ORIGINAL RESEARCH

A vendor-managed inventory model for a three-layer supply chain considering exponential demand, imperfect system, and remanufacturing

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Abstract

This paper presents a vendor-managed inventory model for a three-layer supply chain comprised of suppliers, manufacturers, and retailers. This model considers an imperfect production system with a deterministic percentage of defective and deteriorated items, exponential demand for perishable products, and operational costs. The VMI model maximizes the benefits, assuming a Stackelberg approach to find the optimal values of replenishment lot sizes, production rates, selling prices, and product cycle length. A solution algorithm is developed to find the optimal solutions. A case study of the Dairy Industry for highly perishable products is provided to illustrate the applications, evaluate the performance, and obtain managerial insights. The results show that the percentage of defective and deteriorated items and remanufacturing rates are crucial in replenishment, production, and selling policies. The VMI agreement includes remanufacturing defective products to obtain new dairy derivates. The deteriorated products can be sold as near-expired products or inputs for compost, fertilizers, etc. Finally, some conclusions and future research lines are provided.

Keywords Inventory · VMI · Exponential demand · Deteriorating item · Remanufacturing

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1 Introduction

Efficient inventory management is a significant supply chain management challenge. Therefore, collaboration strategies and cooperation contracts, such as vendor-managed inventory (VMI), are viable approaches to improving inventory levels. This approach reduces inefficiencies in the production system, damages and affectations in the inventory and high costs in the company logistics activities. Then, inventory models allow the optimization of the use of resources and improvement of the production and logistics processes in a supply chain (Salas-Navarro et al., 2017, 2022).

During the last decades, many studies have established a vendor-managed inventory as a collaborative strategy that allows the vendor (manufacturer) to manage the stocks of the buyer (retailer) and to develop the planning and programming of replenishment of products in sale points. Many authors have studied VMI models (see, for instance, Vlist et al., 2007; Darwish & Odah, 2010; Yu et al., 2012; Mateen & Chatterjee, 2015; Mateen et al., 2015; Taleizadeh et al., 2015a; Khan et al., 2016) for a two-layer supply chain. Generally, these studies have considered single-vendor (supplier) and single-buyer (retailer) and established replenishment agreements for joint inventory policies. The approach most used in VMI models is the supply chain for two levels (supplier-retailer).

An imperfect production system generates defective items that affect the inventory levels and optimal lot size because it reduces the number of good items available for sale (Salas-Navarro et al., 2020). Porteus (1986) was the first to incorporate the effect on defective items into EOQ inventory models, followed by Cheng (1989) and Cheng (1991), who included the defective items in EPQ inventory models. Salameh and Jaber (2000) formulated an EOQ inventory model with defective products, and Chiu (2003) incorporated the remanufacturing of defective items as a strategy to reduce non-quality costs. Wee et al. (2007), Eroglu and Ozdemir (2007), and Arindam Roy et al. (2009) considered defective items and that a fraction of these are remanufactured. Kreng and Tan (2011), Lin et al. (2012), Soni and Patel (2012), and Jaggi et al. (2013) assumed defective items and credit policies between supplier and retailer. These studies applied the remanufacturing process to obtain products of intermediate quality that can be sold at a lower selling price.

The deterioration of good items in a real problem is considered an inevitable characteristic of some products, and quality problems generate defective products as outputs from the production process (Lee & Kim, 2014; Tiwari et al., 2022). The deterioration of products is a significant consideration in inventory systems because bacteria, disease, insect damage, and expiration can affect the products while they are stored. Deterioration can occur in perishable products such as food, milk, fruits, vegetables, meats, and other products such as photographic rolls, electronic components, alcohol, perfumes, and gasoline (Khakzad & Gholamian, 2020; Pando et al., 2018).

Whitin (1957) and Ghare and Schrader (1963) were the first to include deteriorated items in inventory models. From these investigations, five reviews have been developed in succession about the advances in the inventory deterioration literature (Bakker et al., 2012; Goyal & Giri, 2001; Janssen et al., 2016; Nahmias, 1982; Raafat, 1991; Taleizadeh et al., 2020). Archetti et al. (2007) developed a vendor-managed inventory-routing problem for a two-layer supply chain comprised of a single supplier and multiple retailers. Coelho et al. (2020) incorporated a multi-attribute approach to extend the multi-depot inventory routing problem. Also, Coelho et al. (2014) and Shaabani (2022) studied the vehicle-routing and inventory management problem, considering single-supplier and multiple-buyer supply chain structures and evaluating perishable inventory.

Some authors have considered variable deteriorated items (Sana, 2010), exponential deteriorated items (Chan et al., 2017), and uncertainty deteriorated items (Li et al., 2017). Other research has incorporated constant deterioration (Shah et al., 2013; Wee & Widyadana, 2013); Sicilia et al., 2014; Kim et al., 2014; Chakraborty et al., 2015; Dye & Yang, 2016; Duan et al., 2016; Chen et al., 2018), and time-varying deterioration (Li et al., 2019). This situation affects the efficiency of the retailer's operation and can incur high costs and reduce profits. Thus, supply chain members should establish strategies that control the processes of product deterioration and quality control to mitigate the negative impact on operational costs.

Many researchers have studied the inventory problem considering the market demand as an exponential function for defective items in economic order quantity (EOQ) inventory models (see, for example, Hollier & Mak, 1983; Ouyang et al., 2005; Sana, 2012; Yadav & Swami, 2013; Salas Navarro et al., 2018). Other authors have developed economic production quantity (EPQ) and EOQ inventory models assuming uncertainty demand that follows an exponential function (Pal et al., 2012; Roy et al., 2012), uniform function (Pal et al., 2013b; Vandana & Kaur, 2019), and normal function (AlDurgam et al., 2017).

Some authors studied imperfect systems with probabilistic demand and deteriorating items consideration. Lin (2013) developed an inventory model with defective products with stochastic demand and lead time. Kitaeva et al. (2017) proposed an inventory model that considers continuous deteriorating and demand-sensitive prices that follow an exponential distribution function. Additionally, Freeman et al. (2018) and Braglia et al. (2019) incorporated uncertain demand by considering randomly generated scenarios from demand distribution. Wang et al. (2021) assumed that the purchasing and the reference price of customers determine customer demand. Furthermore, Poormoaied (2022) developed an inventory model considering a Poisson process for the demand rate, and San-José et al. (2022) included a time-dependent demand function for a discrete inventory cycle system.

Deteriorating items and defective products negatively affect inventory levels in a supply chain. An imperfect production system reduces product availability and the ability to fulfill customer orders, which increases holding inventory and deterioration costs. Companies must then implement strategies to obtain income from selling deteriorating products or remanufacturing defective products as inputs for secondary markets.

The vendor-managed inventory problem for deteriorating items can be studied in supply chains incorporating two or three echelons. This problem includes suppliers, manufacturers, and retailers with imperfect production systems generating defective products.

According to the gaps analysis, the following questions are raised in this research:

- How does an imperfect production system with defective and deteriorated products affect a supply chain?
- How can you optimize the benefits of a 3-layer supply chain in which contracts for inventory policy planning are established through a vendor-managed inventory collaboration scheme?
- How can the defective product remanufacturing strategy generate additional revenue in an imperfect production system?
- How can costs associated with inefficient inventory management processes affect raw material replenishment lot sizes, production rates, sales prices, and product cycle times in a supply chain?

In this paper, a VMI model for a three-layer supply chain is developed. It is assumed a network configuration with multiple suppliers, manufacturers, and retailers that provide, produce and sell perishable items that can deteriorate during storage. This study proposes inventory policies to increase income by selling perishable products in retailers as inputs

for secondary markets. Additionally, an imperfect production system is considered during production time, and a portion of defective products is separated from conforming products through quality inspections. Defective products are remanufactured to obtain secondary products sold at lower prices in supporting industries.

This research aims to develop a VMI model considering an imperfect production system that generates defective and deteriorated items in suppliers, manufacturers, and retailers. This model incorporates the remanufacturing process as a strategy to increase profits. Thus, the VMI model considers the replenishment lot size of raw material, the production rate of products, the selling price of products, and the cycle length of products as decision variables. Additionally, this VMI model includes inventory holding costs, setup costs, screening costs, purchasing costs, deterioration costs, and production costs.

The contributions of this paper that distinguish it from another related research are:

1. This research contributes to inventory management literature because it constitutes a new inventory model that models reality more accurately and establishes a tool for improving company productivity in various economic sectors.
2. The market demand follows a declining exponential function of time and is synchronized with raw material and run-time production.
3. Deteriorated items can occur in agricultural products, electronic components, volatile liquids, drugs, fashion goods, and other products that can suffer deterioration over time.
4. The VMI model optimizes the replenishment lot size of raw materials, production rate of products, selling price, and product cycle length.
5. The authors used the analytical calculation method and differential equations to obtain an optimal solution.
6. The sensitivity analysis includes the performance of inventory holding costs, the percentage of deteriorated items, a fraction of defective items, remanufacturing rates, and selling prices.

The structure of this paper is as follows. Section 2 provides a literature review of relevant research and identifies research gaps. Section 3 presents the problem definition, and Sect. 4 formulates the VMI model, considering the benefits of suppliers, manufacturers, retailers, and supply chains. Section 5 proposes the solution method, Sect. 6 solves the numerical example, and Sect. 7 shows the sensitivity analysis for the benefits functions, decision variables, and managerial implications. Section 8 provides the conclusions, the significant contributions, and future research directions.

2 Literature review

The Vendor-Managed Inventory (VMI) is an integrated approach related to supplier-retail coordination, where the supplier decides on appropriate inventory levels and replenishment within limits according to a contractual agreement between both actors (Darwish & Odah, 2010). The VMI is a policy of collaborative inventory management implemented among manufacturers and retailers, where the first is responsible for making decisions about inventory replenishment (Borade & Sweeney, 2015).

The network configuration of the VMI agreement generally includes vendors and retailers. Yao et al. (2007) developed a VMI model for single-vendor and single-buyer, considering a collaborative approach. Woo et al. (2001) considered a single-manufacturer and multiple-buyers supply chain structure, and Zhang et al. (2007) incorporated different buyer order cycles. Otherwise, Zanoni et al. (2014) proposed a VMI model with consignment stock and

commercial agreements for a single-vendor and single-buyer supply chain to show the total cost as a convex function. However, Wang and Huang (2016) found a calculation error and developed an effective and efficient search algorithm to obtain an approximate solution.

On the other hand, Almehdawe and Mantin (2010) built a VMI model for a single manufacturer and multiple retailers, considering a Stackelberg approach when the manufacturer is the leader when retailers dominate the rules for the game. The VMI system for a two-layer supply chain for single-vendor and two-buyers includes the demand of buyers, transportation cost, ordering cost for vendors, and inventory holding costs (Mateen & Chatterjee, 2015; Rad et al., 2014). Sadeghi and Niaki (2015) studied a bi-objective VMI model for single-vendor and multi-retailers, considering fuzzy demand and minimizing the total cost.

Lee et al., (2016a, 2016b) introduced a VMI model with a coordination approach considering stockout-cost sharing between a supplier and a customer to minimize the total cost. Later, Verma and Chatterjee (2017) presented a VMI model for single-supplier and multi-retailers. The supplier makes decisions related to inventory replenishment for retailers to minimize the total system cost.

Many studies on the VMI approach include the deterioration of items as an unavoidable and ingrained characteristic of perishable products. Also, imperfect production systems and quality problems generate defective products as outputs from production processes (Lee & Kim, 2014). The perishable raw material is an essential consideration in the VMI model because of the slow deterioration of finished products (Fauza et al., 2016; Khan et al., 2016; Yu et al., 2012). The VMI agreements included policies to reduce defective products and disrupt imperfect production systems through a remanufacturing strategy (Bazan et al., 2014; Lee & Cho, 2014). Additionally, Setak and Daneshfar (2014) considered a VMI policy for deteriorated items with stock level-dependent demand rates, and Tat et al. (2015) incorporated a non-instantaneous deteriorating items approach.

