Dynamics of a multiplex neural network with delayed couplings*

Xiaochen MAO[†], Xingyong LI, Weijie DING, Song WANG,
Xiangyu ZHOU, Lei QIAO

Department of Engineering Mechanics, College of Mechanics and Materials,
Hohai University, Nanjing 211100, China

(Received Oct. 6, 2020 / Revised Dec. 16, 2020)

Abstract Multiplex networks have drawn much attention since they have been observed in many systems, e.g., brain, transport, and social relationships. In this paper, the non-linear dynamics of a multiplex network with three neural groups and delayed interactions is studied. The stability and bifurcation of the network equilibrium are discussed, and interesting neural activities of the network are explored. Based on the neuron circuit, transfer function circuit, and time delay circuit, a circuit platform of the network is constructed. It is shown that delayed couplings play crucial roles in the network dynamics, e.g., the enhancement and suppression of the stability, the patterns of the synchronization between networks, and the generation of complicated attractors and multi-stability coexistence.

Key words neural network, time delay, synchronization, coexisting attractor

Chinese Library Classification O241

2010 Mathematics Subject Classification 65D15

1 Introduction

In the past few years, great efforts have been made on the investigations of multiplex networks due to their wide range of applicability in fields such as biology, physics, sociology, epidemiology, and engineering^[1–3]. Multiplex networks are termed in disparate ways, e.g., interacting networks, networks of networks, multilevel networks, and hypergraphs^[2]. In such networks, the nodes are distributed in individual networks, e.g., layers and loops, each of them accounts for a kind of interaction process existing among the nodes. For example, in the case of a *C. elegans* neuronal network, a proper way to describe it is a multiplex graph with two sub-networks, one for the chemical synaptic links and one for the gap junction interactions^[2]. Moreover, due to the cortical connectivity of mammalian brains, the corticocortical network of

* Citation: MAO, X. C., LI, X. Y., DING, W. J., WANG, S., ZHOU, X. Y., and QIAO, L. Dynamics of a multiplex neural network with delayed couplings. *Applied Mathematics and Mechanics (English Edition)*, **42**(3), 441–456 (2021) <https://doi.org/10.1007/s10483-021-2709-6>

[†] Corresponding author, E-mail: maochen@hhu.edu.cn

Project supported by the National Natural Science Foundation of China (Nos. 11872169 and 11472097), the Fundamental Research Funds for the Central Universities of China (No. B200202114), and the Natural Science Foundation of Jiangsu Province of China (No. BK20191295)

©Shanghai University and Springer-Verlag GmbH Germany, part of Springer Nature 2021

the cat can be constructed by modeling each node (cortical area) with a sub-network of interacting neurons^[4–5]. Examples also include air transportation networks, power grids, coupled semiconductor lasers, communication networks, social networks, infectious disease, etc.^[1–4,6–9].

The analysis of dynamical behaviors of multiplex networks is one of the focal issues in complex nonlinear systems, and has found important and extensive applications in science and engineering^[2,10–16]. For example, experimental evidence shows that chaotic behaviors exist in the brain networks and associate with the information processing, cognitive function, memory storage, and retrieval^[10–11]. In multiplex networks, the dynamic processes take place at the same time in interacting sub-networks with different structural and dynamical configurations. The interactions of the sub-networks open up a plethora of fascinating behaviors, e.g., complete/partial synchronization, intra/inter-layer synchronization, cluster synchronization, and explosive synchronization^[7,12–13,17–18]. The phenomena of synchronization in networks are of great importance in nature, and have become focal subjects in many fields^[7,12–13,19–23]. For instance, the synchronization of neurons plays a crucial role in the context of cognition and learning and the pathogenesis of several neurological diseases, e.g., Parkinson's disease and essential tremor^[7,19,21,24]. It is therefore important and essential to investigate the dynamical behaviors arising from the interactions of separate networks.

Since signal propagation is always non-instantaneous, time delays commonly exist in realistic systems^[15,25–29]. In neural networks, for example, the switching speed of action potentials propagating across neuron axons is finite, and the time lapses occurring by both dendritic and synaptic processing are non-negligible^[30–31]. Neglecting the time delay in a dynamical system usually leads to false or even wrong results. In recent years, the investigations on the dynamics of multiplex neural networks with time delays have been witnessed^[17–18,32–37]. Previous studies were mostly devoted to the dynamical characterization of two-coupled networks since such models are fundamental and easily addressable, e.g., two-layer networks^[7,12–13,36,38]. Little attention has been paid to neural networks made up of three or more interconnected networks. However, the multi-coupled structures are ubiquitous in neural systems^[2,9,12]. In brain, many interconnected areas (sub-networks) can be seen, which usually consist of neural assemblies and their couplings represent the structural and functional interactions among them^[9]. For instance, separate parallel neural loops in the cortex or thalamus operate through the basal ganglia and the interplay of them can lead to the generation of tremor oscillations in Parkinson's disease and epilepsy^[32]. Thus, disregarding the multiplex structure of networks may result in misunderstanding of the properties of neural systems.

Motivated by the above discussion, the purpose of this paper is to study the dynamical behaviors of a multiplex network, as shown in Fig. 1. The system consists of three networks, each of which has an arbitrary number of nodes and couplings between single neurons. Different time delays are introduced into the connections between networks.

Fig. 1 Structure of the multiplex network consisting of three sub-networks 1, 2, and 3 (color online)

The remaining part of this paper is organized as follows. In Section 2, the local and global stability and bifurcation of the trivial equilibrium of the network are analyzed. Case studies of numerical simulations are shown in Section 3. A circuit platform is designed to validate the obtained results in Section 4. Finally, conclusions are made in Section 5.

