are the piezoelectric constants, and λ_{11} is the dielectric constant.

By utilizing the technique of conformal mapping, the relationship between the generalized stress and the generalized strain in the ζ -plane can be expressed as

$$\Sigma_\rho + i\Sigma_\theta = B(Z_\rho + iZ_\theta) = e^{i\theta} B \frac{f'(z)}{|\omega'(\zeta)|}, \quad (3)$$

where ρ and θ represent the polar coordinates in the ζ -plane.

For the convenience of analysis, the field resultant force $\mathbf{F}$ along the arc ab including the phonon field resultant force F_σ , the phason field resultant force F_H , and the sum of the normal components of the electric displacement F_D is also introduced, which can be expressed as

$$\begin{aligned} \mathbf{F} &= (F_\sigma \quad F_H \quad F_D)^T \\ &= \int_a^b \left((\sigma_{zx} \quad H_{zx} \quad D_x)^T dy - (\sigma_{zy} \quad H_{zy} \quad D_y)^T dx \right) = -B \operatorname{Re}(f(z))|_a^b. \end{aligned} \quad (4)$$

By considering the equilibrium of an element of a general curved interface with the unit normal vector $\mathbf{n}$ between the regions of hole Ω_E and reinforced layer Ω_L , the equilibrium equation of the interface can be obtained as follows^[38–39]:

$$(\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^2) \cdot \mathbf{n} = \nabla_S \cdot \boldsymbol{\sigma}^S, \quad (5)$$

where $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ are the volume stress tensors in Ω_E and Ω_L , respectively; the right-hand side of Eq. (5) is the surface divergence of the surface stress tensor $\boldsymbol{\sigma}^S$. Equation (5) is the

generalized Young-Laplace equation for solids, which characterizes the interfacial behavior between two different solids with interfacial stresses. For specific geometries of interface, i.e., circular cylindrical coordinates (ρ, θ, z) in the current problem, Eq. (5) can be simplified as^[40]

$$\Sigma_\rho^E - \Sigma_\rho^L = \frac{1}{\rho_1} \frac{\partial \Sigma_\theta^S}{\partial \theta} = \frac{\mathbf{B}^S}{\rho_1} \frac{\partial \mathbf{Z}_\theta^S}{\partial \theta}, \quad (6)$$

where superscripts E, L, and S denote quantities of the nano-elliptic hole, reinforcement layer, and nano-hole surface, respectively, $\Sigma_\rho^E = (0, 0, D_\rho^E)^T$ is an additional generalized stress due to the dielectric property inside the nano-hole, and $\mathbf{B}^S$ is the material matrix of the surface of the nano-hole, which can be determined from atomistic simulations and experimental methods.

By considering the dielectric characteristics inside the nano-elliptical hole as shown in Fig. 1(a), the electric displacement and electric potential induced by the internal electric field can be simplified from Eq. (1) as

$$-\lambda_{11}^E \nabla^2 \phi^E = 0, \quad D_i^E = -\lambda_{11}^E \frac{\partial \phi^E}{\partial x_i}, \quad (7)$$

where λ_{11}^E is the dielectric constant of the medium inside the nano-hole, and $\phi^E = \text{Im}(\Phi^E)$ in which Φ^E satisfies

$$\text{Im}(\Phi^E) = \text{Im}(\Phi^L). \quad (8)$$

3 Exact solutions

For the interfaces, the tangential components of the generalized strains, displacements, and resultant forces at the interface are continuous, i.e.,

$$\begin{cases} \mathbf{Z}_\theta^S = \mathbf{Z}_\theta^L & \text{at } \rho = \rho_1, \\ \mathbf{u}^L = \mathbf{u}^M, \quad \mathbf{F}^L = \mathbf{F}^M & \text{at } \rho = \rho_2. \end{cases} \quad (9)$$

The following single-value condition is satisfied:

$$\Phi^E(\sqrt{n}e^{i\theta}) = \Phi^E(\sqrt{n}e^{-i\theta}). \quad (10)$$

In the nano-elliptic hole, reinforced layer, and matrix, the analytical functions in the ζ -plane can be expanded into the following Laurent series with finite items^[35]:

$$\begin{cases} \mathbf{f}^L = \mathbf{a}^L \zeta + \mathbf{b}^L \zeta^{-1} = \mathbf{a}^L \frac{z + \sqrt{z^2 - c^2}}{2} + \mathbf{b}^L \frac{z - \sqrt{z^2 - c^2}}{2n}, \\ \mathbf{f}^M = \mathbf{a}^M \zeta + \mathbf{b}^M \zeta^{-1} = \mathbf{a}^M \frac{z + \sqrt{z^2 - c^2}}{2} + \mathbf{b}^M \frac{z - \sqrt{z^2 - c^2}}{2n}, \\ \phi^E = e^E \zeta + f^E \zeta^{-1} = e^E \frac{z + \sqrt{z^2 - c^2}}{2} + f^E \frac{z - \sqrt{z^2 - c^2}}{2n}, \end{cases} \quad (11)$$

where $c = \sqrt{a_1^2 - b_1^2} = \sqrt{a_2^2 - b_2^2}$.