The problem of deteriorating items in supply chains incorporates market demand varying rate consideration according to the product type. Taleizadeh et al. (2015a) studied a VMI model for single-vendor and multi-retailers considering deteriorating items, deterministic demand, and sensitive-price demand. Otherwise, Chung et al. (2008) analyzed an inventory system with a reverse material flow supply chain for returned and remanufactured products. Other authors proposed different VMI strategies to reduce ordering and transportation costs considering stochastic demand (Govindan, 2015; Kiesmüller & Broekmeulen, 2010; Mateen et al., 2015). Escuín et al. (2017) considered multi-items and stochastic demand and compared order strategy with the VMI strategy between suppliers and customers.

The joint economic lot-sizing model considers a single-vendor and single-buyer supply chain for a single product, including a consignment agreement and dependent demand (Lee et al., 2017). Therefore, Cai et al. (2017) studied a VMI model for a two-layer supply chain considering uncertainty demand with an exponential distribution function. Then, Kaasgari et al. (2017) investigated a VMI model with perishable products for a one-vendor and multi-buyers supply chain that minimized the total costs and developed a genetic algorithm (GA) and a particle Swarm Optimization (PSC) algorithm for solving the problem.

Recently, studies have addressed different approaches to the VMI problem. Han et al. (2017) examined a VMI model for a three-layer supply chain to minimize transportation and inventory costs. Also, Amirhosseini et al. (2018) incorporated multi-products and defective items. Lee and Cho (2018) considered stockout penalties and holding costs. Therefore, Son and Ghosh (2019) considered VMI agreements and cross-docking strategies at the vendor. Gharaei et al. (2019) included multiple products, multiple buyers, penalties, quality control policies, and VMI agreements. Thus, Weraikat et al. (2019) studied the VMI model to optimize the expired medication cost in hospitals, considering uncertain demand. Wei et al. (2020)

analyzed two types of managed inventory models: a Retailer-Managed Inventory (RMI) and Vendor-Managed Inventory (VMI), considering a Stackelberg approach, where the vendor is the leader, and the retailers are the followers.

Table 1 presents some VMI models with several features. The VMI models include a tri-level or a two-level supply chain, one or multiple members for level, only or multi-products, deteriorated items, defective items, remanufactured products, and type of demand. These inventory models consider a two-level supply chain with only one product and constant demand. Some papers consider multiple members for each level of the supply chain or only one echelon member, stock-dependent demand or exponential demand. Deteriorated, defective, and remanufactured products are the features studied in the VMI models literature.

According to the literature review, an opportunity area is identified to research VMI models in a supply chain configured with more than two levels. Most studies considered deterministic demand, which is not adjusted to fluctuations in the market. For this reason, a new research problem has been identified with the opportunities to develop recent studies that facilitate the planning and execution of logistics activities in a supply chain that present significant challenges for researchers. Additionally, significant research on inventory models has been conducted to examine VMI inventory models, defective items, remanufacturing products, deteriorated items, and exponential demand. These areas recommended that further research include joint optimization of different variables in an EPQ inventory model, imperfect production, and probabilistic demand in a supply chain.

3 Problem definition

This paper considers a Vendor-Managed Inventory system for a three-layer supply chain. The VMI strategy includes collaboration among all supply chain members. A leader decides to replenish raw materials and products to ensure the requirements of customers and optimize the benefits of the supply chain members. Figure 1 presents the VMI model diagram. The manufacturers act as a supply chain leader, assume the retailer replenishment, supervise the inventory level to avoid backorders, and attend to the market demand.

The suppliers store raw materials for sale to manufacturers in cycle time T_{ijs}^S and make quality inspections to find defective items. These items are removed from the quality inventory and sold to other industries with a lower sale price. The manufacturers purchase and store raw materials and transform them into finished products during production time TP_{jm}^M. Currently, the manufacturers produce defective items at a rate of β_{jm}^M detected during the quality inspection. They are removed from good-items inventory to remanufacture them and sell them in other industries with a lower sale price. Manufacturers decide on the inventory levels and replenish the retail stores according to VMI agreements. The retailers purchase and store finished products sold to the final customer. The finished products deteriorate at a rate of κ_{jr}^R during the time T_{jr}^R.

The assumptions used to formulate the vendor managed inventory (VMI) model are presented below:

1. A three-layer supply chain with multiple suppliers, manufacturers, and retailers is considered. The VMI model considers multi-items for raw materials and products for each supply chain member.
2. Considering the VMI strategy, the objective function maximizes the benefit of the manufacturers who decide to supply products and raw materials.

Table 1 Vendor-managed inventory model considerations

Authors	Three level	Two level	One member for level	Multiple member for level	One product	Multi-product	Deteriorated items	Defective items	Remanufactured products	Type of demand
Woo et al. (2001)		X		X	X					Constant
Vlist et al. (2007)		X	X		X					Constant
Zhang et al. (2007)		X		X	X					Constant
Chung et al. (2008)		X	X		X				X	Constant
Darwish and Odah (2010)		X		X	X					Constant
Almehdawe and Mantin (2010)		X		X	X					Constant
Kiesmüller and Broekmeulen (2010)		X		X		X				Bernoulli demand /Poisson demand
Yu et al. (2012)		X		X	X		X			Constant
Bazan et al. (2014)		X	X		X			X	X	Constant
Lee and Kim (2014)		X	X		X		X	X		Constant
Zanoni et al. (2014)		X	X		X					Constant
Lee and Cho (2014)		X	X		X					Poisson demand

Table 1 (continued)

Authors	Three level	Two level	One member for level	Multiple member for level	One product	Multi-product	Deteriorated items	Defective items	Remanufactured products	Type of demand
Hariga et al. (2013)		X		X						Constant
Xiao and Xu (2013)		X	X		X		X			Price and service level- dependent demand
Moncer Hariga et al. (2014)		X		X	X					Constant
Setak and Daneshfar (2014)		X	X		X		X			Stock level-dependent demand
Hong et al. (2016)		X		X	X					Uniform distributed demand
Mateen and Chatterjee (2015)		X		X	X					Constant
Mateen et al. (2015)		X		X	X					Normally distributed demand
Govindan (2015)		X		X	X					Normally distributed demand

Table 1 (continued)

Authors	Three level	Two level	One member for level	Multiple member for level	One product	Multi-product	Deteriorated items	Defective items	Remanufactured products	Type of demand
Taleizadeh, Noori-daryan, & Cárdenas-Barrón (2015)		X		X	X		X			Price-dependent demand
Choudhary and Shankar (2015)		X		X	X		X			Non-stationary stochastic demand
Tat et al. (2015)		X	X		X		X			Constant
Khan et al. (2016)		X	X		X			X		Constant
Lee et al. (2016a, 2016b)		X	X		X					Constant
Fauza et al. (2016)		X		X	X		X			Constant
Cai et al. (2017)		X		X	X					Exponential demand
Lee et al. (2017)		X	X		X					Stock-dependent demand
Verma and Chatterjee (2017)		X		X	X					Constant
Kaasgari et al. (2017)		X		X	X		X			Stock-dependent demand

Table 1 (continued)

Authors	Three level	Two level	One member for level	Multiple member for level	One product	Multi-product	Deteriorated items	Defective items	Remanufactured products	Type of demand
Han et al. (2017)	X			X	X					Stock and price dependent demand
Amirhosseini et al. (2018)		X	X			X		X		Constant
J. Lee and Cho (2018)		X	X							Constant
Gharaei et al. (2019)		X		X		X				Constant
Son and Ghosh (2020)		X		X	X					Constant
Wei et al. (2020)		X	X		X					Inverse demand function
Wangsa et al. (2020)		X	X		X					Normally distributed demand
Sarkar and Giri (2020)		X	X					X		Constant
M. Das Roy and Sana (2021)		X	X		X					Normally distributed demand
Mahdavisharif et al. (2022)		X					X			Uncertainty demand

Table 1 (continued)

Authors	Three level	Two level	One member for level	Multiple member for level	One product	Multi-product	Deteriorated items	Defective items	Remanufactured products	Type of demand
Shaikh et al. (2022)		X	X		X		X			Selling price-dependent demand
Salas-Navarro et al. (2023)	X			X			X			Exponential demand
This paper	X			X		X	X	X	X	Exponential demand
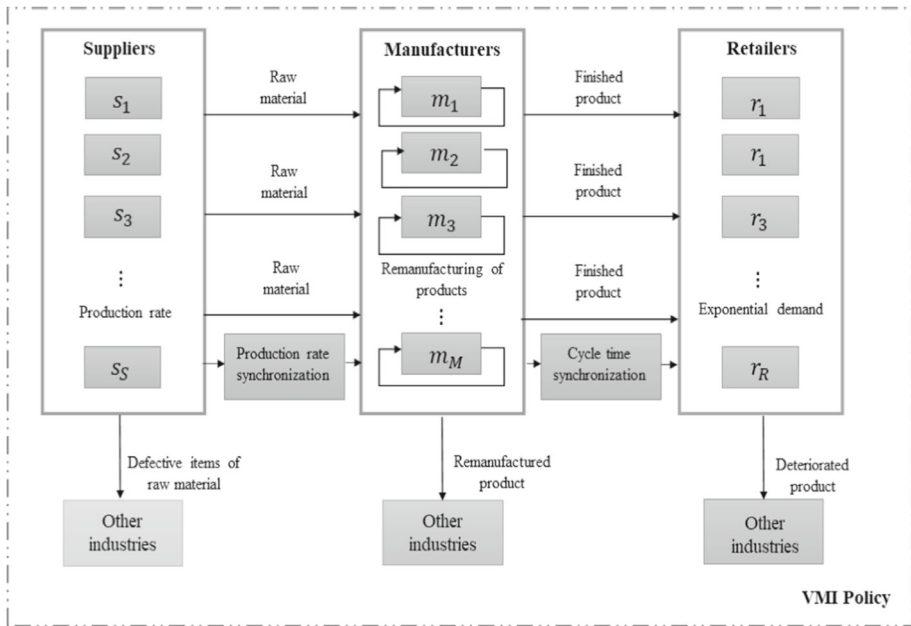
Fig. 1 Diagram of vendor managed inventory system

3. The replenishment rate of the supplier is instantly infinite, but the lot size is finite. Therefore, the lead time is zero.

4. The VMI model is considered an imperfect production system. Quality inspections are considered in suppliers and manufacturers that detect defective raw material items and defective products.

5. Products near the expiration date are considered deteriorated items. Therefore, they are removed from good-product inventory and sold as near-expired products at a lower price. The deterioration rates of finished products are constant and known.

6. The defective items of finished products are remanufactured by the manufacturers and are sold to other secondary industries.

7. The production rate of the suppliers is synchronized with the production rate of the manufacturers so that there are no shortages of raw materials.

8. The retailers' demand is a time-dependent and exponential declining function of time and is given by $DC_{jr}^R(t) = Di_{jr}^R e^{-\lambda_{jr} t}$, where Di_{jr}^R is the products' market demand, $Di_{jr}^R > 0$ and λ_{jr} is constant $0 < \lambda_{jr} < 1$.

9. Shortages are not allowed.

Figure 2 presents the structure of the VMI model for a three-layer supply chain comprised of multiple suppliers, manufacturers, and retailers. Initially, the supplier supplies raw materials at a rate of P_{ijs}^S to satisfy the demand for manufacturers or replenishment lot size, R_{ijs}^S. However, this parameter is affected by the percentage of defective products identified in quality inspections carried out by suppliers, which is presented as $\alpha_{ijs}^S R_{ijs}^S$.