2 Model and stability analysis

As shown in Fig. 1, the network can be described by

$$\begin{cases} \dot{x}_i(t) = \begin{cases} -x_i(t) + \sum_{j=1}^n a_{ij}f(x_j) + r_1g(z_i(t - \tau_1)), & i = 1, \\ -x_i(t) + \sum_{j=1}^n a_{ij}f(x_j), & 2 \leq i \leq n, \end{cases} \\ \dot{y}_i(t) = \begin{cases} -y_i(t) + \sum_{j=1}^n b_{ij}f(y_j) + r_2g(x_i(t - \tau_2)), & i = 1, \\ -y_i(t) + \sum_{j=1}^n b_{ij}f(y_j), & 2 \leq i \leq n, \end{cases} \\ \dot{z}_i(t) = \begin{cases} -z_i(t) + \sum_{j=1}^n c_{ij}f(z_j) + r_3g(y_i(t - \tau_3)), & i = 1, \\ -z_i(t) + \sum_{j=1}^n c_{ij}f(z_j), & 2 \leq i \leq n, \end{cases} \end{cases} \quad (1)$$

where $i, j = 1, 2, 3, \dots, n$, and x_i , y_i , and z_i denote the states of the i th neuron in Networks 1, 2, and 3, respectively. a_{ij} , b_{ij} , and c_{ij} are the connection weights within the networks. r_1 , r_2 , and r_3 are the coupling strengths between the first neuron of each sub-network. τ_i represents the time delay in the couplings. Without loss of generality, the functions f within the individual sub-networks and the function g between sub-networks are absolutely smooth and satisfy $f(0) = 0$ and $g(0) = 0$. In this network, the local kinetics of each node is described by the Hopfield neuron^[39].

The linearization of Eq. (1) at the trivial equilibrium of the network can be written as follows:

$$\dot{u}(t) = -u(t) + L_1 u(t) + r_1 L_2 u(t - \tau_1) + r_2 L_3 u(t - \tau_2) + r_3 L_4 u(t - \tau_3), \quad (2)$$

where

$$u(t) = (x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n, z_1, z_2, z_3, \dots, z_n)^T,$$

$$\alpha_{ij} = a_{ij}f'(0), \quad \beta_{ij} = b_{ij}f'(0), \quad \gamma_{ij} = c_{ij}f'(0), \quad k_l = r_l g'(0),$$

$$L_1 = \begin{pmatrix} A & O & O \\ O & B & O \\ O & O & C \end{pmatrix}, \quad L_2 = \begin{pmatrix} O & O & K \\ O & O & O \\ O & O & O \end{pmatrix}, \quad L_3 = \begin{pmatrix} O & O & O \\ K & O & O \\ O & O & O \end{pmatrix}, \quad L_4 = \begin{pmatrix} O & O & O \\ O & O & O \\ O & K & O \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix},$$

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \vdots & \vdots & & \vdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2n} \\ \vdots & \vdots & & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{pmatrix}.$$

In the above equations, O is an $n \times n$ zero matrix, $i, j = 1, 2, 3, \dots, n$, and $l = 1, 2, 3$.

The characteristic equation of the network is

$$M(\lambda, \tau) = \begin{pmatrix} (\lambda + 1)I - A & O & -k_1 e^{-\lambda \tau_1} K \\ -k_2 e^{-\lambda \tau_2} K & (\lambda + 1)I - B & O \\ O & -k_3 e^{-\lambda \tau_3} K & (\lambda + 1)I - C \end{pmatrix}, \quad (3)$$

where I is an $n \times n$ identity matrix. Thus, the characteristic equation of the network reads

$$\begin{aligned} \Delta(\lambda, \tau) &= |(\lambda + 1)I - A| \cdot |(\lambda + 1)I - B| \cdot |(\lambda + 1)I - C| \\ &\quad - k e^{-\lambda \tau} |(\lambda + 1)I_{n-1} - A_1| \cdot |(\lambda + 1)I_{n-1} - B_1| \cdot |(\lambda + 1)I_{n-1} - C_1| \\ &= A(\lambda)B(\lambda)C(\lambda) - k e^{-\lambda \tau} A_1(\lambda)B_1(\lambda)C_1(\lambda) \\ &= P(\lambda) - k e^{-\lambda \tau} Q(\lambda) = 0, \end{aligned} \quad (4)$$

where I_{n-1} is an $(n-1) \times (n-1)$ identity matrix, and

$$k = k_1 k_2 k_3, \quad \tau = \tau_1 + \tau_2 + \tau_3, \quad P(\lambda) = A(\lambda)B(\lambda)C(\lambda), \quad Q(\lambda) = A_1(\lambda)B_1(\lambda)C_1(\lambda),$$

$$A(\lambda) = |(\lambda + 1)I - A|, \quad B(\lambda) = |(\lambda + 1)I - B|, \quad C(\lambda) = |(\lambda + 1)I - C|,$$

$$A_1(\lambda) = |(\lambda + 1)I_{n-1} - A_1|, \quad B_1(\lambda) = |(\lambda + 1)I_{n-1} - B_1|, \quad C_1(\lambda) = |(\lambda + 1)I_{n-1} - C_1|.$$

In fact, A_1 , B_1 , and C_1 represent the connection matrices of the three sub-networks without the first neuron, respectively, i.e.,

$$A_1 = \begin{pmatrix} \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2n} \\ \alpha_{32} & \alpha_{33} & \cdots & \alpha_{3n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \beta_{22} & \beta_{23} & \cdots & \beta_{2n} \\ \beta_{32} & \beta_{33} & \cdots & \beta_{3n} \\ \vdots & \vdots & & \vdots \\ \beta_{n2} & \beta_{n3} & \cdots & \beta_{nn} \end{pmatrix}, \quad C_1 = \begin{pmatrix} \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2n} \\ \gamma_{32} & \gamma_{33} & \cdots & \gamma_{3n} \\ \vdots & \vdots & & \vdots \\ \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \end{pmatrix}.$$

The stability of the trivial equilibrium of the network can be determined by the root distribution of the characteristic equation (3). When the network has no coupling time delays, the Routh-Hurwitz criteria can be applied. As the time delays vary, let $\lambda = \pm i\omega$ ($\omega > 0$) be the roots of the characteristic equation. By separating the real and imaginary parts of $\Delta(i\omega, \tau) = 0$ and eliminating the harmonic terms, one arrives at

$$D(\omega) = P_R^2(\omega) + P_I^2(\omega) - k^2(Q_R^2(\omega) + Q_I^2(\omega)) = 0,$$

where $P_R(\omega) = \text{Re}(P(i\omega))$, $P_I(\omega) = \text{Im}(P(i\omega))$, $Q_R(\omega) = \text{Re}(Q(i\omega))$, and $Q_I(\omega) = \text{Im}(Q(i\omega))$.

