Bifurcation and dynamic behavior analysis of a rotating cantilever plate in subsonic airflow*

Li MA, Minghui YAO[†], Wei ZHANG, Dongxing CAO

Beijing Key Laboratory of Nonlinear Vibrations and Strength of Mechanical Structures,
College of Mechanical Engineering, Beijing University of Technology,
Beijing 100124, China

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Abstract Turbo-machineries, as key components, have wide applications in civil, aerospace, and mechanical engineering. By calculating natural frequencies and dynamical deformations, we have explained the rationality of the series form for the aerodynamic force of the blade under the subsonic flow in our earlier studies. In this paper, the subsonic aerodynamic force obtained numerically is applied to the low pressure compressor blade with a low constant rotating speed. The blade is established as a pre-twist and pre-setting cantilever plate with a rectangular section under combined excitations, including the centrifugal force and the aerodynamic force. In view of the first-order shear deformation theory and von-Kármán nonlinear geometric relationship, the nonlinear partial differential dynamical equations for the warping cantilever blade are derived by Hamilton's principle. The second-order ordinary differential equations are acquired by the Galerkin approach. With consideration of 1:3 internal resonance and 1/2 sub-harmonic resonance, the averaged equation is derived by the asymptotic perturbation methodology. Bifurcation diagrams, phase portraits, waveforms, and power spectrums are numerically obtained to analyze the effects of the first harmonic of the aerodynamic force on nonlinear dynamical responses of the structure.

Key words subsonic aerodynamic force, asymptotic perturbation method, bifurcation and chaos

Chinese Library Classification O322

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1 Introduction

Blades, which are vital rotating structural units, are widely used in turbo-machinery of aeronautical and aerospace industries. It is well-known that blades are often exposed to harsh working conditions, especially for gas turbine blades. Vibration of blades under airflow excitation has been recognized as one of the major causes of system failure in the engineering field.

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[†] Corresponding author, E-mail: merry_mingming@163.com

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Transverse vibrations of compressor blades often exhibit dynamic behaviors, which may lead to bifurcation phenomena. Dynamic behaviors of blades are very complex due to the coupling effect of fluid-structure interaction and strong nonlinear and unsteady characteristics. To design such structures properly, we need a profound understanding of their dynamic characteristics to avoid some undesirable disasters such as resonance phenomena and flutter.

Blade, a core component of modern rotating machine, endures the coupling effects of the centrifugal force and the aerodynamic load. In order to get a profound understanding of the vibration characteristics of the system, the establishment for a reasonable dynamical model is the most pressing concern for researchers. Plenty of theoretical analyses focus on the vibration characteristics of blades, which have been simplified as Euler-Bernoulli beams^[1] and Timoshenko beams^[2–3] under various conditions.

However, if the blade is thin, the aspect ratio is small, or higher frequencies and modes are needed, the model of the cantilever beam is highly inaccurate. Thus, the cantilever plate or the shell model is needed to describe the rotating blade. Sabuncu and Thomas^[4] numerically investigated the free vibration characteristics of the shrouded pre-twist rotating aerofoil blade without shear deformation and rotary inertia effects. Li and Zhang^[5] observed frequency loci veering phenomena rather than crossing in either rotating homogeneous plates or functionally graded (FG) plates. Mahi et al.^[6] and Bennoun et al.^[7] proposed a refined plate theory and a novel hyperbolic shear deformation theory to study the bending and free vibration of different kinds of composite plates. Xie et al.^[8] put forward a novel breathing model for the blade with cracks of the rotating shaft-disk-blade system subject to bending stresses and centrifugal stresses and compared the vibration responses of this model with two different crack models. Niu et al.^[9] studied free vibrations of a rotating blade, which is modeled as a pre-twisted FG cylindrical cantilever panel reinforced with graphene.

In addition to linear vibrations, some scholars have also studied the nonlinear oscillations of blades. Zhang and Zhao^[10] analyzed the nonlinear oscillations and amplitude-frequency characteristics for the composite laminated rotary plate enduring lateral and horizontal excitations. Chen et al.^[11] discussed the nonlinear steady-state responses of thin-walled structures in supercritical flow. Subrahmanyam et al.^[12] and Yao et al.^[13] studied the effects of the Coriolis terms and several other nonlinear terms on the oscillation and stability of the rotating thin-walled linearly pre-twist blade. Roy and Meguid^[14] found that the centrifugal force field decaying leads to a severe change in the dynamical responses of blades. Wang et al.^[15] explained Hopf bifurcation and saddle-node based on the reduced fluid-structure interaction model of the vortex-induced oscillations for turbine blades. The double Hopf bifurcations, global bifurcations, and multi-pulse chaotic dynamics of the laminated plate under certain internal resonances were investigated^[16–17]. Zhang et al.^[18–19] studied sub-harmonic and super-harmonic resonances combined with different resonance cases of a rotary blade under strong gas pressure. Steady-state responses, saturation, jump, and hysteresis are focused on. Ding et al.^[20] further illustrated the energy transfer between the saturation-like phenomena and the internal resonance modes. Liu et al.^[21] gave phase portraits, time history, Poincaré maps, and bifurcation diagrams with different parameters of an FG cylindrical shell with small initial geometric imperfect defects under complex excitations. Yao et al.^[22] explored the nonlinear transient dynamic responses and steady-state dynamic vibrations of the pre-twisted cylindrical shell model for the aero-engine compressor blade subject to the single point excitation and the uniform distribution excitation.

Enormous studies have been implemented about blades. However, few of them devoted to the nonlinear responses for the pre-twist and presetting rotating cantilever blade under subsonic airflow. The difficulty lies in the fact that we do not have an explicit expression for the blade in subsonic airflow. Therefore, in this paper, the explicit time dependent aerodynamic force expression of the blade in the subsonic airflow is introduced. The nonlinear dynamical behaviors of the blade subject to aerodynamic forces numerically obtained are investigated. Consider-

ing the shear deformation and von-Kármán nonlinear geometric relationship, the equations of motion for the cantilever plate are derived by Hamilton's principle. The asymptotic perturbation methodology is introduced to deduce the averaged equation of the structure in case of 1:3 internal resonance and 1/2 sub-harmonic resonance. Bifurcation and chaotic dynamics of the system are analyzed numerically. Numerical simulation shows that the aerodynamic force has a significant effect on the periodic and chaotic motions of the rotating blade under specific cases. Since we can avoid chaotic motions of the system by changing the aerodynamic force, the large amplitude nonlinear vibration of the blade can be further controlled.

