New groups of solutions to the Whitham-Broer-Kaup equation*

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Abstract The Whitham-Broer-Kaup model is widely used to study the tsunami waves. The classical Whitham-Broer-Kaup equations are re-investigated in detail by the generalized projective Riccati-equation method. 20 sets of solutions are obtained of which, to the best of the authors' knowledge, some have not been reported in literature. Bifurcation analysis of the planar dynamical systems is then used to show different phase portraits of the traveling wave solutions under various parametric conditions.

Key words Whitham-Broer-Kaup equation, travelling wave, bifurcation analysis, projective Riccati-equation method

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1 Introduction

Study of the dynamics and behavior of the tsunami waves is not only of scientific importance but also of significant contribution to the practices in our daily life. A tsunami wave is often characterized as a shallow-water wave since its wavelength is usually far longer than the water depth. Many mathematical models have been developed to describe the shallow water waves in which the typical models include the Korteweg-de Vries (KdV) equation^[1], the Boussinesq equation^[2], the Degasperis-Procesi equation^[3], the Benjamin-Bona-Mahony (BBM) equation^[4], the Kadomtsev-Petviashvili (K-P) equation^[5], and the Whitham-Broer-Kaup (WBK) model^[6]. Particularly, the WBK model is usually used to describe the tsunami wave dynamics under gravity, which is formulated based on the assumption that the fluid is incompressible and irrotational.

Different analytical methods have been developed and used to obtain the solutions of the WBK model and the soliton solutions of nonlinear wave equations, such as the improved

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Riccati equations method^[6–10], the first integral method^[11], the Backlund transformation method^[12], the optimal homotopy asymptotic method^[13], the hyperbolic function method^[14], the homogeneous balance method^[15], the parity-time symmetric potential method^[16], the Darboux method^[17], the variable-coefficient fractional Y-expansion method^[18], and the fractional F-expansion method^[19].

Yan and Zhang^[7] once found some new solutions to the WBK equations by using the generalized projective Riccati-equation method. However, during their solution procedure, they did not treat the linear and nonlinear terms separately, which leads to their solutions questionable.

In this work, we re-visit the WBK model by using the generalized projective Riccati-equation method^[6–10] to obtain the solutions that not only correct the errors in the work of Yan and Zhang^[7] but also include new groups of solutions which are overlooked in the literature. Also, we present the bifurcation analysis to investigate solution multiplicities and their behaviors.

This work is organized as follows. In Section 2, the WBK model^[6] to describe the traveling shallow water wave (tsunami wave) is introduced that is solved by the generalized projective Riccati-equation method^[6–10] as demonstrated in Section 3. We demonstrate 20 sets of solutions of which, to the best of the authors' knowledge, some have not been reported in the literature. In Section 4, we discuss the classification of these sets of solutions, which is followed by the bifurcation analysis of traveling wave solutions as shown in Section 5. In the last section, some conclusions are given.

2 Model and mathematical preliminaries

A sketch of a traveling shallow water wave (tsunami wave) with wavelength l and amplitude a is illustrated in Fig. 1, where h is the undisturbed depth, and λ is the speed of the wave. The WBK model to describe the wave phenomena presented in Fig. 1 can be written as

$$u_t + uu_x + H_x + \beta u_{xx} = 0, \quad (1a)$$

$$H_t + (Hu)_x + \alpha u_{xxx} - \beta H_{xx} = 0, \quad (1b)$$

where u is the horizontal velocity, H is the height deviating from the equilibrium position of the liquid, and α and β are constants that represent different diffusion capabilities of the medium. When $\alpha = 0$ and $\beta \neq 0$, Eq. (1) is the classical long-wave equation that describes shallow water waves with diffusion^[22]. If $\alpha = 1$ and $\beta = 0$, Eq. (1) is the modified Boussinesq equations^[23].