Suppose that $D(\omega) = 0$ has some positive roots ω_j . Then, a set of critical time delays can be obtained by $\tau_{j,l} = (\theta_j + 2l\pi)/\omega_j$ ($l = 0, 1, 2, \dots$), where $\theta_j \in [0, 2\pi)$ yields sets of triangle equations

$$\cos \theta_j = \frac{P_R(\omega_j)Q_R(\omega_j) + P_I(\omega_j)Q_I(\omega_j)}{k|Q(i\omega_j)|^2}, \quad \sin \theta_j = \frac{P_R(\omega_j)Q_I(\omega_j) - Q_R(\omega_j)P_I(\omega_j)}{k|Q(i\omega_j)|^2}.$$

To determine the variation direction of the real part of Eq. (4) with respect to the time delay, the differentiation of λ with respect to τ yields

$$P'(\lambda)\lambda'(\tau) - kQ'(\lambda)\lambda'(\tau)e^{-\lambda\tau} + kQ(\lambda)(\lambda(\tau) + \tau\lambda'(\tau))e^{-\lambda\tau} = 0.$$

Then, one arrives at

$$\operatorname{Re} \left(\frac{d\lambda}{d\tau} \Big|_{\lambda=i\omega} \right) = \operatorname{Re} \left(- \frac{i\omega kQ(i\omega)}{P'(i\omega)e^{i\omega\tau} - kQ'(i\omega) + k\tau Q(i\omega)} \right) = 0.5\omega \frac{D'(\omega)}{|S(i\omega)|^2}, \quad (5)$$

where $S(i\omega) = P'(i\omega)e^{i\omega\tau} - kQ'(i\omega) + k\tau Q(i\omega)$. It follows that

$$\operatorname{sgn} \operatorname{Re} \left(\frac{d\lambda(\tau)}{d\tau} \Big|_{\lambda=i\omega} \right) = \operatorname{sgn} D'(\omega).$$

Thus, the system undergoes finite stability switches, and must become unstable with an increase in the time delay when $D(\omega) = 0$ has positive roots. On the other hand, it is delay-independently stable or unstable for any given time delay when $D(\omega) = 0$ has no positive root^[25,40].

In fact, the stability of the trivial equilibrium of the multiplex network is determined by the characteristic roots of the isolated sub-networks and the product of the coupling strength and the sum of the time delay.

Let $n = 3$, $a_{11} = b_{11} = c_{11} = -1.4$, $a_{12} = b_{12} = c_{12} = 1.3$, $a_{13} = b_{13} = c_{13} = -6$, $a_{21} = b_{21} = c_{21} = 1.1$, $a_{22} = b_{22} = c_{22} = 0$, $a_{23} = b_{23} = c_{23} = 2.6$, $a_{31} = b_{31} = c_{31} = 2.4$, $a_{32} = b_{32} = c_{32} = -2$, $a_{33} = b_{33} = c_{33} = 4$, $r_1 = r_2 = r_3$, $\tau_1 = \tau_2 = \tau_3 = \tau_s$, and $f = g = \tanh$.

Based on the above analysis, the region in the parameter planes indicating the stability of the trivial equilibrium of the triplex network is given in Fig. 2. Along the black and red curves, the characteristic equation of the system has a pair of pure imaginary roots. Moreover, the characteristic equation adds a pair of conjugate roots with positive real parts for each crossing black curves while reduces a pair conjugate roots with positive real parts for each crossing red curves. Besides, the network has delay-independent and delay-dependant stability regions.

Fig. 2 Stability region of the trivial equilibrium of a triplex neural network (color online)

Furthermore, the global asymptotic stability of the trivial equilibrium of the network is discussed as follows. In neural systems, due to the input-output relation of neurons, the activation functions f and g are often assumed to satisfy the normalization, monotonicity, boundedness, and concavity conditions, i.e., $f'(0) = g'(0) = 1$; $f'(v) > 0$ and $g'(v) > 0$ for all $v \in \mathbb{R}$; $-\infty < \lim_{v \rightarrow \pm\infty} f(v) < +\infty$ and $-\infty < \lim_{v \rightarrow \pm\infty} g(v) < +\infty$; $vf''(v) < 0$ and $vg''(v) < 0$ for all $v \neq 0$. The sigmoid functions, e.g., hyperbolic tangent functions, are widely used in neural networks.

Theorem 1 *If $np + q < 1$, the trivial equilibrium of the network is globally asymptotically stable for all time delays, where $p = \max\{|a_{ij}|, |b_{ij}|, |c_{ij}|\}$, and $q = \max\{|r_1|, |r_2|, |r_3|\}$.*

Proof Define the following Lyapunov function:

$$\begin{aligned} V(t) = & \sum_{i=1}^n (x_i^2(t) + y_i^2(t) + z_i^2(t)) + |r_1| \int_{t-\tau_1}^t g^2(z_1(s)) ds \\ & + |r_2| \int_{t-\tau_2}^t g^2(x_1(s)) ds + |r_3| \int_{t-\tau_3}^t g^2(y_1(s)) ds. \end{aligned}$$