Manufacturers receive the raw material from suppliers and start the production process that transforms raw material into a finished product at a production rate of P_{jm}^M. During the
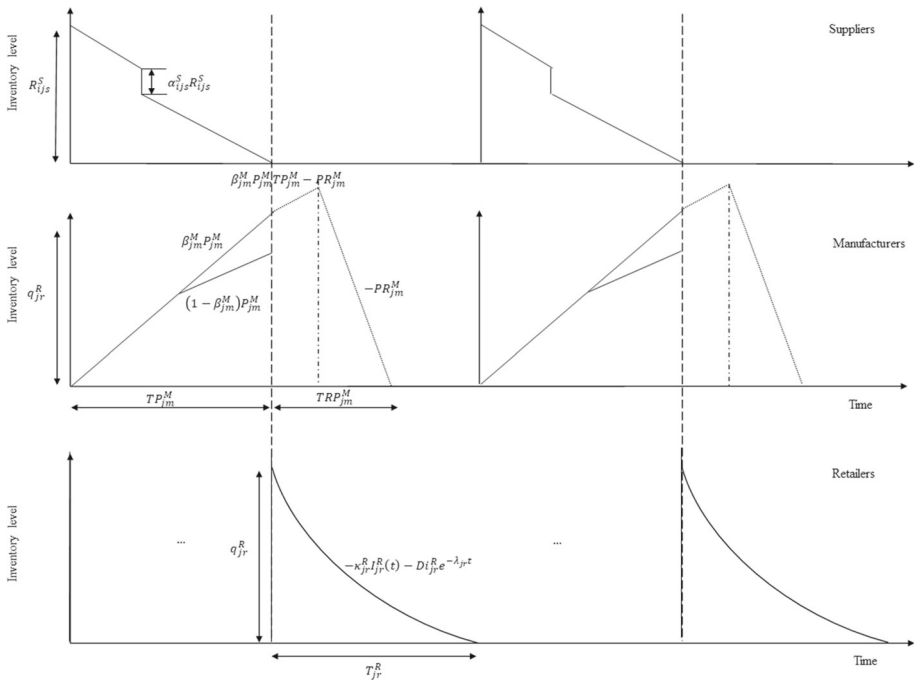
Fig. 2 Structure of the VMI model that considers a three-layer supply chain

production runtime TP_{jm}^M, defective articles are generated with a percentage β_{jm}^M identified in quality inspections and sent to the remanufacturing process. After regular production, defective products are remanufactured during TRP_{jm}^M at a rate PR_{jm}^M and are stored together with conforming regular production to be shipped to other industries.

Retailers purchase the finished product at a demand rate q_{jr}^R at the cycle time T_{jr}^R. The finished products suffer constant deterioration at a rate κ_{jr}^R due to expiration and poor handling during cycle time T_{jr}^R. The deteriorated items are removed from inventory and sold to other industries or near-expired products at a lower price.

Some authors have considered the inventory model with exponential demand. Gross and Harris (1971) developed an inventory model considering random variables that follow an exponential density function. Ouyang et al., (2005) proposed an EPQ inventory model with deteriorating items, exponential demand, and partial backlogging. Also, Tripathi (2012) developed an inventory model with an exponential demand rate with delays in payment periods and backlogging. B. Pal et al., (2013a, 2013b) presented an EPQ inventory model considering stochastic demand that follows an exponential density function for an imperfect production system. Mateen et al. (2015) formulated an inventory model with stochastic demand and backlogging in a two-layer supply chain.

Sajadieh and Larsen (2015) analyzed the optimal inventory level and policies for a two-echelon supply chain, considering variable demand and production rates. Salas Navarro et al. (2018) evaluated the demand as an exponential density function for a three-echelon supply chain. Pal and Mahapatra (2017) incorporated failures, stochastic demand, deteriorated items, and backlogging. Thus, The VMI model considers an exponentially decreasing demand

because the deterioration of products occurs during storage, which reduces the inventory of conforming products.

The following notation is established to develop the VMI model for a three-layer supply chain.

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i Raw materials, $i \in \{1, 2, 3, \ldots I\}$

j Finished products, $j \in \{1, 2, 3, \ldots J\}$

s Suppliers, $s \in \{1, 2, 3, \ldots S\}$

m Manufacturers, $m \in \{1, 2, 3, \ldots M\}$

r Retailers, $r \in \{1, 2, 3, \ldots R\}$

Parameters

P_{ijs}^{S}	Production rate of raw material i for the supplier s to manufacture the product j (units/unit of time)
α_{ijs}^{S}	Fraction of defective items of raw material i for the supplier s to manufacture the product j, where $0 \leq \alpha_{ijs}^{S} \leq 1$
r_{ijs}^{S}	Screening rate of defective items of raw material i for the supplier s to manufacture the product j (units/unit of time)
C_{ijs}^{S}	Screening cost of defective items of raw material i for the supplier s to manufacture the product j (\$/unit)
A_{ijs}^{S}	Setup cost of raw material i for the supplier s to manufacture the product j (\$/cycle)
h_{ijs}^{S}	Holding cost of raw material i for the supplier s to manufacture the product j (\$/unit/unit of time)
PC_{ijs}^{S}	Purchasing cost of raw material i for the supplier s to manufacture the product j (\$/unit)
W_{ijs}^{S}	The selling price of raw material i for the supplier s to manufacture the product j (\$/unit)
WD_{ijs}^{S}	The selling price of defective items of raw material i for the supplier s to manufacture the product j (\$/unit)
PR_{jm}^{M}	Re-manufacturing rate for the product j at the manufacturer m (units/unit of time)
β_{jm}^{M}	Fraction of defective items for the product j at the manufacturer m, where $0 \leq \beta_{jm}^{M} \leq 1$
r_{jm}^{M}	Defective items for the product j at the manufacturer m (units)
C_{jm}^{M}	Screening cost of defective items for the product j at the manufacturer m (\$/unit)
A_{jm}^{M}	Setup cost for the product j at the manufacturer m (\$/cycle)
hr_{jm}^{M}	Holding cost for the raw material required for the product j at the manufacturer m (\$/unit/unit of time)
hp_{jm}^{M}	Holding cost for the product j at the manufacturer m (\$/unit/unit of time)

PC_{ijm}^M	The purchasing cost of raw material i for the product j at the manufacturer m ($/unit)
L_{jm}^M	Labor cost for the product j at the manufacturer m ($)
δ_{jm}^M	Fixed cost for the product j at the manufacturer m ($/unit)
ε_{jm}^M	Remanufacturing cost for the product j at the manufacturer m ($/unit)
W_{jm}^M	The selling price for the product j at the manufacturer m ($/unit)
WD_{jm}^M	The selling price of remanufactured items for the product j at the manufacturer m ($/unit)
Z_{ijm}^M	Units of raw material i required to produce one unit of product j at the manufacturer m (units)
hp_{jr}^R	Holding cost for the product j at the retailer r ($/unit/unit of time)
A_{jr}^R	Setup cost for the product j at the retailer r ($/cycle)
PC_{jr}^R	Purchasing cost for the product j at the retailer r ($/unit)
κ_{jr}^R	Percentage of deteriorated items for the product j at the retailer r
CD_{jr}^R	Deterioration cost for the product j at the retailer r ($/unit)
WD_{jr}^R	Selling price of deteriorated items for the product j at the retailer r ($/unit)
λ_{jr}	A constant that governs the demand for the product j at the retailer r
a_{jr}^R	Initial demand rate of the market for the product j at the retailer r (units/unit of time)
b_{jr}^R	Variation of demand rate for the product j at the retailer r (units/unit of time)

Dependent variables

π_{js}^S	The benefit of the supplier s for the product j ($/unit)
π_{jm}^M	The benefit of the manufacturer m for the product j ($/unit)
π_{jr}^R	The Benefit of the retailer r for the product j ($/unit)
T_{ijs}^S	Replenishment cycle of raw material i at the supplier s to manufacture the product j (unit of time)
T_{jm}^M	Replenishment cycle of product j at the manufacturer m (unit of time)
TP_{jm}^M	Production run-time of product j at the manufacturer m (unit of time)
TPR_{jm}^M	Remanufacturing run-time of product j at the manufacturer m (unit of time)
$I_{ijm}^M(t)$	Inventory level of raw material i to produce j at the manufacturer m at time t (units/unit of time)
$I_{jm}^M(t)$	Inventory level of good items for the product j at the manufacturer m at time t (units/unit of time)

$ID_{jm}^M(t)$	Inventory level of defective items for the product j at the manufacturer m at time t (units/unit of time)
TC_{jr}^R	Replenishment cycle of product j at the retailer j (unit of time)
$I_{jr}^R(t)$	Inventory level of product j at the retailer j at the time t (units/unit of time)
q_{jr}^R	The demand of the product j at the retailer r (units)
Di_{jr}^R	Demand rate of the customers for the product j at the retailer r (units/unit of time)

Decision variables

R_{ijs}^S	Replenishment lot size of raw material i at the supplier s to manufacture the product j (units)
P_{jm}^M	Production rate of product j at the manufacturer m (units/unit of time)
W_{jr}^R	The selling price of the product j at the retailer r (\$/unit)
T_{jr}^R	Cycle length of the product j at the retailer r (unit of time)

4 Formulation of VMI model

A VMI model in the three-layer supply chain is proposed, considering multi-item, multiple suppliers, manufacturers, and retailers. A fraction of defective raw material items, α_{ijs}^S $i \in \{1, 2, \ldots I\}$, are considered in the supplier s for each raw material i. The defective items are sold to other industries with a lower selling price. The manufacturer $m \in \{1, 2, \ldots, M\}$ produces multiple products j from a unit of raw material at a production rate P_{jm}^M.

4.1 Benefit of suppliers

The manufacturer m orders a replenishment lot size R_{ijs}^S of raw material to initiate the production process at a supplier s. The production rate of raw material P_{ijs}^S is considered a dependent variable in the production rate of product function,

$$P_{ijs}^S = \left(1 + \alpha_{ijs}^S\right) P_{jm}^M Z_{ijm}^M. \tag{1}$$

The supplier s does quality inspections during production to screen defective items and remove them from the raw material lot. This situation generates a good-item and defective-item inventory at the supplier during the cycle T_{ijs}^S. Figure 1 presents the inventory level diagram for each layer. The behavior of good items inventory level of raw material, I_{ijs}^S, at

the suppliers is presented in Eq. (2)

$$\frac{dI_{ijs}^S}{dt} = -\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S, \tag{2}$$

Using the initial conditions, $I_{ijs}^S(0) = R_{ijs}^S$, $I_{ijs}^S\left(T_{ijs}^S\right) = 0$ $\quad 0 \le t \le T_{ijs}^S$ $\forall(ijs)$, we obtain

$$T_{ijs}^S = \frac{R_{ijs}^S}{\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S} \quad \forall(ijs). \tag{3}$$

Equation 3 represents the cycle time of raw materials at the suppliers.

The suppliers' revenue provides from the goods item selling of raw material to manufacturers $W_{ijs}^S\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S$, and the defective item selling to other industries is $WD_{ijs}^S \alpha_{ijs}^S P_{ijs}^S$.

