



Equilibrium arrivals to a last-come first-served preemptive-resume queue

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Abstract

We consider a queueing system which opens at a given point in time and serves a finite number of users according to the last-come first-served discipline with preemptive-resume (LCFS-PR). Each user must decide individually when to join the queue. We allow for general classes of user preferences and service time distributions and show existence and uniqueness of a symmetric Nash equilibrium. Furthermore, we show that no continuous asymmetric equilibrium exists, if the population consists of only two users, or if arrival strategies satisfy a mild regularity condition. For an illustrative example, we implement a numerical procedure for computing the symmetric equilibrium strategy based on our constructive existence proof for the symmetric equilibrium. We then compare its social efficiency to that obtained if users are instead served on a first-come first-served (FCFS) basis.

Keywords Queueing · Strategic arrivals · Nash equilibrium · LCFS-PR · FCFS

JEL Classification C72 · D62 · R41

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1 Introduction

In a variety of situations in which multiple users demand a service that is made accessible at a certain time, the initial demand for service often exceeds the capacity to provide it. Examples of such situations include customers returning a product for upgrade or refund, users accessing a website at the release of an online service or the start of a sale, or individuals conducting financial transactions when a bank or stock market opens. To cope with excess demand, the provision of service to users is often managed with a queueing system. The way a queue is managed affects the behaviour of users and, consequently, the waiting time that users face, and inefficient queueing leads to both frustration for the unlucky user and costs to society. Therefore, the study of how strategic users behave when faced with specific queueing systems and of the implied social welfare loss is important for the design and evaluation of queueing systems.

This paper considers a queueing system with a single server that opens at a given point in time. A finite number of users choose independently when to arrive at the system. Users prefer to complete service early rather than later, and they dislike waiting in the queue. The service time requirements of users are identically and independently distributed. The order in which waiting users are served is determined by the Last-Come-First-Served service discipline with preemptive resume (LCFS-PR). The LCFS-PR discipline admits any newly arrived user into service immediately, possibly preempting the service progress of another user. The preempted user on the other hand joins the queue where later arrivals are prioritized over earlier arrivals. When a preempted user re-enters service, her service is resumed from the point of interruption.

Whereas the most frequently used (and studied) discipline is the First-Come First-Served (FCFS) discipline, papers studying equilibrium and efficiency properties of alternative disciplines such as the Last-Come First-Served (LCFS) and LCFS with preemptive resume (LCFS-PR) show that these in some settings provide superior outcomes. In Hassin (1985) and Platz and Østerdal (2017) different environments are studied in which LCFS(-PR) disciplines are shown to be socially optimal for general classes of user preferences and service time distributions. However, in a situation with a finite number of strategic users and LCFS-PR, the question of existence (and uniqueness) of equilibria and of whether LCFS-PR generally outperforms FCFS have remained open. In this paper, we answer the first question affirmatively (with some qualifications) and the last question negatively.

The strategic choices of arrivals to queues have been studied for almost half a century (see e.g. Hassin (2016) and Haviv and Ravner (2021) for extensive surveys). The problem was first approached by considering a fluid model for congestion dynamics that studied the equilibrium arrival behavior of a continuum of users (Vickrey, 1969). In this model, each user must choose his/her arrival time to a continuously open bottleneck, and each user has a preferred time for passing the bottleneck and will incur a cost from being early or late. Similar fluid models have been studied further and extended in various directions, e.g. to treat heterogeneous users (Arnott et al., 1989), elastic user demand (Arnott et al., 1993), and hypercongestion (Verhoef, 2003).

The study of strategic arrivals in queueing systems where the server has a limited service period (i.e. the server admits an opening and/or closing time) was first formulated by Glazer and Hassin (1983). They consider a Poisson-distributed number of identical users with exponential service requirements that arrive at a server with a known opening and closing time and wish to minimize their own waiting time (Glazer & Hassin, 1983). This work showed that in a symmetric equilibrium under FCFS, the users arrive according to a continuous distribution

function that extends over a finite interval before and after the opening time. Several variations of this model have since been considered, e.g., to treat bulk service (Glazer & Hassin, 1987), no arrivals prior to opening (Hassin & Kleiner, 2011; Haviv & Oz, 2018) and discrete arrival times and deterministic service times (Rapoport et al., 2004; Seale et al., 2005; Stein et al., 2007). Whereas the aforementioned studies assume that users only want to minimize their wait in the queue, another body of literature studies environments where users also care about being served at an early time. This type of preference has been modelled as a tardiness cost that increases the later the user is admitted into service. The equilibrium behavior induced by such user preferences has been studied for several variants of assumptions. Specifically, the symmetric equilibrium has been studied for a Poisson-distributed number of identical users with exponential service time requirements and multilinear costs of waiting and tardiness in time, and it has been studied in settings both with and without early arrivals, as have the fluid analogues of these models (Jain et al., 2011; Haviv, 2013; Sherzer & Kerner, 2017). Ravner (2014) studies a model where the customers incur not only congestion (waiting) costs but also penalties for their index of arrival. A complete analysis of the existence and uniqueness of the equilibrium for a general population size with multilinear waiting and tardiness costs and exponential service times showed that there always exists an equilibrium, and that it is in fact symmetric (Juneja & Shimkin, 2013). The existence and uniqueness of a symmetric equilibrium was established for more general classes of utility functions and service time distributions by (Breinbjerg, 2017).

The above-mentioned studies all consider queueing environments that employ the FCFS service discipline.¹ Though the FCFS discipline is intuitively fair and reasonable to most people, it is as mentioned not necessarily the most socially efficient way of settling a queue (Hassin, 1985). In both theoretical analysis (e.g., Glazer and Hassin (1983), Breinbjerg (2017) and others mentioned above) and in empirical experiments (e.g., Rapoport et al. (2004), Seale et al. (2005)), it has been found that under FCFS, users tend to show up (too) early, which may lead to excess waiting time. Therefore, employing a service discipline that induces users to spread out their arrivals in order to avoid arriving at the same time or just before other users, may improve on overall efficiency. In particular, in queueing environments where the server opens at a given point in time, and a continuum of users choose their arrival time in a setting where they incur costs from queueing and being served late, the FCFS discipline provides the lowest level of social efficiency among all work-conserving disciplines, whereas the LCFS discipline provides the highest (Platz & Østerdal, 2017).² Furthermore, empirical support for the greater social efficiency of LCFS compared to FCFS has been established in an experimental setting for a queueing environment with a very small (three-user) population size, where each user chooses arrival time from a finite set of time slots (Breinbjerg et al., 2016).

In this paper, we consider a queueing environment where a finite number of users with identical preferences choose when to arrive at a single-server facility that opens at a commonly known point in time and serves users on a LCFS-PR basis. We allow for general classes of user preferences and service time distributions. We do not allow users to leave the queue once they have arrived, and the system is open until all users have been served. Our main findings are the following: First, we provide a few results on the properties of equilibria in general. Second, we develop a constructive procedure that establishes the existence of a symmetric

¹ Haviv and Oz (2018) consider also other disciplines including LCFS-PR and random order with preemption.

² For the fluid model, it has been shown for varying degrees of random sorting, ranging from FCFS to a completely random service order, that the choice of service discipline does not play a role for the properties of social efficiency if the server is always open, (de Palma & Fosgerau, 2013).

mixed Nash equilibrium for any finite number of users. Third, we show that this is the unique symmetric equilibrium.³ Furthermore, we show that no (continuous) asymmetric equilibrium exists, if the population is of size two, or if arrival strategies have at most a finite number of inflection points. Using a numerical method based on the constructive procedure from the existence proof, we provide an example of a symmetric equilibrium as an illustration. We calculate the social efficiency of the resulting symmetric equilibrium and compare it to the social efficiency when users are served on a first-come first-served basis. The example shows that social efficiency under LCFS-PR may actually be lower than under FCFS when there are only a few users, in contrast to the case of a continuum of users mentioned above.

The paper is organized as follows: Section 2 formalizes the queueing environment and model assumptions. Section 3 defines the relevant notion of an equilibrium, presents the equilibrium properties of the queueing model, and in Sect. 3.3, provides the proof of existence and uniqueness of a symmetric equilibrium. In Sect. 3.4, we provide insights on the (non)-existence of asymmetric equilibria. Section 4 presents a numerical method to compute the symmetric equilibrium and in an example compares the resulting social efficiency with that obtained in a corresponding queueing system that employs the FCFS service discipline. We conclude the paper in Sect. 5 with a brief summary and future research directions. Proofs that require technical notation for the stochastic queueing dynamics are relegated to the Appendix.

2 Model

The queueing model of this paper resembles that analyzed in Breinbjerg (2017) under the FIFO discipline. A finite user population $N = \{1, \dots, n\}$, with $n \geq 2$, must obtain service by a single-server facility. The facility opens for service at time 0 and does not close until all users have been served. The facility serves one user at a time according to a work-conserving LCFS-PR regime. If several users arrive simultaneously, a fair lottery will determine the order of service among them. We assume that a user cannot queue up at the facility before opening time, and we assume that a user cannot leave the queue once arrived. Note that even if early arrivals were allowed, a rational user would never arrive before opening time under a LCFS-PR service discipline, since later arriving users will be prioritized once service starts. Therefore, the equilibrium strategy derived in the present paper would be the same if early arrivals were allowed. Nevertheless, we stick to the assumption arrivals before opening time are not allowed, since it matters for the FIFO discipline that we are comparing with in the example in Sect. 4.

