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ORIGINAL RESEARCH

Liberté, Égalité, Fraternité: a power study in signed networks

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Abstract
Power in human societies is a central phenomenon. Even though, it took ages to understand it and – even more – to measure it. Only in the last decades attempts were made to model power relations and to assign respective power indices to actors in a network. The present work goes a step further. It measures power of actors and groups of actors in networks by means of conditional relations. In a probabilistic framework, such relations are specified as conditionals: Which actor receives power given that the adjacent actor has it, and which actor looses power given that the neighbour dominates. This pattern of power relations allows for an exact calculation of an actor's and groups of actors' power index. The new decision analytics tool for this is maximizing entropy for the whole net and evaluating each actor's influence therein. The new concept is applied to a middle size Kronecker net of clans and subclans operating in a today's society.

Keywords Network analytics · Entropy · Power networks · Power support · Power suppression

1 Introduction

Power in human societies partly is a constructive and partly is a destructive phenomenon. Possibly in all cultures sociologists, philosophers, economists and politicians were concerned

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with the intrinsic What and How of power relations. In other words, it is the very question of how people treat each other. In more recent scientific articles, authors like (Emerson, 1962), (Zegler, 1975), (Witte, 2001) and many others studied the determinants of power. They tried to find out who exerts power on whom, to which degree and by which means. Which resources and which costs will an actor insert, and what is the resistance of the aggrieved party. We cite (Emerson, 1962), p. 32:

> Power (Pab) The power of actor A over actor B is the amount of resistance on the part of B which can be potentially overcome by A.

Other authors like (Bonacich, 1987), (Bozzo & Franceschet, 2016) widen this local view of direct contact of actors to global power relations in networks. From basic insights such as "An actor is powerful if it is connected to powerless actors", cf. Bozzo and Franceschet (2016) on page 76, they deduce mathematical equations to measure power. Literature as to power is almost endless but mostly treats exertion of power *on* somebody (PO). The treatment of power *from* somebody (PF) is of minor interest. In this work, both aspects of power relations will be analysed and integrated into a global model.

Support attitude (PF) is a frequently observed relation in societies. A clique of classmates in high-school, associations or unions, fraternities are typical social groupings with PF-relations. "All for one, one for all". Suppress attitude (PO) is inducement or even manipulation of others' ideas or positions. "Everbody against everybody".

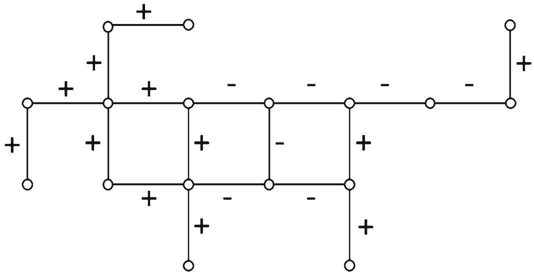
Can these two attitudes be joined in a decision model? Can they be modeled and analysed by tools of Social Network Analysis (SNA)? That is the very question of this article. In other words: Actors might try to suppress each other (PO), or they might try to support each other (PF). In either case, these two behaviourist attitudes are central relations in any human society. We will analyse how they penetrate the whole social structure and, finally, determine each actor's influence, namely its power. The reader familiar with SNA might guess that power is equivalent to one of the centrality concepts: degree centrality, closeness centrality, or betweenness centrality. This guess is wrong. Already Cook et al. in 1983 recognized very clearly the difference, a central actor must be surrounded by weak actors to become powerful; the reader is referred to Cook et al. (1983). There are several occasions and concepts to study actors and groups of actors in networks, perhaps with the aim of detecting or measuring their influential value. For the interested reader, we refer to Can and Alatas (2019), Chen et al. (2020), Saura (2021), Restrepo et al. (2021), and Ghorbani and Azadi (2021).

Graphs – consisting of vertices and edges – are models of the social fabric and first were established by Moreno (1934). But are they able to draw a picture even of positive and negative relations in and between groups of actors? A first step in the right direction might be a signed network (sN), a network with positive and negative edges. Figure 1 shows such an sN with positively linked groups of vertices $\widehat{=}$ actors embedded in and meshed with negative edges. The semantics of these edges might be sympathy ($+$) and antipathy ($-$). Newman (2012) shows on pages 206 ff. that an inconsiderate distribution of $+$ and $-$ can cause contradictions.

In this paper, $+$ and $-$ edges will be modeled by probalitistic conditionals, meaning support (PF) and suppression (PO), respectively. Applying conditionals showed fruitful results in Brenner et al. (2017), Rödder et al. (2019), Dellnitz and Rödder (2020). None of these works treats power in sN, however.

This contribution is organized as follows. The next section shows signed networks and developes conditions for a net to be decomposable or clusterable into support and suppression subnets. Sect. 3.1 provides conditional-probabilistic preliminaries and their application to sN in 3.2. In Sect. 4.1, we define power in conditional-probabilistic nets and illustrate this concept

Fig. 1 A signed network with positive (support) and negative (suppression) edges



in 4.2 for PF-, in 4.3 for PO- and in 4.4 for mixed PF/PO-nets. Mixed nets melt friendly and hostile relations among actors and result in a complex pattern of freedom/bondage on the one hand and equality/disparity on the other. This leads to a sociopolitical contemplation on balance of power in such nets. Section 4.5 widens the conditional forms and shows net transformations when power in a vertex becomes evident. In Sect. 5, power is measured for all 50 vertices in a Kronecker net. Section 6 is a resumé and shows the road ahead.

2 Signed networks

A graph is a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E}$ connecting some of the vertices. Graphs are models of actors $\widehat{=}$ vertices and their relations $\widehat{=}$ edges in a real social fabric. Edges might be undirected, directed, weighted or signed. These characteristic features describe certain properties of the social structure of the real world: Undirected edges represent symmetrical relations like friendship or kinfolk, directed edges stand for transfer of material goods or convictions, opinions, attitudes. Such transfer can be attenuated – material goods might rot, convictions might not convince – and this attenuation is expressed by weights on edges. Signed edges were shown in Fig. 1, they are topic of the remainder of this section.