2 Equations of motion

The structure diagram of the rotary cantilever plate is shown in Fig. 1^[23]. The blade is modeled as a presetting and pre-twist rotating cantilever plate with a rectangular section, which is clamped to a rigid hub with the radius R . The blade rotates with a constant rotating speed Ω_0 . The span length and the chord length of the blade are L and C , respectively, where $L = R_0 - R$. The thickness of the plate is marked as h . The pre-twist angle ϕ_R and the mounting angle θ_r exist at the free end and the fixed end of the plate, respectively. Three coordinate systems, i.e., the inertial coordinate system (i, j, k) , the rotating coordinate system (e_x^0, e_y^0, e_z^0) , and the sectional coordinate system (e_x, e_y, e_z) , are introduced, respectively. The origin of the inertial frame (i, j, k) is the center of the hub. (e_x^0, e_y^0, e_z^0) is located at the edge of the hub. e_x^0 , e_y^0 , and e_z^0 represent the spanwise direction, the chordwise direction, and the thickness direction, respectively.

Fig. 1 Structure diagram of the rotating blade

Based on the research done by Yao et al.^[23], considering the first-order shear deformation and Kirchhoff hypothesis^[24], the dimensionless partial differential governing equations are expressed by the displacements of an arbitrary points on the neutral plane of the plate u_0 , v_0 , and w_0 along the x -, y -, and z -directions. The nonlinear dynamic oscillations of the rotating blade are investigated qualitatively. The dimensionless parameters are introduced as follows:

$$\begin{cases} \bar{y} = \frac{y}{C}, & \bar{A}_{ij} = \frac{(LC)^{\frac{1}{2}}}{Eh^2} A_{ij}, & \bar{B}_{ij} = \frac{(LC)^{\frac{1}{2}}}{Eh^3} B_{ij}, & \bar{D}_{ij} = \frac{(LC)^{\frac{1}{2}}}{Eh^4} D_{ij}, \\ \bar{I}_i = \frac{1}{\rho(LC)^{\frac{(1+i)}{2}}}, & \bar{\Omega} = \Omega \sqrt{\frac{\rho CL}{E}}, & \bar{t} = t \sqrt{\frac{E}{\rho CL}}, & \bar{\theta} = \theta, \\ \bar{\theta}' = L\theta', & \bar{r} = \frac{r}{L}, & \bar{C}_\infty = C_\infty \sqrt{\frac{\rho}{E}}, & \bar{p} = p \sqrt{\frac{\rho}{E}}, & \bar{p}_\infty = \frac{\rho_\infty L^2}{\rho A}. \end{cases} \quad (1)$$

The dimensionless nonlinear governing equations for the structure are described as follows:

δu_0 :

$$\begin{aligned}
& a_{10} \frac{\partial^2 u_0}{\partial x^2} + a_{11} \frac{\partial^2 u_0}{\partial y^2} + a_{12} \frac{\partial^2 v_0}{\partial x \partial y} + a_{13} \frac{\partial w_0}{\partial x} + a_{14} \frac{\partial^2 w_0}{\partial x^2} + a_{15} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \\
& + a_{16} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + a_{17} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} + a_{18} \frac{\partial^2 \varphi_x}{\partial y^2} + a_{19} \frac{\partial^2 \varphi_x}{\partial x^2} + a_{20} \frac{\partial^2 \varphi_y}{\partial x \partial y} \\
& = a_{21} \ddot{u}_0 \cos \Theta + a_{22} \ddot{v}_0 \sin \Theta + a_{23} \ddot{w}_0 \sin \Theta + a_{24} \dot{u}_0 \sin \Theta \cdot \Omega_0 + a_{25} \dot{u}_0 \cos \Theta \cdot \Omega_0 \\
& + a_{26} \dot{w}_0 \cos \Theta \cdot \Omega_0 + a_{27} x \cos \Theta \cdot \Omega_0^2 + a_{28} \Omega_0^2 \cos \Theta + a_{29} u_0 \cdot \Omega_0^2 \cos \Theta \\
& + a_{30} y \Omega_0^2 \sin \Theta + a_{31} v_0 \Omega_0^2 \sin \Theta + a_{32} w_0 \sin \Theta \cdot \Omega_0^2 + a_{33} y \frac{\partial w_0}{\partial x} \cdot \Omega_0^2 \\
& + a_{34} y \frac{\partial u_0}{\partial y} \cdot \Omega_0^2 + a_{35} \left(\frac{C^2}{4} - y^2 \right) \frac{\partial^2 w_0}{\partial x^2} \cdot \Omega_0^2 + a_{36} \left(\frac{C^2}{4} - y^2 \right) \frac{\partial^2 u_0}{\partial y^2} \cdot \Omega_0^2 \\
& + a_{37} x \sin \Theta \cdot \dot{\Omega}_0 + a_{38} \sin \Theta \cdot \dot{\Omega}_0 + a_{39} u_0 \sin \Theta \cdot \dot{\Omega}_0 + a_{40} y \cos \Theta \cdot \dot{\Omega}_0 \\
& + a_{41} v_0 \cos \Theta \cdot \dot{\Omega}_0 + a_{42} w_0 \cos \Theta \cdot \dot{\Omega}_0.
\end{aligned} \tag{2a}$$

δv_0 :

$$\begin{aligned}
& b_{10} \frac{\partial^2 v_0}{\partial x^2} + b_{11} \frac{\partial^2 u_0}{\partial x \partial y} + b_{12} \frac{\partial^2 v_0}{\partial y^2} + b_{13} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} + b_{14} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} + b_{15} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x^2} \\
& + b_{16} \frac{\partial^2 \varphi_x}{\partial x \partial y} + b_{17} \frac{\partial^2 \varphi_y}{\partial x^2} + b_{18} \frac{\partial^2 \varphi_y}{\partial y^2} + b_{19} \frac{\partial^2 w_0}{\partial x \partial y} + b_{20} \frac{\partial w_0}{\partial y} \\
& = b_{21} \ddot{u}_0 \sin \Theta + b_{22} \ddot{v}_0 \cos \Theta + b_{23} \ddot{w}_0 \cos \Theta + b_{24} \dot{u}_0 \cos \Theta \cdot \Omega_0 + b_{25} \dot{v}_0 \sin \Theta \cdot \Omega_0 \\
& + b_{26} \dot{w}_0 \sin \Theta \cdot \Omega_0 + b_{27} x \Omega_0^2 \sin \Theta + b_{28} \Omega_0^2 \sin \Theta + b_{29} \Omega_0^2 \sin \Theta \cdot u_0 + b_{30} y \Omega_0^2 \cos \Theta \\
& + b_{31} v_0 \cos \Theta \cdot \Omega_0^2 + b_{32} w_0 \cos \Theta \cdot \Omega_0^2 + b_{33} (r+x) \frac{\partial w_0}{\partial x} \cdot \Omega_0^2 + b_{34} (r+x) \frac{\partial v_0}{\partial x} \cdot \Omega_0^2 \\
& + b_{35} \left(r(L-x) + \frac{1}{2}(L^2 - x^2) \right) \frac{\partial^2 w_0}{\partial x^2} \cdot \Omega_0^2 + b_{36} \left(r(L-x) + \frac{1}{2}(L^2 - x^2) \right) \frac{\partial^2 v_0}{\partial x^2} \cdot \Omega_0^2 \\
& + b_{37} x \cos \Theta \cdot \dot{\Omega}_0 + b_{38} \cos \Theta \cdot \dot{\Omega}_0 + b_{39} u_0 \cos \Theta \cdot \dot{\Omega}_0 + b_{40} y \sin \Theta \cdot \dot{\Omega}_0 \\
& + b_{41} v_0 \sin \Theta \cdot \dot{\Omega}_0 + b_{42} w_0 \sin \Theta \cdot \dot{\Omega}_0.
\end{aligned} \tag{2b}$$