Substituting Eq. (11) into Eq. (6), the last equation of Eq. (9), and Eq. (10), we obtain the relation between unknown quantities as follows:

$$\begin{cases} \mathbf{b}^L = \rho_1^2 \left(\frac{\mathbf{B}^S}{\rho_1} + \mathbf{B}^L \right)^{-1} \left(\left(\frac{\mathbf{B}^S}{\rho_1} - \mathbf{B}^L \right) \mathbf{a}^L + \begin{pmatrix} 0 \\ 0 \\ -\lambda_{11}^E (e^E + \rho_1^{-2} f^E) \end{pmatrix} \right), \\ f^L = e^L \rho_1^2 - e^E \rho_1^2 + f^E, \quad f^E = n e^E, \end{cases} \quad (12)$$

where $\mathbf{b}^L = (b^L, q^L, f^L)^T$, $\mathbf{a}^L = (a^L, p^L, e^L)^T$, and e^E and f^E are two unknown constants.

When the PQC materials are subject to far-field electromechanical loads, one easily has

$$\mathbf{a}^M = (\varepsilon_{zy}^\infty \quad \omega_{zy}^\infty \quad -E_y^\infty)^T, \quad (13)$$

where ε_{zy}^∞ , ω_{zy}^∞ , and E_y^∞ are the given phonon strain, phason strain, and electric field at infinity, respectively.

By substituting the last two equations of Eq. (12) and Eq. (13) into the first equation of Eq. (12), the constant term e^E can be obtained as follows:

$$e^E = \eta_1 a^L + \eta_2 p^L + \eta_3 e^L, \quad (14)$$

where

$$\begin{aligned} \eta_1 &= \left(2\rho_1^2 \left(\det(\mathbf{B}_{12}) \frac{c_{44}^L e_{15}^S - c_{44}^S e_{15}^L}{\rho_1} - \det(\mathbf{B}_{14}) \frac{R_3^L e_{15}^S - R_3^S e_{15}^L}{\rho_1} \right) \right. \\ &\quad \left. / \left((n - \rho_1^2)(\det(\mathbf{B}_{11}) \times \det(\mathbf{B}_{12}) - \det(\mathbf{B}_{13}) \times \det(\mathbf{B}_{14})) - \lambda_{11}^E (n + \rho_1^2) \right. \right. \\ &\quad \left. \left. \left(\left(\frac{c_{44}^S}{\rho_1} + c_{44}^L \right) \det(\mathbf{B}_{12}) - \det(\mathbf{B}_{14}) \left(\frac{R_3^S}{\rho_1} + R_3^L \right) \right) \right) \right), \\ \eta_2 &= \left(\rho_1^2 \left(\frac{d_{15}^S}{\rho_1} - d_{15}^L \right) \left(\det(\mathbf{B}_{12}) \left(\frac{c_{44}^S}{\rho_1} + c_{44}^L \right) - \det(\mathbf{B}_{14}) \left(\frac{R_3^S}{\rho_1} - R_3^L \right) \right) \right. \\ &\quad \left. - \rho_1^2 \left(\frac{e_{15}^S}{\rho_1} + e_{15}^L \right) \left(\det(\mathbf{B}_{12}) \left(\frac{R_3^S}{\rho_1} - R_3^L \right) - \det(\mathbf{B}_{14}) \left(\frac{K_2^S}{\rho_1} - K_2^L \right) \right) \right) \\ &\quad \left. / \left((n - \rho_1^2)(\det(\mathbf{B}_{11}) \times \det(\mathbf{B}_{12}) - \det(\mathbf{B}_{13}) \times \det(\mathbf{B}_{14})) \right. \right. \\ &\quad \left. \left. - \lambda_{11}^E (n + \rho_1^2) \left(\left(\frac{c_{44}^S}{\rho_1} + c_{44}^L \right) \det(\mathbf{B}_{12}) - \det(\mathbf{B}_{14}) \left(\frac{R_3^S}{\rho_1} + R_3^L \right) \right) \right) \right), \\ \eta_3 &= \left(-2\rho_1^2 \lambda_{11}^L \left(\det(\mathbf{B}_{14}) \left(\frac{R_3^S}{\rho_1} + R_3^L \right) - \det(\mathbf{B}_{12}) \left(\frac{c_{44}^S}{\rho_1} + c_{44}^L \right) \right) \right. \\ &\quad \left. + 2\rho_1^2 \left(\frac{e_{15}^S}{\rho_1} + e_{15}^L \right) \left(e_{15}^L \det(\mathbf{B}_{12}) - d_{15}^L \det(\mathbf{B}_{14}) \right) \right) \\ &\quad \left. / \left((n - \rho_1^2)(\det(\mathbf{B}_{11}) \times \det(\mathbf{B}_{12}) - \det(\mathbf{B}_{13}) \times \det(\mathbf{B}_{14})) \right. \right. \\ &\quad \left. \left. - \lambda_{11}^E (n + \rho_1^2) \left(\left(\frac{c_{44}^S}{\rho_1} + c_{44}^L \right) \det(\mathbf{B}_{12}) - \det(\mathbf{B}_{14}) \left(\frac{R_3^S}{\rho_1} + R_3^L \right) \right) \right) \right). \end{aligned} \quad (15)$$