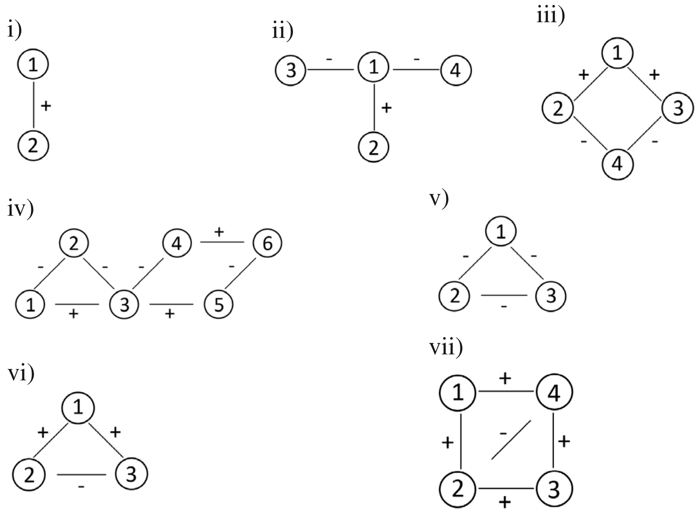
$(\mathcal{V}, \mathcal{E}^+ \dot\cup \mathcal{E}^-)$ is a signed graph, being $\mathcal{E}^+$ the set of positive and $\mathcal{E}^-$ the set of negative edges; see again Fig. 1. To avoid contradictions in signed graphs, we postulate clusterability like in Definition 1 and supplement respective examples.

Definition 1 A signed graph is clusterable if it is decomposable into disjoint induced subgraphs such that

- each subgraph with more than one vertex has only positive edges,
- they are interconnected by only negative edges.

An *induced* subgraph contains all connecting edges from $(\mathcal{V}, \mathcal{E}^+ \dot\cup \mathcal{E}^-)$. An one-vertex subgraph is called degenerated.

Example 1 (clusterable and non-clusterable signed graphs)



The signed graphs (iv), (v), (vi) you find in ((Newman, 2012), p. 206 ff.). Together with (i), (ii), (iii), (vii) they illustrate pretty well the concept of clusterability. (i), (ii), (iii), (iv) obviously satisfy the postulations of Definition 1, (v) has three degenerated subgraphs, only incident with negative edges and hence also meets the definition; (vi) and (vii) do not. The cohesive subgraph in (vi) is not induced and neither is the one in (vii).

To prove clusterability of sN, respective literature provides algorithms. Roughly speaking they build maximal cohesive subnets and then check negative edges therein. If there are no such negative edges, clusterability applies and fails, otherwise.

3 Conditional probabilistic signed networks

3.1 Conditional-probabilistic preliminaries

Let $(\mathcal{V}, \mathcal{E}^+\dot{\cup}\mathcal{E}^-)$ be an sN with $|\mathcal{V}| = \text{n}$. Each vertex $V_i \in \mathcal{V}$ is a boolean variable with $V_i = 1$ or $V_i = 0$. For $V_i = 1$ the vertex is powerful, for $V_i = 0$ it is powerless. $\mathbf{v} = (v_1, v_2, \ldots, v_n) = (V_1 = 1/0, V_2 = 1/0, \ldots, V_n = 1/0)$ are 2^n states or configurations of the net. On $\{\mathbf{v}\}$ we define probability distributions Q; they convey power relations. For this, we impose structuring postulations:

- $Q(V_j = 1 \mid V_i = 1) = 1.$ and $Q(V_i = 1 \mid V_j = 1) = 1.$ $\quad$ for$(i, j) \in \mathcal{E}^+$ (1+)
- $Q(V_j = 0 \mid V_i = 1) = 1.$ and $Q(V_i = 0 \mid V_j = 1) = 1.$ $\quad$ for$(i, j) \in \mathcal{E}^-$ (1-)
- $Q(V_i = 1) = 1.$ $\quad$ for some vertices$i$ (1E)

Conditioned postulations are called conditionals or rules, unconditioned postulations are facts. The two rules in (1+) enforce mutual transfer of power. If V_i is powerful then V_j also is and vice versa. The two rules in (1−) enforce mutual suppression. If vertex V_i is powerful ($V_i = 1$) and is able to assert it on V_j (1.), then V_j is powerless ($V_j = 0$) and vice versa. Facts in (1E) connote evidence of real power rather than a mere "if it is so".

We observe that edges in $\mathcal{E}^+$ support power transfer like in cliques of classmates in high-school, between members of unions or fraternity brothers. "All for one, one for all". On the other hand, adjacent actors incident with vertices from $\mathcal{E}^-$ try to suppress their neighbours. "Everybody against everybody". A vertex might be connected to some neighbours with positive and to others with negative edges, of course. The distribution Q must be chosen thoroughly. Besides respecting the rules or facts in (1+), (1−) or (1E) it must be a prudent representative of all intended conditional structure among the variables V_i, $i = 1, ..., n$. Maximising entropy in Q, $H(Q) = -\sum_{\mathbf{v}} \mathrm{Q}(\mathbf{v}) \log_2 \mathrm{Q}(\mathbf{v})$, and likewise satisfying the rules or facts is the clue for this task. It is called MaxEnt principle and has a very strict axiomatic justification, see Kern-Isberner (1998) or Brenner et al. (2017). H measures mutual independence among the variables V_i. The greater H, the greater independence; and the smaller H, the less independence. This feature of entropy will be very helpful to model power structures in sN.

3.2 Creation of power structures in signed networks

We study three optimisation problems and explain.

$$
\begin{aligned}
Q_+ = {} & \arg\max H(Q) \\
& \text{s.t.} Q(V_j = 1 \mid V_i = 1) = 1. \\
& \quad Q(V_i = 1 \mid V_j = 1) = 1. \qquad \text{for}(i, j) \in \mathcal{E}^+
\end{aligned}
\tag{2+}
$$

$$
\begin{aligned}
Q_- = {} & \arg\max H(Q) \\
& \text{s.t.} Q(V_j = 0 \mid V_i = 1) = 1. \\
& \quad (Q(V_i = 0 \mid V_j = 1) = 1.) \qquad \text{for}(i, j) \in \mathcal{E}^-
\end{aligned}
\tag{2-}
$$

As conditionals in brackets are redundant they can be omitted.

$$
\begin{aligned}
Q_{+-} = {} & \arg\max H(Q) \\
& \text{s.t. like in}(2+)\text{and in}(2-)
\end{aligned}
\tag{2+-}
$$

In (2+), MaxEnt-distribution Q_+ is determined for a net with only positive edges, (2−) treats a net with only negative edges. And finally $(2 + -)$ considers a general sN. The restrictions in these equations postulate respective conditioned probabilities equal to 1., being | the well-known conditional operator.