Strategy of arrival Suppose that each user $i \in N$ independently arrives according to a cumulative distribution function F_i that assigns to each point in time t the probability that i has arrived by time t . We refer to F_i as a (*mixed*) *strategy*. Since users cannot show up before opening time we require $F_i(t) = 0$ for $t < 0$. Moreover we will assume for expositional simplicity that F_i is piecewise absolutely continuous. Let $\mathcal{S}(F_i)$ denote the support of F_i . Thus, if F_i has no jumps, $\mathcal{S}(F_i)$ is the smallest closed set such that $\int_{\mathcal{S}(F_i)} dF_i(t) = 1$. The collection of strategies of all users is given by the *arrival profile* $\mathcal{F} = \{F_i\}_{i \in N}$. The notation \mathcal{F}^{-i} will be used to denote the collection of strategies for all users except user i .

³ When interpreting a symmetric mixed strategy equilibrium, we do not necessarily expect users in real life queueing settings to fully randomize accordingly. Instead, the mixed strategy may be interpreted as a user's expectation about the arrival decisions of others. Empirical support for this interpretation is found in Rapoport et al. (2004) and Stein et al. (2007), who find that overall behaviour in a considered queueing game is represented well by the mixed equilibrium strategy, while it does not reflect individual behaviour.

Time of departure Given an arrival profile \mathcal{F} , we consider the probabilities associated with the time at which a given user has completed her service and departs the system. Assume that the (non-negative) amount of time s required for the facility to complete the service of each user is independently and identically distributed according to an absolutely continuous cumulative distribution function S with $S(s) = 0$ for $s \leq 0$ and its associated probability density function has finite moments.

Let D_i denote the ex-ante *cumulative departure time distribution* for user i induced by S and the LCFS-PR discipline, such that $D_i(d | t, \mathcal{F}^{-i})$ is the probability that user i has departed the system by time $d \in \mathbb{R}$, given that she arrived at time t , and the $n - 1$ other users arrive according to \mathcal{F}^{-i} . Note that $\lim_{d \rightarrow \infty} D_i(d | t, \mathcal{F}^{-i}) = 1$ for all t since the user population is finite, the service time distribution S has finite moments, and LCFS-PR is work-conserving. Note also that $D_i(d | t, \mathcal{F}^{-i}) = 0$ for all $d \leq t$.

Utility function. We assume that all users have identical preferences and that each user prefers early service to later service and dislikes spending time in the queue. To capture such preferences, let $V(t, d)$ be a real-valued function representing the utility of a user who arrives at time t and departs from the system at time $d \geq t$ after waiting in the queue and receiving service for a total of $d - t$ time units. We assume that V is continuous, bounded from above, strictly increasing in t , strictly decreasing in d , and for any $c > 0$ that $V(t, t + c)$ is strictly decreasing in t , with $\lim_{t \rightarrow \infty} V(t, t + c) = -\infty$.

We assume that every user aims to maximize her expected utility with respect to the timing of arrival. For a given collection of strategies \mathcal{F}^{-i} , let U_i denote the expected utility of user i who arrives at time t or according to F_i , when the $n - 1$ other users arrive according to \mathcal{F}^{-i} , i.e.

$$U_i(t, \mathcal{F}^{-i}) = \int_t^\infty V(t, d) dD_i(d | t, \mathcal{F}^{-i}) \quad (1)$$

Here \int is the Lebesgue integral over the cumulative departure time distribution D . If F_i has no jumps, the expected utility of arriving according to strategy F_i is given by

$$\hat{U}_i(F_i, \mathcal{F}^{-i}) = \int_0^\infty U_i(t, \mathcal{F}^{-i}) dF_i. \quad (2)$$

If F_i contains jumps, expected utility is defined by extending (2) in the straightforward way, i.e. where the utility at a jump is weighted with the associated point probability.⁴

A LCFS-PR queueing game is thus represented by a tuple $\mathcal{G} = \langle n, V, S \rangle$.

3 Equilibrium analysis

In this section, we start by defining the notion of an *equilibrium* and establish some general properties of equilibrium arrival profiles in Sect. 3.1. In Section 3.2, we present our main results in Theorem 1. The theorem establishes the existence and uniqueness of a symmetric Nash equilibrium as well as two general properties of such an equilibrium, for any finite number n of users. The proof of Theorem 1 is presented in Sect. 3.3. Subsequently in Section 3.4, we show that no asymmetric equilibrium exists if we impose certain regularity conditions on the arrival strategies of individuals or if $n = 2$.

⁴ For simplicity of exposition, and in the view of Lemma 2 below from which it follows that an equilibrium strategy is everywhere absolutely continuous, we do not write up the extended expression here.

3.1 General properties of equilibrium arrival profiles

To study the strategic arrivals of users in a queueing game \mathcal{G} , we adopt the standard Nash equilibrium concept and say that the arrival profile $\mathcal{F} = \{F_1, \dots, F_n\}$ constitutes an equilibrium, if it holds that no individual user can obtain higher expected utility (2) by changing her arrival strategy unilaterally. Since the expected utility for player i of arriving at time t must be the same for every t which adds to the increment of F_i ,⁵ we may alternatively characterize an equilibrium as follows:

Definition 1 The arrival profile $\mathcal{F} = \{F_1, \dots, F_n\}$ constitutes an equilibrium if for each $i \in N$, $s \geq 0$, and t for which $F(t) > F(w)$ for all $w < t$, that $U_i(t, \mathcal{F}^{-i}) \geq U_i(s, \mathcal{F}^{-i})$.

We start by establishing a result that links the cumulative departure time distribution D and expected utility U .

Lemma 1 Consider a queueing game \mathcal{G} , let $t \in \mathbb{R}$ and let \mathcal{F}^{-i} and $\tilde{\mathcal{F}}^{-i}$ be two distinct arrival profiles. If $D_i(d | t, \mathcal{F}^{-i}) \geq D_i(d | t, \tilde{\mathcal{F}}^{-i})$ for all $d \in \mathbb{R}$, then $U_i(t, \mathcal{F}^{-i}) \geq U_i(t, \tilde{\mathcal{F}}^{-i})$. Furthermore, if strict inequality holds for some d , then $U_i(t, \mathcal{F}^{-i}) > U_i(t, \tilde{\mathcal{F}}^{-i})$.

The lemma follows immediately from first order dominance once we note that the utility function V is monotonically decreasing in the departure time.

Next, we present two general properties that apply to any equilibrium. The first result addresses the continuity of the equilibrium strategies.

Lemma 2 Consider a queueing game \mathcal{G} , and let \mathcal{F} be an equilibrium arrival profile for \mathcal{G} . Then for every $F_i \in \mathcal{F}$ and $t > 0$, we have $F_i(t) = \lim_{s \uparrow t} F_i(s)$.

Proof Suppose for some $t > 0$ and some $i \in N$, that F_i has a point of upwards discontinuity, i.e. $F_i(t) > \lim_{s \uparrow t} F_i(s)$. Then, by the continuity of V , no other players will arrive at or immediately before t in equilibrium. That is, there exists an $\epsilon > 0$ such that none of the other players arrive in the interval $[t - \epsilon, t]$. Since $V(t - \epsilon, t - \epsilon + c) > V(t, t + c)$ for any $c > 0$, we have $U_i(t - \epsilon, \mathcal{F}^{-i}) > U_i(t, \mathcal{F}^{-i})$, i.e. i can increase her expected utility by arriving at $t - \epsilon$ instead of t . This, however, contradicts that \mathcal{F} is an equilibrium profile and proves that no equilibrium strategy can have a point of upwards discontinuity for $t > 0$. Therefore, $F_i(t) = \lim_{s \uparrow t} F_i(s)$ for all $t > 0$ and all $F_i \in \mathcal{F}$. \square

Since F_i is right-continuous by definition, it follows from Lemma 2 that any equilibrium strategy F_i is absolutely continuous at all $t \neq 0$. Thus, if \mathcal{F} is an equilibrium arrival profile, then no strategy in this profile contains jumps except for possibly at $t = 0$. In particular, if no strategy in the equilibrium profile has a jump in $t = 0$, by continuity of V , the expected utility for player i of arriving at time t must be the same for every t in the support of F_i .

The next result establishes that in equilibrium, the users arrive at the queueing system within some bounded interval of time.

Lemma 3 Consider a queueing game \mathcal{G} , and let \mathcal{F} be an equilibrium arrival profile for \mathcal{G} . Then $\mathcal{S}(F_i)$ is a compact set for all $F_i \in \mathcal{F}$.