Then, one arrives at

$$\begin{aligned} \dot{V}(t) = & 2 \sum_{i=1}^n (x_i(t)\dot{x}_i(t) + y_i(t)\dot{y}_i(t) + z_i(t)\dot{z}_i(t)) + |r_1|(g^2(z_1(t)) - g^2(z_1(t - \tau_1))) \\ & + |r_2|(g^2(x_1(t)) - g^2(x_1(t - \tau_2))) + |r_3|(g^2(y_1(t)) - g^2(y_1(t - \tau_3))) \\ \leq & -2 \sum_{i=1}^n (x_i^2 + y_i^2 + z_i^2) + na \sum_{i=1}^n (x_i^2 + f^2(x_i)) + nb \sum_{i=1}^n (y_i^2 + f^2(y_i)) \\ & + nc \sum_{i=1}^n (z_i^2 + f^2(z_i)) + |r_1|(z_1^2 + g^2(z_1)) + |r_2|(x_1^2 + g^2(x_1)) + |r_3|(y_1^2 + g^2(y_1)), \end{aligned}$$

where $a = \max\{|a_{ij}|\}$, $b = \max\{|b_{ij}|\}$, and $c = \max\{|c_{ij}|\}$. Since $vf''(v) < 0$ and $vg''(v) < 0$ for $v \neq 0$, one has $f'(v) \leq f'(0) = 1$ and $g'(v) \leq g'(0) = 1$. In addition, one obtains $f(v_i(t)) = \varphi_i(t)v_i(t)$ and $g(v_i(t)) = \phi_i(t)v_i(t)$, where

$$\varphi_i(t) = \int_0^1 f'(\rho v_i(t)) d\rho, \quad \phi_i(t) = \int_0^1 g'(\rho v_i(t)) d\rho.$$

There exist $\varphi^* \in (0, 1]$ and $\phi^* \in (0, 1]$ such that $\varphi_i(t) \leq \varphi^* \leq 1$ and $\phi_i(t) \leq \phi^* \leq 1$. Hence, one has

$$\dot{V}(t) \leq -2(1 - np - q) \sum_{i=1}^n (x_i^2 + y_i^2 + z_i^2).$$

Then, $\dot{V}(t) < 0$ holds true for $np + q < 1$ when $v \neq 0$. This completes the proof.

3 Case studies

In this section, the activation functions of neurons are chosen as $f = g = \tanh$, which is a typical sigmoid function and has been widely used in neural networks.

Case I $n = 3$, $a_{11} = b_{11} = c_{11} = -1.4$, $a_{12} = b_{12} = c_{12} = 1.3$, $a_{13} = b_{13} = c_{13} = -6$, $a_{21} = b_{21} = c_{21} = 1.1$, $a_{22} = b_{22} = c_{22} = 0$, $a_{23} = b_{23} = c_{23} = 2.6$, $a_{31} = b_{31} = c_{31} = 2.4$, $a_{32} = b_{32} = c_{32} = -2$, $a_{33} = b_{33} = c_{33} = 4$, $r_1 = r_2 = r_3 = 0.17$, and $\tau_1 = \tau_2 = \tau_3 = \tau_s$.

The trivial equilibrium of the network free of coupling time delays is locally asymptotically stable according to the Routh-Hurwitz criteria. Solving the polynomial $D(\omega) = 0$ gives two positive and simple roots $\omega_1 = 3.26$ and $\omega_2 = 3.17$. Then, two sets of critical time delays can be obtained as $\tau_{1,l} = 0.15, 2.08, 4.00, \dots$, and $\tau_{2,l} = 1.26, 3.25, 5.23, \dots$. It is easy to check that $D'(\omega_1) > 0$ and $D'(\omega_2) < 0$ hold. From $\text{sgn Re}\left(\frac{d\lambda(\tau)}{d\tau}\right)\Big|_{\lambda=i\omega} = \text{sgn } D'(\omega)$, the characteristic equation of the network adds a new pair of conjugate roots with positive real parts for each crossing at $\tau_{1,l}$, but reduces a pair of conjugate roots with positive real parts for each crossing at $\tau_{2,l}$. Therefore, the trivial equilibrium of the network is locally asymptotically stable for $\tau \in [0, \tau_{1,0}) \cup \dots \cup (\tau_{2,l}, \tau_{1,l+1}) \cup \dots \cup (\tau_{2,14}, \tau_{1,15})$, and becomes unstable for $\tau \in$

$(\tau_{1,0}, \tau_{2,0}) \cup \dots \cup (\tau_{1,l}, \tau_{2,l}) \cup \dots \cup (\tau_{1,15}, +\infty)$. Figure 3(a) shows that the trivial equilibrium of the network loses its stability and a branch of periodic oscillations arising from the Hopf bifurcation comes into being when $\tau = 0.3 \in (\tau_{1,0}, \tau_{2,0})$. Obviously, the corresponding neurons in sub-networks oscillate with the same waveform and in phase with each other, i.e., $x_i = y_i = z_i$. Figure 3(b) illustrates that the trivial equilibrium remains stable when $\tau = 1.8 \in (\tau_{2,0}, \tau_{1,1})$. Figure 3(c) gives that the trivial equilibrium becomes unstable again and periodic oscillations occur when $\tau = 2.4 \in (\tau_{1,1}, \tau_{2,1})$. As shown in Fig. 3(c), the neurons in the sub-networks move with the same waveform and different phases. The red, green, and blue lines represent the responses of the first neurons in the three sub-networks. It is shown that the network undergoes the stable rest state, completely synchronous periodic oscillation, stable rest state, and periodic oscillations with different phases as the time delay increases.

Fig. 3 Responses of the multiplex network when $a_{11} = b_{11} = c_{11} = -1.4$, $a_{12} = b_{12} = c_{12} = 1.3$, $a_{13} = b_{13} = c_{13} = -6$, $a_{21} = b_{21} = c_{21} = 1.1$, $a_{22} = b_{22} = c_{22} = 0$, $a_{23} = b_{23} = c_{23} = 2.6$, $a_{31} = b_{31} = c_{31} = 2.4$, $a_{32} = b_{32} = c_{32} = -2$, $a_{33} = b_{33} = c_{33} = 4$, $r_1 = r_2 = r_3 = 0.17$, and $\tau_1 = \tau_2 = \tau_3 = \tau_s$ (color online)