For the nets (i) to (v) in Example 1, we stated clusterability; for these, the optimisation problems are feasible. Not so for the nets (vi) and (vii). We observe inconsistancy and hence infeasibility for the respective optimisation problems. Once vertex 1 in net (vi) is powerful this makes vertices 2 *and* 3 powerful. On the other hand a powerful vertex 2 enforces a powerless 3. A contradiction.

Equation 3 shows the optimisation problem for net (iv) in Example 1. Again rules in brackets are redundant and can be omitted.

$$
\begin{aligned}
Q_{+-} \; = {} & \arg\max H(Q) \\
& s.t. \quad Q(V_3 = 1 \mid V_1 = 1) = 1. \quad Q(V_1 = 1 \mid V_3 = 1) = 1. \\
& \quad Q(V_5 = 1 \mid V_3 = 1) = 1. \quad Q(V_3 = 1 \mid V_5 = 1) = 1. \\
& \quad Q(V_6 = 1 \mid V_4 = 1) = 1. \quad Q(V_4 = 1 \mid V_6 = 1) = 1. \\
& \quad Q(V_2 = 0 \mid V_1 = 1) = 1. \quad (Q(V_1 = 0 \mid V_2 = 1) = 1.) \\
& \quad Q(V_2 = 0 \mid V_3 = 1) = 1. \quad (Q(V_3 = 0 \mid V_2 = 1) = 1.)
\end{aligned}
\tag{3}
$$

$$Q(V_4 = 0 \mid V_3 = 1) = 1. \quad (Q(V_3 = 0 \mid V_4 = 1) = 1.)$$
$$Q(V_6 = 0 \mid V_5 = 1) = 1. \quad (Q(V_5 = 0 \mid V_6 = 1) = 1.)$$

The until now generic term "power" needs a mathematical specification. That is what the next section is about.

4 Structural power in sN

4.1 Power in a conditional-probabilistic framework

After solving $(2+-)$ we have Q_{+-}. $Q_{+-}(V_i = 1)$ for $i = 1, ..., n$ then are the probabilities of vertices V_i to be powerful. From textbooks in information theory like (Roman, 1997) or Topsœ1974, we know that $-\log_2 Q_{+-}(V_i = 1)$ is the information the sN recieves when power in vertex V_i becomes evident. The difference $-\log_2 Q_{+-}(V_i = 1) - (-\log_2(1.))$ measures change of all mutual dependencies in the net. The greater this difference, the greater the potential influence of vertex V_i. For more details on this issue, see (Brenner et al. (2017), p. 5). A good reason for Definition 2.

Definition 2 $sp_i = -\log_2 Q_{+-}(V_i = 1)$ is structural power of vertex V_i, $i = 1, ..., n$.

For the little net (ii) in Example 1, we now determine all sp_i, $i = 1, ..., 4$.

Example 2 (structural power of vertices in an sN) After solving $(2+-)$ we have Q_{+-} like in the attached contingency table. It was calculated by means of the expert system shell SPIRIT (2011).

V_1	V_2	V_3	V_4	Q_{+-}	V_1	V_2	V_3	V_4	Q_{+-}	with	$Q_{+-}(V_1 = 1) = Q_{+-}(V_2 = 1) = \frac{1}{5}$
1	1	1	1	0	0	1	1	1	0		$Q_{+-}(V_3 = 1) = Q_{+-}(V_4 = 1) = \frac{2}{5}$
1	1	1	0	0	0	1	1	0	0		
1	1	0	1	0	0	1	0	1	0		
1	1	0	0	$\frac{1}{5}$	0	1	0	0	0		
1	0	1	1	0	0	0	1	1	$\frac{1}{5}$		
1	0	1	0	0	0	0	1	0	$\frac{1}{5}$		
1	0	0	1	0	0	0	0	1	$\frac{1}{5}$		
1	0	0	0	0	0	0	0	0	$\frac{1}{5}$		

In this table, a 0-probability appears for such configurations which contradict the rules. E.g., $Q_{+-}(1, 1, 1, 1) = 0.$ because $V_1 = 1$ and $V_3 = 1$ would contradict the rule $Q(V_3 = 0 \mid V_1 = 1) = 1$. Non 0-probabilities are equal due to maximum entropy. Summing up probabilities of respective configurations yields results on the right side. The vertices V_1 and V_2 show equal power $sp = -\log_2 1/5 = 2.322$ and so do V_3, V_4 with $sp = -\log_2 2/5 = 1.322$. Vertices in cohesive subnets always are of equal power, as will be developed later. So vertices V_3 and V_4 confront the same homogeneous subnet and due to symmetry are equally powerful, too.

Mere PF and PO-nets are of special interest and are studied in the next two sections.

4.2 Structural power in PF-nets

All edges in PF-nets are positive, $V_i \overset{+}{-} V_j$, here represented by two conditionals, cf. (2+). From a graph theoretical point of view, such a net is strongly connected and forms an SCS. For such nets, Brenner et al. (2017) provided propositions which we repeat and explain.

Theorem 1 *Let Q_+ be solution of (2+). Then we have:*

a. $Q_+(V_i = 1) = \frac{1}{2}$ *for all* $i = 1, \ldots, n$.
b. $sp_i = 1$. *for all* $i = 1, \ldots, n$.
c. $H(Q_+) = 1$.

Proof Due to the rules in (2+), all configurations $\mathbf{v}$ with $V_i = 1$ and $V_j = 0$ for some $i \neq j$ have probability 0. Remain configurations $\mathbf{v}_1 = (V_1 = 1, V_2 = 1, ..., V_n = 1)$ and $\mathbf{v}_2 = (V_1 = 0, V_2 = 0, ..., V_n = 0)$. Q_+ has maximal entropy and hence $Q_+(\mathbf{v}_1) = Q_+(\mathbf{v}_2) = \frac{1}{2}$. This implies Theorem 1a. and b. For a proof of c. mind the fact that $H(Q_+) = \frac{1}{2}(-\log_2 \frac{1}{2}) + \frac{1}{2}(-\log_2 \frac{1}{2}) = \frac{1}{2} \cdot 1. + \frac{1}{2} \cdot 1. = 1.$ □

$H(Q_+)$ is the remaining mutual independence in the PF-net. It is small and it fully disappears whenever *any* vertex exerts full power. This benefits them all and all become powerful.