δw_0 :

$$\begin{aligned}
& c_{10} \left(\frac{\partial w_0}{\partial x} \right)^2 \frac{\partial^2 w_0}{\partial x^2} + c_{11} \left(\frac{\partial w_0}{\partial x} \right)^2 \frac{\partial^2 w_0}{\partial y^2} + c_{12} \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} + c_{13} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} \\
& + c_{14} \left(\frac{\partial w_0}{\partial y} \right)^2 \frac{\partial^2 w_0}{\partial y^2} + c_{15} \left(\frac{\partial w_0}{\partial y} \right)^2 \frac{\partial^2 w_0}{\partial x^2} + c_{16} \frac{\partial^2 w_0}{\partial y \partial x} \frac{\partial w_0}{\partial y} + c_{17} \frac{\partial^2 u_0}{\partial x^2} \frac{\partial w_0}{\partial x} + c_{18} \frac{\partial^2 u_0}{\partial y^2} \frac{\partial w_0}{\partial x} \\
& + c_{19} \frac{\partial^2 v_0}{\partial x \partial y} \frac{\partial w_0}{\partial x} + c_{20} \left(\frac{\partial w_0}{\partial x} \right)^2 + c_{21} \frac{\partial^2 \varphi_x}{\partial y^2} \frac{\partial w_0}{\partial x} + c_{22} \frac{\partial^2 \varphi_x}{\partial x^2} \frac{\partial w_0}{\partial x} + c_{23} \frac{\partial^2 \varphi_y}{\partial x \partial y} \frac{\partial w_0}{\partial x} \\
& + c_{24} \frac{\partial^2 u_0}{\partial x \partial y} \frac{\partial w_0}{\partial y} + c_{25} \frac{\partial^2 v_0}{\partial x^2} \frac{\partial w_0}{\partial y} + c_{26} \frac{\partial^2 v_0}{\partial y^2} \frac{\partial w_0}{\partial y} + c_{27} \frac{\partial^2 \varphi_x}{\partial x \partial y} \frac{\partial w_0}{\partial y} + c_{28} \frac{\partial^2 \varphi_y}{\partial x^2} \frac{\partial w_0}{\partial y} \\
& + c_{29} \frac{\partial^2 \varphi_y}{\partial y^2} \frac{\partial w_0}{\partial y} + c_{30} \left(\frac{\partial w_0}{\partial y} \right)^2 + c_{31} \frac{\partial u_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} + c_{32} \frac{\partial^2 v_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} + c_{33} \frac{\partial \varphi_x}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} \\
& + c_{34} \frac{\partial \varphi_y}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} + c_{35} \frac{\partial u_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + c_{36} \frac{\partial v_0}{\partial y} \frac{\partial^2 w_0}{\partial x^2} + c_{37} \frac{\partial \varphi_x}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + c_{38} w_0 \frac{\partial^2 w_0}{\partial x^2} \\
& + c_{39} \frac{\partial^2 \varphi_y}{\partial y} \frac{\partial^2 w_0}{\partial x^2} + c_{40} \frac{\partial u_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + c_{41} \frac{\partial v_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} + c_{42} \frac{\partial \varphi_x}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + c_{43} w_0 \frac{\partial^2 w_0}{\partial y^2} \\
& + c_{44} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} + c_{45} \frac{\partial \varphi_y}{\partial y} \frac{\partial^2 w_0}{\partial y^2} + c_{46} \frac{\partial^2 u_0}{\partial x^2} + c_{47} \frac{\partial^2 v_0}{\partial x \partial y} + c_{48} \frac{\partial^2 \varphi_x}{\partial x^2} + c_{49} \frac{\partial^2 w_0}{\partial x} \\
& + c_{50} \frac{\partial^2 u_0}{\partial x^2} + c_{51} \frac{\partial^2 \varphi_y}{\partial x \partial y} + c_{52} \frac{\partial^2 w_0}{\partial x^2} + c_{53} \frac{\partial \varphi_x}{\partial x} + c_{54} \frac{\partial^2 w_0}{\partial y^2} + c_{55} \frac{\partial \varphi_y}{\partial y} + c_{56} \\
& + c_{57} \sin(\Omega t) + c_{59} \sin(2\Omega t) + c_{58} \cos(\Omega t) + c_{60} \cos(2\Omega t) \\
& = c_{61} \left(r(L-x) + \frac{1}{2}(L^2 - x^2) \right) \frac{\partial w_0}{\partial x} \cdot \Omega_0^2 + c_{62} \left(r(L-x) + \frac{1}{2}(L^2 - x^2) \right) \frac{\partial u_0}{\partial y} \cdot \Omega_0^2 \\
& + c_{63} (r+x) \frac{\partial^2 w_0}{\partial x^2} \cdot \Omega_0^2 + c_{64} (r+x) \frac{\partial^2 u_0}{\partial y^2} \cdot \Omega_0^2 + c_{65} y \frac{\partial w_0}{\partial x} \cdot \Omega_0^2 + c_{66} y \frac{\partial v_0}{\partial x} \cdot \Omega_0^2 \\
& + c_{67} \left(\frac{C^2}{4} - y^2 \right) \frac{\partial^2 w_0}{\partial x^2} \cdot \Omega_0^2 + c_{68} \left(\frac{C^2}{4} - y^2 \right) \frac{\partial^2 v_0}{\partial x^2} \cdot \Omega_0^2 + c_{69} \ddot{v}_0 \sin \theta + c_{70} \ddot{w}_0 \\
& + c_{71} x \ddot{\varphi}_x \cos \Theta + c_{72} x^2 \ddot{w}_0 \cos \Theta + c_{73} x \ddot{\varphi}_y \sin \Theta + c_{74} x \dot{\varphi}_x \Omega_0 + c_{75} x \dot{w}_0 \Omega_0 \\
& + c_{76} x \dot{\varphi}_y \Omega_0 \cos \Theta + c_{77} x \varphi_x \Omega_0^2 \cos \Theta + c_{78} x w_0 \Omega_0^2 \cos \Theta + c_{79} x \varphi_y \Omega_0^2 \sin \Theta \\
& + c_{80} x \Omega_0^2 \sin \Theta + c_{81} x \varphi_x \dot{\Omega}_0 \sin \Theta + c_{82} x^2 w_0 \dot{\Omega}_0 \sin \Theta + c_{83} x \varphi_y \dot{\Omega}_0 \cos \Theta \\
& + c_{84} x \dot{\Omega}_0 \cos \Theta.
\end{aligned} \tag{2c}$$