Fig. 1 Sketch of a traveling shallow water wave (tsunami wave) with wavelength l and amplitude a

Introduce the following scaling transformation^[20]:

$$u(x, t) = u(\xi), \quad H(x, t) = H(\xi) \quad \text{with} \quad \xi = x + \lambda t + c. \quad (2)$$

Substituting Eq. (2) into Eq. (1), we obtain

$$\lambda u' + uu' + H' + \beta u'' = 0, \quad (3a)$$

$$\lambda H' + (Hu)' + \alpha u''' - \beta H'' = 0, \quad (3b)$$

where c is the phase constant. As such, the original WBK model in Eq. (1) for the tsunami wave propagation is reduced to the coupled nonlinear differential equations in Eq. (3) which will be studied in detail in this work.

3 Method and solution

The generalized projective Riccati-equation method^[6–10] is used to solve Eqs. (3), in which $u(\xi)$ and $H(\xi)$ are expressed as

$$u(\xi) = \sum_{i=1}^m \omega^{i-1}(\xi) (A_i \omega(\xi) + B_i \sqrt{\mu_1(1 + \mu_2 \omega^2(\xi))}) + A_0, \quad (4a)$$

$$H(\xi) = \sum_{i=1}^n \omega^{i-1}(\xi) (a_i \omega(\xi) + b_i \sqrt{\mu_1(1 + \mu_2 \omega^2(\xi))}) + a_0. \quad (4b)$$

In Eq. (4), a_i , b_i , A_i , and B_i ($i = 0, 1, 2, \dots$) are constants to be determined, μ_1 and μ_2 are constants, and $\omega(\xi)$ is the perturbed function that satisfies

$$\frac{d\omega}{d\xi} = R(1 + \mu_2 \omega^2). \quad (5)$$

It is easy to find the solution of $\omega(\xi)$ from Eq. (5),

$$\omega(\xi) = \frac{\tanh((- \mu_2)^{\frac{1}{2}}(R\xi + C))}{(-\mu_2)^{\frac{1}{2}}} \quad \text{when } \mu_2 < 0, \quad (6a)$$

$$\omega(\xi) = \frac{\tan(\mu_2^{\frac{1}{2}}(R\xi + C))}{\mu_2^{\frac{1}{2}}} \quad \text{when } \mu_2 > 0, \quad (6b)$$

where R and C are constants.

The orders m and n in Eq. (4) are chosen based on the highest partial differential order of the governing equation (3). In this work, we choose $m = 1$ and $n = 2$. As such, Eq. (4) is reduced to

$$u(\xi) = A_0 + A_1 \omega + B_1 \sqrt{\mu_1(1 + \mu_2 \omega^2)}, \quad (7a)$$

$$H(\xi) = a_0 + a_1 \omega + b_1 \sqrt{\mu_1(1 + \mu_2 \omega^2)} + a_2 \omega^2 + b_2 \omega \sqrt{\mu_1(1 + \mu_2 \omega^2)}. \quad (7b)$$

Substituting the transformations in Eq. (7) into Eq. (8), and collecting the terms with the same power in ω^i ($i = 0, 1, 2, \dots$) and $\omega^j \sqrt{\mu_1 + \mu_1 \mu_2 \omega^2}$ ($j = 0, 1, 2, \dots$), respectively, we get a

set of equations. Forcing these equations to be equal to zero, we have

$$a_1 + \lambda A_1 + A_0 A_1 = 0, \quad (8a)$$

$$2a_2 + 2\beta\mu_2 R A_1 + A_1^2 + \mu_1 \mu_2 B_1^2 = 0, \quad (8b)$$

$$b_2 + \beta\mu_2 R B_1 + A_1 B_1 = 0, \quad (8c)$$

$$b_1 + \lambda B_1 + A_0 B_1 = 0, \quad (8d)$$

$$\lambda a_1 - 2\beta R a_2 + a_1 A_0 + 2\alpha\mu_2 R^2 A_1 + a_0 A_1 + \mu_1 b_2 B_1 = 0, \quad (8e)$$

$$\lambda a_2 + a_2 A_0 + a_1 A_1 - \beta\mu_2 R a_1 + \mu_1 \mu_2 b_1 B_1 = 0, \quad (8f)$$