In Eq. (15), $\mathbf{B}_{1k}$ ($k = 1, 2, \dots, 4$) is the second-order submatrix of matrix $\mathbf{B}_1 = \mathbf{B}^S/\rho_1 + \mathbf{B}^L$, and $\det(\mathbf{B}_{1k})$ is the determinant of the following matrix $\mathbf{B}_{1k}$:

$$\left\{ \begin{aligned} \mathbf{B}_{11} &= \begin{pmatrix} \frac{c_{44}^S}{\rho_1} + c_{44}^L & \frac{e_{15}^S}{\rho_1} + e_{15}^L \\ \frac{e_{15}^S}{\rho_1} + e_{15}^L & -\frac{\lambda_{11}^S}{\rho_1} - \lambda_{11}^L \end{pmatrix}, & \mathbf{B}_{12} &= \begin{pmatrix} \frac{R_3^S}{\rho_1} + R_3^L & \frac{K_2^S}{\rho_1} + K_2^L \\ \frac{e_{15}^S}{\rho_1} + e_{15}^L & \frac{d_{15}^S}{\rho_1} + d_{15}^L \end{pmatrix}, \\ \mathbf{B}_{13} &= \begin{pmatrix} \frac{R_3^S}{\rho_1} + R_3^L & \frac{e_{15}^S}{\rho_1} + e_{15}^L \\ \frac{d_{15}^S}{\rho_1} + d_{15}^L & -\frac{\lambda_{11}^S}{\rho_1} - \lambda_{11}^L \end{pmatrix}, & \mathbf{B}_{14} &= \begin{pmatrix} \frac{c_{44}^S}{\rho_1} + c_{44}^L & \frac{R_3^S}{\rho_1} + R_3^L \\ \frac{e_{15}^S}{\rho_1} + e_{15}^L & \frac{d_{15}^S}{\rho_1} + d_{15}^L \end{pmatrix}. \end{aligned} \right. \quad (16)$$

Therefore, the first equation of Eq. (12) can be simplified to

$$\mathbf{b}^L = \mathbf{Q}_1 \mathbf{a}^L, \quad (17)$$

where

$$\mathbf{Q}_1 = \rho_1^2 \left(\frac{\mathbf{B}^S}{\rho_1} + \mathbf{B}^L \right)^{-1} \left(\left(\frac{\mathbf{B}^S}{\rho_1} - \mathbf{B}^L \right) - \lambda_{11}^E (1 + n\rho_1^{-2}) \mathbf{T} \right) \quad (18)$$

with the matrix $\mathbf{T}$

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_2 \end{pmatrix}. \quad (19)$$

Obviously, the matrix $\mathbf{T}$ satisfies the relationship $\mathbf{T}\mathbf{a}^L = (0, 0, e^E)^T$.

According to the last term in Eq. (9), the following relation is obtained:

$$\mathbf{a}^L \rho_2^2 - \mathbf{b}^L = \mathbf{a}^M \rho_2^2 - \mathbf{b}^M, \quad \mathbf{B}^L (\mathbf{a}^L \rho_2^2 + \mathbf{b}^L) = \mathbf{B}^M (\mathbf{a}^M \rho_2^2 + \mathbf{b}^M). \quad (20)$$

Thus, all the following constant vectors can be determined by $\mathbf{a}^M$:

$$\mathbf{a}^L = 2\mathbf{P}\mathbf{a}^M, \quad \mathbf{b}^M = \mathbf{Q}_2\mathbf{a}^M, \quad \mathbf{b}^L = 2\mathbf{Q}_1\mathbf{P}\mathbf{a}^M, \quad (21)$$

where

$$\mathbf{P} = (\mathbf{I} + (\mathbf{B}^M)^{-1}\mathbf{B}^L + \rho_2^{-2}((\mathbf{B}^M)^{-1}\mathbf{B}^L - \mathbf{I})\mathbf{Q}_1)^{-1}, \quad \mathbf{Q}_2 = \rho_1^2(\mathbf{I} - 2\mathbf{P}) + 2\mathbf{Q}_1\mathbf{P}, \quad (22)$$

in which $\mathbf{I}$ is the third-order identity matrix.