Proof By definition we have $F_i(t) = 0$ for $t < 0$. Thus, the support $\mathcal{S}(F_i)$ of F_i is bounded from below at 0. Moreover, $\mathcal{S}(F_i)$ is also bounded from above. To see this, assume on the

⁵ If strategies are absolutely continuous, this holds for all t in the support of F_i . See the comment after Lemma 2.

contrary that $\inf\{t|F_i(t) = 1\} = \infty$ for some $F_i \in \mathcal{F}$. Now, since $D_i(d|t, \mathcal{F}^{-i}) = 0$ for all $d < t$ and for all $c > 0$ we have $\lim_{t \rightarrow \infty} V(t, t + c) = -\infty$, it follows that if \mathcal{F} represents an equilibrium, then $U_i(t, \mathcal{F}^{-i}) = -\infty$ for all $t \in \mathcal{S}(F_i)$. This, however, leads to a contradiction: Since the user population is finite, the service time distribution S has finite moments, and the LCFS-PR discipline is work-conserving, we must have $U_i(0, \mathcal{F}^{-i}) > -\infty$, a contradiction. The support $\mathcal{S}(F_i)$ is therefore bounded. Since $\mathcal{S}(F_i)$ is closed by definition, it follows immediately from the Heine–Borel theorem that $\mathcal{S}(F_i)$ is compact. \square

3.2 Symmetric equilibrium

We now consider a situation in which \mathcal{F} is a collection of strategies such that $F_i = F_j = F$ for all $F_i, F_j \in \mathcal{F}$. The main results are summarized in the theorem below in which existence and uniqueness of an equilibrium are established, and some general properties of the symmetric equilibrium strategy in a queueing game \mathcal{G} are presented.

Theorem 1 *For any queueing game \mathcal{G} , there exists one and only one strategy F that constitutes a symmetric equilibrium. Moreover, the following properties hold for F :*

- (i) $F(t)$ is continuous at all $t \in \mathbb{R}$ and has $F(0) = 0$.
- (ii) The support $\mathcal{S}(F)$ of F is a closed interval $[0, b]$, for some $b > 0$.

Intuitively speaking, Theorem 1 says that in equilibrium, the users will arrive according to a continuous and strictly increasing distribution function that extends over a finite interval of time starting at the opening time.

3.3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1 which proceeds through several lemmas. We start by noting that, provided that a symmetric equilibrium F exists, F cannot jump at 0 since otherwise a user arriving at 0 would be better off by arriving immediately after 0. The remaining element of Part (i) follows from Lemma 2. The next result addresses the monotonicity of an equilibrium strategy.

Lemma 4 *Consider a queueing game \mathcal{G} , and let F be an equilibrium strategy for \mathcal{G} . Then $\mathcal{S}(F)$ is a connected set.*

Proof Since F is an equilibrium strategy, it follows from Lemma 3 that F has a bounded support $\mathcal{S}(F)$ with supremum $0 < b < \infty$, and it follows from Lemma 2 that F is continuous and, as argued above, $F(0) = 0$. Moreover, since F has no jumps in a symmetric equilibrium, the expected utility of a user i gets the same expected utility from any t in the support of F . This observation is used in some of proofs that follow.

Now, suppose that $\mathcal{S}(F)$ is not a connected set, implying that $\mathcal{S}(F)$ can be covered by the union of two disjoint nonempty open subsets. This implies that there exists an interval $0 \leq t_1 < t_2 \leq b$, with $t_1, t_2 \in \mathcal{S}(F)$ such that $F(t_1) = F(t_2)$. However this leads to a contradiction of the equilibrium definition since $U_i(t_1, F) > U_i(t_2, F)$. To see this, note that any user who arrives at time t_1 will start service instantaneously according to the LCFS-PR service discipline. Since no other users arrive in the time interval $[t_1, t_2]$, and V is strictly decreasing in departure time, it follows that $U_i(t_1, F) > U_i(t_2, F)$. Hence, a strategy F with a support $\mathcal{S}(F)$ that is not a connected set cannot be an equilibrium strategy. \square

Provided that an equilibrium F exists, Part (ii) of Theorem 1 now follows immediately from Lemmas 3 and 4. We next address the existence of an equilibrium strategy for an arbitrary queueing game \mathcal{G} .

Lemma 5 *For any queueing game \mathcal{G} , there exists a strategy F that constitutes a symmetric equilibrium.*

Proof We constructively prove this claim by defining a family of cumulative distribution functions $\{X_b\}_{0 < b < \infty}$, where $X_b(s) = 1$ for all $s \geq b$. We then show that there exists a member of the family $\{X_b\}$ such that the arrival profile $\mathcal{F} = \{X_b, \dots, X_b\}$ is a symmetric equilibrium. We will abuse notation and let $U_i(t, X_b)$ denote the expected utility of arriving at time t , when everyone else arrives according to X_b , i.e., when $\mathcal{F}^{-i} = \{X_b, \dots, X_b\}$. Specifically, we will show that there is a b such that: $U_i(t, X_b) = U_i(s, X_b)$ for all $s, t \in [0, b]$, and $U_i(b, X_b) \geq U_i(q, X_b)$ for all $q \geq b$. Note that any X_b satisfying this criterion will also satisfy the criteria of the equilibrium definition (Definition 1).

From the point of view of user i , we are going to think of b as the earliest point in time, where the $n - 1$ other users have already arrived at the system with certainty. Therefore, if the other $n - 1$ users arrive according to the strategy X_b , then the remaining user i can arrive at time b and start service instantaneously without being preempted, thus obtaining an expected utility of $U_i(b, X_b)$.

We construct the cumulative distribution function X_b as the limit of a convergent and recursive sequence of cumulative distribution functions, $\{X_{b,h} \mid 0 < b < \infty\}_{h \in \mathbb{N}}$, indexed by the non-negative integer h , to be defined in what follows.

For a given $0 < b < \infty$ and $h \in \mathbb{N}$, let $X_{b,h} : [0, \infty) \rightarrow [0, 1]$ be a function where $X_{b,h}(s) = 1$ for all $s \geq b$. In order to define the recursive sequence $\{X_{b,h} \mid 0 < b < \infty\}_{h \in \mathbb{N}}$, we start by introducing some notation.

As before, $U_i(t, X)$ denotes the expected utility for user i of arriving at time t , when everyone else arrives according to a cdf X . Now, for time t and $x \in [0, X(t)]$, we let $\tilde{U}_i(t, X, x)$ denote user i 's expected utility of arriving at t , when the arrival strategy of each of the other $n - 1$ users follows X except that: their probability of having arrived before t is $X(t) - x$, their probability of arriving exactly at t is x , and when arriving at t they will be prioritized before i .⁶ Since we consider only symmetric equilibria, we will for ease of exposition simply denote U_i by U (and \tilde{U}_i by \tilde{U}) in the remainder of this proof.

Next, we define the starting function $X_{b,0}$ such that we ensure that $U(t, X_{b,0}) \geq U(b, X_b)$ for all $t < b$. To achieve this, let I be the cdf defined by $I(t) = 1$ if $t \geq 0$ and $I(t) = 0$ otherwise. We will then consider $\tilde{U}(t, I, x)$ i.e. the expected utility of arriving at time t , when the $n - 1$ other users have arrived before user i with probability $1 - x$, and they arrive at t and be prioritized before i with probability x . Note that $\tilde{U}(t, I, x)$ is strictly decreasing in x . In particular, the higher the probability of other users arriving at t and being prioritized before i , the longer user i is expected to wait in line before service completion.

By varying the probability of the $n - 1$ other users arriving at time t and being prioritized before i , we can determine the maximal jump, x , such that the expected utility, $\tilde{U}(t, I, x)$, is as least as great as the expected utility, $U(b, X_b)$, of arriving at time b and being serviced immediately. We denote this maximal x by $x_{b,0}^t$ and thus define it as:

$$x_{b,0}^t = \max \left\{ x \in [0, 1] \mid \tilde{U}(t, 1, x) \geq U(b, X_b) \right\}. \tag{3}$$

⁶ Note that this corresponds to a situation in which user i arrives at t , and an expected share x of the $n - 1$ other users arrive immediately after t , and all users are serviced on a LCFS-PR basis.

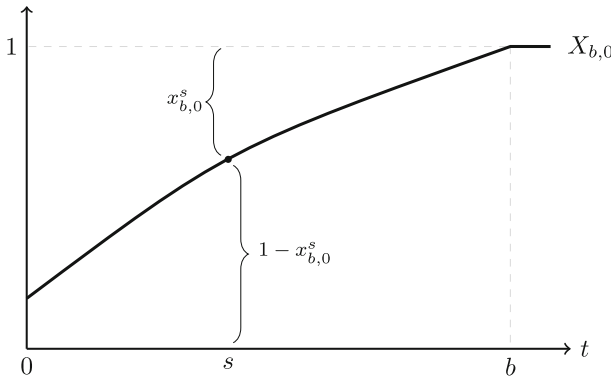


Fig. 1 Example of $X_{b,0}$: The constant b is an arbitrary fixed point in time for which $X_{b,0}(t) = 1$ for all $t \geq b$. $X_{b,0}$ is continuous and strictly increasing over the time interval $[0, b]$

Note that, since the expected utility is strictly decreasing in d , the utility of arriving at $0 \leq t < b$ and being serviced immediately with certainty, is always greater than $U(b, X_b)$. Thus, since $x = 0$ corresponds to the situation where a given user arriving at t is serviced immediately upon arrival, we must have $x_{b,0}^t \geq 0$ for all $0 \leq t \leq b$.