4.3 Structural power in PO-nets

All edges in PO-nets are negative, $V_i \overset{-}{-} V_j$, here represented by two conditionals, cf. (2−). For PO-nets the propositions in Theorem 2 hold.

Theorem 2 *Let Q_- be solution of (2−). Then we have:*

a. *In a complete PO-net, all vertices V_i have equal structural power* $sp_i = \log_2(n+1) > 1$.
b. *Entropy in complete PO-nets equals* $\log_2(n + 1) > 1$.
c. *In incomplete PO-nets, usually the vertices have different structural power.*
d. *Also in incomplete PO-nets, all vertices V_i have structural power* $sp_i > 1$.
e. *Also in incomplete PO-nets, entropy always exceeds* 1.

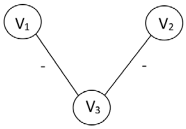
Proof Ad a. Choose an arbitrary V_i. Because of the rules in (2−) for a configuration $\mathbf{v}$, $Q_-(\mathbf{v}) > 0$ holds iff

- $V_i = 1$ and all $V_j = 0$, $j \neq i$ or
- $V_1 = V_2 = ... = V_n = 0$.

The number of these configurations is $n + 1$ and under MaxEnt all probabilities are $\frac{1}{n+1}$. This implies $Q_-(V_i = 1) = \frac{1}{n+1}$ and $sp_i = \log_2(n + 1) > 1$ for all i.
Ad b. Entropy adds up to $(n + 1)\frac{1}{n+1}(-\log_2 \frac{1}{n+1}) = \log_2(n + 1) > 1$.
Ad c. We exemplify.

Example 3 (unequal structural power in incomplete PO-nets)

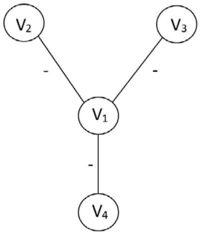
i. For the net



we have

$$sp_1 = sp_2 = \log_2 \tfrac{5}{2}$$
$$sp_3 = \log_2 5$$

ii. For the star



we have

$$sp_2 = sp_3 = sp_4 = \log_2 \tfrac{9}{4}$$
$$sp_1 = \log_2 9$$

Both results confirm our intuition.
Ad d. The proof is bulky and so it is put in the appendix.
Ad e. From b. we know that in a complete PO-net, entropy exceeds 1. Now, for an incomplete PO-net,

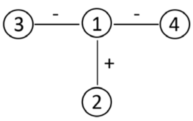
- round it up to a complete net $(\mathcal{V}, \mathcal{E}^-)$ with $H > 1.$,
- regard that $H > 1.$ is $H(Q_-)$ for (2−) with rules linking all vertices to all vertices,
- delete the added rules and solve (2−) again, resulting in $H(\overline{Q}_-)$,
- verify $H(\overline{Q}_-) \geq H(Q_-) > 1.$, as deletion of rules means enlargement of the feasibility set. □

Resuming the last two sections we have the surprising result that PO-nets give more structural power to vertices than PF-nets do. "Everybody against everybody" in our model generates selfdetermination and selfconfidence. "All for one, one for all" in contrast, generates heteronomy. As to entropy, it is small in PF-nets indicating mutual dependence. And it grows in PO-nets alluding to independence. We come back to this issue in Sect. 4.5. Before doing so, we study structural power in general sN including positive and negative edges.

4.4 Structural power in general sN

From Defintion 2 in Sect. 4.1, we have power $sp_i = -\log_2 Q_{+-}(V_i = 1)$ in any general sN, all V_i. In *mere* PF-nets, it always amounts to 1.; see Theorem 1 in Sect. 4.2. This statement does not prevail anymore in general sN. Now, strongly connected subgraphs SCS are embedded in a net of positive and negative edges. This in fact gives all its vertices the same structural power, but not necessarily equal 1. A proof is given in Brenner et al. (2017), Theorem 1. To make this contribution more self-contained, we verify the statement for net ii) of Example 1.

The sN



contains SCS $1 \overset{+}{—} 2$, embedded in a net of negative edges. Q_{+-}, marginal probabilities, and structural power are as follows.

V_1	V_2	V_3	V_4	Q_{+-}	V_1	V_2	V_3	V_4	Q_{+-}
1	1	1	1	0	0	1	1	1	0
1	1	1	0	0	0	1	1	0	0
1	1	0	1	0	0	1	0	1	0
1	1	0	0	$\frac{1}{5}$	0	1	0	0	0
1	0	1	1	0	0	0	1	1	$\frac{1}{5}$
1	0	1	0	0	0	0	1	0	$\frac{1}{5}$
1	0	0	1	0	0	0	0	1	$\frac{1}{5}$
1	0	0	0	0	0	0	0	0	$\frac{1}{5}$

with $Q_{+-}(V_1 = 1) = Q_{+-}(V_2 = 1) = \frac{1}{5}$
$Q_{+-}(V_3 = 1) = Q_{+-}(V_4 = 1) = \frac{2}{5}$
$sp_1 = sp_2 = 2,322$
$sp_3 = sp_4 = 1,322$

Configurations with $(v_1, v_2) = (1, 0)$ and with $(v_1, v_2) = (0, 1)$ have probability 0 because of $1 \overset{+}{\text{—}} 2$. Probability of configurations with (0,0) adds up to $\frac{4}{5}$, the one with (1,1) to $\frac{1}{5}$. So the vertices in the SCS $1 \overset{+}{\text{—}} 2$ show structural power 2.322 rather than 1. like in a mere SCS.

As announced, we now solve (3) by means of the expert system shell (SPIRIT, 2011). The shell supports the architect of a network solving equations (2+), (2−), (2 + −) for hundreds of variables and rules; for the rich applicability of SPIRIT, cf. Rödder et al. (2006). In the present paper, respective variables and rules are shown in Figs. 2 and 3. The figures illustrate the vertices (rectangles) and links between vertices (lines). To identify + and − links, we added the rule tables in the lower part of the two figures. Figure 2 furtheron displays the marginals of Q_{+-} for all variables (bars in the rectangles). The numbers in the vertices of Fig. 3 provide structural power indices $-\log_2 Q_{+-}(V_i = 1)$. V_1, V_3, V_5 are equally powerful and so are V_4, V_6; these groups form strongly connected subgroups. The power of V_2 equals that of V_4, V_6 accidentally, only, due to the structure of the net.

