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ORIGINAL RESEARCH

Reducing incompatibility in a local AHP-group decision making context

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Abstract

In the context of local analytic hierarchy process-group decision making (AHP-GDM), this paper presents a theoretical framework and a semi-automatic procedure for reducing incompatibility between the actors involved in the decision making process and the collective position. The row geometric mean is employed as the prioritisation procedure and the geometric compatibility index (GCOMPI) as the incompatibility measure; individual pairwise comparison matrices are considered as the input of the reduction process, whilst the collective vector is the output. The reduction is attained by slightly modifying, in relative terms, the judgements of the collective pairwise comparison matrix, irrespective of the method used to obtain it, that further improve the GCOMPI. The resulting judgements of the collective matrix and the associated collective priorities are close to the initial collective values. The procedure does not modify the judgements of the initial individual matrices and this simplifies the process of reaching consensus. A simulation analysis is utilised to study the performance of the algorithm along with an illustrative numerical example. The analysis proves that the proposed algorithm is easy to implement and efficient, it provides mathematically closed results and significantly reduces the GCOMPI associated with the precise consistency consensus matrix which is one of the AHP-GDM tools. The framework allows the procedure to be adapted to specific interests.

Keywords Analytic hierarchy process · Group decision making · Incompatibility improvement · Row geometric mean · GCOMPI

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1 Introduction

The analytic hierarchy process (AHP) (Saaty, 1977, 1980) is one of the most widely used multicriteria techniques (Ho and Ma, 2018; Kułakowski, 2020). Among other reasons, this is due to (Aguarón et al., 2021): (i) its potential for addressing multiactor decisions; (ii) its capacity for measuring the actors' consistency when eliciting their preferences; and, (iii) its ability to combine tangible and intangible aspects in formal models. The two methods traditionally employed in local (one criterion) AHP-group decision making (AHP-GDM) contexts (Saaty, 1989; Ramanathan & Ganesh, 1994; Forman & Peniwati, 1998) are the aggregation of individual judgements (AIJ) and the aggregation of individual priorities (AIP). They both use a weighted geometric mean as a synthesis method. As is well known, any mean measure is representative of the collective behaviour if the data does not present a high degree of variability (Saaty & Vargas, 2007). It is therefore necessary to measure the compatibility or proximity between the initial individual positions and the collective position; an acceptable level of incompatibility must also be guaranteed.

Assuming a local AHP-GDM context, where the Row Geometric Mean (RGM) is used for obtaining the local priorities and the geometric compatibility index (GCOMPI) (Escobar et al., 2015; Aguarón et al., 2019) is used for evaluating the incompatibility of the individual positions with regards to the collective position, the paper presents the seminal ideas of the theoretical framework and the semi-automatic procedure proposed by the authors for reducing the incompatibility of the collective matrix (regardless of how it was obtained). This reduction is achieved by slightly modifying, in relative terms, the judgements of the collective matrix that further improve the GCOMPI.

The proposed framework considers relative modifications, instead of absolute, because they better reflect the perceived relevance of changes. When considering small changes, both the judgements and the derived collective priority vector will be close to the initial values, as recommended by Saaty (2003). This framework also provides closed (optimal) results in terms of the judgements that most rapidly reduce incompatibility and the range of values over which the reduction occurs. The expression of the GCOMPI employed in this paper for evaluating incompatibility takes the individual pairwise comparison matrices (PCMs) as the input and the priority vector associated to the collective PCM as the output. Depending on the available information (input and output), other variants of the GCOMPI can be seen in Aguarón et al. (2022), a work that is an extension of this present study and that, following the same theoretical framework, develops and demonstrates analogous results to those presented here, but for the other variants.

The proposed procedure does not modify the judgements of the initial individual matrices, so the continuous intervention by decision makers is not required to corroborate changes made to the judgements in the individual matrices. The paper also includes a simulation study and new mathematical tools that evidence the excellent performance of the algorithm. Finally, it should be noted that the procedure can be adapted to different situations, and therefore has enormous cognitive potential (Moreno-Jiménez & Vargas, 2018).

The paper is structured as follows: Sect. 2 outlines the background of AHP-GDM and the measurement of compatibility; Sect. 3 sets out the theoretical results necessary for the proposal; Sect. 4 describes the procedure for revising judgements and reducing incompatibility, presents a study of its performance, and it also highlights its most outstanding characteristics; Sect. 5 illustrates the procedure by means of a numerical example and justify its potential by comparing its results with those of the Precise Consistency Consensus Matrix (PCCM); and, Sect. 6 highlights the most important conclusions of the study.

2 Background

2.1 AHP in a local group decision making context

AHP methodology consists of three phases (Saaty, 1980): (a) modelling; (b) valuation; and, (c) prioritisation and synthesis. The two methods most commonly employed for obtaining the local priorities are the Eigenvector (EGV) and the Row Geometric Mean (RGM). This paper uses the RGM due to its psychological, mathematical and statistical properties and relationships (Aguarón et al., 2020, 2021).

Considering a PCM of order n as a squared matrix $A = \left(a_{ij}\right)_{n\times n}$ with $a_{ij}a_{ji} = 1$ and $a_{ij} > 0, i, j = 1, \ldots, n$, the priority vector using the RGM (except for the normalization factor) is given by:

$$w_i = \left(\prod_{j=1}^{n} a_{ij}\right)^{1/n} \tag{1}$$

AHP allows measurement of the degree of internal coherence of the decision maker when incorporating their preferences (valuation) through the judgements elicitation process. This internal coherence is known as consistency and guarantees the quality or validity of the priority vector derived from the PCM. For the RGM, Crawford and Williams (1985) advanced an unbiased estimator of the variance of log-errors as a measure of inconsistency. Aguarón and Moreno-Jiménez (2003) referred to this measure as the Geometric Consistency Index (GCI) and established thresholds for the GCI.

AHP stands out for its potential to address multi-actor decisions. Where a number of actors evaluate a set of alternatives according to multiple conflicting criteria (tangible and intangible), Escobar and Moreno-Jiménez (2007) distinguish three situations: (i) Group Decision Making (GDM); (ii) Negotiated Decision Making (NDM); and, (iii) Systemic Decision Making (SDM). In GDM, individuals work together in pursuit of a common goal under the principle of consensus.

In the local AHP-GDM context considered in this work (Altuzarra et al., 2019), consensus refers to the acceptance of the procedure followed to aggregate the individual positions into a collective position (collective matrix of final group priority vector). If the decision makers agree, at the beginning of the Consensus Reaching Process (CRP), on how to obtain the collective position and no intervention is made during the process (AHP-GDM), it is understood that there is an implicit acceptance of the result. If the decision makers directly and continuously participate in obtaining the collective position (AHP-NDM), it is understood that there is an explicit acceptance of that position.

Let $\mathcal{A} = \left\{A^{(k)} = \left(a_{ij}^{(k)}\right)_{n\times n}, k = 1, \ldots, d\right\}$ be a family of PCMs provided, respectively, by d decision makers with weights α_k $\left(\sum_{k=1}^{d} \alpha_k = 1\right)$, and $w^{(k)}, k = 1, \ldots, d$, the priority vectors derived from these matrices using a prioritisation method. The two methods traditionally used in AHP-GDM are the AIP and the AIJ (Aguarón et al., 2019). With AIP, once the individual priorities are obtained, the priority vector for the group is calculated as the normalised (distributive mode) weighted geometric mean of the individual priority vectors, component by component:

$$w_i^{G|P} = \prod_{k=1}^{d} \left(w_i^{(k)}\right)^{\alpha_k} \; i = 1, \ldots, n \tag{2}$$

With AIJ, a collective matrix $A^G = \left(a_{ij}^G\right)_{n \times n}$ is constructed first, where each entry is obtained as $a_{ij}^G = \prod_{k=1}^{d} \left(a_{ij}^{(k)}\right)^{\alpha_k}$. Then, the priority vector $w^{G|J}$ is calculated following a prioritisation method. When the priorities are obtained by means of RGM, both AIJ and AIP provide (Barzilai & Golany, 1994; Escobar et al., 2004) the same collective priority vector ($w_i^{G|P} = w_i^{G|J}$), where

$$w_i^{G|J} = \left(\prod_{j=1}^{n} a_{ij}^G\right)^{1/n} \; i = 1, \ldots, n \tag{3}$$

2.2 Compatibility in AHP-GDM

Irrespective of the method employed to obtain the collective matrix (voting, consensus on the judgements, aggregation, AIJ, AIP, etc.), this matrix cannot be representative of the group position if the individual positions are not homogeneous or their incompatibility is high (Saaty & Vargas, 2007; Scala et al., 2016). It is then necessary to evaluate the compatibility (an objective distance measure) or the agreement (a subjective acceptance that requires personal intervention) between the individual positions, $A^{(k)} = \left(a_{ij}^{(k)}\right)$ or $w^{(k)} = \left(w_1^{(k)}, \ldots, w_n^{(k)}\right)$, $k = 1, \ldots, d$, and the collective position, $w^G = \left(w_1^G, \ldots, w_n^G\right)$. If the group's internal coherence (compatibility or agreement) in choosing its collective position is not reached, the suitability of the aggregation followed is not guaranteed. In this case, the different homogeneous positions of the actors must be identified in order to initiate posterior negotiation processes to achieve final decisions that are as representative as possible (Altuzarra et al., 2019). In addition, from a cognitive perspective (Moreno-Jiménez & Vargas, 2018), the arguments that justify the different positions must be provided.