$$\begin{aligned} \lambda\mu_2 a_1 - 8\beta\mu_2 R a_2 + \mu_2 a_1 A_0 + \mu_2 a_0 A_1 \\ + 8\alpha\mu_2^2 R^2 A_1 + 3a_2 A_1 + 4\mu_1 \mu_2 b_2 B_1 = 0, \end{aligned} \quad (8g)$$

$$A_1 b_1 + \lambda b_2 + A_0 b_2 + a_1 B_1 - \beta\mu_2 R b_1 = 0, \quad (8h)$$

$$\lambda\mu_2 b_1 + \mu_2 A_0 b_1 - 5\beta\mu_2 R b_2 + 2A_1 b_2 + 5\alpha\mu_2^2 R^2 B_1. \quad (8i)$$

With the aid of the software package, Mathematica, we obtain the solutions to Eq. (8) that are listed as follows.

Group 1:

$$\begin{aligned} A_0 = -\lambda, \quad A_1 = -2(\alpha + \beta^2)^{\frac{1}{2}}\mu_2 R, \quad B_1 = 0, \quad a_0 = -2(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2 R^2, \\ a_1 = 0, \quad a_2 = -2(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2^2 R^2, \quad b_1 = 0, \quad b_2 = 0. \end{aligned}$$

Group 2:

$$\begin{aligned} A_0 = -\lambda, \quad A_1 = 2(\alpha + \beta^2)^{\frac{1}{2}}\mu_2 R, \quad B_1 = 0, \quad a_0 = -2(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2 R^2, \\ a_1 = 0, \quad a_2 = -2(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2^2 R^2, \quad b_1 = 0, \quad b_2 = 0. \end{aligned}$$

Group 3:

$$\begin{aligned} A_0 = -\lambda, \quad A_1 = 0, \quad B_1 = -\frac{(\alpha + \beta^2)^{\frac{1}{2}}\mu_2^{\frac{1}{2}}R}{\mu_1^{\frac{1}{2}}}, \quad a_0 = -4(\alpha + \beta^2)\mu_2 R^2, \quad a_1 = 0, \\ a_2 = -\frac{1}{2}(\alpha + \beta^2)\mu_2^2 R^2, \quad b_1 = 0, \quad b_2 = \frac{\beta(\alpha + \beta^2)^{\frac{1}{2}}\mu_2^{\frac{3}{2}}R^2}{\mu_1^{\frac{1}{2}}}. \end{aligned}$$

Group 4:

$$\begin{aligned} A_0 = -\lambda, \quad A_1 = -(\alpha + \beta^2)^{\frac{1}{2}}\mu_2 R, \quad B_1 = -\frac{(\alpha + \beta^2)^{\frac{1}{2}}\mu_2^{\frac{1}{2}}R}{\mu_1^{\frac{1}{2}}}, \\ a_0 = -(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2 R^2, \quad a_1 = 0, \quad a_2 = -(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2^2 R^2, \\ b_1 = 0, \quad b_2 = -\frac{(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2^{\frac{3}{2}}R^2}{\mu_1^{\frac{1}{2}}}. \end{aligned}$$

Group 5:

$$\begin{aligned} A_0 = -\lambda, \quad A_1 = (\alpha + \beta^2)^{\frac{1}{2}}\mu_2 R, \quad B_1 = -\frac{(\alpha + \beta^2)^{\frac{1}{2}}\mu_2^{\frac{1}{2}}R}{\mu_1^{\frac{1}{2}}}, \\ a_0 = -(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2 R^2, \quad a_1 = 0, \quad a_2 = -(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2^2 R^2, \\ b_1 = 0, \quad b_2 = \frac{(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}})\mu_2^{\frac{3}{2}}R^2}{\mu_1^{\frac{1}{2}}}. \end{aligned}$$