$\delta\varphi_x$:

$$\begin{aligned}
 & d_{10} \frac{\partial^2 u_0}{\partial x^2} + d_{11} \frac{\partial^2 u_0}{\partial y^2} + d_{12} \frac{\partial^2 v_0}{\partial x \partial y} + d_{13} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x^2} + d_{14} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x \partial y} + d_{15} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial y^2} \\
 & + d_{16} \frac{\partial w_0}{\partial x} + d_{17} \frac{\partial^2 w_0}{\partial x^2} + d_{18} \frac{\partial^2 \varphi_x}{\partial x^2} + d_{19} \frac{\partial^2 \varphi_x}{\partial y^2} + d_{20} \frac{\partial^2 \varphi_y}{\partial x \partial y} + d_{21} \frac{\partial w_0}{\partial x} + d_{22} \varphi_x \\
 & = d_{23} \ddot{\varphi}_x \cos \Theta + d_{24} x \ddot{w}_0 \cos \Theta + d_{25} \ddot{\varphi}_y \sin \Theta + d_{26} \dot{\varphi}_x \cdot \Omega_0 + d_{27} x \dot{w}_0 \cdot \Omega_0 + d_{28} \dot{\varphi}_y \cos \Theta \cdot \Omega_0 \\
 & + d_{29} \varphi_x \cos \Theta \cdot \Omega_0^2 + d_{30} x w_0 \cos \Theta \cdot \Omega_0^2 + d_{31} \varphi_y \sin \Theta \cdot \Omega_0^2 + d_{32} \sin \Theta \cdot \Omega_0^2 \\
 & + d_{33} \varphi_x \sin \Theta \cdot \dot{\Omega}_0 + d_{34} x w_0 \sin \Theta \cdot \dot{\Omega}_0 + d_{35} \varphi_y \cos \Theta \cdot \dot{\Omega}_0 + d_{36} \cos \Theta \cdot \dot{\Omega}_0. \tag{2d}
 \end{aligned}$$

$\delta\varphi_y$:

$$\begin{aligned}
 & e_{10} \frac{\partial^2 v_0}{\partial x^2} + e_{11} \frac{\partial^2 u_0}{\partial x \partial y} + e_{12} \frac{\partial^2 v_0}{\partial y^2} + e_{13} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial x^2} + e_{14} \frac{\partial w_0}{\partial x} \frac{\partial^2 w_0}{\partial x \partial y} + e_{15} \frac{\partial w_0}{\partial y} \frac{\partial^2 w_0}{\partial y^2} \\
 & + e_{16} \frac{\partial^2 w_0}{\partial x \partial y} + e_{17} \frac{\partial^2 \varphi_x}{\partial x \partial y} + e_{18} \frac{\partial^2 \varphi_y}{\partial x^2} + e_{19} \frac{\partial^2 \varphi_y}{\partial y^2} + e_{20} \frac{\partial w_0}{\partial y} + e_{21} \varphi_y \\
 & = e_{22} \ddot{\varphi}_x \sin \Theta + e_{23} x \ddot{w}_0 \sin \Theta + e_{24} \ddot{\varphi}_y \cos \Theta + e_{25} \dot{\varphi}_x \cos \Theta \cdot \Omega_0 + e_{26} x \dot{w}_0 \cos \Theta \cdot \Omega_0 \\
 & + e_{27} \dot{\varphi}_y \sin \Theta \cdot \Omega_0 + e_{28} \varphi_x \sin \Theta \cdot \Omega_0^2 + e_{29} x w_0 \sin \Theta \cdot \Omega_0^2 + e_{30} \varphi_y \cos \Theta \cdot \Omega_0^2 \\
 & + e_{31} \cos \Theta \cdot \Omega_0^2 + e_{32} \varphi_x \cos \Theta \cdot \dot{\Omega}_0 + e_{33} x w_0 \cos \Theta \cdot \dot{\Omega}_0 + e_{34} \varphi_y \sin \Theta \cdot \dot{\Omega}_0 \\
 & + e_{35} \sin \Theta \cdot \dot{\Omega}_0. \tag{2e}
 \end{aligned}$$

Here, θ is the twist angle of the plate, which is defined as $\theta = \theta_r + (\theta_R/L) \cdot x$; φ_x denotes the rotation of a transverse normal about the y -axis, and φ_y denotes the rotation about the x -axis, where $\varphi_x = -\frac{\partial w_0}{\partial x} + \gamma_{xz}$, and $\varphi_y = -\frac{\partial w_0}{\partial y} + \gamma_{yz}$; Θ is the rotary angle of the plate, which can be written as $\Theta = \Omega_0 \cdot t$. These parameters can be found in Ref. [23].

The Galerkin method, which is derived by the Taylor expansion approach, is proposed for transferring a continuous operator problem to a discrete one. It is a convergent method in mathematics. This method may provide useful qualitative discussion of nonlinear behaviors for a dynamical system. The second-order ordinary differential dimensionless equations for the system are derived by applying the Galerkin method given by Ding et al.^[25]. In this study, only the first two order modes of the later vibration are considered. The lateral displacement $w(x, y, t)$ and the angular displacement functions $\varphi_x(x, y, t)$ and $\varphi_y(x, y, t)$ can be found in Eq. (26) of Ref. [23]. The in-plane displacements are expressed as follows:

$$\begin{cases} u(x, y, t) = u_1(t) \sin \frac{\pi x}{2L} \cos \frac{\pi y}{C} + u_2(t) \sin \frac{3\pi x}{2L} \cos \frac{2\pi y}{C}, \\ v(x, y, t) = v_1(t) \sin \frac{\pi x}{2L} \sin \frac{\pi y}{C} + v_2(t) \sin \frac{3\pi x}{2L} \sin \frac{2\pi y}{C}, \end{cases} \tag{3}$$

where u , v , and w are the displacements of an arbitrary point on the plate along the x -, y -, and z -directions, respectively.