It is therefore necessary to establish (define and characterise) compatibility measures, procedures for their improvement and thresholds that allow validation of the use of collective priorities that represent the individual priorities. Garuti (2020) defines compatibility as the sharing of similar value systems. Compatibility is generally calculated without the personal intervention of the individuals with the exception of the emission of the initial judgements of the PCMs. The published literature (Lipovetsky, 2020; Escobar et al., 2015) describes ordinal and cardinal tools for the assessment of compatibility. Ordinal tools (e.g. ordinal correlation coefficients) work with rankings of the alternatives, but some authors (Garuti, 2012) do not recommend their use in the context of AHP (weighted spaces). Cardinal tools include the S-compatibility (Saaty, 1996), the G-compatibility (Garuti, 2007), the coefficient of multiple determination R^2 that is commonly used in regression analysis (Lipovetsky, 2009); and, the Geometric Compatibility Index (GCOMPI).

Dong et al. (2010), Escobar et al. (2015) and Aguarón et al. (2016) advanced initial proposals of the GCOMPI for the evaluation of the compatibility of individual positions with respect of the collective position. As can be seen in Sect. 3, the expression considered for the GCOMPI in a local context (one criterion) combines the individual PCMs and the collective priority vector. The choice of this expression of the GCOMPI is justified by the fact that it measures the compatibility between the input of the decision makers (individual PCMs) and the output of the group (collective priority vector) used to rank the alternatives and make decisions.

Different approaches have been followed for studying the improvement of incompatibility. They differ with regards to the modification considered for the judgements (in absolute or relative terms), or with regards to the degree of the participation of the actors (automatic, semi-automatic, personal). Most of the procedures in the literature for reducing inconsistency (Dadkhah & Zahedi, 1993) and incompatibility (Dong et al., 2010; Grošelj et al., 2015) include modifications in absolute terms. However, the perception of the importance an actor gives to a change that can lead to consensus positions, that is to say, the acceptability or unacceptability of a modification, is better captured in relative, rather than absolute, terms. In absolute terms, the changes in judgements from 2 to 3 and from 8 to 9 are the same, but the perception of their importance is not; in the first case, the relative change is 50%, in the second, it is just 12.5%.

Modifications in relative terms are in line with the suggestions of Kahneman and Tversky (1979): "the preferences, associated with the same physical magnitude, are relative rather than absolute, depending on the situation of gain or loss, and also on the point of departure" and Grzybowski (2016): "small errors (in terms of absolute values) may significantly change the final rankings if they are big in relation to the true value". In addition, when working in relative terms, the absolute values of the modifications allowed by each decision maker for each judgement, alternative or criterion (bounded confidences) do not have to be provided. These two arguments: (i) suitability to perceive the importance of changes and (ii) lower transaction costs, validate the use of relative changes.

The maximum relative variation allowed for the modification of any judgement is given by the parameter known as permissibility (Aguarón et al., 2021). It considers the attitudes or flexibility of the actors in consensus reaching. This allows them to adapt their initial positions (individual or collective), facilitating the establishment of consensus paths for reaching a more satisfactory final agreement (Altuzarra et al., 2010).

Regarding the degree of actors' participation, automatic procedures are not appropriate in GDM unless they are used to simulate and explore scenarios. CRPs require the personal intervention of the actors, either at the beginning of the process, by setting its degree of flexibility, or throughout, in an interactive manner. The procedure put forward in this current work is semi-automatic; the actors provide at the beginning of the CRP their initial PCMs and establish the permissibility level. However, the procedure can be easily adapted to allow a more personal intervention of the actors in the agreement or consensus searching (interactive procedure).

3 Theoretical results

This section presents the theoretical results (see Appendix A for proofs) that are necessary to develop the procedure to reduce the incompatibility (Sect. 4). In what follows, all the matrices (PCMs) and the priority vectors are of order n, and w^G will refer to the priority vector obtained either applying AIJ or AIP to the matrices of a family $\mathcal{A}$ when using the RGM as the prioritisation method ($w^G = w^{G|J} = w^{G|P}$).

Definition 1 Let $A = \left(a_{ij}\right)$ be a PCM and $u = (u_i)$ be a priority vector. The *Geometric Compatibility Index* between A and u is defined as

$$\mathrm{GCOMPI}(A, u) = \frac{1}{(n-1)(n-2)} \sum_{i,j} \log^2 a_{ij} u_j / u_i \tag{4}$$

Definition 2 Let $\mathcal{A} = \left\{A^{(k)}\right\}$ be a family of PCMs and $u = (u_i)$ be a priority vector. The *Geometric Compatibility Index* between family $\mathcal{A}$ and u is defined as

$$\begin{aligned} \mathrm{GCOMPI}\,(\mathcal{A}, u) &= \sum_{k=1}^{d} \alpha_k \mathrm{GCOMPI}(A^{(k)}, u) \\ &= \frac{1}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i,j} \log^2 a_{ij}^{(k)} u_j / u_i \right) \end{aligned} \tag{5}$$

Remark 1 If A is a PCM and w is its priority vector obtained with the RGM method:

$$\min_u \mathrm{GCOMPI}(A, u) = \mathrm{GCOMPI}(A, w) = GCI(A) \tag{6}$$

where GCI is the Geometric Consistency Index (Aguarón & Moreno-Jiménez, 2003).

Remark 2 If $\mathcal{A} = \left\{A^{(k)}\right\}$ is a family of PCMs, it holds that

$$\min_u \mathrm{GCOMPI}(\mathcal{A}, u) = \mathrm{GCOMPI}(\mathcal{A}, w^G) \tag{7}$$

Remark 3 Let $w = (w_1, \ldots, w_n)$ be a priority vector. It is obvious that

$$\min_A \mathrm{GCOMPI}(A, w) = \mathrm{GCOMPI}(W, w) = 0 \tag{8}$$

where $W = \left(w_{ij}\right) = \left(w_i / w_j\right)$.

Theorem 1 *Let $A = \left(a_{ij}\right)$ and $P = \left(p_{ij}\right)$ be two PCMs and $w = (w_i)$ and $v = (v_i)$ be the corresponding priority vectors associated to A and P obtained with the RGM method. It holds that*

$$\frac{\partial \mathrm{GCOMPI}(A, v)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r / v_s}{w_r / w_s} \tag{9}$$

Theorem 2 *Let $\mathcal{A} = \left\{A^{(k)}\right\}$ be a family of PCMs, $P = \left(p_{ij}\right)$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated to P obtained with the RGM method. It holds that*

$$\frac{\partial \mathrm{GCOMPI}(\mathcal{A}, v)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r / v_s}{w_r^G / w_s^G} \tag{10}$$

From the above result, it is obvious that, when $v = w^G$, all partial derivatives cancel out. This situation corresponds to a critical point and it is easy to see that it is a minimum, consistent with Remark 2. Therefore, any PCM whose priority vector obtained by the RGM method coincides with w^G will provide the minimum value of the GCOMPI.

If the judgement p_{rs} is modified with p'_{rs} as its new value, $t_{rs} = p'_{rs} / p_{rs}$ denotes the relative variation of this judgement, $P' = \left(p'_{ij}\right)$ is the modified PCM and $v' = \left(v'_i\right)$ the associated priority vector obtained with the RGM. The modified values of the GCOMPI are given by the following theorems.

Theorem 3 *In the same conditions of Theorem 1, it holds that*

$$\mathrm{GCOMPI}(A, v') = \mathrm{GCOMPI}(A, v) + \frac{4}{(n-1)(n-2)} \log t_{rs} \left(\frac{\log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r/w_s} \right) \tag{11}$$

Theorem 4 *In the same conditions of Theorem 2, it holds that*

$$\mathrm{GCOMPI}(\mathcal{A}, v') = \mathrm{GCOMPI}(\mathcal{A}, v) + \frac{4}{(n-1)(n-2)} \log t_{rs} \left(\frac{\log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r^G/w_s^G} \right) \tag{12}$$

The following corollaries provide the partial derivatives of the GCOMPI with respect to the relative variation of a judgement (t_{rs}). When small variations are considered, the value of t_{rs} moves around 1, so the values of the derivatives at that point are also presented.

Corollary 1 *In the same conditions of Theorem 1, it is easy to prove that*

$$\frac{\partial \mathrm{GCOMPI}(A, v)}{\partial t_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \left(\frac{2\log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r/w_s} \right) \tag{13}$$

$$\left. \frac{\partial \mathrm{GCOMPI}(A, v)}{\partial t_{rs}} \right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log \frac{v_r/v_s}{w_r/w_s} \tag{14}$$

It follows that the judgement p_{rs} that most decreases the GCOMPI is the judgement for which there is a greater relative difference between ratios v_r/v_s and w_r/w_s. This seems quite logical; if $v_r/v_s > w_r/w_s$ it is necessary to decrease p_{rs}, and increase it otherwise.