Next, by defining the maximal size of this jump for each t , we can construct the function $X_{b,0}$ such that at every point in time $0 < t < b$, the cdf value is given by $1 - x_{b,0}^t$. Thus, we define the starting cdf $X_{b,0}$ of the sequence of recursive functions $X_{b,0}, X_{b,1}, X_{b,2}, \dots$ as follows:

$$X_{b,0}(t) = \begin{cases} 1 - x_{b,0}^t & \text{for } t \in [0, b) \\ 1 & \text{for } t \geq b. \end{cases} \tag{4}$$

Figure 1 graphically illustrates an example of $X_{b,0}$.

Now, since $\tilde{U}(t, I, x) \geq U(b, X_b)$, it must also hold that $U(t, X_{b,0}) \geq U(b, X_b)$. To see this, note that for a given user i , it must be that in the former situation, the probability of another user arriving immediately after i is $x_{b,0}^t$, whereas in the latter case when the others arrive according to $X_{b,0}$, the probability of another user arriving immediately after i is lower, and the arrival strategy will be spread over a larger interval of time. Therefore, the probability of departing the system at any given time is as least as great under $X_{b,0}$, and it follows from Lemma 1 that $U(t, X_{b,0}) \geq \tilde{U}(t, I, x)$. By construction, it therefore follows that $U(t, X_{b,0}) \geq U(b, X_b)$.

Note that if we let $A_0(b)$ denote the latest point in time $t > 0$ such that $X_{b,0}(t) = 0$, if such a t exists, and otherwise let $A_0(b) = 0$, then since the utility function V is strictly decreasing in departure time, it follows by construction that $X_{b,0}$ is strictly increasing over the time interval $[A_0(b), b]$ and non-decreasing over the interval $[0, b]$.

Next, we move on to characterize the recursive statement of $X_{b,h}$ for each $h > 0$ in a similar fashion. First, suppose that $X_{b,h-1}$ has been defined for $h > 0$. Consider a point in time t , $0 \leq t < b$, and let $\tilde{U}(t, X_{b,h-1}, x)$ be the expected utility of a user that arrives at t , when the probability of each of the $n - 1$ other users having already arrived is $X_{b,h-1}(t) - x$, and the probability of each of the other users arriving exactly at time t and being prioritized for service over i is x .

We denote by $x_{b,h}^t$ (where $x_{b,h}^t \leq X_{b,h-1}(t)$) the maximal probability of each of the $n - 1$ other users arriving at time t (the maximal jump at t), when the expected utility $\tilde{U}(t, X_{b,h-1}, x)$ for user i of arriving at t must be at least as high as the expected utility from

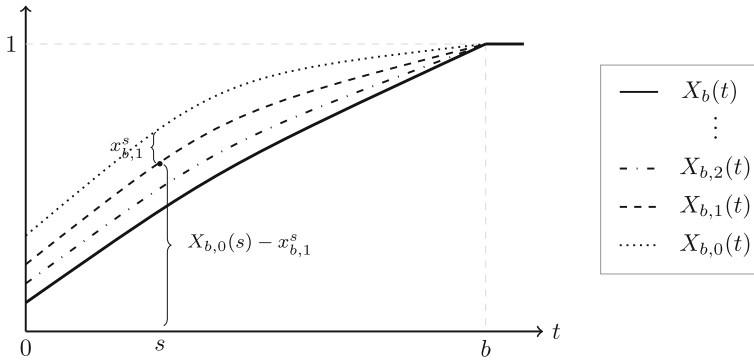


Fig. 2 Example of a recursive sequence $X_{b,0}, X_{b,1}, \dots$: As the number of iterations h increases, the h th recursively stated term $X_{b,h}$ converges towards the limit X_b . Note that X_b is continuous and strictly increasing over $[A(b), b]$, where $A(b) = 0$ in this particular case

arriving at time b and being serviced immediately. The maximal jump is thus defined as:

$$x_{b,h}^t = \max \left\{ x \in [0, X_{b,h-1}(t)] \mid \tilde{U}(t, X_{b,h-1}, x) \geq U(b, X_b) \right\} \tag{5}$$

We define $X_{b,h}(t)$ as follows:

$$X_{b,h}(t) = \begin{cases} X_{b,h-1}(t) - x_{b,h}^t & \text{for } t \in [0, b) \\ 1 & \text{for } t \geq b. \end{cases} \tag{6}$$

As in the previous section, this construction ensures that $U(t, X_{b,h}) \geq \tilde{U}(t, X_{b,h-1}, x) \geq U(b, X_b)$. Note also that if we let $A_h(b)$ denote the latest point in time $t > 0$ where $X_{b,h}(t) = 0$, if such a t exists and otherwise, let $A_h(b) = 0$, then $X_{b,h}$ is, by construction, strictly increasing over the time interval $[A_h(b), b]$.

The recursive process yields the sequence $X_{b,0}, X_{b,1}, X_{b,2}, \dots$ which is bounded and monotonically decreasing with $X_{b,0}(t) \geq X_{b,1}(t) \geq \dots$ over $h \in \mathbb{N}$ and for all $t \in \mathbb{R}$. It thus follows by the monotone convergence theorem that the sequence is convergent. Let $X_b(t) = \lim_{h \rightarrow \infty} X_{b,h}(t)$ denote the limit of the sequence at each t , and let $A(b) = \lim_{h \rightarrow \infty} A_h(b)$. Figure 2 graphically illustrates an example of a recursive sequence $X_{b,0}, X_{b,1}, \dots$ that converges towards the limit X_b .

So far b has been fixed. We now define a family of functions $\{X_b\}_{0 < b < \infty}$ such that for each b , X_b is the limit of the convergent and recursive sequence $\{X_{b,h} \mid 0 < b < \infty\}_{h \in \mathbb{N}}$. For each member of $\{X_b\}$, we examine whether it represents an equilibrium strategy. First, we note that X_b is by construction a cumulative distribution function for any $0 < b < \infty$. Second, note that since X_b must satisfy the criteria that $U_i(t, X_b) \geq U_i(s, X_b)$ for all $s, t \in [0, b]$, we must have $U(0, X_b) = U(A(b), X_b)$. However, this only holds for values of b such that $X_b(0) = 0$ and $A(b) = 0$. The former follows from Lemma 2. To see the latter, note that if $A(b) > 0$, then $X_b(t) = 0$ for $t \in [0, A(b)]$. Then, since no other users arrive in the interval from 0 to $A(b)$, a given user can arrive at $t = 0$ and be serviced immediately without risk of being preempted before time $A(b)$. Therefore, $U(0, X_b) > U(A(b), X_b)$.

We make the following observations:

- (i) For b sufficiently close to 0, $X_b(0) > 0$, implying $A(b) = 0$.

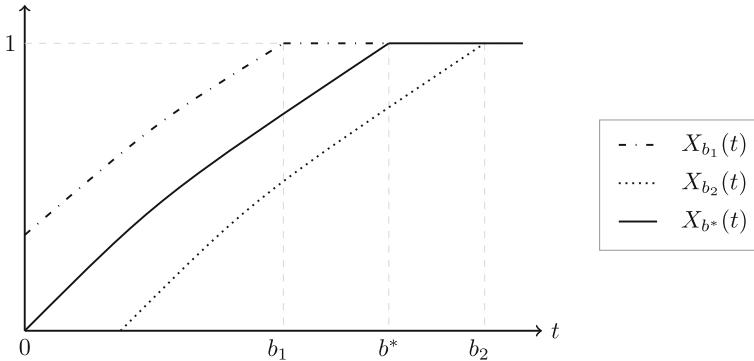


Fig. 3 Example of $\{X_b\}_{0 < b < \infty}$: All members of $\{X_b\}$ for which $b \neq b^*$ yields $X_b(0) > 0$ or $A(b) > 0$. Only the member of $\{X_b\}$ for which $b = b^*$ represents an equilibrium strategy as $X_{b^*}(0) = 0$ and $A(b^*) = 0$

- (ii) For b sufficiently close to ∞ , $A(b) > 0$, and $X_b(t) = 0$ for $t \in [0, A(b)]$.
- (iii) $X_b(0)$ and $A(b)$ are continuous at all b and are monotonically decreasing and monotonically increasing, respectively.

Combining (i), (ii) and (iii), there must exist $b = b^*$ such that $X_{b^*}(0) = 0$ with $A(b^*) = 0$. Finally, we observe that it follows from the construction that X_{b^*} has no jumps and it is limited in how fast it can change, i.e. it is Lipschitz continuous and thus absolutely continuous. It therefore follows that X_{b^*} represents an equilibrium strategy for a symmetric equilibrium. Figure 3 graphically illustrates an example of such X_{b^*} □

We next address the uniqueness of an equilibrium strategy.