Group 6:

$$A_0 = -\lambda, \quad A_1 = 0, \quad B_1 = \frac{(\alpha + \beta^2)^{\frac{1}{2}} \mu_2^{\frac{1}{2}} R}{\mu_1^{\frac{1}{2}}}, \quad a_0 = -4(\alpha + \beta^2) \mu_2 R^2, \\ a_1 = 0, \quad a_2 = -\frac{1}{2}(\alpha + \beta^2) \mu_2^2 R^2, \quad b_1 = 0, \quad b_2 = -\frac{\beta(\alpha + \beta^2)^{\frac{1}{2}} \mu_2^{\frac{3}{2}} R^2}{\mu_1^{\frac{1}{2}}}.$$

Group 7:

$$A_0 = -\lambda, \quad A_1 = -(\alpha + \beta^2)^{\frac{1}{2}} \mu_2 R, \quad B_1 = \frac{(\alpha + \beta^2)^{\frac{1}{2}} \mu_2^{\frac{1}{2}} R}{\mu_1^{\frac{1}{2}}}, \\ a_0 = -(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}}) \mu_2 R^2, \quad a_1 = 0, \quad a_2 = -(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}}) \mu_2^2 R^2, \\ b_1 = 0, \quad b_2 = \frac{(\alpha + \beta^2 - \beta(\alpha + \beta^2)^{\frac{1}{2}}) \mu_2^{\frac{3}{2}} R^2}{\mu_1^{\frac{1}{2}}}.$$

Group 8:

$$A_0 = -\lambda, \quad A_1 = (\alpha + \beta^2)^{\frac{1}{2}} \mu_2 R, \quad B_1 = \frac{(\alpha + \beta^2)^{\frac{1}{2}} \mu_2^{\frac{1}{2}} R}{\mu_1^{\frac{1}{2}}}, \\ a_0 = -(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}}) \mu_2 R^2, \quad a_1 = 0, \quad a_2 = -(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}}) \mu_2^2 R^2, \\ b_1 = 0, \quad b_2 = -\frac{(\alpha + \beta^2 + \beta(\alpha + \beta^2)^{\frac{1}{2}}) \mu_2^{\frac{3}{2}} R^2}{\mu_1^{\frac{1}{2}}}.$$

Group 9:

$$A_0 = -\lambda, \quad A_1 = 0, \\ a_0 = -5(\alpha + \beta^2) \mu_2 R^2 + \mu_1 B_1^2, \quad a_1 = 0, \quad a_2 = -\frac{1}{2} \mu_1 \mu_2 B_1^2, \\ b_1 = 0, \quad b_2 = -\beta \mu_2 R B_1.$$

Since $\omega(\xi)$ has different values as the sign of μ_2 changes, 16 sets of different solutions of Eq. (3) can be obtained from Groups 1 to 8. Also, we notice that the solutions in Group 9 are over-determined as the coefficient B_1 is related to other coefficients such as a_0 , a_2 , and b_2 . In this case, we need to determine B_1 by substituting the solutions in Group 9 into Eq. (3) and enforcing the resulting equations to be zero. By doing so, we find that equality only holds for $B_1 = 0$ when $\mu_2 < 0$. However, three more solutions are obtained with $\mu_2 > 0$, namely,

$$B_1 = 0, \quad B_1 = -\frac{2(\alpha + \beta^2)^{\frac{1}{2}} \mu_2^{\frac{1}{2}} R}{\mu_1^{\frac{1}{2}}}, \quad B_1 = \frac{2(\alpha + \beta^2)^{\frac{1}{2}} \mu_2^{\frac{1}{2}} R}{\mu_1^{\frac{1}{2}}}. \quad (9)$$

This indicates that four more solutions can be derived from Group 9. Therefore, 20 sets of solutions are found for the WBK model.

4 Classification of solutions

When the physical parameters are properly prescribed in Table 1, we can present the solutions obtained above graphically and discuss the properties of those solutions. In Table 1, Case I and Case II correspond to the multiple solutions to the identical equation with the same set of physical parameters when the sign of μ_2 changes. However, Case III coincides with another set of multiple solutions with a different set of physical parameters when the diffusion parameter α goes from 1.5 to -5 .