Corollary 2 *In the same conditions of Theorem 2, it is easy to prove that*

$$\frac{\partial \mathrm{GCOMPI}(\mathcal{A}, v)}{\partial t_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \left(\frac{2\log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r^G/w_s^G} \right) \tag{15}$$

$$\left. \frac{\partial \mathrm{GCOMPI}(\mathcal{A}, v)}{\partial t_{rs}} \right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log \frac{v_r/v_s}{w_r^G/w_s^G} \tag{16}$$

In the case of a family of matrices, the judgement p_{rs} that most decreases the GCOMPI is also the one for which there is a greater relative difference between the ratios v_r/v_s and w_r^G/w_s^G.

Corollary 3 *In the same conditions of Theorem 1, the relative variation of judgement p_{rs} that produces the greatest decrease of* $\mathrm{GCOMPI}(A, v)$ *is*

$$t_{rs}^* = p'_{rs}/p_{rs} = \left(\frac{w_r/w_s}{v_r/v_s} \right)^{n/2} \tag{17}$$

and the variation of the $\mathrm{GCOMPI}(A, v)$ *is*

$$\frac{-n}{(n-1)(n-2)} \log^2 \frac{w_r/w_s}{v_r/v_s} \tag{18}$$

Corollary 4 *In the same conditions of Theorem 2, the relative variation of judgement p_{rs} that produces the greatest decrease of* $\mathrm{GCOMPI}(\mathcal{A}, v)$ *is*

$$t_{rs}^* = p'_{rs}/p_{rs} = \left(\frac{w_r^G/w_s^G}{v_r/v_s} \right)^{n/2} \tag{19}$$

and the variation of the GCOMPI$(\mathcal{A}, v)$ *is*

$$\frac{-n}{(n-1)(n-2)} \log^2 \frac{w_r^G/w_s^G}{v_r/v_s} \tag{20}$$

From the above corollaries, it follows that the the judgement identified in (14) or (16) is that which most rapidly decreases the value of the GCOMPI; it is also the one that allows the greatest reduction in absolute terms.

Remark 4 In contrast to absolute variations, (9) and (10), the gradient associated with the variation in relative terms of the judgement p_{rs} does not depend on p_{rs}; it is given exclusively in terms of the log quadratic discrepancies between the ratios of the elements of the priority vector associated with the collective matrix (P) and that obtained for the family $\mathcal{A}$ $(\log^2 \frac{w_r^G/w_s^G}{v_r/v_s})$. This expression will be used to select the judgements that must be considered for reducing the GCOMPI.

4 Procedure

The results from the previous section give the judgement of the collective matrix that will most rapidly reduce incompatibility and determine its optimal relative variation. The iterative modifications of the judgements of the collective matrix (P) bring it closer to a matrix (P') with the same priority vector (v') as the AIJ matrix (w^G), both vectors obtained with the RGM method. Note that this last vector is the one that minimises the weighted log-quadratic deviation between the individual matrices and a priority vector (see Remark 2).

The value that provides the maximum possible reduction of inconsistency could be far from the initial value in the collective matrix (Khatwani and Kar, 2017). Limiting the intensity of the modifications is a logic criterion to guarantee the validity of the improvement procedure (Saaty, 2003). A parameter of permissibility (Aguarón et al., 2021), ρ, is employed to avoid major modifications of the judgements of the collective matrix in the process of reducing incompatibility. This parameter is incorporated by multiplying the judgements by the factor $1 \pm \rho$ and indicates the maximum relative variation permitted for the modifications of any judgement, and is established by the decision makers or by the facilitator (if there is one).

4.1 The AEM-COM algorithm

Using the RGM as the prioritisation method and the GCOMPI as the incompatibility measure, this section describes a semi-automatic procedure for improving (reducing) the incompatibility between the actors involved in the decision making process and the collective position (matrix) obtained by any method. The procedure considers variations in judgements in relative terms, limited by the permissibility. Expression (16) is used to select the judgement that will be considered for each of the iterations. Expression (19) provides the limit of the variation for this judgment. A modification beyond this value will produce an increase in the GCOMPI.

In order to apply this semi-automatic procedure, it is necessary to provide the weights of the decision makers (α_k), their individual pairwise comparison matrices $(A^{(k)})$, the collective matrix (P) and the parameter of permissibility (ρ). Regarding the collective matrix, the decision makers agree on P using an existing AHP-GDM methodology. This initial matrix is assumed to have an acceptable level of inconsistency which ensures the validity of the collective priority vector that is derived from it.

The AEM-COM (Aguarón, Escobar, Moreno-COMpatibility) algorithm:

Algorithm for improving the GCOMPI **in terms of relative changes**
Inputs: $\mathcal{A} = \{A^{(k)}\}$ a family of pairwise comparison matrices, α_k their respective weights with $\sum_{k=1}^{d} \alpha_k = 1$, P a collective matrix and ρ the permissibility allowed in relative terms for the modification of judgements of P.
Outputs: The updated matrix (P'), its priority vector v' and the GCOMPI($\mathcal{A}$, v').

Step 0. Using RGM, obtain the priority vector v for the matrix P and the priority vector w^G for the matrices of family $\mathcal{A}$.
Let $J = \{(r, s), \text{with } r < s\}$.
Step 1. Evaluate $q_{rs} = \frac{v_r/v_s}{w_r^G/w_s^G}$ and $\log q_{rs}$ for all $(r, s) \in J$.
Step 2. Choose the pair $(r', s') \in J$ for which $\log q_{r's'}$ has the largest absolute value.
Step 3. If $p_{r's'} > 1$ then let $(r, s) = (r', s')$. Otherwise, let $(r, s) = (s', r')$.
Step 4. Modify p_{rs} considering the following value of the relative variation t_{rs} that will depend on the sign of $\log q_{rs}$. Let $t_{rs}^* = q_{rs}^{-n/2}$.

a. If $\log q_{rs} < 0$, use $t_{rs} = \min\left\{1 + \rho, t_{rs}^*\right\}$
b. If $\log q_{rs} > 0$, use $t_{rs} = \max\left\{\frac{1}{1+\rho}, t_{rs}^*\right\}$

Update matrix P with new values $p'_{rs} = p_{rs} t_{rs}$ and $p'_{sr} = 1/p'_{rs}$.
Update $J = J \setminus (r', s')$.
Step 5. Using RGM, obtain the priority vector v' for the matrix P'. Calculate the GCOMPI($\mathcal{A}$, v'). If J is not empty, repeat steps 1 to 4 with $v = v'$. Otherwise, stop and provide P', v' and GCOMPI($\mathcal{A}$, v').

If $\log q_{rs} < 0$ (Step 4a), it is necessary to increase p_{rs} ($t_{rs} \geq 1$) in order to reduce the GCOMPI. In this case, the maximum relative increase delimited by the permissibility and the range of improvement is $t_{rs} = \min\left\{1 + \rho, t_{rs}^*\right\}$. If $\log q_{rs} > 0$ (Step 4b), the value of p_{rs} should be reduced ($t_{rs} \leq 1$). In this situation, permissibility is incorporated as $\frac{1}{1+\rho}$ to keep the property of reciprocity.

If working with judgements between 1/9 and 9, as is usual in the context of AHP (Saaty, 1980), the new values p'_{rs} in Step 4 are limited to the continuous interval [1/9,9]. In what follows, only judgements that fall within this range are considered.

4.2 Performance of the algorithm

It can be verified that, whenever there are judgements that meet the required conditions, the algorithm, by construction, takes $n(n-1)/2$ iterations and reduces the initial GCOMPI. Obviously, small values of permissibility will produce small modifications in the GCOMPI.

A simulation study was undertaken to determine the efficiency of the algorithm by measuring the improvement reached for the GCOMPI in different situations. Efficiency is defined as the reduction achieved with respect to the maximum possible reduction, indicating the improvement in GCOMPI:

$$\text{Efficiency (AEM-COM)} = \frac{\text{GCOMPI}_0 - \text{GCOMPI}_f}{\text{GCOMPI}_0 - \text{GCOMPI}_{min}} \tag{21}$$

where GCOMPI_0 refers to the initial GCOMPI = GCOMPI($\mathcal{A}$, v), GCOMPI_f refers to the final GCOMPI = GCOMPI($\mathcal{A}$, v') and GCOMPI_{min} refers to the minimum GCOMPI = GCOMPI($\mathcal{A}$, w^G).

For the simulation, families of matrices were generated: the first step was to randomly generate a priority vector, u. The vector was randomly perturbed, giving different priority vectors for each decision maker, $u^{(k)}$. The vectors are not too far away from u, but they are sufficiently different from each other for different rankings to exist. From these priority vectors, the corresponding consistent matrices $U^{(k)} = \left(u_i^{(k)}/u_j^{(k)}\right)$ are obtained; these matrices are randomly perturbed (using a lognormal distribution) and then corrected so that the priority vector of $A^{(k)}$ is $u^{(k)}$. The AEM-COM algorithm is applied to the matrices which all have same weights. Note that, without loss of generality, the PCCM (Escobar et al., 2015; Aguarón et al., 2016) is used as the collective matrix. The behaviour of our procedure can therefore be compared with that of the PCCM methodology employed for AHP-GDM.