The inventory holding costs of good-items, H_{ijs}^S, and defective-items, HD_{ijs}^S, are presented in Eqs. (4) and (5), respectively.

$$H_{ijs}^S = \int_0^{T_{ijs}^S} h_{ijs}^S I_{ijs}^S(t)\, dt = h_{ijs}^S \int_0^{T_{ijs}^S} \left[R_{ijs}^S - \left(\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S\right) t \right] dt$$
$$= h_{ijs}^S \left[\frac{\left(R_{ijs}^S\right)^2}{2\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S} \right], \tag{4}$$

$$HD_{ijs}^S = h_{ijs}^S \alpha_{is}^S \left(P_{ijs}^S\right) \left[\frac{P_{ijs}^S}{r_{ijs}^S} \right] = \frac{h_{ijs}^S \alpha_{is}^S \left(P_{ijs}^S\right)^2}{r_{ijs}^S}. \tag{5}$$

The costs of screening, setup, and purchase are presented in Eqs. (6–8), respectively.

$$CQ_{ijs}^S = C_{ijs}^S\left(P_{ijs}^S\right) \tag{6}$$

$$AT_{ijs}^S = A_{ijs}^S T_{ijs}^S = \frac{A_{ijs}^S R_{ijs}^S}{\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S} \tag{7}$$

$$PCO_{ijs}^S = PC_{ijs}^S\left(R_{ijs}^S\right). \tag{8}$$

The benefit function for suppliers comprises revenues, inventory holding cost of good and defective items, screening cost, setup cost, and the purchasing cost. Thus, the benefit function is given by:

$$\pi_{js}^S = W_{ijs}^S \left(1 - \alpha_{ijs}^S\right) P_{ijs}^S + WD_{ijs}^S \alpha_{ijs}^S P_{ijs}^S$$
$$- h_{ijs}^S \left[\frac{\left(R_{ijs}^S\right)^2}{2\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S} \right] - \frac{h_{ijs}^S \alpha_{is}^S \left(P_{ijs}^S\right)^2}{r_{ijs}^S} - C_{ijs}^S\left(P_{ijs}^S\right)$$
$$- \frac{A_{ijs}^S R_{ijs}^S}{\left(1 - \alpha_{ijs}^S\right) P_{ijs}^S} - PC_{ijs}^S R_{ijs}^S. \tag{9}$$

Therefore, the total benefit function of suppliers is given by:

$$\Phi^S = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{s=1}^{S} \pi_{js}^{S}. \tag{10}$$

4.2 Benefit of manufacturers

An imperfect production system at the manufacturer m is considered. During production, quality inspections are done. The defective product is removed to initiate a second transformation process. Therefore, the remanufactured items are sold to related industries at a lower price. A fraction of defective items β_{jm}^{M} during the production run-time TP_{jm}^{M} of the products j at the manufacturer m is included. Thus, the inventory level of finished products at the manufacturer with respect to time is represented by the following differential equation:

$$\frac{dI_{jm}^{M}(t)}{dt} = \left(1 - \beta_{jm}^{M}\right) P_{jm}^{M} \quad 0 \le t \le TP_{jm}^{M}, \tag{11}$$

With the boundary condition $I_{jm}^{M}(0) = 0$. Solving the Eq. (11) yields,

$$I_{jm}^{M}\left(TP_{jm}^{M}\right) = \left(1 - \beta_{jm}^{M}\right) P_{jm}^{M} TP_{jm}^{M} \tag{12}$$

For this case, synchronization between the manufacturer's production run-time $\left(TP_{jm}^{M}\right)$ and the retailer's product cycle length $\left(T_{jr}^{R}\right)$ is considered. Therefore, the inventory level $\left[I_{jm}^{M}\left(TP_{jm}^{M}\right)\right]$ during the production run-time equals to the demand for the retailer's product $\left(q_{jr}^{R}\right)$.

$$I_{jm}^{M}\left(TP_{jm}^{M}\right) = q_{jr}^{R} = \frac{Di_{jr}^{R}}{\left(\kappa_{jr}^{R} - \lambda_{jr}\right)} \left[e^{\left(\kappa_{jr}^{R} - \lambda_{jr}\right) T_{jr}^{R}} - 1\right], \tag{13}$$

Solving the Eq. (13), the production run time of the manufacturer m for the finished product j is

$$TP_{jm}^{M} = \frac{Di_{jr}^{R}}{\left(\kappa_{jr}^{R} - \lambda_{jr}\right)\left(1 - \beta_{jm}^{M}\right) P_{jm}^{M}} \left[e^{\left(\kappa_{jr}^{R} - \lambda_{jr}\right) T_{jr}^{R}} - 1\right]. \tag{14}$$

Also, the inventory level of defective items during the time t is specified by the following differential equation:

$$\frac{dID_{jm}^{M}(t)}{dt} = \beta_{jm}^{M} P_{jm}^{M} \quad 0 \le t \le TP_{jm}^{M}, \tag{15}$$

Considering $ID_{jm}^{M}(0) = 0$, we obtain

$$ID_{jm}^{M}\left(TP_{jm}^{M}\right) = \beta_{jm}^{M} P_{jm}^{M} TP_{jm}^{M}. \tag{16}$$

The following differential equation represents the inventory level of the defective items during the remanufacturing run-time:

$$\frac{dID_{jm}^M(t)}{dt} = \beta_{jm}^M P_{jm}^M T P_{jm}^M - P R_{jm}^M \qquad T P_{jm}^M \le t \le T P_{jm}^M + T R P_{jm}^M, \qquad (17)$$

Using boundary condition, $ID_{jm}^M\left(T P_{jm}^M\right) = \beta_{jm}^M P_{jm}^M T P_{jm}^M$, we have

$$ID_{jm}^M\left(T P_{jm}^M + T R P_{jm}^M\right) = \beta_{jm}^M P_{jm}^M T P_{jm}^M + \beta_{jm}^M P_{jm}^M T P_{jm}^M T R P_{jm}^M - P R_{jm}^M T R P_{jm}^M. \tag{18}$$

Equation (19) models the inventory level of remanufactured products at the manufacturer:

$$\frac{dID_{jm}^M(t)}{dt} = -P R_{jm}^M \qquad T P_{jm}^M \le t \le T P_{jm}^M + T R P_{jm}^M, \qquad (19)$$

With the boundary condition, $ID_{jm}^M\left(T P_{jm}^M + T R P_{jm}^M\right) = 0$. Solving the differential Eq. (19) yields

$$T R P_{jm}^M = \frac{\beta_{jm}^M P_{jm}^M T P_{jm}^M}{P R_{jm}^M}. \tag{20}$$

Equation (20) shows the remanufacturing run-time of the product j at the manufacturer m.

The manufacturer m stores raw material i to manufacture the finished product. The following differential equation represents the inventory level of raw material at the manufacturer m,

$$\frac{dIM_{ijm}^M(t)}{dt} = -\left(1 - \beta_{jm}^M\right) P_{jm}^M \sum_{i=1}^I Z_{ijm}^M \qquad 0 \le t \le T P_{jm}^M, \qquad (21)$$

Using the boundary condition, $IM_{ijm}^M(0) = \left(1 - \alpha_{ijs}^S\right) P_{ijs}^S$, we have

$$IM_{ijm}^M\left(T P_{jm}^M\right) = \sum_{i=1}^I \left(1 - \alpha_{ijs}^S\right) P_{ijs}^S - \left(1 - \beta_{jm}^M\right) P_{jm}^M T P_{jm}^M \sum_{i=1}^I Z_{ijm}^M. \qquad (22)$$

Equation (22) shows the raw materials' inventory level during the production run time for the finished product j.

The manufacturers' revenue from the good item selling of finished products is $W_{jm}^M q_{jr}^R$. The manufacturers' revenue from remanufactured products is $W D_{jm}^M \beta_{jm}^M P_{jm}^M T R P_{jm}^M$.

Equations (23) and (24) show the inventory holding costs of good $\left(H P_{jm}^M\right)$ and defective $\left(H M D_{jm}^M\right)$ items.

$$H P_{jm}^M = \int_0^{T P_{jm}^M} h p_{jm}^M I_{jm}^M(t) dt = h p_{jm}^M\left(1 - \beta_{jm}^M\right) P_{jm}^M\left(T P_{jm}^M\right)^2, \qquad (23)$$

$$HMD_{jm}^M = \left[\int_0^{TP_{jm}^M} hp_{jm}^M ID_{jm}^M(t)dt + \int_{TP_{jm}^M}^{TP_{jm}^M + TRP_{jm}^M} hp_{jm}^M ID_{jm}^M(t)dt \right]$$

$$= \frac{hp_{jm}^M}{2} \left[\beta_{jm}^M P_{jm}^M \left(TP_{jm}^M \right)^2 + \beta_{jm}^M P_{jm}^M TP_{jm}^M \left(TRP_{jm}^M \right) - PR_{jm}^M \left(TRP_{jm}^M \right) \right]. \quad (24)$$

The inventory holding cost of raw material for the manufacturer m during the run-time production is:

$$HM_{jm}^M = \int_0^{TP_{jm}^M} hr_{jm}^M IM_{jm}^M(t)dt$$

$$= hr_{jm}^M \left[TP_{jm}^M \sum_{i=1}^I \left(1 - \alpha_{ijs}^S \right) P_{ijs}^S - \left(1 - \beta_{jm}^M \right) P_{jm}^M \left(TP_{jm}^M \right)^2 \sum_{i=1}^I Z_{ijm}^M \right]. \quad (25)$$

The inventory holding cost of the remanufactured product during the remanufacturing run time is:

$$HMR_{jm}^M = \int_{TP_{jm}^M}^{TP_{jm}^M + TRP_{jm}^M} hp_{jm}^M ID_{jm}^M(t)dt = hp_{jm}^M PR_{jm}^M \left(TRP_{jm}^M \right)^2. \quad (26)$$

The screening and setup costs are $CQ_{jm}^M = C_{jm}^M P_{jm}^M TP_{jm}^M$ and $AT_{jm}^M = A_{jm}^M TRP_{jm}^M$, respectively.

The production cost $\left(CM_{jm}^M \right)$ includes purchasing, labor, and fixed costs. Equation (27) shows the production cost of the manufacturer m for the product j during production run-time,

$$CM_{jm}^M = \sum_{i=1}^I W_{ijs}^S \left(1 - \alpha_{ijs}^S \right) P_{ijs}^S + L_{jm}^M TP_{jm}^M + \delta_{jm}^M P_{jm}^M TP_{jm}^M. \quad (27)$$

Equation (28) models the remanufacturing cost during the remanufacturing run-time.

$$CRP_{jm}^M = \varepsilon_{jm}^M \beta_{jm}^M P_{jm}^M TRP_{jm}^M. \quad (28)$$

The benefit function of manufacturers comprises revenues for selling and the costs of screening, setup, production, and remanufacturing. Also, the inventory holding costs for good items, defective items, raw materials, and remanufactured products are included. Thus, its benefit function of manufacturers is given by:

$$\pi_{jm}^M = \frac{W_{jm}^M Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr} \right)} \left[e^{\left(\kappa_{jr}^R - \lambda_{jr} \right) T_{jr}^R} - 1 \right] + WD_{jm}^M PR_{jm}^M TRP_{jm}^M - hp_{jm}^M \left(1 - \beta_{jm}^M \right) P_{jm}^M \left(TP_{jm}^M \right)^2$$

$$- C_{jm}^M P_{jm}^M TP_{jm}^M - A_{jm}^M TRP_{jm}^M - \left(\sum_{i=1}^I W_{ijs}^S \left(1 - \alpha_{ijs}^S \right) P_{ijs}^S + L_{jm}^M TP_{jm}^M + \delta_{jm}^M P_{jm}^M TP_{jm}^M \right)$$

$$- \varepsilon_{jm}^M \beta_{jm}^M P_{jm}^M TRP_{jm}^M - hr_{jm}^M \left[TP_{jm}^M \sum_{i=1}^I \left(1 - \alpha_{ijs}^S \right) P_{ijs}^S - \left(1 - \beta_{jm}^M \right) P_{jm}^M \left(TP_{jm}^M \right)^2 \sum_{i=1}^I Z_{ijm}^M \right]$$

$$- \frac{hp_{jm}^M}{2} \left[\beta_{jm}^M P_{jm}^M \left(TP_{jm}^M \right)^2 + \beta_{jm}^M P_{jm}^M TP_{jm}^M \left(TRP_{jm}^M \right) - PR_{jm}^M \left(TRP_{jm}^M \right) \right]$$

$$- hp_{jm}^M PR_{jm}^M \left(TRP_{jm}^M \right)^2. \quad (29)$$

Therefore, the total benefit function of manufacturers is given by:

$$\Phi^M = \sum_{m=1}^{M} \sum_{j=1}^{j} \pi_{jm}^M. \tag{30}$$

4.3 Benefit of the retailers

The finished product is delivered instantly to the retailer r in its replenishment cycle T_{jr}^R. These products constantly deteriorated with rate κ_{jr}^R while stored due to expiration and poor handling during the cycle time T_{jr}^R. The deteriorated items are removed from the finished products and sold to other industries at a lower price. The demand is variable and follows an exponential function of time. Thus, the variation in inventory level of the finished product j at the retailer r is described by the following differential equation:

$$\frac{dI_{jr}^R(t)}{dt} = -\kappa_{jr}^R I_{jr}^R(t) - Di_{jr}^R e^{-\lambda_{jr} t} \qquad 0 \le t \le T_{jr}^R. \tag{31}$$