An SCS shows equal marginals for all its vertices and consequently they all have equal structural power. They all are "kindred". Can such an SCS be compressed so as to simplify the net? The remainder of this section gives the answer. For proofs, we recommend (Brenner et al., 2017).

Definition 3 (SCS-compression) Let $\mathcal{V}_{\mathcal{SCS}} \subset \mathcal{V}$ be the vertices of an SCS and $\mathcal{V}_{\mathcal{REM}} = \mathcal{V} \backslash \mathcal{V}_{\mathcal{SCS}}$. For an SCS-compression, we have:

- $\mathcal{V}$ becomes VS $\cup \mathcal{V}_{\mathcal{REM}}$, with VS being a supervertex which substitutes $\mathcal{V}_{\mathcal{SCS}}$.
- All conditionals in SCS vanish, and all conditionals which have connected SCS to some $V_j \in \mathcal{V}_{\mathcal{REM}}$ now connect VS to $V_j \in \mathcal{V}_{\mathcal{REM}}$. Redundant conditionals disappear.

Let $\overline{Q}_{+-}$ be the solution of $(2 + -)$ with vertices VS $\cup \mathcal{V}_{\mathcal{REM}}$ and conditionals like in Definition 3 and let Q_{+-} be the solution of $(2 + -)$ before compression. Then we have the following lemma.

Lemma 1 *a. All $V_k \in \mathcal{V}_{\mathcal{SCS}}$ have equal marginals in Q_{+-}.*
b. VS has the same marginal in $\overline{Q}_{+-}$ as all V_k have in Q_{+-}.
c. All $V_j \in \mathcal{V}_{\mathcal{REM}}$ have the same marginals in $\overline{Q}_{+-}$ and Q_{+-}.
d. The entropies in $\overline{Q}_{+-}$ and Q_{+-} equal.

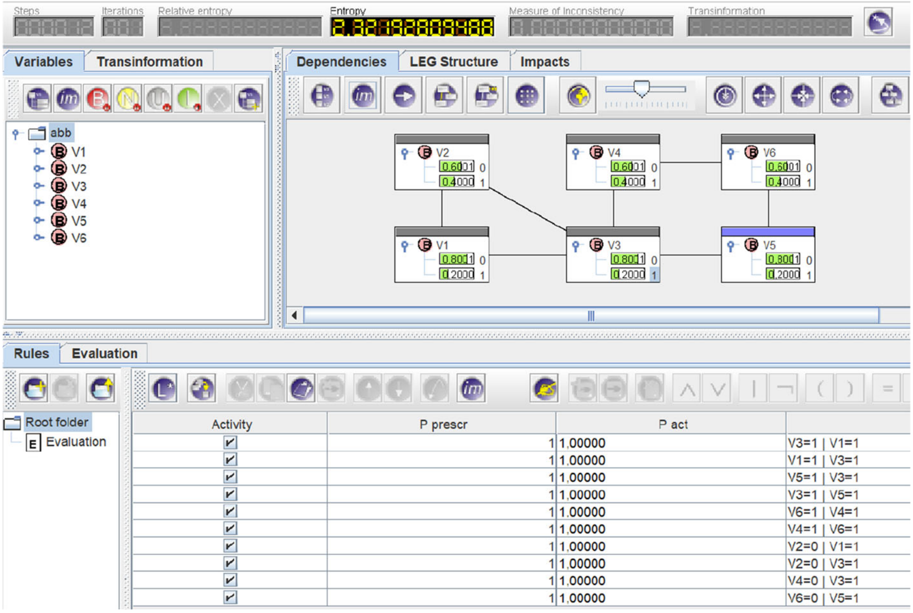


Fig. 2 Example 1iv). Rules (bottom) and marginals for Q_{+-}

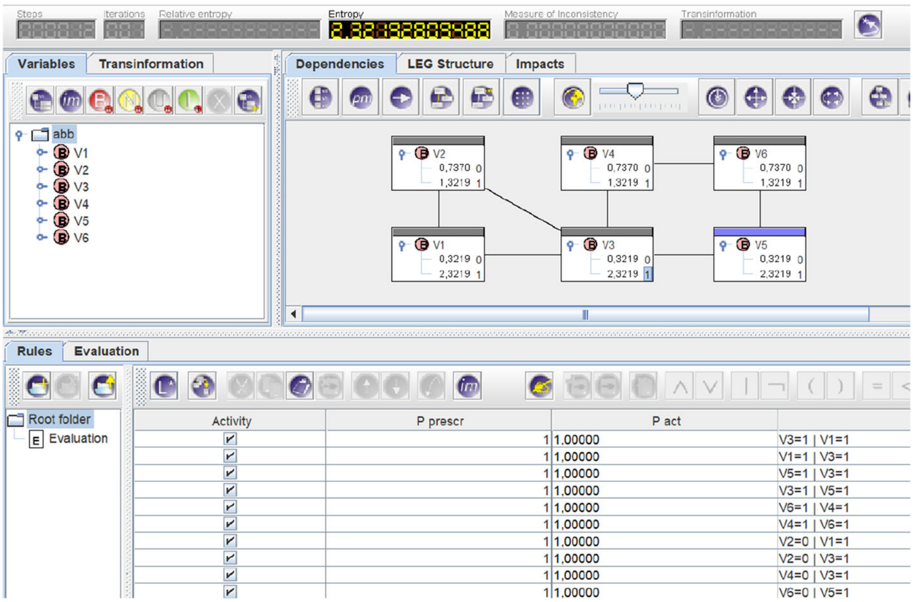


Fig. 3 Example 1v). Rules (bottom) and sp-values for Q_{+-}

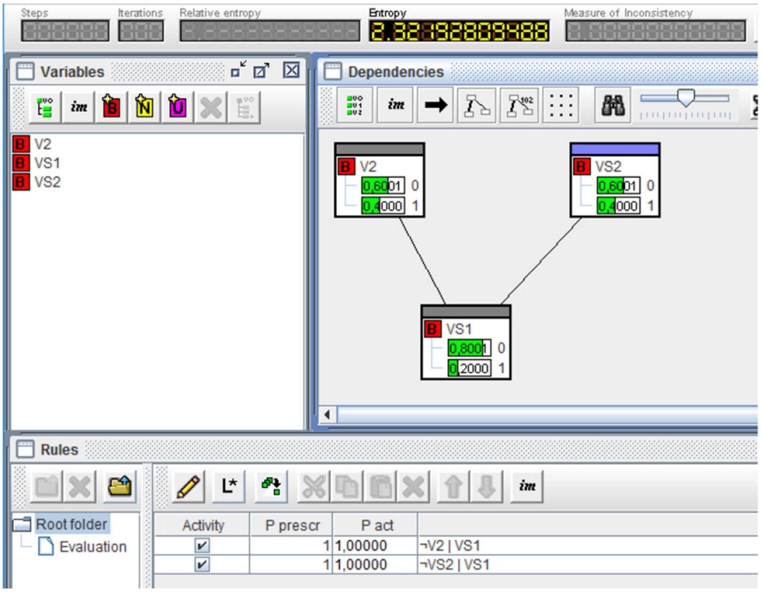


Fig. 4 Example 1iv). Rules (bottom) and marginals in $\overline{Q}_-$ after compression