As previously explained, 10,000 families of matrices were generated for each combination of d (3 to 6) and n (3 to 9). The proposed algorithm was applied to each of them for different permissibility values. The efficiency of the algorithm was calculated for each situation. Table 1 shows the average efficiency for 3 decision makers ($d = 3$) and different values of n and ρ. It can be observed that the greater the permissibility, the greater the average efficiency, for any size of the matrix (n). Permissibility of 5%, which is clearly acceptable, is sufficient to achieve an average efficiency improvement of more than 40% for any given n. For permissibility levels greater than 20%, the average efficiency is higher than 91.8%. In the particular case considered in the numerical example presented in Sect. 5 ($n = 5$ and $\rho = 15\%$), average efficiency is 83.9%.

Table 2 shows the percentage of times that different levels of efficiency are reached for $d = 3$ and $n = 5$, depending on the value of ρ. It can be seen that an efficiency of 10% (adequate in most cases) is achieved for any value of permissibility considered in this table. It can also be observed that a level of efficiency of 60% is achieved 95.8% of the time when $\rho = 15\%$. With the same permissibility, efficiency of 50% was possible 99.6% of the time. For $\rho = 50\%$, the collective matrix resulting from the AEM-COM algorithm has a priority vector that almost coincides with the w^G. In general, the greater the permissibility, the greater is the likelihood of achieving a given efficiency.

Similar results to those presented in Tables 1 and 2 were obtained for the other values of d and n (they have not been included for reasons of space). The results of the simulation study also show that the average differences between the initial and final priorities do not exceed 10% for all combinations of d, n and ρ values.

Table 1 Average efficiency (%) of the algorithm for different values of n and ρ ($d = 3$)

n	ρ							
	5%	10%	15%	20%	25%	30%	40%	50%
3	61.5	84.7	93.8	97.3	98.7	99.2	99.4	99.5
4	46.4	71.9	85.5	92.6	96.2	98.0	99.4	99.8
5	43.2	69.2	83.9	91.8	95.8	97.9	99.4	99.8
6	43.9	70.5	85.3	93.0	96.7	98.4	99.6	99.9
7	45.3	72.5	87.1	94.2	97.4	98.9	99.8	99.9
8	47.0	74.8	88.9	95.3	98.0	99.1	99.8	100.0
9	48.8	76.8	90.4	96.2	98.5	99.4	99.9	100.0

Table 2 Percentage of times different levels of efficiency are reached for different values of ρ ($d = 3$ y $n = 5$)

Efficiency	ρ							
	5%	10%	15%	20%	25%	30%	40%	50%
10%	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
20%	99.6	100.0	100.0	100.0	100.0	100.0	100.0	100.0
30%	82.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0
40%	49.9	98.6	100.0	100.0	100.0	100.0	100.0	100.0
50%	25.7	88.3	99.6	100.0	100.0	100.0	100.0	100.0
60%	13.1	67.4	95.8	99.7	100.0	100.0	100.0	100.0
70%	6.8	45.6	83.3	97.2	99.7	100.0	100.0	100.0
80%	3.2	26.4	62.5	87.4	96.9	99.3	100.0	100.0
90%	1.3	13.6	39.4	66.4	85.1	94.5	99.5	100.0
95%	0.8	8.6	26.9	51.0	72.1	86.1	97.6	99.7

The computational complexity of the algorithm was $o(n^3)$ where n is the size of the matrix. Given the values of n that are usually considered in AHP (from 3 to 9), it is clear that the algorithm is quite efficient. The algorithm has been applied to 1,000 different problems with 9 decision makers ($d = 9$) and 9 alternatives ($n = 9$), obtaining a total execution time of 0.88 s, using an Intel i3 computer.

In short, the new procedure (AEM-COM) for improving the incompatibility measured by the GCOMPI considers relative changes in the judgements of the collective matrix (individual matrices are not modified). It is computationally efficient, easy to implement and can be adapted to fit specific interests; furthermore, it provides closed results and bounds relative changes to guarantee slight modifications in the collective matrix and its priority vector, as recommended by Saaty (2003).

5 Numerical example

The procedure is illustrated with an example from the published literature (Moreno-Jiménez et al., 2009; Escobar et al., 2015; Aguarón et al., 2016; Turón et al., 2019). The problem has $n = 5$ alternatives and $d = 3$ decision makers with weights $\alpha_1 = 5/11$; $\alpha_2 = 4/11$; and $\alpha_3 = 2/11$. The individual pairwise comparison matrices are:

$$A^{(1)} = \begin{pmatrix} 1 & 3 & 5 & 8 & 6 \\ & 1 & 3 & 5 & 4 \\ & & 1 & 3 & 2 \\ & & & 1 & 1/3 \\ & & & & 1 \end{pmatrix}, \; A^{(2)} = \begin{pmatrix} 1 & 3 & 7 & 9 & 5 \\ & 1 & 3 & 7 & 1 \\ & & 1 & 5 & 1/5 \\ & & & 1 & 1/5 \\ & & & & 1 \end{pmatrix}, \; A^{(3)} = \begin{pmatrix} 1 & 5 & 7 & 7 & 5 \\ & 1 & 1 & 5 & 1 \\ & & 1 & 5 & 1/3 \\ & & & 1 & 1/5 \\ & & & & 1 \end{pmatrix}$$

Table 3 Priorities, consistency and compatibility for the individual and collective positions

	w_1	w_2	w_3	w_4	w_5	GCI	GCOMPI
$A^{(1)}$	0.513	0.251	0.115	0.042	0.079	0.143	0.430
$A^{(2)}$	0.520	0.195	0.072	0.030	0.182	0.303	0.564
$A^{(3)}$	0.560	0.135	0.101	0.035	0.168	0.298	0.708
AIJ	0.533	0.208	0.096	0.037	0.125	0.122	0.464
P	0.467	0.255	0.095	0.044	0.139	0.023	0.529

The first of the following collective matrices was obtained using AIJ, the second corresponds to the PCCM which was determined by means of the procedure described in Escobar et al. (2015) and Aguarón et al. (2016):

$$AIJ = \begin{pmatrix} 1 & 3.292 & 6.007 & 8.150 & 5.432 \\ & 1 & 2.457 & 5.651 & 1.878 \\ & & 1 & 3.964 & 1/1.600 \\ & & & 1 & 1/3.964 \\ & & & & 1 \end{pmatrix}, \; PCCM = \begin{pmatrix} 1 & 2.049 & 5.510 & 9.000 & 3.165 \\ & 1 & 3.000 & 6.082 & 1.739 \\ & & 1 & 2.709 & 1/1.467 \\ & & & 1 & 1/2.845 \\ & & & & 1 \end{pmatrix}$$

In general, the collective matrix P can be obtained by any of the procedures allowed for AHP-GDM. In this case, the proposed algorithm (AEM-COM) is applied to improve the incompatibility of the second of the two previous matrices ($P = PCCM$). This collective matrix ensures that the judgements are within the consistency stability intervals for all the actors (Aguarón et al., 2003).

Table 3 details the priorities (obtained with the RGM method) of the three individual and the two collective matrices. It also shows the consistency (GCI) and the compatibility (GCOMPI) indicators for all the matrices. The GCOMPI values in the first 3 rows measure the incompatibility between the individual matrices and the priority vector (v) associated to the collective matrix P. The GCOMPI values in the last 2 rows measure the incompatibility between the family $\mathcal{A}$ and vectors w^G and v, respectively.

As can be seen in Table 3, the GCOMPI value for the collective matrix P is 0.529. The application of the AEM-COM algorithm aims to reduce this value, knowing that the minimum value that can be reached with this indicator for family $\mathcal{A}$ is GCOMPI($\mathcal{A}$, w^G) = 0.464 (Remark 2).

In this illustrative example, it is assumed that the decision makers (or the facilitator) have established a permissibility value $\rho = 15\%$ (they would accept the modification of some judgements of the collective matrix up to 15% of their initial values), and that the judgements are limited to the continuous interval [1/9, 9].

From the priority vector associated to the P matrix (v) and to the AIJ matrix (w^G), the ratio matrix $Q = (q_{ij}) = \left(\frac{v_i/v_j}{w_i^G/w_j^G}\right)$ and the associated matrix log Q that contains the natural logarithms of the q_{ij} are obtained to determine the order of entry of the judgements in the algorithm:

$$Q = \begin{pmatrix} 1.000 & 0.716 & 0.888 & 0.735 & 0.787 \\ 1.397 & 1.000 & 1.241 & 1.026 & 1.099 \\ 1.126 & 0.806 & 1.000 & 0.827 & 0.886 \\ 1.361 & 0.974 & 1.209 & 1.000 & 1.071 \\ 1.271 & 0.910 & 1.129 & 0.934 & 1.000 \end{pmatrix},$$

$$\log Q = \begin{pmatrix} & -0.334 & -0.119 & -0.308 & -0.240 \\ & & +0.216 & +0.026 & +0.094 \\ & & & -0.189 & -0.121 \\ & & & & +0.068 \\ & & & & \end{pmatrix}$$

The maximum value, in absolute terms, corresponds to the judgement $(r', s') = (1, 2)$. As $p_{12} = 2.049 > 1$, this judgement is directly considered, and not its inverse, i.e, $(r, s) = (1, 2)$. This is the judgement that would most rapidly decrease the value of the GCOMPI.