The solution of Eq. (31) with boundary condition $I_{jr}^R(T_{jr}^R) = 0$ is as follows:

$$I_{jr}^R(t) = \frac{Di_{jr}^R e^{-\lambda_{jr} t}}{\left(\kappa_{jr}^R - \lambda_{jr}\right)} \left[e^{\left(\kappa_{jr}^R - \lambda_{jr}\right)\left(T_{jr}^R - t\right)} - 1 \right]. \tag{32}$$

Figure 1 shows that the initial inventory level of the retailers is equal to the demand for retailers q_{jr}^R. Therefore, it is established that the initial condition is equal to the demand of retailers q_{jr}^R, as presented in Eq. (33),

$$I_{jr}^R(0) = q_{jr}^R = \frac{Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)} \left[e^{\left(\kappa_{jr}^R - \lambda_{jr}\right)T_{jr}^R} - 1 \right]. \tag{33}$$

The retailers' revenue is provided by selling good items to customers (See Eq. 34) and deteriorated items to related industries (See Eq. 35).

$$RS_{jr}^R = W_{jr}^R Di_{jr}^R T_{jr}^R. \tag{34}$$

$$
\begin{aligned}
RSD_{jr}^R &= \int_0^{T_{jr}^R} WD_{jr}^R \kappa_{jr}^R I_{jr}^R(t)\, dt \\
&= \frac{WD_{jr}^R Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)} \left[e^{-\lambda_{jr} T_{jr}^R} \left(e^{\kappa_{jr}^R T_{jr}^R} - 1 \right) - \frac{\kappa_{jr}^R}{\lambda_{jr}} \left(1 - e^{-\lambda_{jr} T_{jr}^R} \right) \right].
\end{aligned}
\tag{35}
$$

The inventory holding costs of good items HP_{jr}^R is:

$$
\begin{aligned}
HP_{jr}^R &= \int_0^{T_{jr}^R} hp_{jr}^R I_{jr}^R(t)\, dt \\
&= \frac{hp_{jr}^R Di_{jr}^R}{\kappa_{jr}^R \left(\kappa_{jr}^R - \lambda_{jr}\right)} \left[e^{-\lambda_{jr} T_{jr}^R} \left(e^{\kappa_{jr}^R T_{jr}^R} - 1 \right) - \frac{\kappa_{jr}^R}{\lambda_{jr}} \left(1 - e^{-\lambda_{jr} T_{jr}^R} \right) \right].
\end{aligned}
\tag{36}
$$

The deterioration cost is given by,

$$
\begin{aligned}
HDP_{jr}^R &= \int_0^{T_{jr}^R} CD_{jr}^R \kappa_{jr}^R I_{jr}^R(t)\, dt \\
&= \frac{CD_{jr}^R Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)}\left[e^{-\lambda_{jr}T_{jr}^R}\left(e^{\kappa_{jr}^R T_{jr}^R} - 1\right) - \frac{\kappa_{jr}^R}{\lambda_{jr}}\left(1 - e^{-\lambda_{jr}T_{jr}^R}\right)\right].
\end{aligned}
$$
(37)

The setup cost is given by $AT_{jr}^R = A_{jr}^R T_{jr}^R$, and purchasing cost is modeled by Eq. (38).

$$
PC_{jr}^R = PC_{jr}^R q_{jr}^R = \frac{PC_{jr}^R Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)}\left[e^{\left(\kappa_{jr}^R - \lambda_{jr}\right)T_{jr}^R} - 1\right]
$$
(38)

The benefit function of retailers includes sales revenues and the costs of inventory holding, deterioration, setup, and purchasing. Thus, the benefit function of retailers is given by:

$$
\begin{aligned}
\pi_{jr}^R &= W_{jr}^R Di_{jr}^R T_{jr}^R + \frac{WD_{jr}^R Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)}\left[e^{-\lambda_{jr}T_{jr}^R}\left(e^{\kappa_{jr}^R T_{jr}^R} - 1\right) - \frac{\kappa_{jr}^R}{\lambda_{jr}}\left(1 - e^{-\lambda_{jr}T_{jr}^R}\right)\right] \\
&\quad - \frac{hp_{jr}^R Di_{jr}^R}{\kappa_{jr}^R\left(\kappa_{jr}^R - \lambda_{jr}\right)}\left[e^{-\lambda_{jr}T_{jr}^R}\left(e^{\kappa_{jr}^R T_{jr}^R} - 1\right) - \frac{\kappa_{jr}^R}{\lambda_{jr}}\left(1 - e^{-\lambda_{jr}T_{jr}^R}\right)\right] \\
&\quad - \frac{CD_{jr}^R Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)}\left[e^{-\lambda_{jr}T_{jr}^R}\left(e^{\kappa_{jr}^R T_{jr}^R} - 1\right) - \frac{\kappa_{jr}^R}{\lambda_{jr}}\left(1 - e^{-\lambda_{jr}T_{jr}^R}\right)\right] - A_{jr}^R T_{jr}^R \\
&\quad - \frac{PC_{jr}^R Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)}\left[e^{\left(\kappa_{jr}^R - \lambda_{jr}\right)T_{jr}^R} - 1\right].
\end{aligned}
$$
(39)

Therefore, the total benefit function of retailers is given by:

$$
\Phi^R = \sum_{j=1}^J \sum_{r=1}^R \pi_{jr}^R.
$$
(40)

4.4 VMI model

The VMI strategies assume a Stackelberg approach (Von Stackelberg, 2011) among the supply chain members. In this system, one of the members acts as a leader who manages and makes decisions related to inventory replenishment for the supply chain. The others are the followers. Therefore, the VMI model considers the manufacturers leaders because this layer has direct relationships, such as raw material suppliers and market retailers. This approach allows for developing requirement and demand forecasts and joint replenishment for the supply chain members. The VMI model for a three-layer supply chain is represented by the following:

$$
\mathrm{Max}\,\Phi^M = \sum_{j=1}^J \sum_{m=1}^M \pi_{jm}^M,
$$
(41)

Subject to:

$$R_{ijs}^S \geq Z_{ijm}^M q_{jr}^R \qquad \forall i,j,s,r \tag{42}$$

$$P_{jm}^M \geq Di_{jr}^R \qquad \forall j,m,r \tag{43}$$

$$W_{jr}^R \geq PC_{jr}^R \qquad \forall j,r. \tag{44}$$

Equation (41) includes the terms $e^{-\lambda_{jr} T_{jr}^R}$ and $e^{\kappa_{jr}^R T_{jr}^R}$, which makes mathematical optimization difficult. Then, the deterioration rate $\left(\kappa_{jr}^R\right)$ and the constant that governs the retailers' demand $\left(\lambda_{jr}\right)$ are approximated to a Taylor series (Chen, 2017; Misra, 1975), as follows.

$$e^{-\lambda_{jr} T_{jr}^R} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - \lambda_{jr} T_{jr}^R + \frac{\left(\lambda_{jr} T_{jr}^R\right)^2}{2!}, \tag{45}$$

$$e^{\kappa_{jr}^R \left(T_{jr}^R\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 + \kappa_{jr}^R \left(T_{jr}^R\right) + \frac{\left(\kappa_{jr}^R T_{jr}^R\right)^2}{2!}, \tag{46}$$

Substituting Eqs. (45) and (46) into Eq. (33), we have:

$$\begin{aligned} q_{jr}^R = \frac{Di_{jr}^R}{\left(\kappa_{jr}^R - \lambda_{jr}\right)} & \left[-\lambda_{jr} T_{jr}^R + \frac{\left(\lambda_{jr} T_{jr}^R\right)^2}{2} + \kappa_{jr}^R \left(T_{jr}^R\right) - \lambda_{jr} \kappa_{jr}^R \left(T_{jr}^R\right)^2 \right. \\ & + \frac{\kappa_{jr}^R (\lambda_{jr})^2 \left(T_{jr}^R\right)^3}{2} + \frac{\left(\kappa_{jr}^R T_{jr}^R\right)^2}{2} \\ & \left. - \frac{\lambda_{jr} \left(\kappa_{jr}^R\right)^2 \left(T_{jr}^R\right)^3}{2} + \frac{\left(\kappa_{jr}^R\right)^2 (\lambda_{jr})^2 \left(T_{jr}^R\right)^4}{4} \right] \end{aligned} \tag{47}$$

The production rate of raw materials is synchronized with the production run time of manufacturers. Therefore, no shortage of raw material is generated.

$$P_{ijs}^S = \left(1 + \alpha_{ijs}^S\right) P_{jm}^M Z_{ijm}^M. \tag{48}$$

The demand rate of the customers is a linear function of the selling price of products at the retailers, as shown in Eq. (49).

$$Di_{jr}^R = a_{jr}^R + b_{jr}^R \left(W_{jr}^R\right) \tag{49}$$

5 Solution method

The Stackelberg approach is used to solve the problem above (Von Stackelberg, 2011). The leader makes decisions about the competitive strategy in the market for the followers. This approach has been considered by Taleizadeh et al., (2015a, 2015b), Panda et al. (2015), Lan et al. (2018), Salas Navarro et al. (2018), Wei et al. (2020). They have developed inventory

models in the supply chain, considering a supply chain member as the leader and the followers. In this case, the manufacturers are examined as the leaders. The suppliers and retailers are the followers. Consequently, manufacturers process raw materials from suppliers into finished products, which retailers sell. Manufacturers plan product replenishment and establish the optimal values of decision variables. Based on these values, the suppliers and retailers optimize their decisions.

According to Chen (2017), the deteriorated rates for perishable items are small values between 0.01 and 0.1, and it is possible to apply the Taylor series approximation of the exponential terms considering the first three terms of the series. In this way, the benefit functions are mathematically optimized. A solution algorithm is implemented to maximize the objective function and find the optimal decision variables, which substitute into benefit functions of suppliers and retailers.

Step 1. Input the production rate of the raw material function (Eq. 49) and the demand rate of the customers (Eq. 50).
Step 2. Use the Taylor series approximation for exponential terms in Eqs. (45) and (46), and substitute into Eqs. (19) and (29).
Step 3. Determine the total benefits functions in Eqs. (10), (20), and (30) considering Taylor series approximation for exponential terms.
Step 4. Calculate the new total benefit functions of the suppliers Φ^S, manufactures Φ^M, and retailers Φ^R.
Step 5. Obtain optimal solution values of R_{ijs}^S, P_{jm}^M, W_{jr}^R, and T_{jr}^R. Maximize Φ^M (Eq. 41) considering the constraints Eqs. (42–44).
Step 6. Substitute the optimal solution values of R_{ijs}^S, P_{jm}^M, W_{jr}^R, and T_{jr}^R into the new total benefit functions obtained by step 4 and determine Φ^S and Φ^R.

The method used to solve the model combined the analytical calculation method and differential equations. The optimization model results were obtained from the algorithm designed according to a numerical example. The proposed algorithm was programmed using Wolfram Mathematica 11.0 in an Intel® i7-model PC at 2.4 GHz (6.0 GB RAM), and the solution was obtained in approximately 1 s. The results of the model are presented with a numerical example.

6 Numerical example

In order to illustrate the solution algorithm, a numerical example is presented and solved. This case considers a VMI model in a three-layer Dairy Industry supply chain of the Department of Atlántico, Colombia. Two types of dairy derivatives that are highly perishable are considered. The network configuration includes two suppliers of raw materials and inputs, two manufacturers of dairy derivates (two types of cheese), and two retailers of finished products (see Tables 2, 3, 4).

Each supplier provides two types of raw materials delivered to the manufacturers. Each manufacturer produces two types of dairy derivates according to the demand rate for retailers. The defective items generated during the production runtime are remanufactured to obtain a new product (whey). This product is sold in other industries at a lower sale price. These products can be reused as raw materials or inputs to manufacture other products. Manufacturers decide on inventory levels for retailers and replenish the retail stores according to VMI agreements. Retailers purchase and store finished products sold to the final customer.