The transverse ordinary differential dimensionless equations of the rotating structure are deduced by using the Galerkin method as follows:

$$\begin{aligned} & \ddot{w}_1 + (\alpha_1 + f_{11} \cos(\Omega_0 t) + f_{12} \sin(\Omega_0 t))w_1 + \alpha_2 \dot{w}_1 + \alpha_3 w_1^2 + \alpha_4 w_1 w_2 + \alpha_5 w_1 w_2^2 + \alpha_6 w_1^2 w_2 \\ & + \alpha_7 w_1^3 + \alpha_8 w_2^2 + \alpha_9 w_2^3 + (\alpha_{10} + f_{13} \cos(\Omega_0 t) + f_{14} \sin(\Omega_0 t))w_2 + \alpha_{11} \dot{w}_2 + \alpha_{12} \ddot{w}_2 \\ = & \alpha_{13} + f_{15} \cos(\Omega_0 t) + f_{16} \sin(\Omega_0 t) + g_1 (f_2 \sin(\Omega_1 t) + f_1 \cos(\Omega_1 t) \\ & + f_4 \sin(2\Omega_1 t) + f_3 \cos(2\Omega_1 t)), \end{aligned} \quad (4a)$$

$$\begin{aligned} & \ddot{w}_2 + (\beta_1 + f_{21} \cos(\Omega_0 t) + f_{22} \sin(\Omega_0 t))w_2 + \beta_2 \dot{w}_2 + \beta_3 w_2^2 + \beta_4 w_2 w_1 + \beta_5 w_2 w_1^2 + \beta_6 w_2^2 w_1 \\ & + \beta_7 w_2^3 + \beta_8 w_1^2 + \beta_9 w_1^3 + (\beta_{10} + f_{23} \cos(\Omega_0 t) + f_{24} \sin(\Omega_0 t))w_1 + \beta_{11} \dot{w}_1 + \beta_{12} \ddot{w}_1 \\ = & \beta_{13} + f_{25} \cos(\Omega_0 t) + f_{26} \sin(\Omega_0 t) + g_2 (f_2 \sin(\Omega_1 t) + f_1 \cos(\Omega_1 t) + f_4 \sin(2\Omega_1 t) \\ & + f_3 \cos(2\Omega_1 t)), \end{aligned} \quad (4b)$$

where Ω_1 is the frequency of the aerodynamic force^[23].

3 Perturbation analysis

The perturbation method, which is called the small parameter expansion method, aims to find an asymptotic solution to a problem, especially a nonlinear problem. In Eq. (5), the square and cubic nonlinear terms are presented. Thus, it is reasonable to exploit the asymptotic perturbation methodology^[26] to derive the average equations, which guarantees the accuracy of the solution to a certain extent.

Considering 1/2 sub-harmonic resonance and 1:3 internal resonance, the relationship between ω_1 and ω_2 for the structure is written as follows:

$$3\omega_1 = \omega_2, \quad \omega_1 = \frac{1}{2}\Omega + \varepsilon^2 \sigma_1, \quad \omega_2 = \frac{3}{2}\Omega + \varepsilon^2 \sigma_2, \quad (5)$$

where σ_1 and σ_2 are two detuning parameters, and Ω is the frequency of the external excitation.

By introducing the following time scale transformation, the positive rational number q is constant in the following deduction:

$$\tau = \varepsilon^q t. \quad (6)$$

The power series solutions of the equations for the blade under 1/2 sub-harmonic resonance and 1:3 internal resonance are written as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} \varepsilon^{r_n} \psi_n(\tau, \varepsilon) e^{-in\frac{\Omega}{2}t}, \quad (7a)$$

$$y(t) = \sum_{n=-\infty}^{\infty} \varepsilon^{r_n} \phi_n(\tau, \varepsilon) e^{-in\frac{3\Omega}{2}t}. \quad (7b)$$

In the above expression, if $n \neq 0$, then $r_n = |n|$, otherwise $r_0 = r$. In the following derivation, we assume that the value of r_n does not change.

Functions $\psi_n(\tau, \varepsilon)$ and $\phi_n(\tau, \varepsilon)$ are given by

$$\psi_n(\tau, \varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^i \psi_n^{(i)}(\tau), \quad (8a)$$

$$\phi_n(\tau, \varepsilon) = \sum_{i=0}^{+\infty} \varepsilon^i \phi_n^{(i)}(\tau). \quad (8b)$$

Suppose when $\varepsilon \rightarrow 0$, the limits of $\psi_n(\tau, \varepsilon)$ and $\phi_n(\tau, \varepsilon)$ exist. Meanwhile, we define $\psi_n^{(0)} = \psi_n$, $\phi_n^{(0)} = \phi_n$ for $n \neq 1$ and $\psi_1^{(0)} = \psi$, $\phi_1^{(0)} = \phi$ for $n = 1$. When $n = 2$, we obtain the following derivative form^[26]:

$$\frac{d}{dt}(\psi_n e^{-in\frac{\Omega}{2}t}) = \left(-in\frac{\Omega}{2}\psi_n + \varepsilon^q \frac{d\psi_n}{d\tau}\right) e^{-in\frac{\Omega}{2}t}, \quad (9a)$$

$$\frac{d}{dt}(\phi_n e^{-in\frac{3\Omega}{2}t}) = \left(-in\frac{3}{2}\Omega\phi_n + \varepsilon^q \frac{d\phi_n}{d\tau}\right) e^{-in\frac{3\Omega}{2}t}. \quad (9b)$$

To solve the coefficients $\psi_n(\tau, \varepsilon)$ and $\phi_n(\tau, \varepsilon)$, substitute the scale transformation, Eq. (5), and Eq. (7) into Eq. (4). Then, the equations for each n order harmonic and certain orders of approximate values on the perturbation parameter ε are derived as follows.

When $n = 0$ and $r = 2$, the following expressions are obtained:

$$\begin{aligned} \psi_0 = & -\frac{4}{\Omega^2}(2\alpha_3\psi_1\psi_1^* - 2f_4\sin(\Omega_1t)\cos(\Omega_1t) + 2\alpha_8\phi_1\phi_1^* - \alpha_{20}\sin(\Omega_0t) \\ & - \alpha_{19}\cos(\Omega_0t) - 2g_1f_3\cos^2(\Omega_1t) - \alpha_{18} + g_1f_3), \end{aligned} \quad (10a)$$

$$\begin{aligned} \phi_0 = & -\frac{4}{9\Omega^2}(2\beta_3\phi_1\phi_1^* - 2f_{14}\sin(\Omega_1t)\cos(\Omega_1t) + 2\beta_8\psi_1\psi_1^* + f_{13} \\ & - \beta_{19}\cos(\Omega_0t) - \beta_{20}\sin(\Omega_0t) - 2f_{13}\cos^2(\Omega_1t)), \end{aligned} \quad (10b)$$

where the items with asterisk are the complex conjugate of the corresponding items.