Since $\log q_{12} = -0.334 < 0$, the judgement must be increased. The optimal relative variation is determined by $t^*_{12} = q_{12}^{-n/2} = 2.307$. This would mean an increase of 130.7%, greater than a permissibility level of 15%, so the relative variation of the judgement will be determined by the permissibility:

$$t_{12} = \min\left\{1 + \rho, t^*_{12}\right\} = \min\{1.15, 2.307\} = 1.15$$

The new value of the judgement is $p'_{12} = 1.15 \times p_{12} = 2.356$ (then, $p'_{21} = 1/2.356$), and the associated incompatibility measure of matrix P' for the first iteration is $\text{GCOMPI}' = 0.5146$.

Table 4 summarises the iterations of the procedure. In the second iteration, the selected judgement $(1, 4)$ is not modified because its new value would be outside the range $[1/9, 9]$. In this example, the modifications of the judgements in the rest of the iterations (except the last one) have been determined by the permissibility (15%) and not by the value that provides the maximum possible reduction of inconsistency (t^*_{rs}); the average relative change of the judgements is 11.26%.

After considering all the judgements (10 iterations) the final pairwise comparison matrix, P', its associated priority vector, v', and the corresponding values of the GCI and the GCOMPI are:

$$P' = \begin{pmatrix} 1.000 & 2.356 & 6.336 & 9.000 & 3.640 \\ 0.424 & 1.000 & 2.609 & 6.110 & 1.512 \\ 0.158 & 0.383 & 1.000 & 3.115 & 0.784 \\ 0.111 & 0.164 & 0.321 & 1.000 & 0.306 \\ 0.275 & 0.661 & 1.275 & 3.272 & 1.000 \end{pmatrix}$$

$$v' = (0.496, 0.229, 0.098, 0.041, 0.136) \quad GCI(P') = 0.0423 \quad \text{GCOMPI}(\mathcal{A}, v') = 0.4814$$

As the algorithm advances it is possible to measure the improvements achieved for the GCOMPI and to obtain measurements of the proximity between the initial and final priority vectors, v and v' (see Table 5). Efficiency, defined in expression (21), can be adapted to reflect how the GCOMPI improves at each iteration:

$$\text{Efficiency}_i = \frac{\text{GCOMPI}_0 - \text{GCOMPI}_i}{\text{GCOMPI}_0 - \text{GCOMPI}_{min}}$$

where GCOMPI_i refers to the GCOMPI at iteration i.

The maximum differences in relative terms and the value of the G-compatibility index (Garuti, 2007; Garuti & Salomon, 2012) are calculated to measure the distances between the priority vectors. The vectors are said to be highly compatible when the value of the latter indicator is above 0.90.

$$\text{Max. Rel. Dif.}(x, x') = \max_i \left| \frac{x'_i - x_i}{x_i} \right| * 100 \qquad G(x, y) = \sum_{i=1}^{n} \frac{\min(x_i, y_i)}{\max(x_i, y_i)} \frac{x_i + y_i}{2}$$

Table 4 Information on the iterations of the procedure for $\rho = 15\%$

Iteration	GCOMPI_0	(r, s)	p_{rs}	$\log q_{rs}$	Incr/decr	limit DM	limit	t_{rs}	a'_{rs}	GCOMPI'
1	0.5289	(1, 2)	2.049	−0.334	Increase	1.150	2.307	1.150	2.356	0.5146
2	0.5146	(1, 4)	9	−0.280	Increase	1.150	2.015	1.000	9.000	0.5146
3	0.5146	(1, 5)	3.165	−0.212	Increase	1.150	1.699	1.150	3.640	0.5060
4	0.5060	(3, 4)	2.709	−0.189	Increase	1.150	1.606	1.150	3.115	0.4985
5	0.4985	(2, 3)	3	0.160	Decrease	0.870	0.671	0.870	2.609	0.4924
6	0.4924	(1, 3)	5.510	−0.119	Increase	1.150	1.346	1.150	6.336	0.4881
7	0.4881	(4, 5)	2.845	−0.068	Increase	1.150	1.186	1.150	3.272	0.4863
8	0.4863	(3, 5)	1.467	0.093	Decrease	0.870	0.792	0.870	1.275	0.4832
9	0.4832	(2, 5)	1.739	0.067	Decrease	0.870	0.847	0.870	1.512	0.4814
10	0.4814	(2, 4)	6.082	−0.002	Increase	1.150	1.005	1.005	6.110	0.4814

Table 5 Evolution of the GCOMPI efficiency and measures of proximity between priority vectors ($\rho = 15\%$)

Iteration	GCOMPI	Efficiency (%)	Max Rel Dif (%)	G
0	0.5289			
1	0.5146	22.0	3.4	0.980
2	0.5146	22.0	3.4	0.980
3	0.5060	35.2	4.3	0.963
4	0.4985	46.8	4.4	0.962
5	0.4924	56.2	6.7	0.952
6	0.4881	62.8	7.7	0.944
7	0.4863	65.6	8.0	0.947
8	0.4832	70.3	7.9	0.940
9	0.4814	73.1	10.2	0.938
10	0.4814	73.1	10.1	0.938

Table 5 shows that the application of the proposed procedure significantly reduces incompatibility, resulting in an efficiency rate of 73.1% (percentage of the maximum possible reduction) with slight modifications of the collective priorities (Max. Rel. Dif. $(v, v') = 10.1\%$ and G(v, v') = 0.938). This demonstrates that the new approach has significantly improved the incompatibility of the decision makers regarding the collective position obtained with the PCCM methodology. If, instead of completing all the iterations of the algorithm, a specific reduction in GCOMPI had been set, e.g. 60%, the algorithm would have been finished on the 6th iteration, when the target had been reached (with Max. Rel. Dif. (v, v') = 7.7% and G(v, v') = 0.944). With regards to the relative changes of the collective priority vectors, the maximum differences during the process remain below 10.2%. The G values indicate that for all the iterations, the initial and the updated priority vectors are highly compatible ($G > 0.9$). In particular, the G value between the initial and the final priority vectors is $G(v, v') = 0.938$. In terms of the compatibility of the collective vectors (v and v') with w^G, the procedure has made it possible to move from a compatible vector v – $G(v, w^G) = 0.874 \geq 0.85$ – to a highly compatible vector v' – $G(v', w^G) = 0.929 \geq 0.9$: v' is highly compatible with v and w^G.

With the initial individual PCMs, the proposed procedure (AEM-COM) has made it possible (Table 6) for the three decision makers to improve their compatibility with respect to the final collective position (v'). All three have improved the category considered on the scale associated with Garuti's G index (Garuti, 2017).

Table 7 shows the results obtained by applying the procedure with different values of the permissibility (ρ between 5% and 50%). As might be expected, it can be observed that the greater the permissibility, the greater is the % of reduction of the GCOMPI (efficiency), but

Table 6 G-compatibility indexes between individual and collective priority vectors

	$G(w^{(i)}, v)$	$G(w^{(i)}, v')$
$w^{(1)}$	0.885	0.900
$w^{(2)}$	0.837	0.873
$w^{(3)}$	0.782	0.827

Table 7 GCOMPI improvement and measures of proximity between priority vectors ($\rho = 5\%$ to 50%)

ρ	GCOMPI	Efficiency (%)	Max Rel Dif (%)	G
5%	0.5084	31.5%	4.2	0.976
10%	0.4928	55.6	7.3	0.956
15%	0.4814	73.1	10.1	0.938
20%	0.4736	85.1	12.4	0.921
25%	0.4690	92.2	14.0	0.907
30%	0.4661	96.7	15.4	0.896
40%	0.4656	97.4	15.7	0.893
50%	0.4644	99.3	16.8	0.884

also the greater is the distance between the initial and final priority vectors. The improvement in incompatibility increases rapidly when permissibility is low and slows down when permissibility is higher, maintaining small distances between the priority vectors. For example, for a permissibility level of $\rho = 25\%$, a reduction in the value of the GCOMPI of 92.2% is achieved, with just 14% maximum difference between the priority vectors. The associated value of the G index is 0.907, indicating high compatibility between the initial and final priority vectors. For higher values of permissibility ($25\% < \rho \leq 50\%$), the value of G indicates compatibility ($G > 0.85$). It should also be mentioned that, for each value of ρ, the inconsistency measure (GCI) of the resulting P' matrix does not exceed the threshold allowed for ensuring the validity of the priority vectors.

If the permissibility constraint is removed (a situation not in accordance with the proposal), efficiencies of 71.72% and 94.66% would be achieved in the first and second iterations. The substantial improvement of incompatibility occurs at the cost of greater maximum relative changes in judgements (130.7%) and priorities (19.52%), which might not be accepted.