Table 2 Values of parameters of suppliers (COP)

Values of parameters	Supplier 1	Supplier 2
Product 1	$\alpha^S_{111} = 0.130$, $\alpha^S_{211} = 0.100$, $r^S_{111} = 15320$, $r^S_{211} = 13230$, $C^S_{111} = \$0.20$, $C^S_{211} = \$0.23$, $A^S_{111} = \$250$, $A^S_{211} = \$245$, $h^S_{111} = \$0.30$, $h^S_{211} = \$0.31$, $PC^S_{111} = \$5.26$, $PC^S_{211} = \$5.57$, $W^S_{111} = \$89.5$, $W^S_{211} = \$80.6$, $WD^S_{111} = \$17.4$, $WD^S_{211} = \$17.7$	$\alpha^S_{112} = 0.180$, $\alpha^S_{212} = 0.130$, $r^S_{112} = 16790$, $r^S_{212} = 15340$, $C^S_{112} = \$0.26$, $C^S_{212} = \$0.23$, $A^S_{112} = \$248$, $A^S_{212} = \$251$, $h^S_{112} = \$0.29$, $h^S_{212} = \$0.29$, $PC^S_{112} = \$5.12$, $PC^S_{212} = \$5.18$, $W^S_{112} = \$89.0$, $W^S_{212} = \$89.2$, $WD^S_{112} = \$17.3$, $WD^S_{212} = \$17.3$
Product 2	$\alpha^S_{121} = 0.150$, $\alpha^S_{221} = 0.160$, $r^S_{121} = 15170$, $r^S_{221} = 15500$, $C^S_{121} = \$0.21$, $C^S_{221} = \$0.22$, $A^S_{121} = \$248$, $A^S_{221} = \$246$, $h^S_{121} = \$0.17$, $h^S_{221} = \$0.26$, $PC^S_{121} = \$5.42$, $PC^S_{221} = \$5.32$, $W^S_{121} = \$90.8$, $W^S_{221} = \$99.0$, $WD^S_{121} = \$17.7$, $WD^S_{221} = \$17.3$	$\alpha^S_{122} = 0.160$, $\alpha^S_{222} = 0.170$, $r^S_{122} = 15530$, $r^S_{222} = 16960$, $C^S_{122} = \$0.20$, $C^S_{222} = \$0.23$, $A^S_{122} = \$251$, $A^S_{222} = \$249$, $h^S_{122} = \$0.28$, $h^S_{222} = \$0.30$, $PC^S_{122} = \$5.50$, $PC^S_{222} = \$5.26$, $W^S_{122} = \$97.5$, $W^S_{222} = \$99.5$, $WD^S_{122} = \$16.9$, $WD^S_{222} = \$17.4$

The maximum total benefit of the manufacturers is $\Phi^{M*} = $ COP\$3, 492, 220, 299.22. Then, the benefit for suppliers is $\Phi^{S*} = $ COP\$7, 768, 807.61, and the benefit of retailers is $\Phi^{R*} = $ COP\$672, 928, 462.64. Table 5 presents the optimal values of $R^S_{ijs}, P^M_{jm}, W^R_{jr}$, and T^R_{jr}.

7 Sensitivity analysis

This section presents the sensitivity analysis for the VMI model to evaluate parameters such as inventory holding cost, percentage of deteriorated items, the fraction of defective items, and selling prices. The sensitivity analysis shows the effects of parameter changes on the decision variables, increasing or decreasing by 25% and 50%. The results are presented in Table 6.

7.1 Sensitivity analysis of supply chain benefits

The benefit of suppliers (Φ^S) is sensitive to changes in a fraction of defective raw material (α^S_{ijs}) and inventory holding cost (h^S_{ijs}) for suppliers (see Table 6). If α^S_{ijs} increases by 25%, the benefit of suppliers decreases by 41.5% (see Fig. 3). Therefore, the suppliers should establish standardized processes that guarantee the quality of their products and avoid incurring high

Table 3 Values of parameters of manufacturers (COP),

Values of parameters	Manufacturer 1	Manufacturer 2
Product 1	$PR_{11}^M = 79, \beta_{11}^M = 0.090, r_{11}^M = 145,$ $C_{11}^M = \$0.0110, A_{11}^M = \$19.40,$ $hp_{11}^M = \$2.0, L_{11}^M = \$48.0, \delta_{11}^M = \$0.05,$ $\varepsilon_{11}^M = \$0.030, W_{11}^M = \$305,$ $WD_{11}^M = \$153, Z_{111}^M = 1.0,$ $Z_{211}^M = 0.5, hr_{11}^M = \$0.67$	$PR_{12}^M = 85, \beta_{12}^M = 0.099, r_{12}^M = 149,$ $C_{12}^M = \$0.0110, A_{12}^M = \$15.20,$ $hp_{12}^M = \$2.3, L_{12}^M = \$45.0,$ $\delta_{12}^M = \$0.06, \varepsilon_{12}^M = \$0.040,$ $W_{12}^M = \$308, WD_{12}^M = \$151,$ $Z_{112}^M = 1.0,$ $Z_{212}^M = 0.5, hr_{12}^M = \$0.77$
Product 2	$PR_{21}^M = 80, \beta_{21}^M = 0.099, r_{21}^M = 145,$ $C_{21}^M = \$0.0130, A_{21}^M = \$18.40,$ $hp_{21}^M = \$2.0, L_{21}^M = \$54.0, \delta_{21}^M = \$0.07,$ $\varepsilon_{21}^M = \$0.030, W_{21}^M = \$312,$ $WD_{21}^M = \$155, Z_{121}^M = 1.0,$ $Z_{221}^M = 0.5, hr_{21}^M = \$0.67$	$PR_{22}^M = 77, \beta_{22}^M = 0.090, r_{22}^M = 149,$ $C_{22}^M = 0.0090, A_{22}^M = \$20.00, hp_{22}^M =$ $\$2.1, L_{22}^M = \$55.0, \delta_{22}^M = \$0.05,$ $\varepsilon_{22}^M = \$0.030,$ $W_{22}^M = \$313, WD_{22}^M = \$154,$ $Z_{122}^M = 1.0,$ $Z_{222}^M = 0.5, hr_{22}^M = \$0.70$

Table 4 Values of parameters of retailers (COP)

Values of parameters	Manufacturer 1	Manufacturer 2
Product 1	$a_{11}^R = 6000, b_{11}^R = 160, hp_{11}^R = \$0.080,$ $A_{11}^R = \$255, PC_{11}^R = \$305, \kappa_{11}^R = 0.050,$ $CD_{11}^R = \$122.40,$ $WD_{11}^R = \$100, \lambda_{11} = 0.025$	$a_{12}^R = 6250, b_{12}^R = 160,$ $hp_{12}^R = \$0.100,$ $A_{12}^R = \$245, PC_{12}^R = \$308,$ $\kappa_{12}^R = 0.050,$ $CD_{12}^R = \$120.80,$ $WD_{12}^R = \$98, \lambda_{12} = 0.025$
Product 2	$a_{21}^R = 5990, b_{21}^R = 170, hp_{21}^R = \$0.090,$ $A_{21}^R = \$250, PC_{21}^R = \$312, \kappa_{21}^R = 0.050,$ $CD_{21}^R = \$123.60,$ $WD_{21}^R = \$101, \lambda_{21} = 0.025$	$a_{22}^R = 6200, b_{22}^R = 160,$ $hp_{22}^R = \$0.100,$ $A_{22}^R = \$240, PC_{22}^R = \$313,$ $\kappa_{22}^R = 0.050,$ $CD_{22}^R = \$122.80,$ $WD_{22}^R = \$103, \lambda_{22} = 0.025$

Table 5 Decision variable results

Decision variables	Results
Replenishment lot size of raw material i at the supplier s to manufacture the product j (units)	$R_{111}^S = 689, 578.76$, $R_{112}^S = 696, 470.31$, $R_{121}^S = 744, 352.12$, $R_{122}^S = 698, 631.25$, $R_{211}^S = 345, 796.01$, $R_{212}^S = 349, 241.75$, $R_{221}^S = 372, 882.98$, $R_{222}^S = 350, 322.28$
Production rate of product j at the manufacturer m (units/time unit)	$P_{11}^M = 64, 269.78$, $P_{12}^M = 64, 764.62$, $P_{21}^M = 68, 509.59$, $P_{22}^M = 65, 122.66$
Selling price of product j at the retailer r (\$/unit)	$W_{11}^R = \$361.70$, $W_{12}^R = \$363.21$, $W_{21}^R = \$365.89$, $W_{22}^R = \$365.83$
Cycle length of product j at the retailer r (time unit)	$T_{11}^R = 10.01$, $T_{12}^R = 10.04$, $T_{21}^R = 10.12$, $T_{22}^R = 10.01$

costs that negatively affect their profits. Also, the benefit of suppliers decreases by decreasing the selling price of raw materials. In addition, the benefit of suppliers is sensitive to changes in parameters of the other echelon, such as re-manufacturing rates and the percentage of deteriorated items (see Table 6).

The benefit of manufacturers (Φ^M) increases by decreasing the inventory holding costs (see Table 6). However, it affects the suppliers because reducing inventory holding costs can store more products and generate many suppliers requirements. The benefit of manufacturers (Φ^M) and retailers (Φ^M) are sensitive to increments by the fraction of defective products β_{jm}^M (See Fig. 4). The defective items are remanufactured to obtain whey. These products are sold to other industries, generating additional income for manufacturers. Therefore, an increase in selling prices by 50% yields 8.2% of the benefit of manufacturers and 7.6% of the benefit of retailers. Also, the growth of the remanufacturing rates decreases the manufacturer profit. This situation can affect the capacity of production plants. If PR_{jm}^M decreases by 25%, the benefit of manufacturers increases by 10.4% (see Fig. 5).

The benefit of retailers is sensitive to variation in the percentage of deteriorated items and deteriorating costs (see Table 6). A growth of 50% of expired products increases by 20.1% of the profits (see Fig. 6). The proposed policy generates growth of incomes due to sales of perishable products or expiring at a lower price in promotions.

7.2 Sensitivity analysis of decision variables

Raw material replenishment lot size is sensitive to variations of % defective raw material and raw material prices. The defective raw material can be reduced, improving the quality control processes in the supplier. Therefore, a reduction of W_{ijs}^S by 25% increases R_{ijs}^S by 9.2% (See Table 7). This situation reduces the benefits of suppliers by 59.3% (See Table 6). Then, a raw material price reduction policy should not be implemented in suppliers. Otherwise, the increment of defective products increases the replenishment lot size because the reworking process requires more raw material. Additionally, the remanufacturing rate, percentage of

Table 6 Results of benefits for the VMI model

Benefits			
% of change	Suppliers (%)	Manufacturers (%)	Retailers (%)
− 50%	− 28.1	5.0	7.6
− 25%	− 28.1	5.0	7.6
α_{ijs}^{S}	0.0	0.0	0.0
25%	− 41.5	7.2	11.1
50%	− 28.9	5.0	7.6
− 50%	7.4	0.0	0.0
− 25%	3.7	0.0	0.0
h_{ijs}^{S}	0.0	0.0	0.0
25%	− 3.7	0.0	0.0
50%	− 7.4	0.0	0.0
− 50%	− 64.9	5.1	7.7
− 25%	− 59.3	6.9	10.4
W_{ijs}^{S}	0.0	0.0	0.0
25%	− 10.0	5.0	7.6
50%	7.9	5.0	7.6
− 50%	− 40.0	− 5.5	10.6
− 25%	− 30.8	− 1.6	8.3
β_{jm}^{M}	0.0	0.0	0.0
25%	− 25.6	10.8	2.8
50%	7.4	23.8	− 2.3
− 50%	− 29.2	8.0	7.9
− 25%	− 41.2	8.7	11.1
hp_{jm}^{M}	0.0	0.0	0.0
25%	− 28.3	3.7	7.6
50%	− 27.7	2.2	7.5
− 50%	− 28.3	1.8	7.6
− 25%	− 28.3	3.4	7.6
W_{jm}^{M}	0.0	0.0	0.0
25%	− 28.9	6.7	7.8
50%	− 28.3	8.2	7.6

Table 6 (continued)

Benefits

% of change	Suppliers (%)	Manufacturers (%)	Retailers (%)
− 50%	− 40.4	8.7	4.4
− 25%	− 25.8	10.4	7.0
PR^R_{jm}	0.0	0.0	0.0
25%	− 42.0	3.8	11.3
50%	− 28.9	− 0.7	7.7
− 50%	− 40.3	7.1	− 80.6
− 25%	− 0.1	0.0	− 13.9
κ^R_{jr}	0.0	0.0	0.0
25%	− 43.5	7.6	17.7
50%	− 35.2	6.2	20.1
− 50%	0.0	0.0	21.5
− 25%	0.0	0.0	10.8
CD^R_{jr}	0.0	0.0	0.0
25%	0.0	0.0	− 10.8
50%	0.0	0.0	− 21.5
− 50%	0.0	0.0	0.3
− 25%	0.0	0.0	0.1
hp^R_{jr}	0.0	0.0	0.0
25%	0.0	0.0	− 0.1
50%	0.0	0.0	− 0.3
− 50%	− 32.7	5.8	19.8
− 25%	− 31.0	5.5	15.1
λ^R_{jr}	0.0	0.0	0.0
25%	− 40.8	7.2	− 0.6
50%	− 27.9	5.0	− 31.8
− 50%	0.0	0.0	− 0.9
− 25%	0.0	0.0	− 0.4
WD^R_{jr}	0.0	0.0	0.0
25%	0.0	0.0	0.4
50%	0.0	0.0	0.9

deteriorated products, and the constant market demand increased by 25%, increasing R^S_{ijs} by 9% (see Fig. 7).