Lemma 6 *For any queueing game \mathcal{G} , there exists at most one strategy F that constitutes a symmetric equilibrium.*

Proof We prove this by contradiction. Let F and \tilde{F} be two distinct symmetric equilibrium strategies such that $F \neq \tilde{F}$. Let $b = \min\{t \mid F(t) = 1\}$ and $\tilde{b} = \min\{t \mid \tilde{F}(t) = 1\}$. It then follows from Theorem 1 that both are strictly increasing with supports $[0, b]$ and $[0, \tilde{b}]$, respectively. We distinguish between three cases:

$b < \tilde{b}$: It immediately follows that $U(t, F) > U(t, \tilde{F})$ for all $t \in [0, b]$. Let $s = \max\{t \mid F(t) = \tilde{F}(t), 0 \leq t < b\}$ be the latest point in time at which the two strategies intersect. Note that s exists and is uniquely determined since F and \tilde{F} are continuous, $F(0) = \tilde{F}(0)$, and $b < \tilde{b}$. It then follows that the expected share of users arriving from time s up until time b is strictly larger under F than under \tilde{F} . Therefore, $D(d \mid s, F) \leq D(d \mid s, \tilde{F})$ for all d with strict inequality at some d , implying that $U(s, F) < U(s, \tilde{F})$ by Lemma 1. This contradicts the assumption that F provides higher expected utility than \tilde{F} and proves that F and \tilde{F} cannot both be equilibrium strategies.

$b > \tilde{b}$: The case is symmetric to that of $b < \tilde{b}$ and thus omitted.

$b = \tilde{b}$: In this case, it immediately follows that $U(t, F) = U(t, \tilde{F}) = U(b, F)$ for all $t \in [0, b]$. Let F be the symmetric equilibrium strategy constructed from the procedure in Lemma 5 and recall that $F(0) = \tilde{F}(0) = 0$. Then by construction, F first-order stochastically dominates \tilde{F} in the sense that $F(t) \geq \tilde{F}(t)$ for all $t \in [0, b]$ with strict inequality at some t . To see this recall that $F(t) = \lim_{h \rightarrow \infty} X_{b,h}(t)$ for

the recursive sequence $X_{b,0}, X_{b,1}, \dots, X_{b,h}$ with $X_{b,0}(t) \geq X_{b,1}(t) \geq X_{b,2}(t), \dots$. From the definition of $X_{b,0}$, it immediately follows that $\tilde{F}(t) \leq X_{b,0}(t)$ for every $t \in [0, b]$, since otherwise, $U(t, \tilde{F}) > U(b, \tilde{F})$. Next, to show that $\tilde{F}(t) \leq X_{b,1}(t)$, assume on the contrary that $X_{b,0}(s) > \tilde{F}(s) > X_{b,1}(s)$ for some $s \in]0, b[$. Let $\tilde{U}(s, X_{b,0}, X_{b,0} - X_{b,1})$ denote the expected utility of arriving at s , when the probability of another user having already arrived is $X_{b,1}(s)$, and the probability for each of the other $n - 1$ users of arriving immediately after s is $X_{b,0}(s) - X_{b,1}(s)$. Then it follows from the procedure that $\tilde{U}(s, X_{b,0}, X_{b,0} - X_{b,1}) \geq U(b, F) = U(b, \tilde{F})$. However, since $\tilde{F}(s) > X_{b,1}(s)$, and $X_{b,0}(t) > \tilde{F}(t)$ for all $t \in [0, b]$, it follows that under \tilde{F} , fewer users will arrive after s , and they will arrive at a slower rate than under F . Therefore, $U(s, \tilde{F}) > \tilde{U}(s, X_{b,0}, X_{b,0} - X_{b,1}) \geq U(b, \tilde{F})$, which contradicts that \tilde{F} is an equilibrium strategy. Thus, $\tilde{F}(t) \leq X_{b,1}(t)$ for all $t \in [0, b]$. Recursively applying this argument for each element in the sequence, we arrive at the desired result. It now follows that $D(d \mid 0, F) \leq D(d \mid 0, \tilde{F})$ for all d with strict inequality at some d , implying $U(0, F) < U(0, \tilde{F})$. This contradicts that $U(t, F) = U(t, \tilde{F})$ for all $t \in [0, b]$.

To conclude, there cannot exist two distinct strategies F and \tilde{F} with $F \neq \tilde{F}$ that both constitute a symmetric equilibrium. □

Lemmas 5 and 6 complete the proof of Theorem 1.

3.4 Asymmetric equilibria

Let \mathcal{F} be an equilibrium arrival profile. Then we say that \mathcal{F} is an *asymmetric* equilibrium, if there exists $F_i, F_j \in \mathcal{F}$ such that $F_i \neq F_j$. In this section, we show that no continuous asymmetric equilibrium exists, if $n = 2$, or if arrival strategies cannot have an infinite number of inflection points.

Theorem 2 *Let $\mathcal{G} = \langle n, V, S \rangle$ be a queueing game. If $n = 2$, then no continuous asymmetric equilibrium exists.*

Proof To prove the theorem by contradiction, assume that \mathcal{F} is an asymmetric equilibrium for some queueing game \mathcal{G} with $N = \{1, 2\}$ and where F_1 and F_2 are continuous. It follows from the equilibrium definition and continuity that $U_i(s, \mathcal{F}^{-i}) = U_i(t, \mathcal{F}^{-i})$ for all $s, t \in \mathcal{S}(F_i)$ and $i \in \{1, 2\}$. Next, we show that $\mathcal{S}(F_1) = \mathcal{S}(F_2)$. To arrive at a contradiction, let $b = \max\{b_1, b_2\}$ and assume that there exists an $s \in]0, b[$ and an $\epsilon > 0$ such that a) $[s, s + \epsilon] \subset \mathcal{S}(F_1)$, $[s, s + \epsilon] \subset [0, b] \setminus \mathcal{S}(F_2)$, or b) $[s, s + \epsilon] \subset [0, b] \setminus \mathcal{S}(F_1)$, $[s, s + \epsilon] \subset \mathcal{S}(F_2)$.

Consider case a). Since there is no risk of user 2 arriving in the time interval $[s, s + \epsilon]$ it follows that $D_1(d \mid s, F_2) \geq D_1(d \mid s + \epsilon, F_2)$ for all d with strict inequality at some d , which then (due to Lemma 1) implies that $U_1(s, F_2) > U_1(s + \epsilon, F_2)$. This contradicts that F_1 is an equilibrium strategy. A symmetric argument applies to case b). Therefore, we must have $b_1 = b_2 = b$ and $\mathcal{S}(F_1) = \mathcal{S}(F_2) = [0, b]$, which in turn implies $U_1(t, F_2) = U_2(t, F_1) = U_1(b, F_2)$, for all $t \in [0, b]$.

It remains to be shown that $F_1 = F_2$. Since $U_1(t, F_2) = U_1(b, F_2)$, for all $t \in [0, b]$, the arrival strategy F_2 is also a best response to F_2 , implying that $\mathcal{F}' = \{F_2, F_2\}$ must be a symmetric equilibrium strategy. However, applying the symmetric argument for F_1 implies that $\mathcal{F}'' = \{F_1, F_1\}$ is also a symmetric equilibrium, thereby contradicting Theorem 1. □

Next, if we allow for any finite number of users but restrict the possible set of strategies to include only cdf which are continuous and that cannot cross each other an infinite number of

times, then again no asymmetric equilibria exist. This means that although we cannot rule out existence of asymmetric equilibria in general, these must involve either jump discontinuities or more than two users and highly irregular strategy functions.

Theorem 3 *Let $\mathcal{G} = \langle n, V, S \rangle$ be a queueing game, and restrict the set of admissible strategies to those that are continuous and piecewise concave or convex with at most a finite number of inflection points. Then no asymmetric equilibrium exists.*

Proof To prove the theorem by contradiction, assume that \mathcal{F} is an asymmetric equilibrium for some queueing game \mathcal{G} , and assume that every $F_i \in \mathcal{F}$ has a finite number of inflection points. As before, it follows from the equilibrium definition that $U_i(s, \mathcal{F}^{-i}) = U_i(t, \mathcal{F}^{-i})$ for all $s, t \in \mathcal{S}(F_i)$ and all $i \in N$. Furthermore, the expected utility in equilibrium must be the same for all users. To see this, let $b_i = \min\{t | F_i(t) = 1\}$ for all $i \in N$ and assume on the contrary that there exists a pair of users $i, j \in N$, such that $U_j(b_j, \mathcal{F}^{-j}) > U_i(b_i, \mathcal{F}^{-i})$. Then there exists an $\epsilon > 0$ sufficiently small such that $U_i(b_j + \epsilon, \mathcal{F}^{-i}) > U_i(b_i, \mathcal{F}^{-i})$, contradicting equilibrium.

From Lemma 3, we know that there exists an earliest point in time $b = \max_i b_i$, such that all n users have arrived with certainty. Let $N_b \subseteq N$ be the set of users i for whom $b_i = b$. Furthermore, let $s < b$ be the earliest point in time such that all users in $N \setminus N_b$ have arrived with certainty. Next, consider two users i, j such that $i \in N_b, j \in N \setminus N_b$. Then $b_i = b$ and $b_j = s$. Now there exists $0 < \epsilon < b - s$ such that $U_i(s + \epsilon, \mathcal{F}^{-i}) > U_j(s, \mathcal{F}^{-j}) = U_i(b, \mathcal{F}^{-i})$, where the inequality follows since i by arriving immediately after s preempts any user currently in service and furthermore faces a lower risk of being preempted herself, since one user less can potentially arrive in the time interval from s to b , thereby contradicting equilibrium. Thus, for \mathcal{F} to be an equilibrium arrival profile, the upper bound of $\mathcal{S}(F_i)$ must be the same for all $i \in N$. That is, there exists a b such that $b_i = b$ for all $i \in N$.