Table 1 The corresponding physical parameters used in our analysis

Case	Group	μ_1	μ_2	α	β	R	C	λ
I	1–9	1	1	1.5	2	1	1	0.5
II	1, 2	1	–1	1.5	2	1	1	0.5
III	3, 6	1	–1	–5	2	1	1	0.5

We first consider the solutions given in Group 1. As illustrated in Fig. 2, two different real solutions can be obtained as the physical parameters denoted in Case I and Case II are respectively used. Clearly, the profiles of u and H with negative μ_2 are Fisher-type and soliton-type, respectively. However, the profiles of u and H with positive μ_2 are periodic with infinite discontinuities, which clearly cannot be used to describe water waves.

Fig. 2 Solutions of Group 1 when $\mu_1 = 1$, $|\mu_2| = 1$, $\alpha = 1.5$, $\beta = 2$, $R = 1$, $C = 1$, and $\lambda = 0.5$

Similarly, for the solutions in Group 2, two different real solutions can be obtained when the parameters of Case I and Case II in Table 1 are used. They possess the similar behaviors as those of the solutions in Group 1, as shown in Fig. 3.

For the solutions in Group 3, we notice that one real solution can be obtained when the physical parameters of Case I in Table 1 are used, as shown in Figs. 4(a) and (b). Nevertheless, in this case, the profiles of u and H are discontinuous, which indicates that they are not suitable to describe the water waves. No real solution can be found with the parameters of Case II in Table 1. However, if the diffusion parameter α is changed from 1.5 to -5 , one real solution can be obtained for negative μ_2 (the parameters of Case III in Table 1), as shown in Fig. 4(c) and Fig. 4(d). In this case, the profiles of u and H are soliton-type that can describe water waves.

Fig. 3 Solutions of Group 2 when $\mu_1 = 1$, $|\mu_2| = 1$, $\alpha = 1.5$, $\beta = 2$, $R = 1$, $C = 1$, and $\lambda = 0.5$

Fig. 4 Solutions of Group 3 when $\mu_1 = 1$, $|\mu_2| = 1$, $\beta = 2$, $R = 1$, $C = 1$, and $\lambda = 0.5$

The similar properties of solutions in Group 6 are observed. Only one real solution is captured for positive μ_2 when using the parameters of Case I in Table 1, while no solution is available with the parameters of Case II. Nevertheless, when μ_2 is negative, a real solution can be obtained as α changes from 1.5 to -5 , as shown in Fig. 5. Once again, when the profiles of u and H are discontinuous, they are not useful to describe water waves. When the parameters of Case III are used, the profiles of u and H are soliton-type, which might be able to be used to describe the water waves.

Fig. 5 Solutions of Group 6 when $\mu_1 = 1$, $|\mu_2| = 1$, $\beta = 2$, $R = 1$, $C = 1$, and $\lambda = 0.5$

Only one real solution can be generated from Groups 4, 5, 7, and 8 with the parameters of Case I in Table 1. No solutions can be found with the parameters of Case II or Case III. The profiles of u and H in Groups 4, 5, 7, and 8 are shown in Fig. 6. It can be seen easily that these u and H are discontinuous that cannot be used to describe the water waves.

When Group 9 is under the consideration, we find that there are two real solutions as the parameters denoted in Case I. The profiles of u and H shown in Fig. 7 are discontinuous that cannot be used to describe the water waves.

In summary, though real solutions can be found for all 9 groups of solutions when the parameters of Case I listed in Table 1 are used, none of them are suitable for the description of the water waves as the profiles of u and H are discontinuous. When the parameters of Case II are used, only Group 1 and 2 have real solutions, and the profiles of u and H are Fish-type and soliton-type, respectively. When the parameters of Case III are used, only Groups 3 and 6 have real solutions, and the profiles of u and H are soliton-type. As such, these 4 cases can be used to describe the water waves.