The production rate at manufacturers does not significantly vary with changes in the parameters studied. Figure 8 shows the performance of P^M_{jm} with variations of α^S_{ijs}, PR^M_{jm}, κ^R_{jr}, and λ^R_{jr}. These parameters decreased by 50%, increasing P^M_{jm} by 0.25% (see Table 7). Then, the reduction of defective raw material and remanufacturing rates improves the performance of manufacturers and retailers.

Fig. 3 The benefit of suppliers with respect fraction of defective raw material
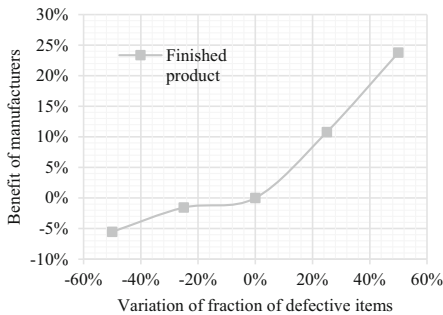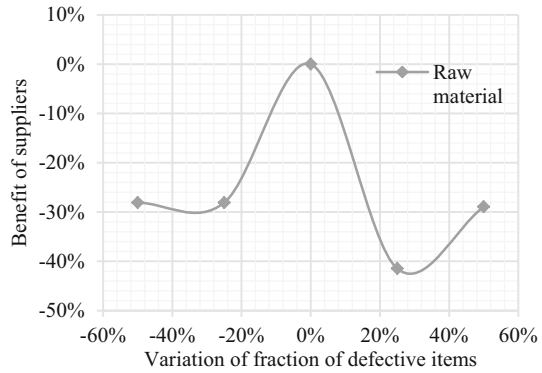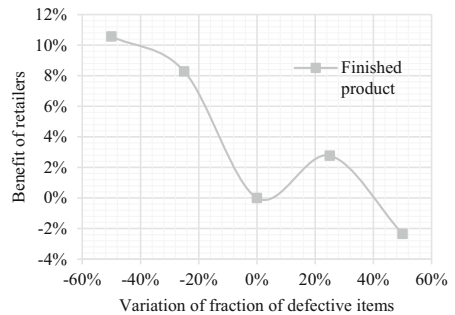
Fig. 4 The benefit of manufacturers and retailers with respect fraction of defective product

a)   b)

Fig. 5 The benefit of manufacturers with respect variation of remanufacturing rate
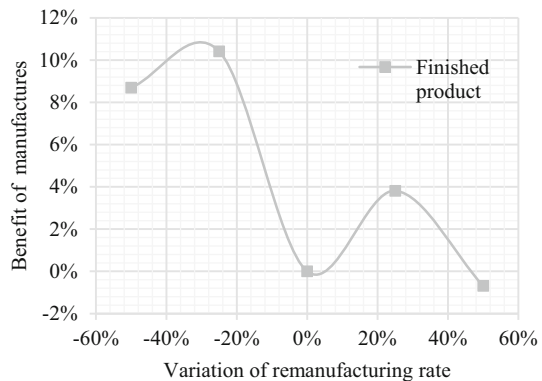
The selling price at retailers is sensitive to changes in remanufacturing rates, deteriorated products, selling prices, and market demand (see Fig. 9). A reduction of the percentage of defective products by 25% and 50%, decreasing the selling prices by 0.1% and 0.2% (see Table 7), respectively. Then, it improves manufacturer productivity and increases retailer profit (see Table 6). Otherwise, an imperfect production system generates defective and

Fig. 6 The benefit of retailers with respect percentage of deteriorated items
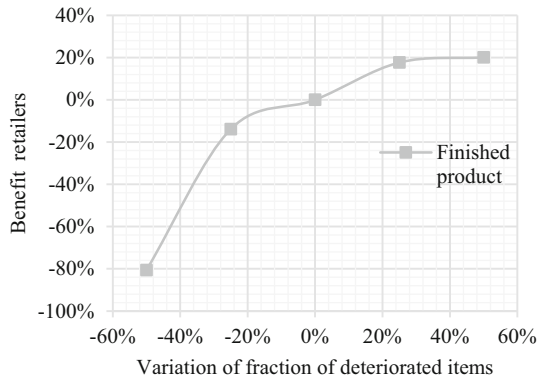
deteriorating items that increase the selling price of products (dairy derivatives) and affect the market demand. Therefore, retailers must improve warehouse management and inventory handling practices to avoid increased damaged products, generate high costs, and affect profits.

Product cycle time is sensitive to deteriorated rates and inventory holding costs. Then, reducing these parameters improves manufacturer and retailer benefits (see Table 6). Additionally, T_{jr}^R is easily affected by defective items percentage and constant market demand (see Fig. 10).

A reduction by 50% of deteriorated rates increases T_{jr}^R by 14.1% and decrease the retailer benefit by 80.6%. A growing cycle length of products affects the retailer benefits. The market demand $\left(\lambda_{jr}^R\right)$ increased by 25% and 50%, growth T_{jr}^R by 11.2% and 8.7%. This condition reduces the benefit of retailers by 0.6% and 31.8%, respectively. Then, the inventory policy should be managed to reduce the cycle length of products to satisfy customer requirements just in time and establish accurate demand forecasts.

The sensitivity analysis shows that the percentage of defective raw materials $\left(\alpha_{ijs}^S\right)$, defective products $\left(\beta_{jm}^M\right)$, and deteriorated items $\left(\kappa_{jr}^R\right)$ represent an imperfect production system. The decision variables studied are sensitive to these parameters. Otherwise, selling prices at manufacturers is a decisive parameter in the replenishment, production, and selling policies. Then, the VMI agreement must establish an agreed selling price for manufactured products to avoid variations that affect suppliers and retailers.

The proposed model considered a declining exponential function of time for the market demand. The parameter λ_{jr}^R (constant market demand) is examined. Figure 10 shows that market demand is closely associated with cycle times. A growth of around 25% in λ_{jr}^R could generate an 11.2% increase in product cycle times.

7.3 Managerial implications

An imperfect production system can generate a fraction of defective items that affect the lot size of raw materials and products. This situation affects the supply chain to the extent that reliability and compliance with customer requirements are reduced. Then, the supply chain members should implement quality control strategies and improve the production process and warehouse storage management to reduce high levels of defective items that affect supply

Table 7 Results of the decision variables for the VMI model

% of change	Decision variables			
	R_{ijs}^S (%)	P_{jm}^M (%)	W_{jr}^R (%)	T_{jr}^R (%)
− 50%	6.4	0.2	0.1	6.0
− 25%	6.4	0.2	0.1	6.0
α_{ijs}^S	0.0	0.0	0.0	0.0
25%	9.2	0.3	0.2	8.6
50%	6.4	0.2	0.1	5.9
− 50%	0.0	0.0	0.0	0.0
− 25%	0.0	0.0	0.0	0.0
h_{ijs}^S	0.0	0.0	0.0	0.0
25%	0.0	0.0	0.0	0.0
50%	0.0	0.0	0.0	0.0
− 50%	6.4	0.2	0.1	6.0
− 25%	9.2	0.3	0.2	8.6
W_{ijs}^S	0.0	0.0	0.0	0.0
25%	6.4	0.2	0.1	6.0
50%	6.4	0.2	0.2	6.0
− 50%	9.2	0.3	− 0.2	9.0
− 25%	6.4	0.2	− 0.1	6.2
β_{jm}^M	0.0	0.0	0.0	0.0
25%	6.2	0.2	0.3	2.4
50%	− 1.4	0.0	0.1	− 1.4
− 50%	6.4	0.2	0.2	6.0
− 25%	9.3	0.3	0.3	8.6
hp_{jm}^M	0.0	0.0	0.0	0.0
25%	6.4	0.2	0.1	6.0
50%	6.2	0.2	0.0	6.0
− 50%	6.3	0.2	0.0	6.0
− 25%	6.3	0.2	0.1	6.0
W_{jm}^M	0.0	0.0	0.0	0.0
25%	6.5	0.2	0.2	6.1
50%	6.8	0.2	0.2	6.3
	Decision variables			
% of change	R_{ijs}^S (%)	P_{jm}^M (%)	W_{jr}^R (%)	T_{jr}^R (%)
− 50%	9.5	0.3	0.3	3.7

Table 7 (continued)

% of change	Decision variables			
	R^S_{ijs} (%)	P^M_{jm} (%)	W^R_{jr} (%)	T^R_{jr} (%)
− 25%	6.3	0.2	0.3	5.7
PR^R_{jm}	0.0	0.0	0.0	0.0
25%	9.2	0.3	0.1	8.7
50%	6.3	0.2	0.0	6.1
− 50%	8.3	0.3	0.2	14.1
− 25%	0.0	0.0	0.0	3.4
κ^R_{jr}	0.0	0.0	0.0	0.0
25%	10.1	0.3	0.2	5.8
50%	8.7	0.2	0.1	− 1.7
− 50%	0.0	0.0	0.0	0.0
− 25%	0.0	0.0	0.0	0.0
CD^R_{jr}	0.0	0.0	0.0	0.0
25%	0.0	0.0	0.0	0.0
50%	0.0	0.0	0.0	0.0
− 50%	0.0	0.0	0.0	0.0
− 25%	0.0	0.0	0.0	0.0
hp^R_{jr}	0.0	0.0	0.0	0.0
25%	0.0	0.0	0.0	0.0
50%	0.0	0.0	0.0	0.0
− 50%	8.0	0.2	0.1	0.6
− 25%	7.3	0.2	0.1	3.4
λ^R_{jr}	0.0	0.0	0.0	0.0
25%	8.8	0.3	0.2	11.2
50%	5.9	0.2	0.1	8.7
− 50%	0.0	0.0	0.0	0.0
− 25%	0.0	0.0	0.0	0.0
WD^R_{jr}	0.0	0.0	0.0	0.0
25%	0.0	0.0	0.0	0.0
50%	0.0	0.0	0.0	0.0

chain benefits. A strategy to increase the performance of the production system includes remanufacturing defective items and selling them to other industries, generating additional income for producers.