It remains to be shown that $F_i = F_j$ for all pair of users $i, j \in N$. To prove this by contradiction, consider $i, j \in N$ with $F_i \neq F_j$, and let \underline{t} be the latest point in time such that the two strategies cross and such that there exists an $s > \underline{t}$ where the strategies differ. Furthermore, let $\bar{t} > \underline{t}$ denote the earliest point in time where the two strategies cross after \underline{t} . Since $F_i(0) = F_j(0) = 0, F_i(b) = F_j(b) = 1, F_i \neq F_j$, and each arrival strategy has only finitely many inflection points, we know that \underline{t} and \bar{t} exist. Then, one of the following cases must hold:

- (i) $F_i(t) \geq F_j(t)$ for all $t \in [\underline{t}, b]$, with strict inequality for $t \in]\underline{t}, \bar{t}[$
- (ii) $F_i(t) \leq F_j(t)$ for all $t \in [\underline{t}, b]$, with strict inequality for $t \in]\underline{t}, \bar{t}[$

For case (i), we distinguish between two subcases depending on whether there exists an $s, \underline{t} < s < \bar{t}$ such that $s \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$:

- (a) There exists $s, \underline{t} < s < \bar{t}$ such that $s \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$. Since $F_i(b) - F_i(t) \leq F_j(b) - F_j(t)$ for all $t \in [s, b]$ with strict inequality at some t , it follows that $D_i(d | s, \mathcal{F}^{-i}) \leq D_j(d | s, \mathcal{F}^{-j})$ for all $d > s$, with strict inequality at some d . Hence $U_i(s, \mathcal{F}^{-i}) < U_j(s, \mathcal{F}^{-j})$, and since $s \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$, this contradicts that $U_i(t, \mathcal{F}^{-i}) = U_j(t, \mathcal{F}^{-j})$ for all $t \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$ and all i, j .
- (b) $s \notin \mathcal{S}(F_i)$ or $s \notin \mathcal{S}(F_j)$ for all $s, \underline{t} < s < \bar{t}$. Then there must exist an $\epsilon > 0$ such that $t \notin \mathcal{S}(F_i)$ for all $t \in]\bar{t} - \epsilon, \bar{t}[$ whereas $\bar{t} - \epsilon \in \mathcal{S}(F_i)$. Furthermore, there exists $\epsilon' < \epsilon$ such that $[\bar{t} - \epsilon', \bar{t}[\subset \mathcal{S}(F_j)$. Together, this implies that there exists $\delta > 0$ (sufficiently small) such that $U_j(\bar{t} - \epsilon + \delta, \mathcal{F}^{-j}) > U_i(\bar{t} - \epsilon, \mathcal{F}^{-i})$. To see this note first that by arriving at $\bar{t} - \epsilon + \delta$, user j will preempt any service progress of user i . Second, since $\mathcal{F}^{-i} = (\mathcal{F}^{-\{i,j\}}, F_j)$ and $\mathcal{F}^{-j} = (\mathcal{F}^{-\{i,j\}}, F_i)$, users i and j will face the

same arrival patterns of users in $N \setminus \{i, j\}$, but whereas j does not risk user i arriving in the interval $]\bar{t} - \epsilon, \bar{t}[$, user i faces a positive risk of j arriving before \bar{t} . Therefore, $U_j(\bar{t} - \epsilon + \delta, \mathcal{F}^{-j}) > U_i(\bar{t} - \epsilon, \mathcal{F}^{-i}) = U_j(\bar{t} - \epsilon', \mathcal{F}^{-j})$, where the equality holds since $\bar{t} - \epsilon \in \mathcal{S}(F_i)$ and $\bar{t} - \epsilon' \in \mathcal{S}(F_j)$. This, however, contradicts that F_j is an equilibrium strategy.

Symmetric arguments holds for case (ii), thereby proving that no asymmetric equilibrium exists. □

4 Computational approach

This section develops an approach to numerically compute a symmetric equilibrium strategy for a queueing game \mathcal{G} under exponential service times and provides an illustrative example (Sect. 4.1). We subsequently compute the social efficiency of the computed equilibrium example and compare it to the social efficiency obtained in Breinbjerg (2017), where users are served on a FCFS basis and cannot arrive before opening time (Sect. 4.2). We restrict attention to games with two users and exponential service times, due to its tractable numerical solutions for the cumulative departure time distribution. Haviv and Oz (2018) consider a model where users have a linear waiting cost and no tardiness cost, and show that for any game with two users and exponential service times, the resulting symmetric equilibrium of a processor sharing discipline yields socially optimal strategies.

We start by deriving an expression for the departure time distribution D in this two-user LCFS-PR queueing game.

Lemma 7 *Consider a queueing game \mathcal{G} in which $n = 2$, and S is independently, identically and exponentially distributed. Let F be a strategy with $b = \min\{t \mid F(t) = 1\} < \infty$, and let I_a denote the size of a jump discontinuity of F at point a . Then the cumulative departure time distribution D can be expressed as*

$$\begin{aligned}
 D(d \mid t, F) &= \sum_{a \leq t} I_a G(d - t; \mu) + \int_0^t f_+(a) G(d - t; \mu) da \\
 &+ \sum_{t < a < b} I_a (G(a - t; \mu) G(d - t; \mu) + (1 - G(a - t; \mu)) H(d - a; 2, \mu)) \\
 &+ \int_t^b f_+(a) (G(a - t; \mu) G(d - t; \mu) + (1 - G(a - t; \mu)) H(d - a; 2, \mu)) da
 \end{aligned}$$

for each $t \geq 0$, where

G is the exponential cumulative distribution function, and H is the cumulative distribution function of the Erlang distribution, i.e.:

$$\begin{aligned}
 G(x; \mu) &= \begin{cases} 1 - e^{-\mu x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \\
 H(x; k, \mu) &= \begin{cases} 1 - \sum_{m=0}^{k-1} \frac{1}{m!} e^{-\mu x} (\mu x)^m & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}
 \end{aligned}$$

for any $x \in \mathbb{R}$.

The proof is postponed to Appendix A.1 as it requires additional notation to describe the stochastic queueing processes. Specifically, we prove Lemma 7 by defining some basic

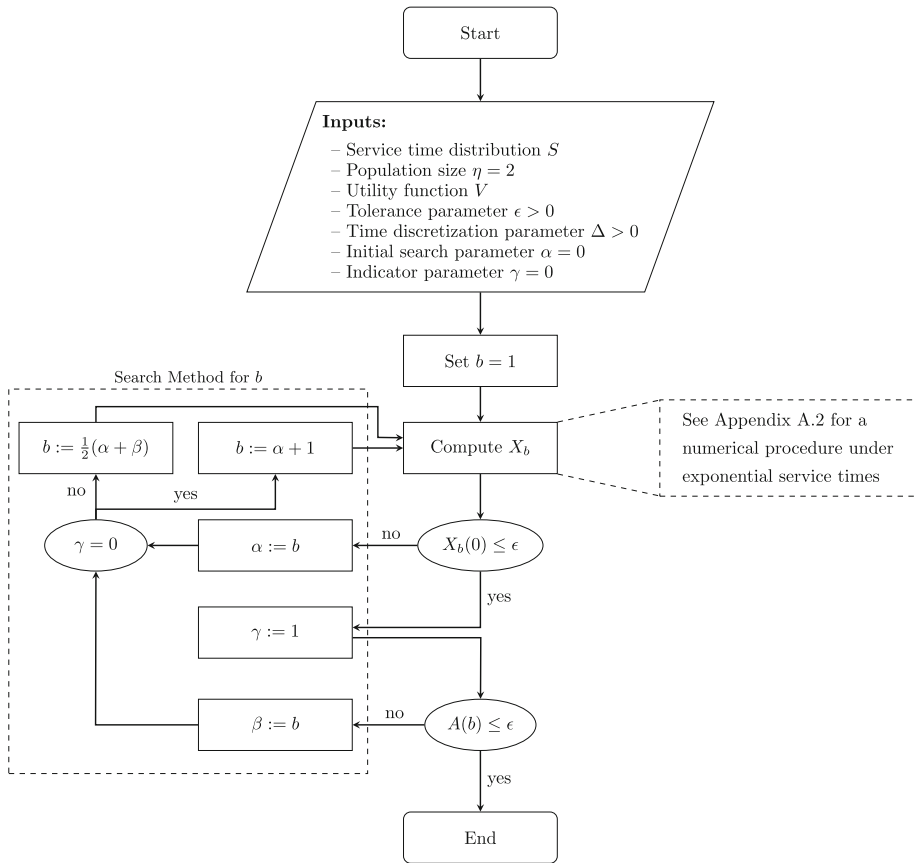


Fig. 4 Flowchart of the numerical procedure: Each geometric shape represents an action within the method. That is, the rounded squares are the start and ending, the trapezium is the exogenous inputs, the squares are steps in the process, and circles are binary decisions (yes/no) based on a question. The arrows indicate the flow from one action to another. Note that := is the assignment operator that changes the value of an existing variable

queueing relations for our system and deriving the cumulative departure time process using sample-path techniques. Such a sample-path approach is commonly used in the queueing literature to describe the transient states of a queueing system. As an example, Juneja and Shimkin (2013) use a similar sample-path approach to derive relevant queueing relations for a corresponding queueing game where users are served on a FCFS basis.