To show the effects of the property of the coupling on the network dynamics, let the parameters within the sub-networks be the same as those in the above case study and the time delay $\tau_s = 0.1$. Figure 4(a) gives the asynchronous periodic oscillations with $x_i = -y_i = z_i$ when $k_1 = -k_2 = -k_3$. As shown in Fig. 4(a), the neurons of Networks 1 and 3 oscillate with the same waveform and in phase with each other, but the neurons of Networks 1 and 2 move with the same waveform and half a period out of phase with each other. Figure 4(b) illustrates that the neurons of Networks 1 and 2 move synchronously, but the neurons of Networks 1 and 3 oscillate out-of-phase. Figure 4(c) shows the asynchronous periodic oscillations with $x_i = -y_i = -z_i$ when $k_1 = k_2 = -k_3$. In this case, the neurons of Networks 2 and 3 move synchronously, but the neurons of Networks 1 and 2 oscillate out-of-phase. As shown in Fig. 4, the red solid, green dashed, and blue dotted curves represent the responses of the first neurons

Fig. 4 Responses of the network when $\tau_s = 0.1$: (a) asynchronous periodic oscillation with $x_i = -y_i = z_i$; (b) asynchronous periodic oscillation with $x_i = y_i = -z_i$; (c) asynchronous periodic oscillation with $x_i = -y_i = -z_i$ (color online)

in the sub-networks. It is interesting that the network exhibits four patterns of synchronization between networks for the same product of coupling strengths, as depicted in Figs. 3(a) and 4. It follows that the property of the coupling can be used to regulate the patterns of synchronization between sub-networks.

Case II $n = 3$, $a_{11} = b_{11} = c_{11} = 1.5$, $a_{12} = b_{12} = c_{12} = 2.9$, $a_{13} = b_{13} = c_{13} = 0.7$, $a_{21} = b_{21} = c_{21} = -2$, $a_{22} = b_{22} = c_{22} = 1.18$, $a_{23} = b_{23} = c_{23} = 0$, $a_{31} = b_{31} = c_{31} = 2.98$, $a_{32} = b_{32} = c_{32} = -10$, $a_{33} = b_{33} = c_{33} = 0.47$, $r_1 = r_2 = r_3 = 0.2$, and $\tau_1 = \tau_2 = \tau_3 = \tau_s$.

Figure 5 shows the coexistence of two period-2 orbits and two period-4 oscillations under different initial conditions when $\tau_s = 0.1$. It is seen that the period-2 orbits on the x_1 - x_2 plane are colored in blue and red lines, while the period-4 trajectories are colored in purplish red and black lines. The blue, red, purplish red, and black lines correspond to the initial conditions, which are defined as $(x_1(0), x_2(0), x_3(0), y_1(0), y_2(0), y_3(0), z_1(0), z_2(0), z_3(0))$, IC1 (0.5, 0.1, 0.2, 0.7, 0.8, 0.3, 0.6, 0.9, 0.4), IC2 (-0.5, -0.1, -0.2, -0.7, -0.8, -0.3, -0.6, -0.9, -0.4), IC3 (0.5, -0.1, 0.2, 0.7, 0.8, 0.3, 0.6, -0.9, 0.4), and IC4 (-0.5, 0.1, -0.2, -0.7, -0.8, -0.3, -0.6, 0.9, -0.4), respectively. By increasing the time delay $\tau_s = 0.6$, Fig. 6 illustrates the coexistence of a new pair of period-2 oscillations and two separated chaotic motions. As shown in Fig. 7, the period-2 responses disappear and a pair of period-4 solutions come into being when $\tau_s = 0.8$. Figure 8 gives the coexistence of two chaotic motions and two period-2 responses for $\tau_s = 1.2$. As shown in Figs. 5–8, different types of multiple coexisting attractors are observed when the coupling time delays vary.

Figure 9 gives the bifurcation diagram as a function of the coupling time delay τ_s . The Poincaré section is defined by $\Sigma = \{(\tau_s, x_1) : (x_2 = 0, \dot{x}_2 > 0)\}$. When the coupling time delay

Fig. 5 Phase trajectories of the network when $\tau_s = 0.1$: (a) two period-2 oscillations under the initial conditions IC1 and IC2; (b) two period-4 orbits under the initial conditions IC3 and IC4 (color online)

Fig. 6 Phase trajectories of the network when $\tau_s = 0.6$: (a) two period-2 oscillations under the initial conditions IC1 and IC2; (b) two chaotic motions under the initial conditions IC3 and IC4 (color online)

Fig. 7 Phase trajectories of the network when $\tau_s = 0.8$: (a) two period-4 oscillations under the initial conditions IC1 and IC2; (b) two chaotic motions under the initial conditions IC3 and IC4 (color online)

varies, the multiplex network exhibits interesting and complicated behaviors, e.g., multi-periodic orbits and chaotic attractors, showing that time delay can be used to regulate the dynamic performances of networks, including the generation and transition of different complex oscillations.

Fig. 8 Phase trajectories of the network when $\tau_s = 1.2$: (a) two separated chaotic attractors under the initial conditions IC1 and IC2; (b) two coexisting period-2 orbits under the initial conditions IC3 and IC4 (color online)

Fig. 9 Bifurcation diagrams on the Poincaré section under the initial conditions IC1, IC2, IC3, and IC4 when the coupling time delay varies (color online)

4 Circuit implementation

The circuit is constructed based on Hopfield neuron circuit, transfer function circuit, and time delay circuit. As shown in Fig. 10, the Hopfield neuron circuit unit is achieved by inverting adder circuit, integral circuit, and inverting circuit. Blocks with $\tanh$ / $-\tanh$ represent the positive/negative hyperbolic tangent function circuit between neurons. The hyperbolic tangent function circuit unit is constructed by crucial bipolar transistors, operational amplifiers, potentiometers, and ± 15 V direct current (DC) voltage sources^[41–42]. Blocks delay1, delay2, and delay3 are time delay circuits. The time delay circuit unit can be achieved by a resistor-capacitor

Fig. 10 A circuit for the multiplex Hopfield neural network (color online)