6 Conclusions and recommendations

Assuming a local AHP-GDM context and the Row Geometric Mean as the prioritisation method, this paper presents a theoretical framework and a semi-automatic procedure (AEM-COM) for improving the representativity (incompatibility) of the collective pairwise comparison matrix, irrespective of the method by which the collective matrix is obtained. It should be noted that the AEM-COM procedure does not require the continuous intervention of decision makers and does not modify the individual matrices; it only modifies the judgements of the collective matrix. It is based on the consideration of relative changes and on the slight modification of the judgements of the collective matrix that further improve the GCOMPI. The maximum relative variation allowed for the modification of any judgement is bounded by the parameter known as permissibility (ρ).

The input data for the procedure comprise the weights of the decision makers, their individual pairwise comparison matrices, the collective matrix (obtained using any CRP) and the parameter ρ. To apply the procedure for reducing incompatibility, it is necessary to check that the initial collective matrix has an acceptable level of inconsistency. If it does not, inconsistency should be reduced (Aguarón et al., 2021) until it reaches the threshold necessary to guarantee the validity of the collective vector.

It may be sufficient to consider a small value of permissibility, since the empirical studies have shown that even when using small values, significant reductions of the GCOMPI are achieved. If the reduction is not sufficient and further improvement is desired, an increase of the value of ρ is suggested. Performing a sensitivity analysis of the permissibility parameter may provide relevant information on the critical points and the decision opportunities of the resolution process. The initial value of ρ can be determined by consensus among the decision makers. If there is no consensus, it is recommended to take the minimum of the individual permissibility values ($\rho = \min \rho_k$) to solve the problem. In both cases an implicit acceptance of the final result is assumed.

The analysis of the behaviour of the algorithm and the numerical example used to illustrate the new methodology give a clear idea of its practical potential. The algorithm is computationally efficient, easy to implement, and can be adapted to allow the consideration of several variants.

The proposed framework makes it possible to incorporate specific interests of the decision makers as additional restrictions, as a veto or the imposition of rigid constraints. The modifications of the judgements can be limited so that they do not lie outside of a specific range (Saaty's scale, priority or consistency stability intervals...). In addition, the algorithm can be adapted to incorporate the personal intervention of decision makers to a greater extent, not only at the beginning of the CRP. The interactive variant would require more time and effort.

Other stopping rules based on thresholds for efficiency, or based on distances between collective priority vectors can be contemplated. For example, with the G-compatibility index (Garuti, 2007), the algorithm would stop when the distance between the initial and final collective positions is lower than a value fixed in advance. The combination of the information provided by the efficiency and the Garuti G measure would allow the establishment of thresholds for the GCOMPI. In case there is different information available (input and/or output), compatibility can be measured by other GCOMPI expressions (Aguarón et al., 2022).

The range of possibilities for adapting the procedure gives it enormous cognitive potential. These variants and other possible extensions of the current work, including new measures for compatibility and direct thresholds for the GCOMPI that would make it operative, will be the subject of future research.

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Appendix A. Proofs of the theorems

In what follows GCOMPI(x, y) is denoted as $G(x, y)$. It will be also assumed that the priority vectors obtained with the RGM method are standardized being the product of its components equal to the unit.

Proof of Remark 2

$$\frac{\partial G(\mathcal{A}, v)}{\partial v_r} = \frac{\partial}{\partial v_r}\sum_{k=1}^{d}\alpha_k G(A^{(k)}, v) = \sum_{k=1}^{d}\alpha_k \frac{\partial G(A^{(k)}, v)}{\partial v_r}$$
$$= \frac{1}{(n-1)(n-2)}\sum_{k=1}^{d}\left(\alpha_k \frac{\partial}{\partial v_r}\sum_{i,j=1}^{n}\log^2 \frac{a_{ij}^{(k)} v_j}{v_i}\right)$$

The only terms in the inner sum that depend on v_r are those that are in row r or column s. It is obvious that $\log^2 a_{ij} v_j/v_i = \log^2 a_{ji} v_i/v_j$ so we only consider twice the terms $\log^2 a_{rj} v_j/v_r$:

$$\frac{\partial G(\mathcal{A}, v)}{\partial v_r} = \frac{2}{(n-1)(n-2)}\sum_{k=1}^{d}\left(\alpha_k \frac{\partial}{\partial v_r}\sum_{j=1}^{n}\log^2 \frac{a_{rj}^{(k)} v_j}{v_r}\right)$$
$$= \frac{-4}{(n-1)(n-2)}\frac{1}{v_r}\sum_{k=1}^{d}\left(\alpha_k \sum_{j=1}^{n}\log \frac{a_{rj}^{(k)} v_j}{v_r}\right)$$
$$= \frac{-4}{(n-1)(n-2)}\frac{1}{v_r}\sum_{j=1}^{n}\left(\sum_{k=1}^{d}\alpha_k \left(\log a_{rj}^{(k)} + \log v_j - \log v_r\right)\right)$$

Taking into account that $\sum_k \alpha_k = 1$ and that $a_{ij}^G = \prod_{k=1}^{d}\left(a_{ij}^{(k)}\right)^{\alpha_k}$ and assuming that v has been normalised as $\prod_j v_j = 1$, then:

$$\frac{\partial G(\mathcal{A}, v)}{\partial v_r} = \frac{-4}{(n-1)(n-2)}\frac{1}{v_r}\sum_{j=1}^{n}\left(\log a_{rj}^G + \log v_j - \log v_r\right)$$
$$= \frac{-4}{(n-1)(n-2)}\frac{1}{v_r}\left(\log \prod_{j=1}^{n} a_{rj}^G - n\log v_r\right)$$

And

$$\frac{\partial G(\mathcal{A}, v)}{\partial v_r} = 0 \to v_r = \left(\prod_{j=1}^{n} a_{rj}^G\right)^{1/n} = w_r^{G|J} = w_r^G$$

In other words, a critical point exists when the vector v is the same as the priority vector obtained by applying the RGM method to the matrix AIJ derived from the matrices of family $\mathcal{A}$.

To verify that this point corresponds to a minimum, it is enough to note that the second partial derivatives are:

$$\frac{\partial^2 G(\mathcal{A}, v)}{\partial v_r^2} = \frac{4}{(n-1)(n-2)} \frac{1}{v_r^2} \left[\left(\log \prod_{j=1}^{n} a_{rj}^G - n \log v_r \right) + n \right]$$

With the above extreme condition, the parenthesis is cancelled and the second partial derivatives are positive. The cross partial derivatives are zero, so the Hessian matrix is positive-definite and the critical point corresponds to a minimum. □

As a previous step to the proof of Theorem 1, Lemmas 1 and 2 are included.

Lemma 1 *Let $A = (a_{ij})$ and $P = (p_{ij})$ with $i, j = 1, \ldots, n$ be two PCMs and $v = (v_i)$ with $i = 1, \ldots, n$ be the priority vector associated to P obtained with the RGM method.*
The derivatives of the discrepancies $e_{ij} = a_{ij} v_j / v_i$ are given by:

$$\begin{aligned}
&\frac{\partial e_{rs}}{\partial p_{rs}} = -\frac{2}{n} \frac{e_{rs}}{p_{rs}} && \frac{\partial e_{sr}}{\partial p_{rs}} = \frac{2}{n} \frac{e_{sr}}{p_{rs}} \\
&\frac{\partial e_{rj}}{\partial p_{rs}} = -\frac{1}{n} \frac{e_{rj}}{p_{rs}} \quad j \neq s && \frac{\partial e_{sj}}{\partial p_{rs}} = \frac{1}{n} \frac{e_{sj}}{p_{rs}} \quad j \neq r \\
&\frac{\partial e_{is}}{\partial p_{rs}} = -\frac{1}{n} \frac{e_{is}}{p_{rs}} \quad i \neq r && \frac{\partial e_{ir}}{\partial p_{rs}} = \frac{1}{n} \frac{e_{ir}}{p_{rs}} \quad i \neq s
\end{aligned}$$

Proof For error e_{rs} we have:

$$e_{rs} = a_{rs} \frac{v_s}{v_r} = a_{rs} \left(\frac{p_{s1} \cdots p_{sr} \cdots p_{ss} \cdots p_{sn}}{p_{r1} \cdots p_{rr} \cdots p_{rs} \cdots p_{rn}} \right)^{1/n} = a_{rs} p_{rs}^{-2/n} \left(\frac{p_{s1} \cdots p_{sr} \cdots p_{ss} \cdots p_{sn}}{p_{r1} \cdots p_{rr} \cdots p_{rs} \cdots p_{rn}} \right)^{1/n}$$

And taking the derivative

$$\begin{aligned}
\frac{\partial e_{rs}}{\partial p_{rs}} &= -\frac{2}{n} a_{rs} p_{rs}^{-1-2/n} \left(\frac{p_{s1} \cdots p_{sr} \cdots p_{ss} \cdots p_{sn}}{p_{r1} \cdots p_{rr} \cdots p_{rs} \cdots p_{rn}} \right)^{1/n} = \\
&= -\frac{2}{n} \frac{a_{rs}}{p_{rs}} p_{rs}^{-2/n} \left(\frac{p_{s1} \cdots p_{sr} \cdots p_{ss} \cdots p_{sn}}{p_{r1} \cdots p_{rr} \cdots p_{rs} \cdots p_{rn}} \right)^{1/n} = -\frac{2}{n} \frac{a_{rs}}{p_{rs}} \frac{v_s}{v_r} = -\frac{2}{n} \frac{e_{rs}}{p_{rs}}
\end{aligned}$$

