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FOREWORD

Nonlinear dynamics and game-theoretic modeling in economics and finance

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Abstract
In this foreword to the Special Issue "Nonlinear dynamics and game-theoretical modeling in economics and finance" we review the contributions in the issue highlighting the economic results and the connections with aspects of dynamic analysis. Indeed, the common theme of the contributions is the focus on system dynamics and the latest analytical techniques. This issue is devoted to celebrating the 70th birthday of Professor Laura Gardini, who inspired a generation of scholars in the study of discrete-time systems and global methods of analysis.

Keywords Evolutionary games · Oligopoly games · Differential games · Networks · Nonlinear dynamics · Bifurcations

Nonlinear dynamics and deterministic chaos (also referred to as the *butterfly effect*) gained attention after Lorenz's, 1963 seminal contribution and are initially observed in economic models such as duopoly games (Rand, 1978), overlapping generations (Day, 1983), Keynesian models (Day & Shafer, 1985) and models of economic growth (Benhabib & Day, 1982). Subsequently, the discipline that studies nonlinear dynamics and chaos in economic and financial models grew with remarkable contributions in oligopoly games (Puu, 1991),

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asset pricing models (Day & Huang, 1990; Chiarella, 1992), adaptive rational-equilibrium dynamics (Brock & Hommes, 1997, 1998). These contributions inspired the literature on nonlinear economic dynamics in the last two decades, see Anufriev et al. (2018) for a survey.

Restricting the attention to game theory in economics, nonlinear dynamics emerge in oligopoly games (Bischi et al., 2010), minority games (Dindo, 2005), differential games (Hinloopen et al., 2017), and evolutionary games (Hofbauer & Sigmund, 1988, 2003; Hofbauer & Weibull, 1996). The main drivers of nonlinear dynamics in oligopoly games are non-perfect expectation schemes or adaptive-adjustment processes (Bischi et al., 2000, 2007; Bischi & Kopel, 2001). Evolutionary games with nonlinear dynamics find applications in environmental economics (Antoci et al., 2011), resource economics (Bischi et al., 2013, 2015a; Bischi & Lamantia, 2009; Lamantia & Radi, 2015), financial markets (Blaurock et al., 2023), housing markets (Dieci et al., 2018; Schmitt & Westerhoff, 2022), models with expectations and learning (Anufriev et al., 2013; Radi, 2017; Anufriev & Kopányi, 2018), industrial organization (Kopel et al., 2014; Lamantia & Radi, 2018), markets with incomplete information (Droste et al., 2002; Bischi et al., 2015b), production-location problems (Radi et al., 2021), segregation dynamics (Radi & Gardini, 2018; Harting & Radi, 2020).

The common denominator of all these contributions is the qualitative theory of nonlinear dynamical systems (Kuznetsov, 1998; Guckenheimer & Holmes, 1983; Devaney, 1989; Bischi et al., 2016). First proposed at the beginning of the 20th century, this theory offers the prevalent methodology and a very effective tool for the description of the time evolution of economic, financial, and social systems, as well as their interaction with the natural environment and the related concerns of sustainable development. All in all, this approach offers a set of methods and techniques to study the existence and stability of static as well as dynamic equilibrium configurations, optimal policies, persistence, resilience, emerging properties, stable fluctuations along periodic or chaotic attracting sets, robustness, structural stability, local and global bifurcations, hysteresis, multistability, and path dependence with the related study of basins attraction.

The qualitative theory of nonlinear dynamical systems provides a unifying approach and a suitable language to understand the complex time evolution of strongly interconnected communities, often influenced by the decisions of strategically interacting agents and by environmental variables and constraints. Nevertheless, the description and study of such complex systems require a multidisciplinary approach, including competencies in economics, psychology, biology, physics, and statistics. Moreover, the investigation of the models proposed, as well as the related policy implications, require methods based on a wide spectrum of mathematical tools, ranging from the qualitative theory of discrete and continuous dynamical systems to game theory, from optimization and optimal control to network theory, from evolutionary games and population dynamics to agent-based modeling, as well as advanced numerical techniques.

This special issue of *Annals of Operations Research* contains, among other works, a selection of refereed papers presented at the 11th edition of the international Dynamic Models in Economics and Finance (MDEF) Workshop held at the Department of Economics, Society, Politics of the University of Urbino (Italy) in September 2022. The tradition of the MDEF Workshop was launched in 2000 by Laura Gardini with a group of other scholars, and the current issue is dedicated to Laura's 70th birthday. Laura worked (in the early years) to investigate homoclinic chaos emerging in smooth systems, mainly in discrete time (Gardini, 1994). Later, she focused on the bifurcation structures emerging in non-smooth discrete-time dynamical systems (Sushko et al., 2006, 2016a, b; Sushko & Gardini, 2010). Particularly relevant are the results and methods for studying one-dimensional non-smooth

maps summarized in Avrutin et al. (2019) and their applications in economics (Matsuyama et al., 2016; Sushko et al., 2014). Laura has investigated (recently) two-dimensional piecewise-smooth discrete time dynamical systems (Gardini et al., 2022c), and their applications in macroeconomics (Gardini et al., 2023), finance (Gardini et al., 2022a, b), and oligopoly games (Gardini & Radi, 2023).