4.1 Numerical procedure and an example

We now present a numerical method which can be used to compute the symmetric equilibrium strategy in practice. The method is a discretized variant of the constructive proof of Lemma 5. Figure 4 depicts a flowchart of the general numerical procedure. For a given set of inputs, the method performs a search for the value b that induces a function X_b which constitutes an equilibrium strategy. Note that the number of required iterations for the search of b to converge is a function of the tolerance parameter ϵ . For any equilibrium strategy with $b \neq 1$,

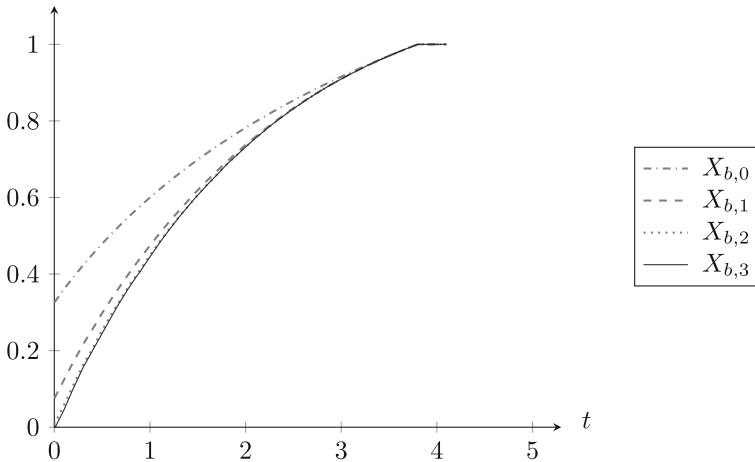


Fig. 5 Numerically computed equilibrium strategy: An approximated symmetric equilibrium in the queueing game \mathcal{G} , where: $n = 2$; $V(t, d) = -d^{0.5}(d-t)^{0.8}$; S is identical, independently and exponentially distributed with rate $\mu = 1$; $\Delta = 0.1$; and $\epsilon = 0.01$. The function $X_{b,3}$ represents an equilibrium strategy in the sense that it approximates the convergent limit of the recursive sequence $\{X_{b,h}\}_{h \in \mathbb{N}}$ with respect to the tolerance parameter ϵ

the search method (which combines a linear and binary search) requires multiple, and possibly many, iterations of b before convergence.

Next, we apply the numerical procedure to compute an equilibrium strategy for a population of size two under exponential service times and for a specific utility function. Figure 5 depicts the equilibrium strategy for the utility function $V(t, d) = -d^{0.5}(d-t)^{0.8}$. The figure illustrates a recursive sequence $\{X_{b,h}\}_{h \in \mathbb{N}}$ for $b = 3.8$ that represents an equilibrium strategy. Intuitively speaking, the symmetric equilibrium prescribes a strategy such that each user arrives according to a continuous and strictly increasing distribution function that extends over the interval from the opening time and up until time 3.8.

4.2 Social inefficiency

We measure the social inefficiency of the equilibrium under LCFS-PR by comparing the aggregate expected utility in the Nash equilibrium to that of a (first-best) socially optimal solution in which there is no waiting time or idle time. That is, similarly to Juneja and Shimkin (2013), we consider the case where the central planner is able to schedule arrivals based on observed service completions.⁷ For the considered queueing game with only two users, the socially optimal solution is one in which one user starts service at time 0, and the other starts service immediately after the departure of the first user with no idleness at the server. Formally, let \mathbf{W} denote the (random) sum of the two users' utilities in the socially optimal solution. Let \mathbf{S}_1 and \mathbf{S}_2 be the (random) independent, identically and exponentially distributed service time requirements of the users. The expected value of \mathbf{W} conditional on

⁷ Another plausible option is to assume that the arrival times must be prescheduled, with no feedback on service completions. However, finding the socially optimal schedule for this problem is a hard global optimization problem and can typically only be solved using heuristic or approximation algorithms. See the related discussion by Juneja and Shimkin (2013) for their model.

Table 1 Equilibrium utility and social inefficiency of LCFS-PR and FCFS

	U^*	$E[\mathbf{W}]$	$\frac{2U^*}{E[\mathbf{W}]}$
LCFS-PR	-2190	-2.129	2057
FCFS	-1925	-2.129	1808

Approximated values for a queueing game with $n = 2$, $V(t, d) = -d^{0.5}(d - t)^{0.8}$, S exponentially distributed with rate $\mu = 1$, and $\Delta = 0.1$, $\epsilon = 0.01$ under the LCFS-PR and FCFS service discipline, respectively

S_1 and S_2 is then given by

$$E[\mathbf{W} \mid S_1, S_2] = V(0, S_1) + V(S_1, S_1 + S_2),$$

where E is the expectation operator. Note that the sum of two independent and identically exponentially distributed variables follows an Erlang distribution with shape 2. Let $g(x; \mu)$ and $h(x; \mu)$ denote the density function at x for the exponential and Erlang distribution with rate μ and shape 2, respectively. Then the total expected utility for the socially optimal solution is given by

$$E[\mathbf{W}] = \int_0^\infty \int_0^\infty (V(0, s) + V(s, z)) g(s; \mu) h(z; \mu) ds dz. \tag{7}$$

Let U^* denote the expected utility for any of the users as induced by the equilibrium strategy. Then, we may consider the ratio between the total expected utility in equilibrium and in the socially optimal solution as a measure of the social inefficiency of the equilibrium solution:

$$\frac{2U^*}{E[\mathbf{W}]} \tag{8}$$

Table 1 reports the approximated value of this ratio in the specific queueing game considered in Figure 5, for which $n = 2$, the utility function is given by $V(t, d) = -d^{0.5}(d - t)^{0.8}$, and the service time requirement S is exponentially distributed with rate 1.

The table also reports the approximated ratio obtained in Breinbjerg (2017), where users are facing a similar queueing system but are served on a FCFS basis. Breinbjerg (2017) finds that the symmetric equilibrium prescribes a strategy such that both users arrive at opening time, i.e., $t = 0$, with certainty. When comparing the two approximated values, we find that the FCFS queueing discipline yields a lower efficiency loss in equilibrium compared to that of the LCFS-PR queueing discipline. The example thus shows that for the case of two users, social efficiency under LCFS-PR may actually be lower than under FCFS. This example in favor of FCFS should be contrasted with, e.g., the case of many small users where the equilibrium utility under FCFS and LCFS provide a lower and higher bound on equilibrium social efficiency, respectively (Platz & Østerdal, 2017).

5 Conclusion

We have examined the strategic choices of a population of users that independently choose when to arrive at a queueing system that employs the LCFS-PR service discipline, when each user prefers earlier service and dislikes spending time in the queue. Our main contribution consists of establishing the existence and uniqueness of a symmetric mixed Nash equilibrium for general classes of preferences and service time distributions. Whereas the constructive

procedure provided in the existence proof advises an approach to solve the problem that could in principle be applied for any n , the computations quickly become cumbersome. We provide a numerical method to compute the equilibrium in a two-user setting and present an example.

The numerical example shows that the LCFS-PR service discipline may provide incentives for arrival profiles that lead to lower social efficiency compared to the incentives provided by the FCFS discipline. A likely explanation for this is the additional inefficiency caused by the property of preemption. As any newly arrived user may preempt the service progress of another user, the users must in equilibrium arrive according to a distribution function that extends over an ‘excessively’ large interval of time, in order to mitigate the expected disutility of being preempted after arrival. A further and more comprehensive study of the impact of preemption on social efficiency proposes an interesting avenue for future research. In particular, it remains an open question to determine the optimal service discipline in the current setting. Furthermore, the differences in social efficiency induced by the FCFS and LCFS-PR service disciplines could be studied in more detail. This could be done for various combinations of user population sizes, service times distributions (e.g. non-exponential distributions with decreasing hazard rate, heavy tailed, etc.) and utility functions. By establishing existence and uniqueness of a symmetric equilibrium under general preferences and service time requirements, we have provided a solid starting point for the search for comprehensive methods to numerically solve the problem for varying populations sizes and specific user preferences.

Finally, the LCFS-PR queueing game may be further extended in other important directions. One is the consideration of heterogeneous (multiclass) users. This has for example been considered in a FCFS queueing environment by Guo and Hassin (2012). A second direction is the consideration of queueing games with multiple servers. This has recently been considered by Haviv and Ravner (2015) who examine a multi-server system with no queue buffer, where users are interested in maximizing the probability of obtaining service. A third direction is the consideration of a stochastic number of users.⁸ This would mean that each user does not know how many players there are in the game. However, such extension would require a comprehensive analysis beyond the scope of the present paper.

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⁸ For instance, Glazer and Hassin (1983), Glazer and Hassin (1987), Hassin and Kleiner (2011), Juneja and Shimkin (2013), Haviv (2013), Haviv (2013), Sherzer and Kerner (2017) and Haviv and Oz (2018) allow for a stochastic number of users in their models.