When $n = 2$, considering the derivative form of Eq. (9) yields

$$\psi_2 = -\frac{2}{3} \frac{g_1f_1 - 2\alpha_3\psi_1^2 + Ig_1f_2 - 2\alpha_4\psi_1^*\phi_1}{\Omega^2}, \quad (11a)$$

$$\phi_2 = \frac{4}{27} \frac{\beta_3\phi_1^2}{\Omega^2}. \quad (11b)$$

When $n = 1$ and $q = 2$, the equations can be established. We obtain the following equations:

$$\begin{aligned} & -I\Omega D_1(\psi_1) + 2\alpha_3\psi_0\psi_1 + \alpha_4\phi_0\psi_1 - \frac{1}{2}I\alpha_2\psi_1\Omega + \Omega\sigma_1\psi_1 + 2\alpha_3\psi_2\psi_1^* \\ & + 3\alpha_7\psi_1^2\psi_1^* + \alpha_4\phi_1\psi_2^* + \alpha_6\psi_1^*\phi_1 + 2\alpha_5\psi_1\phi_1\phi_1^* = 0, \end{aligned} \quad (12a)$$

$$\begin{aligned} & -3I\Omega D_1(\phi_1) + 2\beta_3\phi_0\phi_1 + \beta_4\phi_1\psi_0 + 2\beta_3\phi_2\phi_1^* + 3\beta_7\phi_1^2\phi_1^* + 2\beta_8\psi_1\psi_2 \\ & + 3\Omega\sigma_2\phi_1 + 2\beta_5\phi_1\psi_1\psi_1^* + \beta_9\psi_1^3 - \frac{3}{2}I\beta_{14}\phi_1\Omega - \frac{3}{2}I\beta_2\phi_1\Omega = 0. \end{aligned} \quad (12b)$$

Substitute Eqs. (10) and (11) into Eq. (12). Then, the differential equations about ψ_1 and ϕ_1 are obtained as follows:

$$\begin{aligned} D_1(\psi_1) = & (h_1f_2 + Ih_2f_1)\phi_1 + (Ih_3 - I\sigma_1 + h_4)\psi_1 + (h_5f_2 + Ih_6f_1)\psi_1^* \\ & + Ih_7\phi_1\phi_1^*\psi_1 + Ih_8\psi_1^*\psi_1^2 + Ih_9\phi_1\psi_1^{*2}, \end{aligned} \quad (13a)$$

$$D_1(\phi_1) = (Ik_1 - I\sigma_2 + k_2)\phi_1 + (k_3f_2 + Ik_4f_1)\psi_1 + Ik_5\phi_1\psi_1\psi_1^{*2} + Ik_6\phi_1^*\phi_1^2 + Ik_7\psi_1^3. \quad (13b)$$

Let ψ_1 and ϕ_1 be in the following forms:

$$\psi_1 = \frac{1}{2}a_1 e^{i\varphi_1}, \quad (14a)$$

$$\phi_1 = \frac{1}{2}a_2 e^{i\varphi_2}. \quad (14b)$$

The following polar coordinate averaged equation is acquired by extraction of the real and imaginary parts after substituting Eq. (14) into Eq. (13):

$$\begin{aligned} \dot{a}_1 = & h_4 a_1 - h_5 f_2 a_1 + \cos(\varphi_1 - \varphi_2) a_2 f_2 h_1 + \sin(\varphi_1 - \varphi_2) a_2 f_1 h_2 \\ & + 4a_1^2 \cos^2 \varphi_1 \sin(\varphi_1 - \varphi_2) a_2 h_9 - a_1^2 \sin(\varphi_1 - \varphi_2) a_2 h_9 + 2a_1^2 \sin \varphi_2 \cos \varphi_1 a_2 h_9 \\ & + \sin(2\varphi_1) a_1 f_1 h_6 + 2(\cos \varphi_1)^2 a_1 f_2 h_5, \end{aligned} \quad (15a)$$

$$\begin{aligned} a_1 \dot{\varphi}_1 = & h_8 a_1^3 + h_7 a_2^2 - a_1 \sigma_1 + a_1 h_3 - \sin(\varphi_1 - \varphi_2) a_2 f_2 h_1 - \sin(2\varphi_1) a_1 f_2 h_5 \\ & - a_1^2 \cos(\varphi_1 - \varphi_2) a_2 h_9 - 2a_1^2 \cos \varphi_2 \cos \varphi_1 a_2 h_9 + 4a_1^2 \cos^2 \varphi_1 \cos(\varphi_1 - \varphi_2) a_2 h_9 \\ & + \cos(\varphi_1 - \varphi_2) a_2 f_1 h_2 + \cos(2\varphi_1) h_6 a_1 f_1, \end{aligned} \quad (15b)$$

$$\begin{aligned} \dot{a}_2 = & -4a_1^3 \cos^2 \varphi_1 \sin(\varphi_1 - \varphi_2) k_7 + a_1^3 \sin(\varphi_1 - \varphi_2) k_7 - 2a_1^3 k_7 \sin \varphi_2 \cos \varphi_1 \\ & - \sin(\varphi_1 - \varphi_2) k_4 a_1 f_1 + \cos(\varphi_1 - \varphi_2) k_3 a_1 f_2 + k_2 a_2, \end{aligned} \quad (15c)$$

$$\begin{aligned} a_2 \dot{\varphi}_2 = & k_6 a_2^3 + \sin(\varphi_1 - \varphi_2) k_3 a_1 f_2 + \cos(\varphi_1 - \varphi_2) k_4 a_1 f_1 - a_1^3 \cos(\varphi_1 - \varphi_2) k_7 \\ & - 2a_1^3 k_7 \cos \varphi_2 \cos \varphi_1 + 4a_1^3 \cos^2 \varphi_1 \cos(\varphi_1 - \varphi_2) k_7 + k_1 a_2 - a_2 \sigma_2 + k_5 a_1^2. \end{aligned} \quad (15d)$$

When $\dot{a}_1 = \dot{a}_2 = \dot{\varphi}_1 = \dot{\varphi}_2 = 0$, the following equations of the steady-state motions of the system are obtained:

$$\begin{aligned} & (h_5 f_2 a_1 - h_4 a_1)^2 + (h_8 a_1^3 + h_7 a_2^2 - a_1 \sigma_1 + a_1 h_3)^2 \\ = & (\cos(\varphi_1 - \varphi_2) a_2 f_2 h_1 + \sin(\varphi_1 - \varphi_2) a_2 f_1 h_2 + 4a_1^2 \cos^2 \varphi_1 \sin(\varphi_1 - \varphi_2) a_2 h_9 \\ & - a_1^2 \sin(\varphi_1 - \varphi_2) a_2 h_9 + 2a_1^2 \sin \varphi_2 \cos \varphi_1 a_2 h_9 + \sin(2\varphi_1) a_1 f_1 h_6 \\ & + 2(\cos \varphi_1)^2 a_1 f_2 h_5)^2 + (\sin(\varphi_1 - \varphi_2) a_2 f_2 h_1 + \sin(2\varphi_1) a_1 f_2 h_5 + a_1^2 \cos(\varphi_1 - \varphi_2) a_2 h_9 \\ & + 2a_1^2 \cos \varphi_2 \cos \varphi_1 a_2 h_9 - 4a_1^2 \cos^2 \varphi_1 \cos(\varphi_1 - \varphi_2) a_2 h_9 - \cos(\varphi_1 - \varphi_2) a_2 f_1 h_2 \\ & + \cos(2\varphi_1) h_6 a_1 f_1)^2, \end{aligned} \quad (16a)$$

$$\begin{aligned} & (k_2 a_2)^2 + (a_2 \sigma_2 - k_6 a_2^3 - k_1 a_2 - k_5 a_1^2)^2 \\ = & 4a_1^2 (\cos \varphi_1)^2 k_7 k_4 f_1 - 2a_1^2 \sin(2\varphi_1) k_7 k_3 f_2 + k_7^2 a_1^4 - 2a_1^2 k_7 k_4 f_1 + k_3^2 + f_2^2 + k_4^2 f_1^2. \end{aligned} \quad (16b)$$