SCS-compression does not change marginals and consequently it does not change structural power. SCS-compression is power-invariant. Repeating compression for all SCS in $(\mathcal{V}, \mathcal{E}^+ \dot{\cup}\, \mathcal{E}^-)$ yields a mere PO-net $(\overline{\mathcal{V}}, \overline{\mathcal{E}^-})$ with all results from Theorem 2. In particular, each vertex has power > 1 and we have $H > 1$.

Example 4 (SCS-compressions) Figure 2 shows the graph for Example 1iv) before compression. $\{V_1, V_3, V_5\}$ is an SCS and so is $\{V_4, V_6\}$. Marginals in each SCS coincide, just as structural power; see Fig. 3. Compression of both SCS like in Definition 3 results in graph, rules and mariginals like in Fig. 4 and in structural power like in Fig. 5.

SCS-compression is a suitable means to reduce a network without loss of structural information. In Sect. 5, we will reduce a network from 50 to 21 vertices maintaining all relevant net structure.

Perhaps this is the moment for some sociopolitical contemplations on our so far results.

Liberté, Égalité, Fraternité

chanted the masses during the french revolution.

Liberty, Equality, Fraternity

are incorporated in almost all constitutions of modern democracies. How do these claims combine with our theory of power?

- In a society exclusively formed by PF-relations, all actors are equal, see Theorem 1. H as the measure of independence is small; all actors' power situation is the same: If one of them is powerful, they all are. If one of them is powerless, they all are, too. This leaves no room for individual freedom: Égalité, Fraternité.
- In a society exclusively formed by PO-relations, each actor's power index exceeds 1. Each of them is free to suppress its neighbours, and vice versa, once power becomes evident. H as the global measure of such freedom $\widehat{=}$ independence – as per Theorem 2 – also exceeds 1.: Liberté.

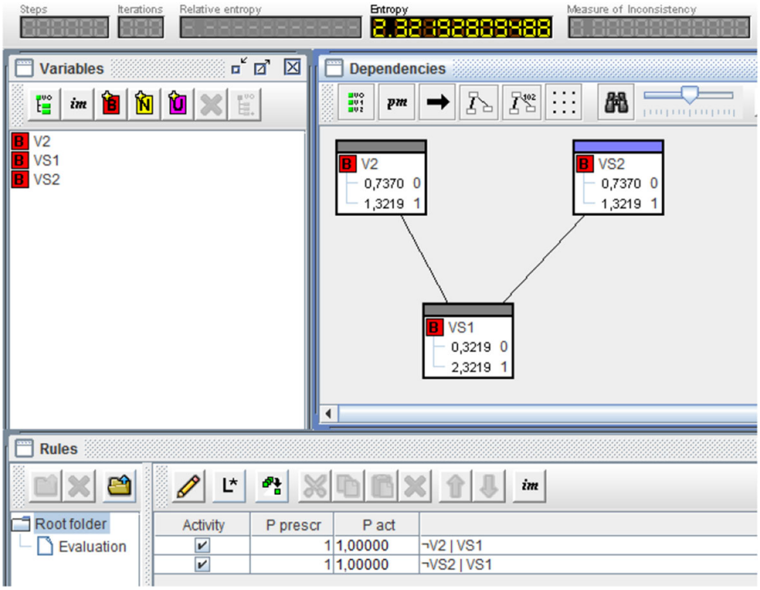


Fig. 5 Example 1iv). Rules (bottom) and sp-values in $\overline{Q}_-$ after compression

- In a society formed by PF- and PO-relations, equality holds for members of an SCS. And their structural power increases as they draw it from the overlying PO-net. H always exceeds 1: Liberté, Égalité, Fraternité.

4.5 Power evidence in general sN

After solving $(2 + -)$ with result Q_{+-}, the question arises: how does power evidence in a vertex influence the whole network. What if a vertex exerts power on its neigbours – not potentially but really? Please fold back to Figs. 2 and 3. When adding, e.g., the fact $Q(V_3 = 1) = 1.$, the net alterations are like those in Figs. 6 and 7. In Fig. 6, the fact $V_3 = 1$ becomes evident (the red bar) implying in certain $V_1 = 1$, $V_5 = 1$ and in certain $V_2 = 0$, $V_4 = 0$, $V_6 = 0$. The fact changed the whole net as it should be.

In Fig. 7, a 0 for $V_i = 1$ means full exploitation of power potential, ∞ indicates a contradiction for an additional evidence of this fact.

For a more ambitious study of multiple evidence, see Dellnitz and Rödder (2020), Sect. 3.

5 Power, dependence and independence in a middle size Kronecker graph

Literature research in the field of sN leads us to

- directed signed graphs; see Kim et al. (2018). That is not the focus of our paper.
- the question of how to generate sN, but unfortunately with disregard of clusterability of respective nets; cf. Jung et al. (2020).