(RC) low-pass filter consisting of operational amplifiers, capacitors, and resistors^[34,43]. The circuit state equations can be established as follows:

$$\begin{cases}
 R_{x1}C_{x1} \frac{dX_1}{dt'} = -\frac{R_{x4}}{R_{x7}}X_1 + \frac{R_{x4}}{R_{x8}}f(X_1) + \frac{R_{x4}}{R_{x9}}f(X_2) + \frac{R_{x4}}{R_{x10}}f(X_3) + \frac{R_{x4}}{R_{x11}}f(Z_1(t' - \tau'_1)), \\
 R_{x2}C_{x2} \frac{dX_2}{dt'} = -\frac{R_{x5}}{R_{x12}}X_2 + \frac{R_{x5}}{R_{x13}}f(X_1) + \frac{R_{x5}}{R_{x14}}f(X_2) + \frac{R_{x5}}{R_{x15}}f(X_3), \\
 R_{x3}C_{x3} \frac{dX_3}{dt'} = -\frac{R_{x6}}{R_{x16}}X_3 + \frac{R_{x6}}{R_{x17}}f(X_1) + \frac{R_{x6}}{R_{x18}}f(X_2) + \frac{R_{x6}}{R_{x19}}f(X_3), \\
 R_{y1}C_{y1} \frac{dY_1}{dt'} = -\frac{R_{y4}}{R_{y7}}Y_1 + \frac{R_{y4}}{R_{y8}}f(Y_1) + \frac{R_{y4}}{R_{y9}}f(Y_2) + \frac{R_{y4}}{R_{y10}}f(Y_3) + \frac{R_{y4}}{R_{y11}}f(X_1(t' - \tau'_2)), \\
 R_{y2}C_{y2} \frac{dY_2}{dt'} = -\frac{R_{y5}}{R_{y12}}Y_2 + \frac{R_{y5}}{R_{y13}}f(Y_1) + \frac{R_{y5}}{R_{y14}}f(Y_2) + \frac{R_{y5}}{R_{y15}}f(Y_3), \\
 R_{y3}C_{y3} \frac{dY_3}{dt'} = -\frac{R_{y6}}{R_{y16}}Y_3 + \frac{R_{y6}}{R_{y17}}f(Y_1) + \frac{R_{y6}}{R_{y18}}f(Y_2) + \frac{R_{y6}}{R_{y19}}f(Y_3), \\
 R_{z1}C_{z1} \frac{dZ_1}{dt'} = -\frac{R_{z4}}{R_{z7}}Z_1 + \frac{R_{z4}}{R_{z8}}f(Z_1) + \frac{R_{z4}}{R_{z9}}f(Z_2) + \frac{R_{z4}}{R_{z10}}f(Z_3) + \frac{R_{z4}}{R_{z11}}f(Y_1(t' - \tau'_3)), \\
 R_{z2}C_{z2} \frac{dZ_2}{dt'} = -\frac{R_{z5}}{R_{z12}}Z_2 + \frac{R_{z5}}{R_{z13}}f(Z_1) + \frac{R_{z5}}{R_{z14}}f(Z_2) + \frac{R_{z5}}{R_{z15}}f(Z_3), \\
 R_{z3}C_{z3} \frac{dZ_3}{dt'} = -\frac{R_{z6}}{R_{z16}}Z_3 + \frac{R_{z6}}{R_{z17}}f(Z_1) + \frac{R_{z6}}{R_{z18}}f(Z_2) + \frac{R_{z6}}{R_{z19}}f(Z_3),
 \end{cases} \quad (6)$$

where X_j , Y_j , and Z_j are the output voltages in the circuit, f is the hyperbolic tangent function, R_{xi} , R_{yi} , and R_{zi} represent resistors, C_{xi} , C_{yi} , and C_{zi} stand for capacitors, τ'_1 , τ'_2 , and τ'_3 denote time delays in the blocks delay1, delay2, and delay3, respectively, $j = 1, 2, 3$, $i = 1, 2, 3, \dots, 19$, and

$$\begin{aligned} R_{x1} &= R_{y1} = R_{z1} = 1 \text{ k}\Omega, & R_{x2} &= R_{y2} = R_{z2} = 1 \text{ k}\Omega, \\ R_{x3} &= R_{y3} = R_{z3} = 1 \text{ k}\Omega, & R_{x4} &= R_{y4} = R_{z4} = 10 \text{ k}\Omega, \\ R_{x5} &= R_{y5} = R_{z5} = 10 \text{ k}\Omega, & R_{x6} &= R_{y6} = R_{z6} = 10 \text{ k}\Omega, \\ R_{x7} &= R_{y7} = R_{z7} = 10 \text{ k}\Omega, & R_{x8} &= R_{y8} = R_{z8} = 6.67 \text{ k}\Omega, \\ R_{x9} &= R_{y9} = R_{z9} = 3.45 \text{ k}\Omega, & R_{x10} &= R_{y10} = R_{z10} = 14.29 \text{ k}\Omega, \\ R_{x11} &= R_{y11} = R_{z11} = 50 \text{ k}\Omega, & R_{x12} &= R_{y12} = R_{z12} = 10 \text{ k}\Omega, \\ R_{x13} &= R_{y13} = R_{z13} = 5 \text{ k}\Omega, & R_{x14} &= R_{y14} = R_{z14} = 8.47 \text{ k}\Omega, \\ R_{x16} &= R_{y16} = R_{z16} = 10 \text{ k}\Omega, & R_{x17} &= R_{y17} = R_{z17} = 3.36 \text{ k}\Omega, \\ R_{x18} &= R_{y18} = R_{z18} = 1 \text{ k}\Omega, & R_{x19} &= R_{y19} = R_{z19} = 21.27 \text{ k}\Omega, & R_{gi} &= 1 \text{ k}\Omega, \\ C_{x1} &= C_{y1} = C_{z1} = C_{x2} = C_{y2} = C_{z2} = C_{x3} = C_{y3} = C_{z3} = 1 \text{ }\mu\text{F}. \end{aligned}$$

It is easy to check that Eq. (1) is the dimensionless form of the circuit equation (6), where

$$t = \frac{1}{R_{x1}C_{x1}}t', \quad \tau_j = \frac{1}{R_{x1}C_{x1}}\tau'_j.$$

Moreover, the connections of R_{x15} , R_{y15} , and R_{z15} should be removed when

$$a_{23} = b_{23} = c_{23} = 0.$$

Figures 11–14 show the phase portraits of the output voltages in the circuit based on the Multisim electronic circuit simulator. As shown in Figs. 11–14, when the coupling time delay varies, the circuit exhibits different patterns of multiple coexisting attractors, which are consistent with the phenomena given in Figs. 5–8.