The dairy industry presents challenges in improving production processes. Consolidating the new production of whey from the surplus generated in the production process of dairy products that do not meet the required quality levels can be remanufactured. The remanufacturing processes of perishable products contribute to developing new product lines or using these products in industries related to the dairy sector. Additionally, the dairy industry
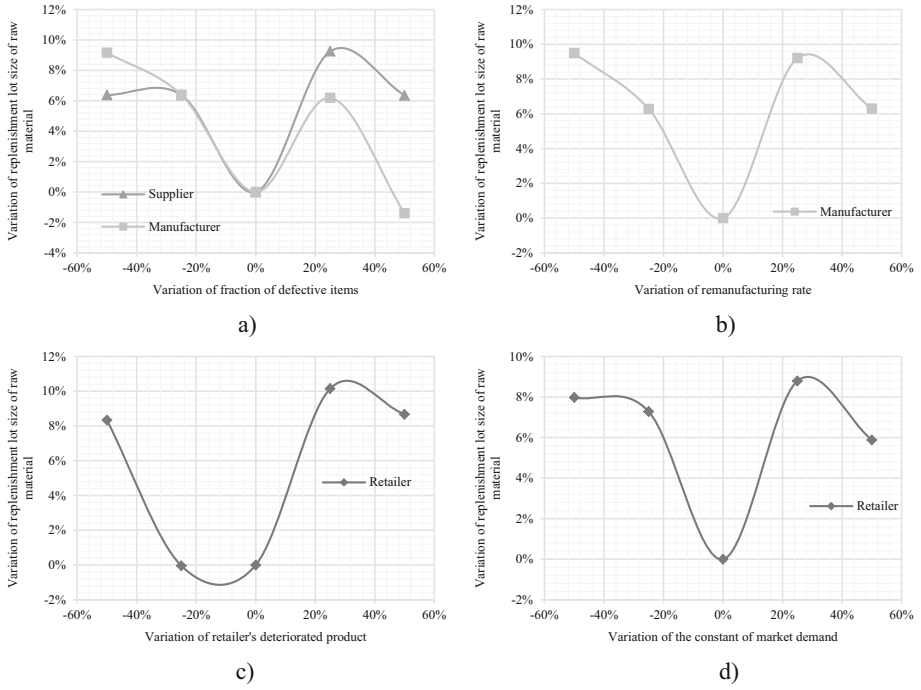
Fig. 7 Variation of replenishment lot size of raw material
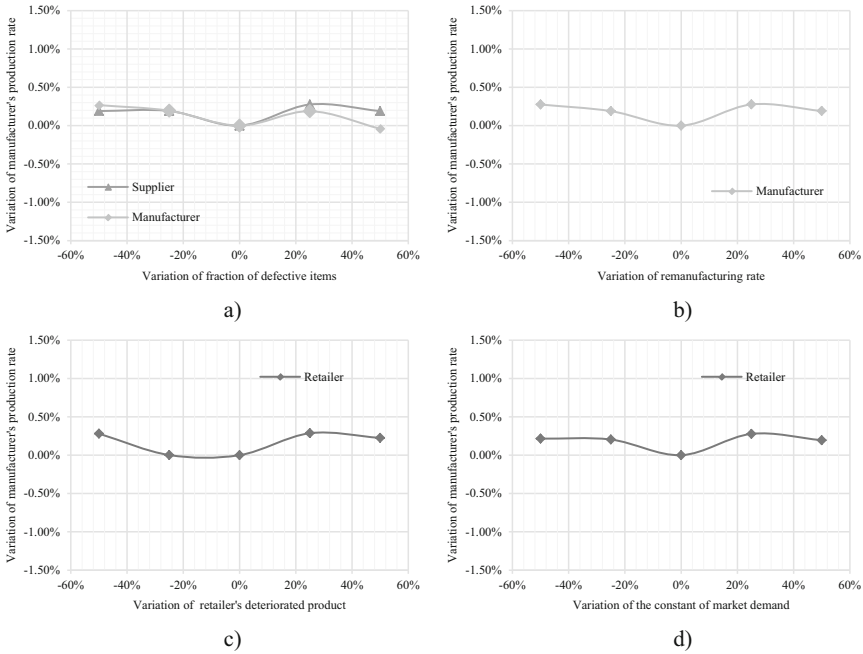
Fig. 8 Variation of production rate of product
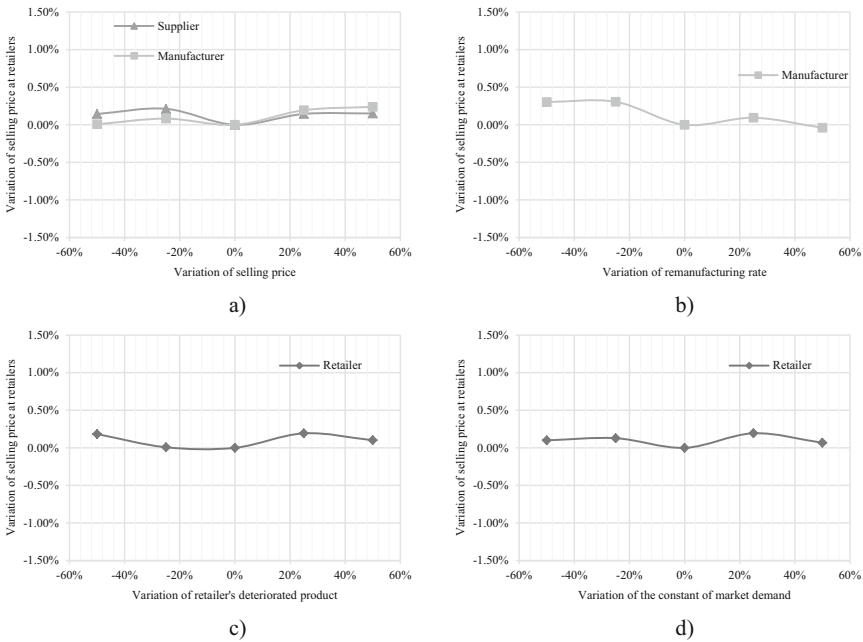
Fig. 9 Variation of selling price at retailers
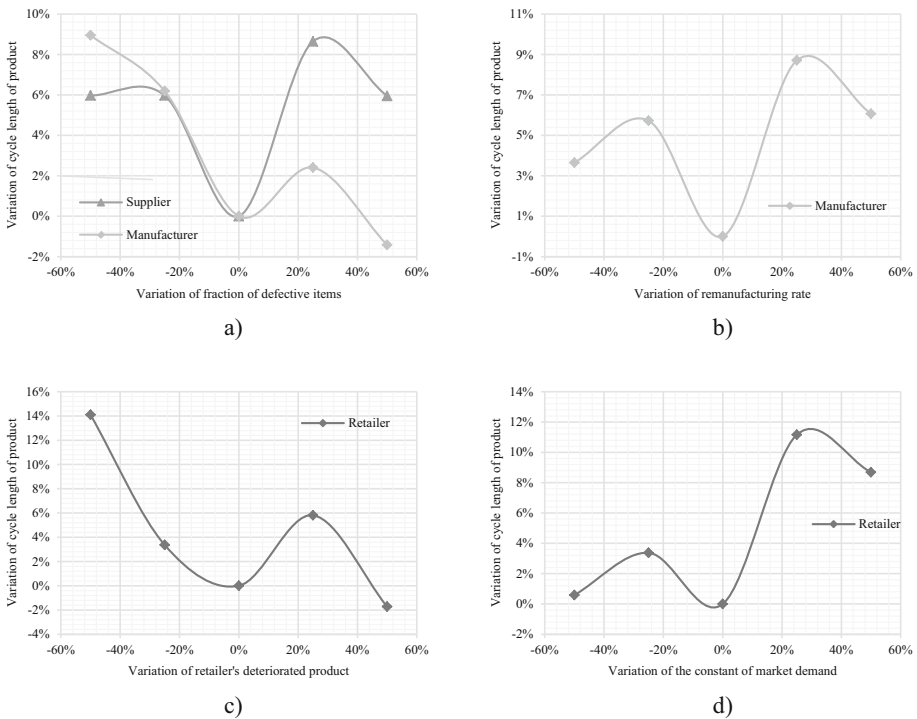
Fig. 10 Variation of cycle length of product

must improve the planning of operations in the supply chain, specifically between the links of suppliers and manufacturers, to reduce defective products that generate high reprocessing costs.

Highly perishable products show an exponential decrease in demand because, as time goes by, they can deteriorate due to expiration, poor handling, and storage. A strategy to reduce deteriorated products consists of improving warehouse management and inventory handling practices to avoid increased damaged products, generating high costs and affecting profits. Additionally, the inventory policy should be managed to reduce the cycle length of products to satisfy customer requirements just in time and establish accurate demand forecasts.

The VMI policy can be applied in a three-level supply chain comprised of suppliers, manufacturers and retailers. The manufacturers assume the role of leaders in the supply chain for decision-making regarding levels of inventories at each level, sales forecasts and production planning. Also, it must establish confidentiality agreements allowing real-time information sharing on sales, inventories, production, and other organizational processes.

VMI agreements in the supply chain allow the development of collaborative strategies to improve the performance of an imperfect production system. VMI collaboration schemes facilitate inventory management of perishable products by allowing the manufacturer to quickly replenish, control and withdraw products due to market deterioration to avoid shortages. Also, implementing reworking processes for defective products to generate new products increases the raw material requirements and the sale of new products in the market. Therefore, it streamlines operations throughout the supply chain.

8 Conclusions and future research

This study considered a supply chain comprised of multiple suppliers, manufacturers, and retailers. A new Vendor-Managed Inventory (VMI) model is presented for a three-layer supply chain considering different types of raw materials, different perishable products, and defective items. This VMI model includes an imperfect production system at the manufacturer and incorporates the remanufacturing process to increase profits. Thus, the VMI model considers the replenishment lot size of raw material, the production rate of products, the selling price of products, and the cycle length of products as decision variables. Additionally, this VMI model includes inventory holding costs, setup costs, screening costs, purchasing costs, deterioration costs, and production costs.

The main challenge for optimizing the VMI model is the exponential terms included in the deteriorated rates since mathematical optimization is difficult to handle. According to Chen (2017), the deteriorated rates for perishable items are small values between 0.01 and 0.1. It is possible to apply the Taylor series approximation of the exponential terms considering the first three terms of the series. In this way, the benefit functions are mathematically optimized. A solution algorithm is implemented to maximize the objective function and obtain the optimal values of decision variables.

VMI policy allows control of inventory levels in the supply chain and establishes cooperation agreements to improve an imperfect production system. This approach shows promotional strategies for selling defective raw materials, deteriorated products, and remanufacturing defective products. This study constitutes a novelty in analyzing an imperfect production system with defective and deteriorated products, generating alternatives for remanufacturing, and selling these products to other industries to maximize profits in a

Stackelberg approach. Also, VMI models for three supply chain levels have yet to addressed multiple members at each level or multiple items of raw materials and finished products.

The VMI model results show that the benefits of the supply chain members can be maximized by establishing a joint inventory policy. This approach includes replenishing raw materials and products and the cost of inventory and operation for an imperfect production system. This VMI model assumes the manufacturers are leaders who decide about replenishment and optimizing decision variables. This approach allows collaboration among members of different levels of a supply chain, sharing information on sales, operational costs, quality conditions of the production system, and situations that affect the supply chain productivity.

This study considers a deteriorating process for perishable products such as dairy derivates with an expiration date that reduces their useful life. During storage, the products can deteriorate due to expiration or handling and costs associated with time maintenance. Then, the demand rate presents an exponential decrease with respect to time because the deterioration process in perishable products reduces the inventory of good items. This VMI model can be applied to any supply chain of perishable products.

The deterioration of perishable products increases costs and affects profits due to increased cycle times to maintain higher inventory levels. Then, the retailers can increase their revenues by selling deteriorated items to related industries like composting and fertilizers or promoting near-expire products. Also, the increased percentage of defective raw materials negatively impacts supplier benefits. It is recommended that suppliers establish standardized processes that guarantee the quality of their products and avoid incurring high costs that negatively affect their profits. On the other hand, it is possible to increase profits by implementing a remanufacturing process for defective products sold to related industries. This approach generates additional income that increases manufacturer profits.

The VMI system for three-layer supply chains with exponential demand denotes the beginning of new lines of research around the studied problem. The first line of research can be addressed from the imperfect production system, including machine failures and breakdowns, preventive maintenance, and remanufacturing defective products at suppliers and manufacturers. The second line could be in the retail echelon, adding credit policies and payment terms, contracts, total backlogging, partial backlogging, demand depending on sales effort, and non-zero lead time. The third line of research can be addressed by considering a supply chain for perishable products, including stochastic deterioration of raw materials and finished products at all levels of the supply chain. This approach considers the impact of this on the entire chain and establishes strategies that better manage perishable products in inventories. The fourth line of research adds elements of sustainability to the supply chain.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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