Fig. 6 Solutions of Groups 4, 5, 7, and 8 when $\mu_1 = 1$, $\mu_2 = 1$, $\alpha = 1.5$, $\beta = 2$, $R = 1$, $C = 1$, and $\lambda = 0.5$

5 Bifurcation analysis

To demonstrate different phase portraits of the traveling wave solutions, we use the bifurcation theory of planar dynamical systems^[20] to analyze the nonlinear system (3). Integrating Eqs. (3a) and (3b) with respect to ξ once, we obtain

$$\lambda u + \frac{1}{2}u^2 + H + \beta u' = g_1, \quad (10a)$$

Fig. 7 Solutions of Group 9 when $\mu_1 = 1$, $\mu_2 = 1$, $\alpha = 1.5$, $\beta = 2$, $R = 1$, $C = 1$, and $\lambda = 0.5$

$$\lambda H + H u + \alpha u'' - \beta H' = g_2, \quad (10b)$$

where g_1 and g_2 are integral constants. Substituting Eqs. (3a) and (10a) into Eq. (10b), we obtain

$$u'' = \frac{1}{2(\alpha + \beta^2)}(u^3 + 3\lambda u^2 + 2(\lambda^2 - g_1)u + 2(g_2 - g_1\lambda)). \quad (11)$$

Multiplying u' on both sides of Eq. (11) and integrating the resulting equation with respect to ξ one more time, this equation is converted into a Hamiltonian system as follows:

$$\mathcal{H}(u, u') = \frac{1}{2}u'^2 - \frac{1}{2(\alpha + \beta^2)}\left(\frac{1}{4}u^4 + \lambda u^3 + (\lambda^2 - g_1)u^2 + 2(g_2 - g_1\lambda)u\right), \quad (12)$$

where $\mathcal{H}(u, u')$ denotes the Hamiltonian function.

According to the bifurcation theory of planar dynamical systems, the different phase portraits of traveling waves with different parametric conditions can be obtained through the correlation between the waveform and the orbital phase diagram^[21]. It is known that the solitary wave, the kink or anti-kink waveform, and the periodic wave are associated with the homoclinic orbit, the heteroclinic orbit, and the periodic orbit, respectively. To this knowledge, when the parameters corresponding to different types of traveling waves given by the WBK model as listed in Table 2 are used, the phase portraits of the different waves are achieved, as shown in Fig. 8.

Table 2 Parameters used in bifurcation analysis

Case	λ	α	β	g_1	g_2
I	2	1	0	1	-1
II	1.5	-5	2	1	1
III	0.5	1	2	1	-1

Fig. 8 Phase portraits of typical wave solutions

6 Conclusions

The classical Whitham-Broer-Kaup model has been re-examined thoroughly by using the generalized projective Riccati-equation method. A total of 20 sets of solutions have been obtained, and to the best of the authors' knowledge, some of them have not been reported previously. Real solution patterns have been presented graphically. The bifurcation analysis of the planar dynamical systems has been used to show different phase portraits of the traveling wave solutions under different parametric conditions. This work can potentially contribute to studying the behavior of tsunami waves that are usually characterized as the shallow-water waves.