The works proposed in this special issue exhibit features such as endogenous oscillations, stability, and bifurcations in a wide spectrum of applications to economics and social systems, ranging from environmental and behavioral economics to dynamic and evolutionary games including differential games and games on networks. Many applications address boundedly rational expectations and adaptive behavior to describe social conflicts and dilemmas, interaction between populations and the environment, and real and financial markets with the interplay between individual behavior at the micro-level and collective behaviors at the macro-level. This view emphasizes, in particular, two key features arising from nonlinear dynamics: (i) the role of boundedly rational behavior based on partial information and conditioned upon psychological biases, and (ii) the strategic interactions among heterogeneous agents and the resulting aggregate behaviors which, in turn, provide feedback to individual decisions. Also on the methodological aspects, many works deal with innovative approaches, with particular reference to non-smooth systems and the related local and global analysis methodologies of the resulting dynamical systems.

In this vein, *Commendatore et al.* propose a new economic geography (NEG) model with local pollution and differential labor, with low-skilled workers able to migrate and high-skilled entrepreneurs able to choose where to work and live. The underlying assumptions give rise to a footloose entrepreneur model, with piecewise dynamics for entrepreneurs' residential choice and production location, in which non-smooth bifurcations and global analysis of the model are mainly emphasized. Also in *Radi et al.*, the duopoly model is formulated as a piecewise-smooth discrete-time dynamical system, in this case because of the ambiguity aversion of the firms, which leads, in an otherwise simple model, to multiple equilibria, periodic and chaotic dynamics through border collision bifurcations. *Mignot et al.* develop a nonlinear duopoly model in which the heuristic expectation formation and learning behavior of two boundedly rational firms may engender complex dynamics. Remaining in the oligopolistic framework, *Buccella et al.* elaborate on Katz and Shapiro's (1985) model of consumption externalities on networks reformulated for corporate socially responsible (CSR) firms. The authors address the issue of commitment decision and study all possible subgame perfect Nash equilibria. *Murakami* analyzes the long-term perspective of optimal research and development in the presence of product life cycles. The dynamic features of consumer theory with behavioral biases such as imitation of a single consumer to a reference group are analyzed by *Bischi and Tramontana*. With an analogous behavioral focus, *Campisi et al.* investigate the complex dynamics arising from a behavioral exchange rate discontinuous model with heterogeneous agents and explain the emergence of chaos induced by a sentiment index, which affects the way (optimistic or pessimistic) investors make their trading decisions. *Matsumoto and Szidarovszky* consider a Kaldorian business cycle model with time shifts and study how time-delayed and time-advanced terms affect stability and the occurrence of economic fluctuations.

The issues related to lockdown policies are the focus of *Gubar et al.* By analyzing different lockdown policy structures, the authors show both the benefits of a regulatory lockdown policy and its dependency on the particular socioeconomic structure to be managed.

The issue of social distancing is also addressed by *La Torre et al.* within a susceptibles-infectives-susceptibles (SIS) framework, where the biological traits of the pathogen follow random shocks. They show that social distancing could also be beneficial for reducing the support of the steady-state distribution of the epidemiological outcome. *Cavalli et al.* propose a dynamic model of technology adoption with pollution evolution, in which the production system, represented by a population of firms, can adopt green or dirty technologies by following evolutionary dynamics. The role of taxation, with subsequent reinvestment of taxes in abatement, is the key tool to steering the system toward sustainable outcomes, although over-taxation of dirty technology may not be beneficial as multiple steady states may emerge leading to suboptimal outcomes. Along a similar line, *Buccella et al.* consider an oligopoly with differentiated technologies and fiscal incentives for actions that reduce environmental impact. Again, the role of the regulator is central and can greatly alter the social and environmental outcome, which can also be characterized by persistent and endogenous oscillations typical of nonlinear dynamics. *Grassetti et al.* propose a growth model with co-evolution of the natural system, the quality of which is measured by a synthetic indicator. In their analysis, a central role is played by the regulator, which can impose a severe reduction of the capital available for production in the productive sector in proportion to the expected damages to economic activity. Again, multiple equilibira and possible poverty traps are highlighted, which can be mitigated by consumer awareness campaigns for more responsible economic choices. Maladaptation is the focus of *Antoci et al.* in a north-south evolutionary model, in which the south may be exposed to greater vulnerability to environmental degradation, while the north can resist this kind of vulnerability. In this framework, the authors show how the system can be trapped in a Pareto-dominated equilibrium and place the integration of environmental risk and cross-country cooperation into economic growth choices at the center of the debate.

The natalist bias of pollution control hypothesis, related to a shift from production to procreation as a result of emission taxation, is the focus of *Cafferata and Dávila-Fernández*. They investigate an overlapping generations growth model to infer that this assumption does not necessarily hold if the effects of such policies are evaluated over a medium/long term. The nonlinear dynamics analysis also shows that this framework is compatible with persistent endogenous fluctuations in human capital accumulation and pollution. Optimal water resource management is addressed in *Caravaggio et al.*, who formalize this problem in a leader-follower differential game setting, in which the aspect of buyer heterogeneity is also taken into account, showing which pricing design is preferable in this context. The problem of evaluating undermining projects is the object of *Hozman et al.*, who develop the evaluation model via real options theory and propose a numerical evaluation via the discontinuous Galerkin approach. *Merlone et al.* propose an extension of the inspection game to the case of continuous strategies and also present a possible dynamic version of this game.

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