For error e_{sr} we use the relation $e_{sr} = 1/e_{rs}$:

$$\frac{\partial e_{sr}}{\partial p_{rs}} = \frac{\partial e_{sr}}{\partial e_{rs}} \frac{\partial e_{rs}}{\partial p_{rs}} = \frac{-1}{e_{rs}^2} \left(-\frac{2}{n} \right) \frac{e_{rs}}{p_{rs}} = \frac{2}{n} \frac{e_{sr}}{p_{rs}}$$

The term e_{rj} with $j \neq r$ can be expressed as:

$$e_{rj} = a_{rj} \frac{v_j}{v_r} = a_{rj} \frac{v_j}{(p_{r1} \cdots p_{rs} \cdots p_{rn})^{1/n}} = a_{rj} p_{rs}^{-1/n} \frac{v_j}{\prod_{k \neq s} p_{rk}^{1/n}}$$

And taking the derivative we have

$$\frac{\partial e_{rj}}{\partial p_{rs}} = -\frac{1}{n} a_{rj} p_{rs}^{-1-1/n} \frac{v_j}{\prod_{k \neq s} p_{rk}^{1/n}} = -\frac{1}{n} \frac{a_{rj}}{p_{rs}} \frac{v_j}{\prod_k p_{rk}^{1/n}} = -\frac{1}{n} \frac{a_{rj}}{p_{rs}} \frac{v_j}{v_r} = -\frac{1}{n} \frac{e_{rj}}{p_{rs}}$$

The other derivatives can be demonstrated analogously □

Lemma 2 *Under the same conditions as the previous lemma*

$$\prod_{j=1}^{n} e_{rj} = \frac{w_r^n}{v_r^n} \tag{A.1}$$

where w is the priority vector associated to A obtained with the RGM method.

Proof Based on the definition of the errors (e_{ij}) and assuming that the priority vectors are standardized so that the product is equal to the unit:

$$\prod_{j=1}^{n} e_{rj} = \prod_{j=1}^{n} \frac{a_{rj} v_j}{v_r} = \frac{1}{v_r^n} \left(\prod_{j=1}^{n} a_{rj} \right) \left(\prod_{j=1}^{n} v_j \right) = \frac{w_r^n}{v_r^n}$$

Proof of Theorem 1 The only terms of $G(A, v)$, see expression (4), that depend on p_{rs} are those that are in rows r, s or columns r, s. As $e_{ij} = a_{ij} v_j / v_i$, it is obvious that $\log^2 e_{ij} = \log^2 e_{ji}$ so we only consider two-time terms e_{rs}, e_{rj} with $j \neq s$ and e_{sj} with $j \neq r$:

$$\begin{aligned}
\frac{\partial G(A, v)}{\partial p_{rs}} &= \frac{2}{(n-1)(n-2)} \frac{\partial}{\partial p_{rs}} \left(\log^2 e_{rs} + \sum_{j \neq r,s} \log^2 e_{rj} + \sum_{j \neq r,s} \log^2 e_{sj} \right) \\
&= \frac{2}{(n-1)(n-2)} \left[2 \log e_{rs} \frac{1}{e_{rs}} \left(-\frac{2}{n} \frac{e_{rs}}{p_{rs}} \right) + 2 \sum_{j \neq r,s} \log e_{rj} \frac{1}{e_{rj}} \left(-\frac{1}{n} \frac{e_{rj}}{p_{rs}} \right) \right. \\
&\quad \left. + 2 \sum_{j \neq r,s} \log e_{sj} \frac{1}{e_{sj}} \left(\frac{1}{n} \frac{e_{sj}}{p_{rs}} \right) \right] \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{n p_{rs}} \left[-2 \log e_{rs} - \sum_{j \neq r,s} \log e_{rj} + \sum_{j \neq r,s} \log e_{sj} \right]
\end{aligned} \tag{A.2}$$

Using that $\log e_{rs} = -\log e_{sr}$ and $\log e_{rr} = \log e_{ss} = 0$ in (A.2):

$$\begin{aligned}
\frac{\partial G(A, v)}{\partial p_{rs}} &= \frac{4}{(n-1)(n-2)} \frac{1}{n p_{rs}} \left[-\sum_{j=1}^{n} \log e_{rj} + \sum_{j=1}^{n} \log e_{sj} \right] \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{n p_{rs}} \log \frac{\prod_{j=1}^{n} e_{sj}}{\prod_{j=1}^{n} e_{rj}}
\end{aligned} \tag{A.3}$$

Substituting expresion (A.1) from Lemma 2 in (A.3),

$$\frac{\partial G(A, v)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{n p_{rs}} \log \frac{w_s^n / v_s^n}{w_r^n / v_r^n} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r / v_s}{w_r / w_s}$$

□

Proof of Theorem 2 From Definition 2, we have:

$$\begin{aligned}
\frac{\partial G(\mathcal{A}, v)}{\partial p_{rs}} &= \frac{\partial}{\partial p_{rs}} \sum_{k=1}^{d} \alpha_k G(A^{(k)}, v) \\
&= \sum_{k=1}^{d} \alpha_k \frac{\partial G(A^{(k)}, v)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \sum_{k=1}^{d} \alpha_k \log \frac{v_r / v_s}{w_r^{(k)} / w_s^{(k)}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \sum_{k=1}^{d} \alpha_k \left(\log \frac{v_r}{v_s} - \log w_r^{(k)} + \log w_s^{(k)} \right) \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \left(\log \frac{v_r}{v_s} - \sum_{k=1}^{d} \alpha_k \log w_r^{(k)} + \sum_{k=1}^{d} \alpha_k \log w_s^{(k)} \right) \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \left(\log \frac{v_r}{v_s} - \log \prod_{k=1}^{d} \left(w_r^{(k)} \right)^{\alpha_k} + \log \prod_{k=1}^{d} \left(w_s^{(k)} \right)^{\alpha_k} \right) \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \left(\log \frac{v_r}{v_s} - \log w_r^{G|P} + \log w_s^{G|P} \right) \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r/v_s}{w_r^{G|P}/w_s^{G|P}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r/v_s}{w_r^{G}/w_s^{G}}
\end{aligned}
$$

□

Lemma 3 *Let $P = (p_{ij})$ with $i, j = 1, \ldots, n$ be a PCM. If the judgement p_{rs} ($r \neq s$) changes to p'_{rs}, the new priority vector obtained with the RGM method is given by (except for the normalisation factor):*

$$
\begin{aligned}
v'_i &= v_i \qquad \forall\, i \neq r, s \\
v'_r &= v_r t_{rs}^{1/n} \\
v'_s &= v_s t_{rs}^{-1/n}
\end{aligned}
$$

where $t_{ij} = \frac{p'_{ij}}{p_{ij}}$, $\quad i \neq j$.

Proof When $i \neq r, s$, p'_{rs} is not included in the value of v_i ($v_i = (\prod_{j=1}^{n} p_{ij})^{1/n}$), and the value v_i stays the same ($v'_i = v_i$). The value of v'_r is obtained as

$$
v_r'^{\,n} = p_{r1} \ldots p'_{rs} \ldots p_{rn} = p_{r1} \ldots p_{rs} \ldots p_{rn} \frac{p'_{rs}}{p_{rs}} = v_r^n \frac{p'_{rs}}{p_{rs}} = v_r^n t_{rs}
$$

and that of v_s, in the same way, as

$$
v_s'^{\,n} = v_s^n \frac{p'_{sr}}{p_{sr}} = v_s^n t_{sr} = v_s^n t_{rs}^{-1}
$$

□

Lemma 4 *Let $A = \left(a_{ij}\right)$ and $P = \left(p_{ij}\right)$ with $i, j = 1, \ldots, n$ be two PCMs and $v = (v_i)$ with $i = 1, \ldots, n$ be the priority vector associated to P obtained with the RGM method. If the judgement p_{rs} changes to p'_{rs} ($r \neq s$), the new error terms, $e'_{ij} = a_{ij} v'_j / v'_i$ in the expression of $G(A, v)$ are:*

$$
\begin{aligned}
& e'_{ij} = e_{ij} \quad \forall i, j \neq r, s && \\
& e'_{rj} = e_{rj} t_{rs}^{-1/n} \quad \forall j \neq r, s && e'_{jr} = e_{jr} t_{rs}^{1/n} \quad \forall j \neq r, s \\
& e'_{sj} = e_{sj} t_{rs}^{1/n} \quad \forall j \neq r, s && e'_{js} = e_{js} t_{rs}^{-1/n} \quad \forall j \neq r, s \\
& e'_{rs} = e_{rs} t_{rs}^{-2/n} && e'_{sr} = e_{rs} t_{rs}^{2/n}
\end{aligned}
$$

where $t_{rs} = \frac{p'_{rs}}{p_{rs}}$.