A Appendix

A.1 Proof of Lemma 7

We start the proof by the following observation: A user’s waiting time when queueing under the LCFS-PR service discipline is independent of the queue length she faces upon arrival, since the discipline allows the user to suspend the whole queue until after she completes her service. Thus, without loss of generality, we may say that the user arrives at an idle server. A user’s waiting time in a LCFS-PR queue is then identical to the period of time between when the user arrives to an empty system and when she departs, leaving behind an empty queue. A user therefore only cares about the expected share of $n - 1$ users that may arrive after her arrival and their respective service time requirements.

To capture such a situation, we start by introducing some notation. Let \mathbf{A} denote the (random) arrival time of one of the two users, and let $\{\mathbf{S}_j\}_{j \in \{1,2\}}$ be a sequence of (random) service time requirements, such that \mathbf{S}_j is the service time of the j th user to start service. For any user $i \in \{1, 2\}$, let \mathbf{R}_i denote the (random) residual service time of user i , if i is preempted prior to service completion. Moreover, let $\mathbf{D}_i(t)$ denote the (random) departure time of user i , when she arrives at time t , and the other user arrives at \mathbf{A} . The departure time of user i satisfies for each $t \geq 0$ the following sample path relation:

$$\mathbf{D}_i(t) = \begin{cases} \mathbf{S}_2 + t & \text{if } \mathbf{A} \leq t \\ \mathbf{1}_{\{\mathbf{S}_1+t \leq \mathbf{A}\}}(\mathbf{S}_1 + t) + \mathbf{1}_{\{\mathbf{S}_1+t > \mathbf{A}\}}(\mathbf{A} + \mathbf{S}_2 + \mathbf{R}_i) & \text{if } \mathbf{A} > t \end{cases}$$

Intuitively, the sample path above describes the possible outcomes, depending on whether user i is preempted or not prior to service completion. That is, in the event of $[\mathbf{A} \leq t]$, user i possibly preempts the user already residing in the queue and completes her service after \mathbf{S}_2 time units. In the event of $[\mathbf{A} > t]$, user i is the first to arrive at the system and is possibly preempted prior to service completion. That is, in the event that $[\mathbf{S}_1 + t \leq \mathbf{A}]$, user i departs the system at time $\mathbf{S}_1 + t$ before the other user arrives at \mathbf{A} . Otherwise, if $[\mathbf{S}_1 + t > \mathbf{A}]$, then user i is preempted and does not depart the system until the other user has completed service, and i has completed her residual service requirement, i.e. she departs at time $\mathbf{A} + \mathbf{S}_2 + \mathbf{R}_i$.

Fix a strategy F , and let $\mathbf{A} \sim F$. Moreover, let $\mathbf{S}_j \sim G$ for any j where G is the exponential cumulative distribution function, such that for any $x \in \mathbb{R}$

$$G(x; \mu) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and $g(x; \mu)$ denotes the density of G at x . We characterize the probability of the event $[\mathbf{D}_i(t) \leq d]$ conditional on \mathbf{A} , \mathbf{S}_1 and \mathbf{S}_2 , which equals zero for any $d < t$, and is otherwise given by

$$\begin{aligned} \Pr \{ \mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A} \} &= \mathbb{E} \left[\mathbf{1}_{\{\mathbf{D}_i(t) \leq d\}} \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A} \right] \\ &= \mathbf{1}_{\{\mathbf{A} \leq t\}} G(d - t; \mu) \\ &\quad + \mathbf{1}_{\{\mathbf{A} > t\}} \left[\mathbf{1}_{\{\mathbf{S}_1+t \leq \mathbf{A}\}} G(d - t; \mu) + \mathbf{1}_{\{\mathbf{S}_1+t > \mathbf{A}\}} H(d - \mathbf{A}; 2, \mu) \right] \end{aligned}$$

for any $0 \leq t \leq d$, where H denotes the cumulative distribution function of the Erlang distribution defined by

$$H(x; k, \mu) = \begin{cases} 1 - \sum_{m=0}^{k-1} \frac{1}{m!} e^{-\mu x} (\mu x)^m & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Note that the memoryless property of the exponential distribution implies that the distribution of the residual service times does not depend on how long a user has been in service prior to preemption, since the remaining time is still probabilistically the same as at her arrival time. That means that in case she is preempted, user i 's departure time is the sum of two independent, identically and exponentially distributed variables (or equivalently, Erlang distributed with shape 2) with location at time \mathbf{A} . Consequently, the conditional probability $\Pr \{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\}$ is independent of \mathbf{S}_2 .

We next characterize D by marginalizing out the variables \mathbf{A} and \mathbf{S}_1 such that

$$\begin{aligned} D(d \mid t, F) &= \Pr \{\mathbf{D}_i(t) \leq d\} \\ &= \int_{S(F)} \int_{S(G)} \Pr \{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} dG(s - t; \mu) dF(a) \\ &= \int_0^b \int_0^\infty \Pr \{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} g(s - t; \mu) ds dF(a) \end{aligned}$$

for each $t \geq 0$ where all integrals are Lebesgue integrals, and the supremum of the support $S(F)$ is given by b . Since F might have points of discontinuity, let I_a denote the jump size of F at the point in time a , so

$$I_a = \begin{cases} F(a) - \lim_{s \uparrow a} F(s) & \text{if } F(a) - \lim_{s \uparrow a} F(s) > 0 \\ 0 & \text{otherwise} \end{cases}$$

for any $a \in \mathbb{R}$. Then we may express D as follows

$$\begin{aligned} D(d \mid t, F) &= \sum_{a \leq b} I_a \int_0^\infty \Pr \{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} g(s - t; \mu) ds \\ &\quad + \int_0^b \int_0^\infty \Pr \{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} g(s - t; \mu) f_+(a) ds da. \end{aligned}$$

We next insert the expression for $\Pr \{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\}$ and divide the expression in the two intervals of $(-\infty, t]$ and $(t, b]$, respectively:

$$\begin{aligned} D(d \mid t, F) &= \sum_{a \leq t} I_a G(d - t; \mu) + \int_0^t f_+(a) G(d - t; \mu) da \\ &\quad + \sum_{t < a \leq b} I_a \left(\int_0^a g(s - t; \mu) G(d - t; \mu) ds + \int_a^\infty g(s - t; \mu) H(d - a; 2, \mu) ds \right) \\ &\quad + \int_t^b f_+(a) \left(\int_0^a g(s - t; \mu) G(d - t; \mu) ds + \int_a^\infty g(s - t; \mu) H(d - a; 2, \mu) ds \right) da. \end{aligned}$$

The claim of Lemma 7 now follows immediately once we note that $\int_0^a g(s - t; \mu) ds = G(a - t; \mu)$ and $\int_a^\infty g(s - t; \mu) ds = 1 - G(a - t; \mu)$. \square

A.2 Numerical procedure

We here present a numerical method to compute X_b for a given b . Note that $V, n, \mu, \epsilon, \Delta$ and b are exogenous inputs to the procedure:

1. Let $\mathcal{T}_\Delta^b = \{t \in \{0, 1, 2, \dots\} : t\Delta < b\}$ be a discretization of the interval $[0, b)$ wrt. Δ .
2. Let $X_{b,h}(s) = 1$ for all $s \geq b$ and all $h \in \mathbb{N}$

3. Compute $U(b, X_{b,h}) = \int_b^\infty V(b, d)dD(d | b, X_{b,h})$ according to the expression of D in Lemma 7 (note that $U(b, X_{b,h})$ is the same for all $h \in \mathbb{N}$).⁹
4. Let $h = 0$ and sequentially compute $X_{b,0}(t)$ for each $t \in \mathcal{T}_\Delta^b$ according to equation (4).
5. Assign $h := h + 1$ and sequentially compute $X_{b,h}(t)$ for each $t \in \mathcal{T}_\Delta^b$ according to equation (6).
6. If $X_{b,h}(t) - X_{b,h-1}(t) \leq \epsilon$ for all $t \in \mathcal{T}_\Delta^b$, then let $X_b = X_{b,h}$ and stop the procedure.
7. Else, go back to step (5) and begin the next iteration of h .

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⁹ We can further reduce the expression of D in the context of the constructive procedure in Lemma 5. That is, for a given user the procedure considers the other user’s strategy to be such that only one jump occurs immediately after the point in time that she arrives. Thus, the D in this context can be expressed as

$$\begin{aligned}
 D(d | t, F) &= F(t)G(d - t; \mu) + \lim_{s \downarrow t} (F(s) - F(t))H(d - t; 2, \mu) \\
 &+ \int_t^b f_+(a) [G(a - t; \mu)G(d - t; \mu) + (1 - G(a - t; \mu))H(d - a; 2, \mu)] da.
 \end{aligned}$$

Since H and G are everywhere continuous and have $\lim_{a \downarrow t} G(a - t; \mu) = 0$ and $\lim_{a \downarrow t} H(a - t; 2, \mu) = 0$, respectively, we also note that D is continuous and everywhere differentiable with respect to d . Let $D'_i(d | t, F) = \frac{d}{dd} D(d | t, F)$ be the derivative of D with respect to d , then $U(t, F) = \int_t^\infty V(t, d)D'_i(d | t, F)dd$.

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