Along the x -axis, the phonon stresses, phason stresses, and electric displacements of the electrically permeable nano-elliptical hole in the reinforcement layer and matrix are finally obtained as

$$\begin{cases} \begin{pmatrix} \sigma_{zy}^L(x, 0) \\ H_{zy}^L(x, 0) \\ D_y^L(x, 0) \end{pmatrix} = \mathbf{B}^L \left(\left(1 + \frac{x}{\sqrt{x^2 - c^2}} \right) \mathbf{I} + \frac{4}{c^2} \left(1 - \frac{x}{\sqrt{x^2 - c^2}} \right) \mathbf{Q}_1 \right) \mathbf{P} (\mathbf{B}^M)^{-1} \begin{pmatrix} \sigma_{zy}^\infty \\ H_{zy}^\infty \\ D_y^\infty \end{pmatrix}, \\ \begin{pmatrix} \sigma_{zy}^M(x, 0) \\ H_{zy}^M(x, 0) \\ D_y^M(x, 0) \end{pmatrix} = \frac{1}{2} \mathbf{B}^M \left(\left(1 + \frac{x}{\sqrt{x^2 - c^2}} \right) \mathbf{I} + \frac{4}{c^2} \left(1 - \frac{x}{\sqrt{x^2 - c^2}} \right) \mathbf{Q}_2 \right) (\mathbf{B}^M)^{-1} \begin{pmatrix} \sigma_{zy}^\infty \\ H_{zy}^\infty \\ D_y^\infty \end{pmatrix}. \end{cases} \quad (23)$$

Furthermore, the generalized stresses at points A and B in Fig.1(a) of the electrically permeable nano-elliptical hole are expressed as follows:

$$\begin{cases} (\sigma_{zy}^A \ H_{zy}^A \ D_y^A)^T = \frac{1 + \beta_1}{\beta_1} \mathbf{B}^L (\mathbf{I} - \rho_1^{-2} \mathbf{Q}_1) \mathbf{P} (\mathbf{B}^M)^{-1} (\sigma_{zy}^\infty \ H_{zy}^\infty \ D_y^\infty)^T, \\ (\sigma_{zy}^B \ H_{zy}^B \ D_y^B)^T = \frac{1 + \beta_2}{2\beta_2} \mathbf{B}^M (\mathbf{I} - \rho_2^{-2} \mathbf{Q}_2) (\mathbf{B}^M)^{-1} (\sigma_{zy}^\infty \ H_{zy}^\infty \ D_y^\infty)^T, \end{cases} \quad (24)$$

where $\beta_1 = b_1/a_1$ and $\beta_2 = b_2/a_2$. In the case of an electrically impermeable boundary condition and the dielectric coefficient $\lambda_{11}^E = 0$, Eq. (18) can be simplified as

$$\mathbf{Q}_1 = \rho_1^2 \left(\frac{\mathbf{B}^S}{\rho_1} + \mathbf{B}^L \right)^{-1} \left(\frac{\mathbf{B}^S}{\rho_1} - \mathbf{B}^L \right). \quad (25)$$

As special cases, the present solutions can be reduced to the previous results, such as a nano-elliptical hole in piezoelectric materials^[31–32]. Besides, some new results including the reinforced Griffith nano-crack and the reinforced nano-circular hole in PQC, piezoelectric (PE), and QC materials for two kinds of electrical boundary conditions can also be derived from the present results, which are neglected here.

4 Numerical examples

In this section, the effects of the surface layer properties, the reinforcing layer properties, and the thickness of reinforced layer on the field concentration near the nano-elliptical hole and the field intensity factors near the nano-crack tip in PQC are analyzed. The material properties of the matrix and surface layer are listed in Tables 1 and 2, respectively^[13,31].