Proof If the judgement p_{rs} is modified, with p'_{rs} as its new value, the only priorities corresponding to matrix P which are modified are v_r and v_s, so the only errors which will be modified are those corresponding to any index r or s.

Let us consider the error e_{rj} with $j \neq r, s$. From definition of errors and the previous lemma:

$$e'_{rj} = \frac{a_{rj} v_j}{v'_r} = \frac{a_{rj} v_j}{v_r t_{rs}^{1/n}} = e_{rj} {t_{rs}}^{-1/n}$$

The errors are reciprocal, that is to say, $e_{ji} = 1/e_{ij}$, then $e'_{jr} = 1/e'_{rj} = t_{rs}^{1/n}/e_{rj} = e_{jr} t_{rs}^{1/n}$ for all $j \neq r, s$.

Analogously, the relationships for e_{sj} and e_{js} are proved. For e_{rs}, we have:

$$e'_{rs} = \frac{a_{rs} v'_s}{v'_r} = \frac{a_{rs} v_s {t_{rs}}^{-1/n}}{v_r t_{rs}^{1/n}} = \frac{a_{rs} v_s}{v_r} {t_{rs}}^{-2/n} = e_{rs} t_{rs}^{-2/n}$$

With this expression, we automatically obtain that of e_{sr}. □

Proof of Theorem 3 The GCOMPI expression uses the aggregation of terms that depend on errors (e_{rs}). Thus, its variation depends exclusively on the modification of the elements located in the rows or columns r and s. Therefore, it can be written as:

$$\Delta G = \Delta_r G + \Delta_s G + \Delta_{rs} G$$

where $\Delta_r G$ indicates the variation due to the elements with any index equal to r except for the e_{rs}, $\Delta_s G$ indicates the variation due to the elements with any index equal to s except for e_{rs}, and $\Delta_{rs} G$ indicates the variation due to the term e_{rs}.

The addends that appear in the G expression are $\log^2 e_{ij}$, so the term e_{ij} and its reciprocal e_{ji} contribute the same amount. Therefore, it is sufficient to operate with half of the terms and afterwards multiply the result by two.

Let us consider the term $\Delta_r G$. This represents the variation due to the elements, different from e_{rs}, that are in row r or in column r. Given that the elements of the row and the column are reciprocal, it is sufficient to operate with one of them, i.e., the row:

$$\Delta_r G = 2 \sum_{j \neq r,s} (\log^2 e'_{rj} - \log^2 e_{rj})$$

We will now operate only with the numerator of the G expression, leaving the denominator to be included at the end of the proof. Developing the value of e'_{rj} we have:

$$\Delta_r G = 2 \sum_{j \neq r,s} (\log^2 \frac{e_{rj}}{t_{rs}^{1/n}} - \log^2 e_{rj})$$

and operating

$$\begin{aligned}\Delta_r G &= 2 \sum_{j \neq r,s} (\log \frac{e_{rj}}{t_{rs}^{1/n}} + \log e_{rj})(\log \frac{e_{rj}}{t_{rs}^{1/n}} - \log e_{rj}) \\ &= 2 \sum_{j \neq r,s} (\log \frac{e_{rj}^2}{t_{rs}^{1/n}})(\log \frac{1}{t_{rs}^{1/n}}) = 2 \log \frac{1}{t_{rs}^{1/n}} \sum_{j \neq r,s} \log \frac{e_{rj}^2}{t_{rs}^{1/n}}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{n}\log t_{rs}\left(2\sum_{j\neq r,s}\log e_{rj} - \sum_{j\neq r,s}\log t_{rs}^{1/n}\right)\\
&= -\frac{2}{n}\log t_{rs}\left[2\sum_{j\neq r,s}\log e_{rj} - (n-2)\log t_{rs}^{1/n}\right]\\
&= -\frac{2}{n}\log t_{rs}\left[2\sum_{j=1}^{n}\log e_{rj} - 2\log e_{rs} - (n-2)\log t_{rs}^{1/n}\right] \qquad \text{(A.4)}
\end{aligned}
$$

Using Lemma 2, (A.4) can be written as as

$$
\begin{aligned}
\Delta_r G &= -\frac{2}{n}\log t_{rs}\left[2\log\prod_{j=1}^{n} e_{rj} - 2\log e_{rs} - (n-2)\log t_{rs}^{1/n}\right]\\
&= -\frac{2}{n}\log t_{rs}\left[2\log\frac{w_r^n}{v_r^n} - 2\log e_{rs} - (n-2)\log t_{rs}^{1/n}\right]\\
&= \frac{2}{n}\log t_{rs}\left[2\log\frac{v_r^n}{w_r^n} + 2\log e_{rs} + (n-2)\log t_{rs}^{1/n}\right]
\end{aligned}
$$

Following an analogous process to calculate $\Delta_s G$, we obtain:

$$
\Delta_s G = \frac{2}{n}\log t_{rs}\left[2\log\frac{w_s^n}{v_s^n} + 2\log e_{rs} + (n-2)\log t_{rs}^{1/n}\right]
$$

Finally, it is necessary to calculate $\Delta_{rs}G$:

$$
\begin{aligned}
\Delta_{rs}G &= 2\left(\log^2 e'_{rs} - \log^2 e_{rs}\right) = 2\left(\log^2 e_{rs}t_{rs}^{-2/n} - \log^2 e_{rs}\right)\\
&= 2\left(\log e_{rs}t_{rs}^{-2/n} + \log e_{rs}\right)\left(\log e_{rs}t_{rs}^{-2/n} - \log e_{rs}\right) = 2\left(\log e_{rs}^2 t_{rs}^{-2/n}\right)\log t_{rs}^{-2/n}\\
&= -\frac{4}{n}\log t_{rs}\left(2\log e_{rs} - 2\log t_{rs}^{1/n}\right)
\end{aligned}
$$

Then, the total variation of G is:

$$
\begin{aligned}
\Delta G &= \Delta_r G + \Delta_s G + \Delta_{rs} G\\
&= \frac{2}{n}\log t_{rs}\left[2\log\frac{v_r^n}{w_r^n} + 2\log e_{rs} + (n-2)\log t_{rs}^{1/n} + 2\log\frac{w_s^n}{v_s^n} + 2\log e_{rs} + (n-2)\log t_{rs}^{1/n}\right.\\
&\quad\left. -2\left(2\log e_{rs} - 2\log t_{rs}^{1/n}\right)\right] = \frac{2}{n}\log t_{rs}\left(2\log\frac{v_r^n w_s^n}{w_r^n v_s^n} + 2n\log t_{rs}^{1/n}\right)\\
&= 4\log t_{rs}\left(\log\frac{v_r w_s}{w_r v_s} + \log t_{rs}^{1/n}\right) = 4\log t_{rs}\left(\log\frac{v_r/v_s}{w_r/w_s} + \frac{\log t_{rs}}{n}\right)
\end{aligned}
$$

Finally, including the denominator, we have

$$
\Delta G = \frac{4}{(n-1)(n-2)}\log t_{rs}\left(\log\frac{v_r/v_s}{w_r/w_s} + \frac{\log t_{rs}}{n}\right)
$$

and then

$$
G(A, v'(t_{rs})) = G(A, v) + \frac{4}{(n-1)(n-2)}\log t_{rs}\left(\log\frac{v_r/v_s}{w_r/w_s} + \frac{\log t_{rs}}{n}\right)
$$

□

Proof of Theorem 4. Analogous to proof of Theorem 2 □

Proof of Corollary 1. Immediate from Theorem 3 □

Proof of Corollary 2. Immediate from Theorem 4 □

Lemma 5 *Let $A = (a_{ij})$ and $P = (p_{ij})$ with $i, j = 1, \ldots, n$ be two PCMs and $w = (w_i)$ and $v = (v_i)$ with $i = 1, \ldots, n$ be the corresponding priority vectors associated to A and P obtained with the RGM method. It holds that*

$$\frac{\partial^2 G(A, v)}{\partial t_{rs}^2} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}^2} \left[\frac{2}{n} - \left(\frac{2 \log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r/w_s} \right) \right] \tag{A.5}$$

Proof Immediate from Corollary 1. □

Proof of Corollary 3 From expression (13) of Corollary 1:

$$\frac{\partial G}{\partial t_{rs}} = 0 \Rightarrow \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \left(\frac{2 \log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r/w_s} \right) = 0$$

Therefore

$$\frac{2 \log t_{rs}}{n} + \log \frac{v_r/v_s}{w_r/w_s} = 0 \tag{A.6}$$

and then

$$t_{rs}^* = p'_{rs}/p_{rs} = \left(\frac{w_r/w_s}{v_r/v_s} \right)^{n/2}$$

From (A.5) and (A.6) it is easy to check that it is a minimum.
Finally, replacing t_{rs}^* in expression (11)

$$\Delta G(A, v) = \frac{-n}{(n-1)(n-2)} \log^2 \frac{w_r/w_s}{v_r/v_s}$$

□

Proof of Corollary 4. Analogous to proof of Corollary 3. □

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