Let ψ_1 and ϕ_1 in Eq. (13) be rewritten in the following forms:

$$\psi_1 = x_1 + ix_2, \quad (17a)$$

$$\phi_1 = x_3 + ix_4. \quad (17b)$$

Substitute the above formula into Eq. (13). Then, the corresponding averaged equations in

the Cartesian coordinates are given as follows:

$$\begin{aligned} \dot{x}_1 = & (f_2 h_5 + h_4)x_1 + (\sigma_1 - h_3 + f_1 h_6)x_2 + f_2 h_1 x_3 - f_1 h_2 x_4 \\ & - h_8 x_2(x_1^2 + x_2^2) - h_9 x_4(x_1^2 - x_2^2) - h_7 x_2(x_3^2 + x_4^2) + 2h_9 x_1 x_2 x_3, \end{aligned} \quad (18a)$$

$$\begin{aligned} \dot{x}_2 = & (h_3 - \sigma_1 + f_1 h_6)x_1 + (h_4 - f_2 h_5)x_2 + f_1 h_2 x_3 + f_2 h_1 x_4 + h_8 x_1(x_1^2 + x_2^2) \\ & + h_9 x_3(x_1^2 - x_2^2) + h_7 x_1(x_3^2 + x_4^2) + 2h_9 x_1 x_2 x_4, \end{aligned} \quad (18b)$$

$$\begin{aligned} \dot{x}_3 = & f_2 k_3 x_1 - f_1 k_4 x_2 + k_2 x_3 + (\sigma_2 - k_1)x_4 - k_7 x_2(3x_1^2 - x_2^2) - k_5 x_4(x_1^2 + x_2^2) \\ & - k_6 x_4(x_3^2 + x_4^2), \end{aligned} \quad (18c)$$

$$\begin{aligned} \dot{x}_4 = & f_1 k_4 x_1 + f_2 k_3 x_2 + (k_1 - \sigma_2)x_3 + k_2 x_4 + k_7 x_1(x_1^2 - 3x_2^2) + k_5 x_3(x_1^2 + x_2^2) \\ & + k_6 x_3(x_3^2 + x_4^2). \end{aligned} \quad (18d)$$

4 Numerical simulation

4.1 Numerical simulation of frequency-response

The variation trends of the stiffness characteristic of the steady-state rotating cantilever blade enduring the aerodynamic force in the series form in the subsonic airflow are analyzed numerically in this section. Considering the operating condition of the compressor blade, it is obvious that the nonlinear responses of the blade can be affected by various factors, such as the linear parameters, the nonlinear parameters, and the aerodynamic force. Therefore, their effects on the frequency-responses for the plate are analyzed by numerical simulation.

Considering the weakly coupled condition, namely, let $a_2 = 1$ in Eq. (16a) and $a_1 = 1$ in Eq. (16b), the frequency-response curves of the cantilever blade structure for the first two order modes are obtained, as shown in Fig. 2, which illustrates that the plate performs hard spring properties. The effects of different factors on the frequency-responses of the cantilever plate are depicted in Fig. 3. Due to the nonlinearity, the multi-valued phenomenon is exhibited in Fig. 3.

Fig. 2 Frequency-response curves: (a) uncoupled frequency-response curve of the 1st order mode; (b) uncoupled frequency-response curve of the 2nd order mode (color online)

Figures 3(a), 3(b), 3(e), and 3(f) show that, when the first harmonics of f_1 , h_6 , and k_3 increase, respectively, the nonlinear vibration amplitudes and the response bandwidths of the first two modes both increase. When the nonlinear parameters $|h_8|$ and $|k_6|$ increase, the dynamical response amplitudes decrease. Moreover, the larger $|h_8|$ and $|k_6|$ are, the greater the curvatures of the curves become, as shown in Figs. 3(c) and 3(d).

With the decrease in the frequencies σ_1 and σ_2 , the frequency-response curves jump down to lower amplitudes for soft spring properties while jump up to higher amplitudes for hard spring properties. The signs of h_8 and k_6 can change the spring characteristics of the system. When the

Fig. 3 Frequency-response curves for the cantilever rotating blade: (a) effects of the peak of the first harmonic of the aerodynamic force on frequency-response curves for the 1st order mode; (b) effects of the first harmonic of the aerodynamic force on frequency-response curves of the 2nd order mode; (c) effects of the nonlinear parameter of the dynamic equations of the blade on frequency-response curves of the 1st order mode; (d) effects of the nonlinear parameter of the dynamic equations of the blade on frequency-response curves of the 2nd order mode; (e) effects of the linear parameter on frequency-response curves of the 1st order mode; (f) effects of the linear parameter on frequency-response curves of the 2nd order mode (color online)

nonlinear parameters h_8 and k_6 are positive, the system exhibits hard spring characteristics. However, when h_8 and k_6 are negative, the system has soft spring characteristics. When the signs of h_8 and k_6 are opposite, the corresponding curves are symmetrical about the vertical axis.

4.2 Numerical simulation of nonlinear behaviors

Besides the steady-state analysis, the nonlinear dynamical behaviors of the compressor blade subject to the aerodynamic load and the centrifugal force are also numerically investigated based on Eq.(18). Except for the first harmonic of the aerodynamic force f_1 or f_2 , keep other parameters and the initial condition unchanged so as to investigate the effects of the first harmonic of the aerodynamic force on nonlinear oscillations of the rotating plate. When f_1 or f_2 changes in a certain interval, bifurcation diagrams, in which complicated oscillation behaviors can be found, are numerically depicted based on the Poincaré map theory and Runge-Kutta algorithm. Unless otherwise stated, the parameters are set as $\sigma_1 = 1.3842$, $\sigma_2 = 1.2312$, $h_1 = 0.313$, $h_2 = 0.285$, $h_3 = -1.127$, $h_4 = -0.918$, $h_5 = -0.049$, $h_6 = -0.1833$, $h_7 = 0.151$, $h_8 = 0.0995$, $h_9 = 0.829$, $k_1 = 0.798$, $k_2 = -0.964$, $k_3 = 0.0465$, $k_4 = -1.4016$, $k_5 = 0.002$, $k_6 = 0.3293$, $k_7 = 0.525$, $f_2 = 5$, $x_{10} = 0.3$, $x_{20} = -0.0101$, $x_{30} = -0.26$, and $x_{40} = 0.2$, and f_1 varies in the interval of $f_1 \in [0, 4]$. Thus, Fig. 4 is obtained.