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References

- [1] GARDNER, C. S., GREENE, J. M., KRUSKAL, M. D., and MIURA, R. M. Korteweg-devries equation and generalizations. VI. methods for exact solution. *Communications on Pure and Applied Mathematics*, **27**(1), 97–133 (1974)
- [2] LÜ, X., WANG, J. P., LIN, F. H., and ZHOU, X. W. Lump dynamics of a generalized two-dimensional Boussinesq equation in shallow water. *Nonlinear Dynamics*, **91**, 1249–1259 (2018)
- [3] LUNDMARK, H. and SZMIGIELSKI, J. Multi-peakon solutions of the Degasperis-Procesi equation. *Inverse Problems*, **19**(6), 1241–1245 (2003)
- [4] BENJAMIN, T. B., BONA, J. L., and MAHONY, J. J. Model equations for long waves in nonlinear dispersive systems. *Philosophical Transactions of the Royal Society of London, Series A, Mathematical and Physical Sciences*, **272**(1220), 47–78 (1972)
- [5] DATE, E., JIMBO, M., KASHIWARA, M., and MIWA, T. Quasi-periodic solutions of the orthogonal KP equation — transformation groups for soliton equations V. *Publications of the Research Institute for Mathematical Sciences*, **18**(3), 1111–1119 (1982)
- [6] ZHANG, P. New exact solutions to breaking soliton equations and Whitham-Broer-Kaup equations. *Applied Mathematics and Computation*, **217**(4), 1688–1696 (2010)
- [7] YAN, Z. Y. and ZHANG, H. Q. New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water. *Physics Letters A*, **285**(5-6), 355–362 (2001)
- [8] HUANG, D. J. and ZHANG, H. Q. Variable-coefficient projective Riccati equation method and its application to a new $(2 + 1)$ -dimensional simplified generalized Broer-Kaup system. *Chaos, Solitons & Fractals*, **23**(2), 601–607 (2005)
- [9] DAI, C. Q. and ZHANG, J. F. New types of interactions based on variable separation solutions via the general projective Riccati equation method. *Reviews in Mathematical Physics*, **19**(2), 195–226 (2007)
- [10] WANG, B. H., LU, P. H., DAI, C. Q., and CHEN, Y. X. Vector optical soliton and periodic solutions of a coupled fractional nonlinear Schrödinger equation. *Results in Physics*, **17**, 103036 (2019)
- [11] GUO, L. H. and ZHOU, R. Travelling wave solutions of a class of space-time fractional Whitham-Broer-Kaup equations. *Journal of Jilin University (Science Edition)*, **55**(1), 7–12 (2017)
- [12] FAN, E. G. and ZHANG, H. Q. Backlund transformation and exact solutions for Whitham-Broer-Kaup equations in shallow water. *Applied Mathematics and Mechanics (English Edition)*, **19**(8), 713–716 (1998) <https://doi.org/10.1007/BF02457745>
- [13] HAQ, S. and ISHAQ, M. Solution of coupled Whitham-Broer-Kaup equations using optimal homotopy asymptotic method. *Ocean Engineering*, **84**, 81–88 (2014)
- [14] XIE, F. D., YAN, Z. Y., and ZHANG, H. Q. Explicit and exact traveling wave solutions of Whitham-Broer-Kaup shallow water equations. *Physics Letters A*, **285**(1-2), 76–80 (2001)
- [15] WANG, M. L. Solitary wave solutions for variant Boussinesq equations. *Physics Letters A*, **199**(3-4), 169–172 (1995)
- [16] DAI, C. Q., FAN, Y., and WANG, Y. Y. Three-dimensional optical solitons formed by the balance between different-order nonlinearities and high-order dispersion/diffraction in parity-time symmetric potentials. *Nonlinear Dynamics*, **98**, 489–499 (2019)
- [17] DAI, C. Q. and ZHANG, J. F. Controlling effect of vector and scalar crossed double-Ma breathers in a partially nonlocal nonlinear medium with a linear potential. *Nonlinear Dynamics*, **100**, 1621–1628 (2020)
- [18] WU, G. Z. and DAI, C. Q. Nonautonomous soliton solutions of variable-coefficient fractional nonlinear Schrödinger equation. *Applied Mathematics Letters*, **106**, 106365 (2020)
- [19] WANG, B. H. and WANG, Y. Y. Fractional white noise functional soliton solutions of a wick-type stochastic fractional NLSE. *Applied Mathematics Letters*, **110**, 106583 (2020)
- [20] LI, J. B. and ZHANG, L. J. Bifurcations of traveling wave solutions in generalized Pochhammer-Chree equation. *Chaos, Solitons & Fractals*, **14**(4), 581–593 (2002)
- [21] LERMAN, L. M. Complex dynamics and bifurcations in a Hamiltonian system having a transversal homoclinic orbit to a saddle focus. *Chaos*, **1**(2), 174–180 (1991)
- [22] KUPERSHMIT, B. A. Mathematics of dispersive water waves. *Communications in Mathematical Physics*, **99**(1), 51–73, (1985)
- [23] ABLOWITZ, M. J. and CLARKSON, P. A. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge (1991)