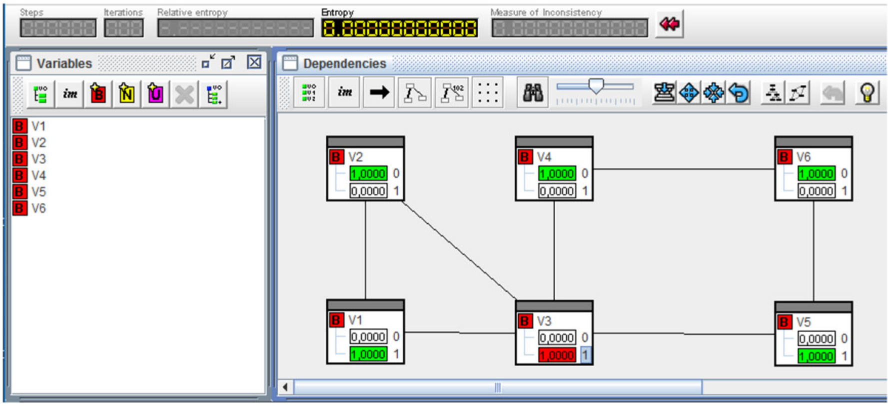


Fig. 6 Example 1iv). Evident fact $V_3 = 1$ and marginals

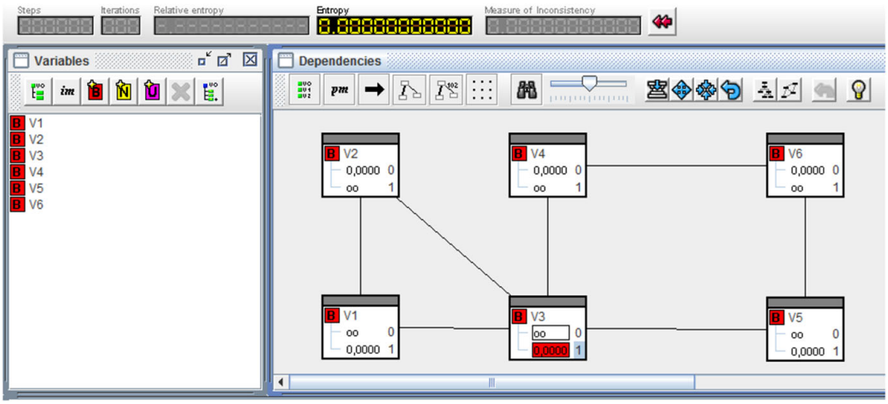


Fig. 7 Example 1iv). Evident fact $V_3 = 1$ and sp-values

- the very nice work about political ties and opposition between indigenous tribes of the central highlands in New Guinea from Read (1954).

The disregard of clusterability in those publications makes the construction of a new net mandotory so as to meet all requirements of our theory.

The net under consideration is supposed to be composed by criminal clans and subclans in any nowadays country. A clan consists of subclans, which support each other. “All for one, one for all”. When a subclan finds out about a quick police raid, it informs the whole clan. Not so between different hostile clans. With faked news they try to damage each other. “Everybody against everybody”.

We opt for the construction of a 64 $\times$ 64 Kronecker-net as follows.

1. With the initiator $\begin{pmatrix} .8 & .6 \\ .5 & .3 \end{pmatrix}$ and applying 5 iterations, we determine a 64 $\times$ 64 stochastic Kronecker matrix. For details see (Leskovec et al. (2010), pages 998 ff.). As the initiator is asymmetric, the matrix also is.
2. Construct a matrix A of negative edges by means of uniform random numbers: Make $A(i, j) = -1, i \neq j$ if the respective entry in the Kronecker matrix exceeds the random number; and make $A(i, j) = 0$, otherwise. A is asymmetric, in general.

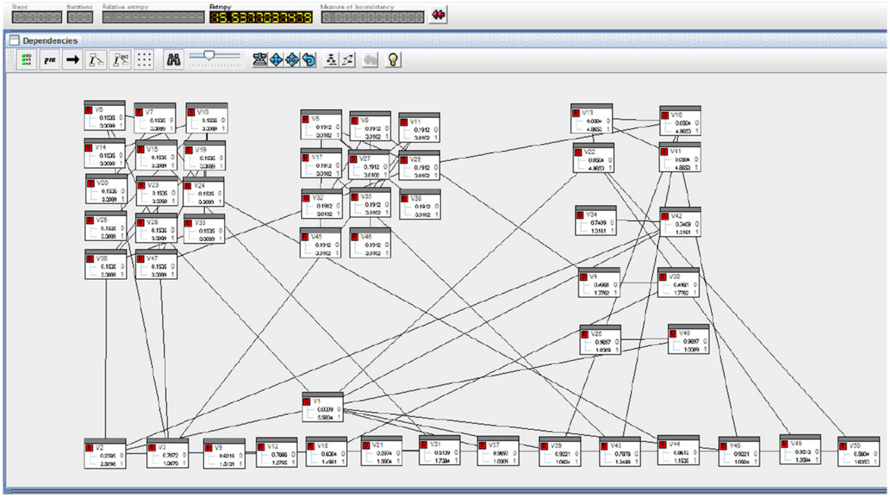


Fig. 8 A signed Kronecker graph of 50 vertices (actors) and their structural power

3. Make $B(i, j), i \neq j$, equal to the minimum of $A(i, j)$ and $A(j, i)$. B then is a symmetric matrix of negative adjacencies, forming a PO-net.
4. Choose vertices at random and suround them by SCS, also at random.
5. Check all SCS in the net. Eliminate negative links between vertices within each SCS, thus forming a clusterable sN.

With this process, we produced 28 negative edges, 28 positive edges and 14 isolated vertices; the latter are eliminated. Further characteristics of the net read:

- 15 degenerated SCS with 1 vertex,
- 3 SCS with 2 vertices,
- 1 SCS with 4 vertices,
- 1 SCS comprising 11 vertices,
- 1 SCS comprising 14 vertices.

Figure 8 shows the signed Kronecker graph plus structural power of each vertex. We arranged the vertices in such a way that the SCS are apparent:

- the 14 vertices SCS top left,
- the 11 vertices SCS top middle,
- the 4 vertices SCS top right, followed by the 2 vertices SCS,
- the 15 degenerated SCS bottom.

All vertices in an SCS have equal power as we developed earlier. We highlight the most important results.

- The degenerated SCS V_1 has greatest structural power $sp = 5.56$; it is negative adjacent to the 14 vertices SCS, to the 4 vertices SCS, to the upper and lower 2 vertices SCS and some degenerated SCS. These negative adjacencies make V_1 powerful. Evidencing $V_1 = 1$ makes them all powerless. Evidencing $V_1 = 1$ does reach neither the degenerated V_{12} nor V_{44}, however. V_1 and V_{12} or V_{44} are no "neighbours". There is no negative edge linking them. Wether a powerful V_1 makes V_{12} or V_{44} powerful or powerless, the system does not know. It makes the structural power of V_{12} a neutral 1. The same holds for V_{44}.