Table 1 Material properties of PQC and PE material (PZT-5) matrix

Property	PQC	PZT-5	Property	PQC	PZT-5
$c_{44}/(\text{N}\cdot\text{m}^{-2})$	50×10^9	21.1×10^9	$R_3/(\text{N}\cdot\text{m}^{-2})$	1.2×10^9	—
$K_2/(\text{N}\cdot\text{m}^{-2})$	0.3×10^9	—	$e_{15}/(\text{C}\cdot\text{m}^{-2})$	−0.138	12.3
$\lambda_{11}/(\text{C}^2\cdot\text{N}^{-2}\cdot\text{m}^2)$	82.6×10^{-12}	8.107×10^{-9}	$d_{15}/(\text{C}\cdot\text{m}^{-2})$	−0.16	—

Table 2 Material properties of the surface layer

$c_{44}^S/(\text{N}\cdot\text{m}^{-1})$	$K_2^S/(\text{N}\cdot\text{m}^{-1})$	$\lambda_{11}^S/(\text{C}^2\cdot\text{N}^{-1}\cdot\text{m}^{-1})$	$R_3^S/(\text{N}\cdot\text{m}^{-1})$	$e_{15}^S/(\text{C}\cdot\text{m}^{-1})$	$d_{15}^S/(\text{C}\cdot\text{m}^{-1})$
2	0.6	0	0.8	5×10^{-9}	0

Figure 2 shows the variations of the stress and electric displacement intensity factors near the nano-crack tip induced by remotely mechanical loads in a reinforced piezoelectric material with the crack length a_1 under two electrical boundary conditions for $a_2/a_1 = 1.2$ and $\beta_2 = b_2/a_2 = \sqrt{11}/6$. It can be found from Figs. 2(a) and 2(b) that the stress and electric displacement intensity factors in piezoelectric materials under the electrically impermeable boundary condition agree well with the results obtained by Xiao et al.^[31] when $L_C = 1$ (without reinforcement).

Fig. 2 Stress and electric displacement intensity factors induced by mechanical loads at the nano-crack tip in a reinforced piezoelectric material under two electrical boundary conditions (color online)

Besides, the stress and electric displacement intensity factors in piezoelectric materials under the electrically permeable boundary condition are the same as those obtained by Guo and Li^[32] when $L_C = 1$ (see Figs. 2(c) and 2(d)). Moreover, the soft reinforcement ($L_C < 1$) could decrease the magnitude of the stress and electric displacement intensity factors near the nano-crack tip, while the rigid reinforcement ($L_C > 1$) could increase the magnitude of the stress and electric displacement intensity factors near the nano-crack tip, indicating that the soft reinforcement in piezoelectric materials can reduce the stress concentration near the crack tip. When the crack size becomes large, the field intensity factors with surface effect approach to the classical results (without surface effect).

Figure 3 shows the effects of the thickness of reinforced layer a_2/a_1 on the phonon stress,

Fig. 3 Variations of the phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced PQC composite with the thickness of reinforced layer a_2/a_1 (color online)

phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced PQC composite under two electrical boundary conditions for given $a_1 = 5$ nm and $L_C = 0.5$. It can be observed that an electrically permeable nano-crack with the surface effect displays a different behavior from an electrically impermeable one. In general, three fields including the phonon field, phason field, and electric field are uncoupled in the classical elasticity theory, but they are coupled with the surface effect for the electrically impermeable case.

Figure 4 shows the effects of the material property of reinforcement L_C on the phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced PQC composite under two electrical boundary conditions for given $a_1 = 5$ nm, $a_2/a_1 = 1.2$, and $\beta_2 = b_2/a_2 = \sqrt{11}/6$. It can be seen that for an electrically permeable nano-crack, the field

Fig. 4 Variations of the phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip in a reinforced PQC composite with the material property of reinforcement layer L_C (color online)

intensity factors induced by electrical loadings are always zero, but they are greatly dependent on the material property of reinforcement L_C under the mechanical loadings of phonon field. By contrast, the field intensity factors induced by mechanical or electrical loadings are always dependent on L_C for an electrically impermeable nano-crack.

Figure 5 shows the variations of the phonon stress, phason stress, and electric displacement in the reinforced layer (short for RL in Fig. 5) and matrix (short for M in Fig. 5) of 1D hexagonal PQC along the x -axis direction under far-field mechanical and electrical loads for given $\lambda_{11}^E = 8.85 \times 10^{-12} \text{ C}^2 \cdot \text{N}^{-1} \cdot \text{m}^2$, $L_C = 0.5$, $a_1 = 5 \text{ nm}$, $a_2 = 6 \text{ nm}$, $\beta_1 = 0.5$, and $\beta_2 = \sqrt{69}/3$. It can be found that the phonon stress, phason stress, and electric displacement have a big jump across the interface between the reinforced layer and the matrix.