Fig. 11 Phase portraits of output voltages in the circuit when $\tau'_s = 0.1 \text{ ms}$ (color online)

5 Conclusions

In biological and physiological systems, the interacting neural networks are crucial for the function of the brain and the efficient processing of information. The dynamical behaviors of

Fig. 12 Phase portraits of output voltages in the circuit when $\tau'_s = 0.6$ ms (color online)

Fig. 13 Phase portraits of output voltages in the circuit when $\tau'_s = 0.8$ ms (color online)

Fig. 14 Phase portraits of output voltages in the circuit when $\tau'_s = 1.2$ ms (color online)

multiplex neural systems are revealed through an example of three coupled networks, each of which has an arbitrary number of neurons. In the parameter plane of the product of coupling strength and the sum of time delays, the delay-independent and delay-dependent regions of the network equilibrium are shown. By regarding the sum of the coupling time delay as the pa-

parameter, various dynamical phenomena are observed, e.g., multiple stability switches, different patterns of periodic oscillations, the coexistence of two period-2 and two period-4 responses, the coexistence of two period-2 orbits and two chaotic motions, and the coexisting period-4 and chaotic attractors. Moreover, it is found that the excitatory and inhibitory couplings can induce complete and partial synchronization between sub-networks for the same product of coupling strengths. An electronic circuit is designed, and the phenomena agree with the revealed results. The obtained results in this paper can lead to a broader understanding in the mechanisms of the rhythms and complex evolution patterns of neural systems.

Acknowledgements The authors thank the anonymous reviewers for their helpful comments and suggestions that have helped to improve the presentation.

References

- [1] BATTISTON, F., NICOSIA, V., and LATORA, V. Structural measures for multiplex networks. *Physical Review E*, **89**, 032804 (2014)
- [2] BOCCALETTI, S., BIANCONI, G., CRIADO, R., DEL GENIO, C. I., GOMEZ-GARDENES, J., ROMANCE, M., SENDINA-NADAL, I., WANG, Z., and ZANIN, M. The structure and dynamics of multilayer networks. *Physics Reports*, **544**, 1–122 (2014)
- [3] GOMEZ, S., DIAZ-GUILERA, A., GOMEZ-GARDENES, PEREZ-VICENTE, C. J., MORENO, Y., and ARENAS, A. Diffusion dynamics on multiplex networks. *Physical Review Letters*, **110**, 028701 (2013)
- [4] ZHOU, C. S., ZEMANOVA, L., ZAMORA-LOPEZ, G., HILGETAG, C. C., and KURTHS, J. Structure-function relationship in complex brain networks expressed by hierarchical synchronization. *New Journal of Physics*, **9**, 178 (2007)
- [5] SUN, W. G., WANG, R. B., WANG, W. X., and CAO, J. T. Analyzing inner and outer synchronization between two coupled discrete-time networks with time delays. *Cognitive Neurodynamics*, **4**, 225–231 (2010)
- [6] BENTLEY, B., BRANICKY, R., BARNES, C. L., CHEW, Y. L., YEMINI, E., BULLMORE, E. T., VERTES, P. E., and SCHAFER, W. R. The multilayer connectome of *Caenorhabditis elegans*. *PLoS Computational Biology*, **12**, e1005283 (2016)
- [7] NIKITIN, D., OMELCHENKO, I., ZAKHAROVA, A., AVETIAN, M., FRADKOV, A. L., and SCHOLL, E. Complex partial synchronization patterns in networks of delay-coupled neurons. *Philosophical Transactions of the Royal Society A*, **377**, 20180128 (2019)
- [8] CARDILLO, A., ZANIN, M., GOMEZ-GARDENES, ROMANCE, M., GARCIA DEL AMO, A. J., and BOCCALETTI, S. Modeling the multi-layer nature of the European Air Transport Network: resilience and passengers re-scheduling under random failures. *European Physical Journal-Special Topics*, **215**, 23–33 (2013)
- [9] BATTISTON, F., NICOSIA, V., CHAVEZ, M., and LATORA, V. Multilayer motif analysis of brain networks. *Chaos*, **27**, 047404 (2017)
- [10] KORN, H. and FAURE, P. Is there chaos in the brain? II. experimental evidence and related models. *Comptes Rendus Biologies*, **326**, 787–840 (2003)
- [11] HU, X. Y., LIU, C. X., LIU, L., NI, J. K., and YAO, Y. P. Chaotic dynamics in a neural network under electromagnetic radiation. *Nonlinear Dynamics*, **91**, 1541–1554 (2018)
- [12] ZHOU, L. L., TAN, F., YU, F., and LIU, W. Cluster synchronization of two-layer nonlinearly coupled multiplex networks with multi-links and time-delays. *Neurocomputing*, **359**, 264–275 (2019)
- [13] MAJHI, S., PERC, M., and GHOSH, D. Chimera states in a multilayer network of coupled and uncoupled neurons. *Chaos*, **27**, 073109 (2017)
- [14] YU, Y. and WANG, Q. Y. Oscillation dynamics in an extended model of thalamic-basal ganglia. *Nonlinear Dynamics*, **98**, 1065–1080 (2019)
- [15] MA, J., YANG, Z. Q., YANG, L. J., and TANG, J. A physical view of computational neurodynamics. *Journal of Zhejiang University-Science A*, **20**, 639–659 (2019)