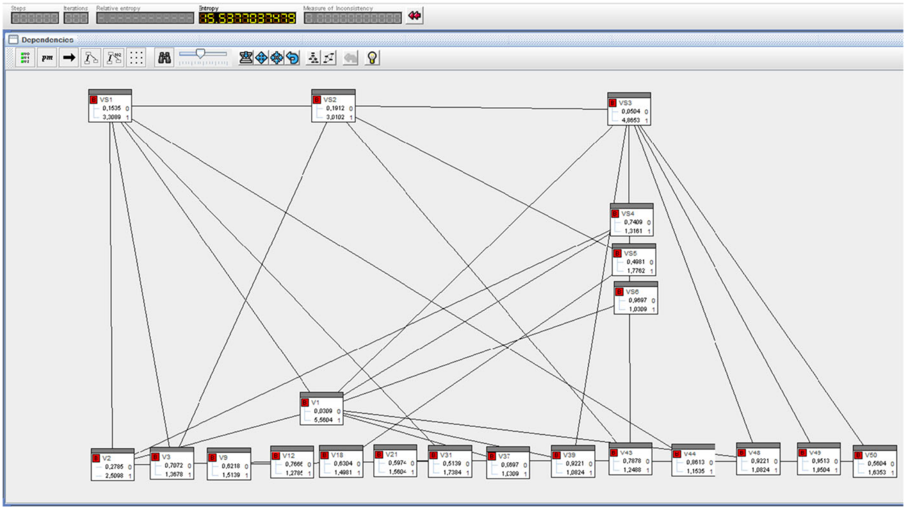


Fig. 9 Compressed network from Fig. 8

- After V_1, the next powerful SCS is the one with 4 vertices and $sp = 4.8$, each. Followed by the SCS with 14 vertices, then with 11 vertices, etc. We notice that structural power is not a function of an SCS's cardinality but rather of its dominance over others.
- The entropy as a measure of conditional independence in the net amounts to $H = 15.538$. Each evidence of power in a vertex reduces the entropy. When evidencing V_1, then V_{29} as a member of the 11 vertices SCS, then the degenerated SCSs V_{12}, V_{18}, V_{31}, V_{44}, V_{49}, entropy becomes 0. All conditional independence vanishes.

Now, we study the consequences of mathematical results for the criminial clans, as announced:

- All subclans in an SCS support each other and have equal structural power.
- Structural power between different clans varies significantly: from $sp = 5.56$ for the degenerated V_1, via $sp = 4.87$ for the 4 vertices SCS and $sp = 3.31$ for the 14 vertices SCS to $sp = 1.03$ for V_{37}.
- Even the small clan V_1 or the small clan of 4 subclans can damage many other clans. Once they find out about a quick police raid they broadcast true news to friends and scatter fake news to enemies.

Figure 8 contains 50 (sub-)clans and hence might be somewhat labyrinthine. In Definition 3 and the subsequent lemma, we developed the concept of compression. An SCS can be compressed to a supervertex and such compression is power-invariant. If we repeat such compression for all SCS, the result is a mere PO-net; see again Sect. 4.4 right after the lemma. Figure 9 shows all supervertices of our network.

For a better transparency, we arranged them similar as in Fig. 8. We verify power invariance: the structural power of a superclan equals that of each subclan before compression. And entropy as a measure of mutual independence does not change, either.

This kind of analyses in signed networks gives new insights beyond a mere description of their characteristics, such as number of negative/ positive edges, number of triads, etc. It shows the power pattern all over the net and might be the basis for reshaping such pattern.

Hopefully, we are on the right way towards a deeper understanding of power determinants and power emergents in modern societies.

6 Summary and future work

Measuring power in a social fabric was the focus of the following paper: “An entropy-based framework to analyse structural power and alliances in social networks”; see Dellnitz and Rödder (2020). There power is derived from mutual suppression as a driving force among actors. In the present paper, we go a step further and consider mutual suppression and mutual support, modeled by negative and positive edges in a graph. Even in such complex signed networks, power is measurable for each actor $\widehat{=}$ vertex. But for this, nets must be balanced or – more precisely – must be clusterable. Power then is the degree of influence of an actor upon the whole net rather than upon its neighbours, only. To be able to calculate this power in bigger nets, we need an optimisation software called (SPIRIT, 2011).

Clusterability in signed networks is a pretty rigid demand. Roughly speaking: My friends (mutual support) should not be enemies (mutual supression). We generate such a net with the property of clusterability and apply it to a group of 50 criminal (sub-)clans which partly support each other and partly try to damage each other. To make the net realistic, we use the technique of stochastic Kronecker matrices.

As we mentioned above, is clusterability a rigid demand, but necessary for a respective determination of power indices. Here immediately the question arises whether this rigid demand can be weakened. Is it possible to calculate entropy-based power indices even when clusterability is missing?

In the present paper, we study nets with positive and negative edges representing friendship and enmity. Are there further relations which can be embedded into an entropy-based framework? To answer these questions is a must and a rule for our future work.

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Data availability The datasets generated during and analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interests All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

Appendix A: Proof of Theorem 2d

Proof In an incomplete PO-net, all vertices have a structural power > 1.
Choose an arbitrary V_i and an arbitrary adjacent V_j.

1. A PO-net allows positive probabilities only if not $V_i = 1$, $V_j = 1$ for adjacent vertices.
2. If a configuration with $V_i = 1$ shows a positive probability, then also the configuration with $V_i = 0$ and identical, otherwise, does.
3. Choose configuration $\bar{\mathbf{v}}$ with $V_i = 1$, $V_j = 1$, $V_k = 0$, all $V_k \in \mathcal{V}\backslash\{V_i, V_j\}$. $Q_-(\bar{\mathbf{v}}) = 0$, as V_i, V_j are adjacent. Now, make $\bar{\mathbf{v}} \rightarrow \bar{\bar{\mathbf{v}}}$ by changing $V_i = 1$ to $V_i = 0$, resulting in $Q_-(\bar{\bar{\mathbf{v}}}) > 0$.
4. Because of the MaxEnt principle all positive probabilities in Q_- equal.
5. $\sum_{\mathbf{v}\text{ with } V_i=1} Q_-(\mathbf{v}) < \sum_{\mathbf{v}\text{ with } V_i=0} Q_-(\mathbf{v})$
   $\Longrightarrow Q_-(V_i = 1) < Q_-(V_i = 0)$
   $\Longrightarrow Q_-(V_i = 1) < \frac{1}{2}$
   $\Longrightarrow -\log_2(Q_-(V_i = 1)) > 1$

□

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