Figure 6 displays the effects of the aspect ratio of the nano-elliptical hole and the reinforced

Fig. 5 Variations of the phonon stress, phason stress, and electric displacement along the x -axis under mechanical and electrical loadings (color online)

Fig. 6 Effects of the aspect ratio of the elliptical hole and the reinforced layer on the phonon stress, phason stress, and electric displacement at points A and B under mechanical loadings (color online)

layer on the phonon stress, phason stress, and electric displacement at points A and B in 1D hexagonal PQC under far-field mechanical loads for given $a_2/a_1 = 1.2$ and $L_C = 0.5$. It can be observed that the magnitudes of the phonon stress, phason stress, and electrical displacement at points A and B decrease with the increase in the aspect ratio of the elliptical hole and the reinforced layer. The field concentration at the point A is smaller than that at the point B , which is attributed to the soft reinforced layer surrounding the hole.

Figure 7 plots the variations of the phonon stress, phason stress, and electric displacement around the nano-circular hole in 1D hexagonal PQC under far-field mechanical and electrical loads for given $a_2/a_1 = 1.2$ and $L_C = 0.5$. It is seen that the phonon stress, phason stress, and electric displacement around the circular hole are symmetrical about the y -axis ($\theta = \pi/2$). Furthermore, these three fields are nonzero at $\theta = \pi/2$ with surface effect, which are different

Fig. 7 Variations of the phonon stress, phason stress, and electric displacement around the circular hole under mechanical and electrical loadings (color online)

from the classical results. The phonon stress concentration induced by mechanical loadings (see Fig. 7(a)) and the electric displacement concentration induced by electrical loadings (see Fig. 7(f)) occur at $\theta = 0$ and π , but the other field concentrations occur at $\theta = \pi/2$. It is interesting to note that the effects of the dielectric constant inside the hole on the phonon stress, phason stress, and electric displacement for an electrically permeable hole induced by electrical loadings are larger than those induced by mechanical loadings as compared with the electrically impermeable case. Besides, under the electrical loadings, the field concentration for an electrically permeable hole is smaller than that for an electrically impermeable one.

5 Conclusions

Based on the surface/interface model, an anti-plane shear problem of a reinforced nano-elliptical hole in 1D hexagonal PQC composites is considered under both electrically permeable

and impermeable boundary conditions. Exact solutions in closed-form of the phonon stresses, phason stresses, and electric displacements in the matrix and reinforced layer are derived by the complex variable method. The phonon stress, phason stress, and electric displacement intensity factors near the nano-crack tip are further obtained when the elliptical hole is reduced to a crack. Some useful features can be found from the numerical examples as follows.

(I) The soft reinforcement could decrease the magnitudes of stress and electric displacement intensity factors near the nano-crack tip, while the rigid reinforcement could increase the magnitudes of stress and electric displacement intensity factors near the nano-crack tip.

(II) The electrically permeable nano-hole or nano-crack with surface effects shows a different behavior from an electrically impermeable one. The phonon field, phason field, and electric field are coupled with surface effects for the electrically impermeable boundary condition, which are also different from the classical elasticity results.

(III) The material property and thickness of the reinforced layer have great effects on the phonon field, phason field, and electric field around the nano-holes and nano-cracks.

(IV) The phonon stress, phason stress, and electric displacement of the nano-elliptical hole are discontinuous at the interface between the matrix and the reinforcement layer. The stress concentration around both the hole and the reinforced layer always decreases with the increasing aspect ratio of the hole and the reinforced layer.

(V) Under electrical loadings, the dielectric characteristics inside holes should be considered. Thus, the electrically permeable boundary conditions should be adopted in the analysis.

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References

- [1] SHECHTMAN, D., BLECH, I., GRATIAS, D., and CAHN, J. W. Metallic phase with long-range orientational order and no translational symmetry. *Physical Review Letters*, **53**, 1951–1953 (1984)
- [2] BELLISSARD, J., IOCHUM, B., SCOPPOLA, E., and TESTARD, D. Spectral properties of one-dimensional quasi-crystals. *Communications in Mathematical Physics*, **125**(3), 527–543 (1989)
- [3] OVIOD'KO, I. A. Plastic deformation and decay of dislocations in quasi-crystals. *Materials Science and Engineering A*, **154**(1), 29–33 (1992)
- [4] WOLLGARTEN, M., BEYSS, M., and URBAN, K., LIEBERTZ, H., and KOSTER, U. Direct evidence for plastic deformation of quasicrystals by means of a dislocation mechanism. *Physical Review Letters*, **71**(4), 549–552 (1993)
- [5] FAN, T. Y. and MAI, Y. W. Elasticity theory, fracture mechanics, and some relevant thermal properties of quasicrystalline materials. *Applied Mechanics Reviews*, **57**, 325–343 (2004)
- [6] EISENHAMMER, T., HAUGENEDER, A., and MAHR, A. High-temperature optical properties and stability of selective absorbers based on quasicrystalline AlCuFe. *Solar Energy Materials and Solar Cells*, **54**(1-4), 379–386 (1998)
- [7] SÁNCHEZ, A., DE BLAS, F. J. G., ALGABA, J. M., ALVAREZ, J., VALLÉS, P., GARCÍA-POGGIO, M. C., and AGÜERO, A. Application of quasicrystalline materials as thermal barriers in aeronautics and future perspectives of use for these materials. *MRS Proceedings*, **553**, 447–458 (1998)
- [8] INOUE, A. and KIMURA, H. High-strength aluminum alloys containing nanoquasicrystalline particles. *Materials Science and Engineering A*, **286**(1), 1–10 (2000)
- [9] FU, Y. Q., AN, L. T., ZHOU, F., ZHAO, Y. Y., YANG, D. M., and GAO, Y. Al-Cu-Cr quasicrystalline coatings prepared by low power plasma spraying. *Surface and Coatings Technology*, **202**(20), 4964–4970 (2008)