-
- [16] ZHOU, J., LIU, Z. R., and XIANG, L. Global dynamics of delayed bidirectional associative memory (BAM) neural networks. *Applied Mathematics and Mechanics (English Edition)*, **26**(3), 327–335 (2005) <https://doi.org/10.1007/BF02440083>
 - [17] KACHHVAH, A. D. and JALAN, S. Delay regulated explosive synchronization in multiplex networks. *New Journal of Physics*, **21**, 015006 (2019)
 - [18] SAWICKI, J., OMELCHENKO, I., ZAKHAROVA, A., and SCHOELL, E. Delay controls chimera relay synchronization in multiplex networks. *Physical Review E*, **98**, 062224 (2018)
 - [19] HAMMOND, C., BERGMAN, H., and BROWN, P. Pathological synchronization in Parkinson’s disease: networks, models and treatments. *Trends in Neurosciences*, **30**, 357–364 (2007)
 - [20] MAO, X. C. and WANG, Z. H. Stability, bifurcation, and synchronization of delay-coupled ring neural networks. *Nonlinear Dynamics*, **84**, 1063–1078 (2016)
 - [21] TANG, Y., QIAN, F., GAO, H. J., and KURTHS, J. Synchronization in complex networks and its application — a survey of recent advances and challenges. *Annual Reviews in Control*, **38**, 184–198 (2014)
 - [22] HAN, F., GU, X. C., WANG, Z. J., FAN, H., CAO, J. F., and LU, Q. S. Global firing rate contrast enhancement in E/I neuronal networks by recurrent synchronized inhibition. *Chaos*, **28**, 106324 (2018)
 - [23] FAN, D. G., ZHENG, Y. H., YANG, Z. C., and WANG, Q. Y. Improving control effects of absence seizures using single-pulse alternately resetting stimulation (SARS) of corticothalamic circuit. *Applied Mathematics and Mechanics (English Edition)*, **41**(9), 1287–1302 (2020) <https://doi.org/10.1007/s10483-020-2644-8>
 - [24] LIANG, S. and WANG, Z. H. Controlling a neuron by stimulating a coupled neuron. *Applied Mathematics and Mechanics (English Edition)*, **40**(1), 13–24 (2019) <https://doi.org/10.1007/s10483-019-2407-8>
 - [25] HU, H. Y. and WANG, Z. H. *Dynamics of Controlled Mechanical Systems with Delayed Feedback*, Springer-Verlag, Heidelberg (2002)
 - [26] CHEN, L. X. and CAI, G. P. Design method of multiple time-delay controller for active structural vibration control. *Applied Mathematics and Mechanics (English Edition)*, **30**(11), 1405–1414 (2009) <https://doi.org/10.1007/s10483-009-1106-z>
 - [27] STEPAN, G. Delay effects in brain dynamics. *Philosophical Transactions of the Royal Society A*, **367**, 1059–1062 (2009)
 - [28] FLUNKERT, V., FISCHER, I., and SCHOELL, E. Dynamics, control and information in delay-coupled systems. *Philosophical Transactions of the Royal Society A*, **371**, 20120465 (2013)
 - [29] WANG, Y. P., CONG, Y. H., and HU, G. D. Delay-dependent stability of linear multistep methods for differential systems with distributed delays. *Applied Mathematics and Mechanics (English Edition)*, **39**(12), 1837–1844 (2018) <https://doi.org/10.1007/s10483-018-2392-9>
 - [30] BALDI, P. and ATIYA, A. F. How delays affect neural dynamics and learning. *IEEE Transactions on Neural Networks*, **5**, 612–621 (1994)
 - [31] MAO, X. C., SUN, J. Q., and LI, S. F. Dynamics of delay-coupled FitzHugh-Nagumo neural rings. *Chaos*, **28**, 013104 (2018)
 - [32] CAMPBELL, S. A., EDWARDS, R., and VAN DEN DRIESSCHE, P. Delayed coupling between two neural network loops. *SIAM Journal on Applied Mathematics*, **65**, 316–335 (2005)
 - [33] HSU, C. H. and YANG, T. S. Periodic oscillations arising and death in delay-coupled neural loops. *International Journal of Bifurcation and Chaos*, **17**, 4015–4032 (2007)
 - [34] MAO, X. C., ZHOU, X. Y., SHI, T. T., and QIAO, L. Dynamical analysis of coupled bidirectional FitzHugh-Nagumo neuronal networks with multiple delays. *Journal of Computational and Nonlinear Dynamics*, **14**, 061002 (2019)
 - [35] SINGH, A., GHOSH, S., JALAN, S., and KURTHS, J. Synchronization in delayed multiplex networks. *Europhysics Letters*, **111**, 30010 (2015)
 - [36] CHENG, C. Y. Induction of Hopf bifurcation and oscillation death by delays in coupled networks. *Physics Letters A*, **374**, 178–185 (2009)

- [37] SONG, Y. L. and XU, J. Inphase and antiphase synchronization in a delay-coupled system with applications to a delay-coupled FitzHugh-Nagumo system. *IEEE Transactions on Neural Networks and Learning Systems*, **23**, 1659–1670 (2012)
- [38] XU, X., YU, D., and WANG, Z. Inter-layer synchronization of periodic solutions in two coupled rings with time delay. *Physica D*, **396**, 1–11 (2019)
- [39] HOPFIELD, J. J. Neurons with graded response have collective computational properties like those of two-state neurons. *Proceedings of the National Academy of Sciences of the United States of American*, **81**, 3088–3092 (1984)
- [40] HASSARD, B. D., KAZARINOFF, N. D., and WAN, Y. H. *Theory and Application of Hopf Bifurcation*, Cambridge University Press, Cambridge (1981)
- [41] DUAN, S. K. and LIAO, X. F. An electronic implementation for Liao’s chaotic delayed neuron model with non-monotonous activation function. *Physics Letters A*, **369**, 37–43 (2007)
- [42] BAO, B. C., QIAN, H., XU, Q., CHEN, M., WANG, J., and YU, Y. J. Coexisting behaviors of asymmetric attractors in hyperbolic-type memristor based Hopfield neural network. *Frontiers in Computational Neuroscience*, **11**, 81 (2017)
- [43] ABLAY, G. Novel chaotic delay systems and electronic circuit solutions. *Nonlinear Dynamics*, **81**, 1795–1804 (2015)