Figures 4(a) and 4(b) describe the bifurcation diagrams about the first harmonic of the aerodynamic force f_1 to the lateral displacements x_1 and x_3 , respectively, when f_1 varies in the interval of $f_1 \in [0, 4]$. Select f_2 as the control parameter to detect the impact of f_2 on the nonlinear vibrations of the rotating blade. Figures 5(a) and 5(b) describe the bifurcations of f_2 to the lateral displacements x_1 and x_3 when f_1 is chosen as $f_1 = 4$, respectively. It can be observed from Figs. 4 and 5 that chaotic and periodic motions occur in the vibration of the blade. The diagrams show that the structure is sensitive to the first harmonics of the aerodynamic forces f_1 and f_2 . Figures 4 and 5 illustrate that, with the increase in the first harmonic of the aerodynamic force, the blade performs complex dynamical behaviors.

Fig. 4 Bifurcation diagrams for $f_1 \in [0, 4]$ about the first harmonic of the aerodynamic force f_1 : (a) bifurcation diagram about f_1 to the 1st lateral displacement x_1 ; (b) bifurcation diagram about f_1 to the 2nd lateral displacement x_3

Fig. 5 Bifurcation diagrams for $f_2 \in [0, 16]$ about the first harmonic of the aerodynamic force f_2 : (a) bifurcation diagram about f_2 to the 1st lateral displacement x_1 ; (b) bifurcation diagram about f_2 to the 2nd lateral displacement x_3

The chaotic and periodic responses can be identified by several conventional criteria. Thus, the bifurcation diagrams, waveforms, phase portraits, and power spectra are utilized to further verify the existence of the chaotic and periodic motions of the blade. Figure 6 illustrates that the blade performs period-1 motion as the first harmonic of the aerodynamic force f_1 is chosen as $f_1 = 0.9$. Figure 7 shows that the system exhibits period-3 motion as the first harmonic of the aerodynamic force f_1 is chosen as $f_1 = 1.98$. The bifurcation diagrams in Fig. 4 demonstrate that, as f_1 increases, the multi-periodic motion of the blade changes with the chaotic motion. Due to the limited space, only two diagrams are depicted here. When the first harmonic of

Fig. 6 The period-1 motion of the blade when $f_1 = 0.9$: (a) the phase portrait in the x_1x_2 -plane; (b) the waveform for x_1 ; (c) the phase portrait in the x_3x_4 -plane; (d) the waveform for x_3 ; (e) the three-dimensional phase portrait; (f) the power spectrum

the aerodynamic f_1 is changed as $f_1 = 3$, the system performs multi-periodic motion, as shown in Fig. 8. Figure 9 exhibits the chaotic motion of the system when the parameter f_1 is selected as $f_1 = 3.91$. It can be seen in Fig. 5 that the multi-periodic motion of the plate changes with its chaotic motion when f_2 is larger than 2.6. When f_2 is chosen as $f_2 = 2.6$, the system exhibits the multi-periodic motion, as shown in Fig. 10. When f_2 increases to 15, the system performs chaotic motion, as shown in Fig. 11.

Fig. 7 The period-3 motion of the blade when $f_1 = 1.98$: (a) the phase portrait in the x_1x_2 -plane; (b) the waveform for x_1 ; (c) the phase portrait in the x_3x_4 -plane; (d) the waveform for x_3 ; (e) the three-dimensional phase portrait; (f) the power spectrum

According to the numerical simulation, the system performs complex dynamic behaviors when the parameters f_1 and f_2 change in a certain region. The first harmonic of the aerodynamic force significantly affects the dynamical vibrations of the blade.

Fig. 8 The multi-periodic motion of the blade when $f_1 = 3$: (a) the phase portrait in the x_1x_2 -plane; (b) the waveform for x_1 ; (c) the phase portrait in the x_3x_4 -plane; (d) the waveform for x_3 ; (e) the three-dimensional phase portrait; (f) the power spectrum

5 Conclusions

In this paper, the traditional piston theory is not used. The nonlinear vibration of the plate rotating with a constant speed subject to the aerodynamic load in series form is studied. The nonlinear partial differential governing equations of the rotating blade are derived by using Hamilton's principle. Ordinary differential equations are obtained by discretizing the nonlinear partial differential governing equations with the Galerkin method. Asymptotic perturbation methodology is used to obtain the averaged equation for 1:3 internal resonance and 1/2 subharmonic resonance rotating blade. The nonlinear dynamics and frequency-responses for the blade are investigated numerically.

Fig. 9 The chaotic motion of the blade when $f_1 = 3.91$: (a) the phase portrait in the x_1x_2 -plane; (b) the waveform for x_1 ; (c) the phase portrait in the x_3x_4 -plane; (d) the waveform for x_3 ; (e) the three-dimensional phase portrait; (f) the power spectrum

Jumping of the solutions is described clearly in frequency-response curves. The effects of various factors on the frequency-response curves show that the system spring property is determined by whether the certain nonlinear parameter is larger or smaller than zero. The increase in the aerodynamic force and linear parameters results in the existence of the hard spring characteristic of the blade. Meanwhile, the increase in the aerodynamic force and linear parameters causes the increase in the vibration amplitudes and the response bandwidth of the first two modes. However, when the nonlinear parameters increase, the amplitude of the dynamic responses decreases.

Based on Eq. (18), bifurcation diagrams, phase portraits, waveforms, and power spectra are obtained to demonstrate that periodic motions and chaotic motions occur in the nonlinear

Fig. 10 The multi-periodic motion of the blade when $f_2 = 2.6$: (a) the phase portrait in the x_1x_2 -plane; (b) the waveform for x_1 ; (c) the phase portrait in the x_3x_4 -plane; (d) the waveform for x_3 ; (e) the three-dimensional phase portrait; (f) the power spectrum

vibrations of the rotating blade under certain conditions. The bifurcation diagrams, phase portraits, waveforms, and power spectra numerically depicted demonstrate that system nonlinear oscillations can be easily affected by the first harmonic of the aerodynamic force. Period-1 motion, period-3 motion, multi-periodic motion, and chaotic motion occur when f_1 and f_2 are chosen in specified intervals. From mathematical points of view, in the certain initial condition, the appearance of amplitude-modulated periodic and chaotic vibrations for original systems is explained by the bifurcation diagrams that we have obtained. The large-amplitude vibrations of the blade with a constant rotating speed that may lead to undesirable damage of the aero-engine can be controlled by adjusting the aerodynamic force. Therefore, it is of great significance to investigate the nonlinear behaviors of blades to reduce the probability of aero-engine failure.

Fig. 11 The chaotic motion of the blade when $f_2 = 15$: (a) the phase portrait in the x_1x_2 -plane; (b) the waveform for x_1 ; (c) the phase portrait in the x_3x_4 -plane; (d) the waveform for x_3 ; (e) the three-dimensional phase portrait; (f) the power spectrum

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