- [10] BAE, D. H., LEE, M. H., KIM, K. T., KIM, W. T., and KIM, D. H. Application of quasicrystalline particles as a strengthening phase in Mg-Zn-Y alloys. *Journal of Alloys and Compounds*, **342**(1), 445–450 (2002)
- [11] LEVINE, D., LUBENSKY, T. C., OSTLUND, S., RAMASWAMY, S., STEINHARD, P. J., and TONER, J. Elasticity and dislocations in pentagonal and icosahedral quasicrystals. *Physical Review Letters*, **54**(14), 1520–1523 (1985)
- [12] WANG, X. and PAN, E. Analytical solutions for some defect problems in 1D hexagonal and 2D octagonal quasicrystals. *Pramana — Journal of Physics*, **70**(5), 911–933 (2008)
- [13] LI, X. Y., LI, P. D., WU, T. H., SHI, M. X., and ZHU, Z. W. Three-dimensional fundamental solutions for one-dimensional hexagonal quasicrystal with piezoelectric effect. *Physics Letters A*, **378**(10), 826–834 (2014)
- [14] YANG, J. and LI, X. Analytic solutions of problem about a circular hole with a straight crack in one-dimensional hexagonal quasicrystals with piezoelectric effects. *Theoretical and Applied Fracture Mechanics*, **82**, 17–24 (2016)
- [15] BAI, Q. M. and DING, S. H., An anti-plane problem of cracks at edges of regular hexagonal holes in 1D hexagonal piezoelectric quasicrystals (in Chinese). *Applied Mathematics and Mechanics*, **40**(10), 1071–1080 (2019)
- [16] JIANG, L. J. and LIU, G. T. The interaction between a screw dislocation and a wedge-shaped crack in one-dimensional hexagonal piezoelectric quasicrystals. *Chinese Physics B*, **26**(4), 245–251 (2017)
- [17] SLADEK, J., SLADEK, V., KRAHULEC, S., ZHANG, C., and WUNSCHE, M. Crack analysis in decagonal quasicrystals by the MLPG. *International Journal of Fracture*, **181**(1), 115–126 (2013)
- [18] WANG, Z. and RICOEUR, A. Numerical crack path prediction under mixed-mode loading in 1D quasicrystals. *Theoretical and Applied Fracture Mechanics*, **90**, 122–132 (2017)
- [19] FAN, C. Y., YUAN, Y. P., PAN, Y. B., and ZHAO, M. H. Analysis of cracks in one-dimensional hexagonal quasicrystals with the heat effect. *International Journal of Solids and Structures*, **120**(1), 146–156 (2017)
- [20] ZHAO, M. H., DANG, H. Y., FAN, C. Y., and CHEN, Z. T. Analysis of a three-dimensional arbitrarily shaped interface crack in a one-dimensional hexagonal thermo-electro-elastic quasicrystal bi-material, part 1: theoretical solution. *Engineering Fracture Mechanics*, **179**, 59–78 (2017)
- [21] TUPHOLME, G. E. One-dimensional piezoelectric quasicrystals with an embedded moving, non-uniformly loaded shear crack. *Acta Mechanica*, **228**(2), 547–560 (2017)
- [22] ZHOU, Y. B. and LI, X. F. A Yoffe-type moving crack in one-dimensional hexagonal piezoelectric quasicrystals. *Applied Mathematical Modelling*, **65**, 148–163 (2019)
- [23] HU, K. Q., JIN, H., YANG, Z. J., and CHEN, X. Interface crack between dissimilar one-dimensional hexagonal quasicrystals with piezoelectric effect. *Acta Mechanica*, **230**, 2455–2474 (2019)
- [24] LI, L. H., CUI, X. W., and GUO, J. H. Interaction between a screw dislocation and an elliptical hole with two asymmetrical cracks in a one-dimensional hexagonal quasicrystal with piezoelectric effect. *Applied Mathematics and Mechanics (English Edition)*, **41**(6), 899–908 (2020) <https://doi.org/10.1007/s10483-020-2615-6>
- [25] LOBODA, V., KOMAROV, O., BILYI, D., and LAPUSTA, Y. An analytical approach to the analysis of an electrically permeable interface crack in a 1D piezoelectric quasicrystal. *Acta Mechanica*, **231**, 3419–3433 (2020)
- [26] LI, Y., ZHAO, M. H., QIN, Q. H., and FAN, C. Y. Analysis solution method for 3D planar crack problems of two-dimensional hexagonal quasicrystals with thermal effects. *Applied Mathematical Modelling*, **69**, 648–664 (2019)
- [27] WANG, X. and SCHIAVONE, P. Elastic field near the tip of an anticrack in a decagonal quasicrystalline material. *Applied Mathematics and Mechanics (English Edition)*, **41**(3), 401–408 (2020) <https://doi.org/10.1007/s10483-020-2582-8>

-
- [28] XU, W., GALANO, M., and AUDEBERT, F. Nanoquasicrystalline Al-Fe-Cr-Ti alloy matrix/g-Al₂O₃ nanocomposite powders: the effect of the ball milling process. *Journal of Alloys and Compounds*, **701**(15), 342–349 (2017)
 - [29] GALANO, M., AUDEBERT, F., GARCIA ESCORIAL, A., STONE I. C., and CANTOR, B. Nanoquasicrystalline Al-Fe-Cr-based alloys, part II, mechanical properties. *Acta Materialia*, **57**(17), 5120–5130 (2009)
 - [30] ALI, F., SCUDDINO, S., ANWAR, M. S., SHAHID, R. N., SRIVASTAVA, V. C., UHLENWINKEL, V., STOICA, M., VAUGHAN, G., and ECKERT, J. Al-based metal matrix composites reinforced with Al-Cu-Fe quasicrystalline particles: strengthening by interfacial reaction. *Journal of Alloys and Compounds*, **607**, 274–279 (2014)
 - [31] XIAO, J. H., XU, Y. L., and ZHANG, F. C. A rigorous solution for the piezoelectric materials containing elliptic cavity or crack with surface effect. *ZAMM — Journal of Applied Mathematics and Mechanics*, **96**(5), 633–641 (2016)
 - [32] GUO, J. H. and LI, X. F. Surface effects on an electrically permeable elliptical nano-hole or nano-crack in piezoelectric materials under anti-plane shear. *Acta Mechanica*, **229**, 4251–4266 (2018)
 - [33] CHEN, F. M. and CHAO, C. K. Stress analysis of an infinite plate with a coated elliptic hole under a remote uniform heat flow. *Journal of Thermal Stresses*, **31**(7), 599–613 (2008)
 - [34] HU, K. Q. and CHEN, Z. Exact solution for an elliptical cavity or crack in a reinforced magnetoelectroelastic material under anti-plane shear. *Theoretical and Applied Fracture Mechanics*, **86**, 317–331 (2016)
 - [35] GUO, J. H., HE, L. T., LIU, Y. Z., and LI, L. H. Anti-plane analysis of a reinforced nano-elliptical cavity or nano-crack in a magnetoelectroelastic matrix with surface effect. *Theoretical and Applied Fracture Mechanics*, **107**, 102553 (2020)
 - [36] GURTIN, M. E. and MURDOCH, A. I. A continuum theory of elastic material surfaces. *Archive for Rational Mechanics and Analysis*, **57**(4), 291–323 (1975)
 - [37] GURTIN, M. E. and MURDOCH, A. I. Surface stress in solids. *International Journal of Solids and Structures*, **14**(6), 431–440 (1978)
 - [38] POVSTENKO, Y. Z. Theoretical investigation of phenomena caused by heterogeneous surface tension in solids. *Journal of the Mechanics and Physics of Solids*, **41**, 1499–1514 (1993)
 - [39] DUAN, H. L., WANG, J., HUANG, Z. P., and KARIHALOO, B. L. Eshelby formalism for nano-inhomogeneities. *Proceedings of the Royal Society A*, **461**, 3335–3353 (2005)
 - [40] CHEN, T., CHIU, M. S., and WENG, C. N. Derivation of the generalized Young-Laplace equation of curved interfaces in nanoscaled solids. *Journal of Applied Physics*, **100**, 074308 (2006)