



On stratified sampling for estimating coalitional values

A. Saavedra-Nieves¹

Accepted: 24 October 2022 / Published online: 6 November 2022
© The Author(s) 2022

Abstract

This paper addresses two sampling methodologies to respectively estimate the Owen value and the Banzhaf–Owen value for TU-games with a priori unions. Both proposals are based on stratified sampling on the set of those coalitions that are compatible with the system of unions according to their cardinalities. These sampling methodologies are analysed in terms of the theoretical properties and of the establishment of bounds for the absolute error from a statistical point of view. Finally, we evaluate the performance of these tools on several real well-known examples in the literature.

Keywords Multi-agent systems · Games with a priori unions · Coalitional values · Sampling

1 Introduction

The notion of cooperation bases any real multi-agent interactive situation. The collaboration among those agents involved in this class of problems is often presented as an essential requirement to achieve a common goal. From a mathematical point of view, transferable utility games (or *TU-games*) are the tools for modelling this kind of situations. Besides, allocating the profits or the costs resulting from the cooperation becomes a relevant issue to be dealt with in cooperative game theory. In this line of research, coalitional values have received much attention in the game-theoretical literature.

Formally, a *coalitional value* assigns an allocation vector for each TU-game that implicitly confers a certain compromise to be enforced for all players in the case of being accepted. The *Shapley value* (Shapley, 1953) and the *Banzhaf value* (Banzhaf, 1964) are considered two of the most relevant examples of coalitional values in literature, that are obtained by weighty averaging the marginal contributions of a player to those coalitions to which it does not belong. However, these coalitional values do not contemplate the possibility of existing a structure of coalitions that have previously committed themselves to cooperate in a TU-game, as in many real problems. For instance, different situations in politics, logistics or focused on cost sharing address this assumption. To model them, transferable utility games with a priori unions (or *TU-games with a priori unions*) are used as an extension of TU-games to this

✉ A. Saavedra-Nieves
alejandro.saavedra.nieves@usc.es

¹ Departamento de Estatística, Análise Matemática e Optimización, Universidade de Santiago de Compostela, 15705 Santiago de Compostela, Spain

context. Analogously, the called *Owen value* (Owen, 1977) is introduced as the extension of the Shapley value and Owen (1982) introduces the *Banzhaf–Owen value* as the corresponding extension of the Banzhaf value.

Undoubtedly, it is well known how difficult it is to exactly calculate the Shapley value and the Banzhaf value when the number of agents involved significantly increases. By its definition, the total amount of marginal contributions to be evaluated exponentially increases. However, this drawback is not unrelated to the context of TU-games with a priori unions. Under this perspective, Deng and Papadimitriou (1994) study the coalitional values from a complexity theoretic point of view. Many references focus on potential applications of coalitional values by providing, for specific classes of games, the procedure to polynomially compute them by consequently reducing the associated computational complexity. Among others, Algaba et al. (2019) provides an extensive analysis of the computational aspects and the applications of the Shapley value. However, this is not the only reference that focuses on this task. Firstly, we mention those linked to a cost setting. For instance, Littlechild and Owen (1973) and Vázquez-Brage et al. (1997) obtain a polynomial expression for the Shapley and the Owen values for sharing the costs by the planes' operation in airport games, respectively. In maintenance of railway infrastructures, Costa (2016) and Fragnelli et al. (2000) provide the exact expressions for the above-mentioned coalitional values as cost sharing mechanisms. Another approach is the one introduced by Moretti et al. (2007) and Lucchetti et al. (2010), that characterize the Banzhaf value and the Shapley for microarray games and that apply them to identify those relevant genes in cancer diseases. In addition to the above, coalitional values have been widely used in voting situations in the literature. The proposals of Owen (1972, 1975), based on direct enumeration and approximation methodologies, are used by Leech (2002, 2003) for computing power indices for TU-games. Algaba et al. (2003) use generating functions to efficiently compute power indices for weighted multiple majority games. After, Algaba et al. (2007) and Alonso-Mejide and Bowles (2005) make use of these procedures to rank the members of the International Monetary Fund (IMF) and of the enlarged European Union according to the power that several coalitional values indicate, respectively. Alonso-Mejide et al. (2009) does the same task with the members of the European Union.

Sampling techniques (Cochran, 2007) are an alternative for estimating the coalitional values when the complexity of their exact computation increases enough. It needs to be stressed that the above-mentioned coalitional values are obtained as the expected value of marginal contributions by considering alternative probability mass functions. Using this idea, the usage of sampling methodologies for estimating the population mean of a random variable seems natural.

The use of sampling techniques in this context is not new and they have been applied in a wide variety of papers. The initial reference to be mentioned is Mann and Shapley (1960), that estimate the Shapley value for voting situations. Extending this approach for general TU-games, Castro et al. (2009) and Fernández-García and Puerto-Albandoz (2006) provide an algorithm to estimate the Shapley value based on simple random sampling with replacement (in what follows, *SRS_{wr}*). Besides, Bachrach et al. (2010) also use this sampling technique to estimate the Banzhaf value for simple games. Under the existence of a system of a priori unions, Saavedra-Nieves et al. (2018) describe the algorithm to estimate the Owen value also based on *SRS_{wr}*. Besides, Saavedra-Nieves and Fiestras-Janeiro (2021) add the non-replacement hypothesis to *SRS_{wr}* for the Banzhaf–Owen value estimation. Alternatively, other approaches, as the one introduced in Benati et al. (2019) and based on the stochastic approximation of deterministic games, have been used. However, stratified sampling is used by Castro et al. (2017), Fernández-García and Puerto-Albandoz (2006) and Maleki (2015)

to approximate the Shapley value as a guarantee to reduce the variance of the estimators obtained under simple random sampling with replacement.

The main goal of this work is to propose two sampling methods to respectively approximate the Owen value and the Banzhaf–Owen value for a large class of TU-games with a priori unions, generalizing those results in Maleki (2015) and extending them to the case of the Banzhaf's values. Both mechanisms are based on stratified sampling techniques on the set of compatible coalitions with a system of a priori unions. The idea that bases their definition is the complete division of the population of compatible coalitions according to the sizes of the unions and the coalitions in the union that they are composed of. The use of stratified sampling in this setting can be naturally justified for the following two features:

- When simple random sampling with replacement is used to estimate, for instance, the Banzhaf–Owen value, all coalitions compatible with the partition are equally likely to be extracted, regardless of which stratum they belong to. It is noteworthy that, according to the criterion of defining the strata by the cardinality of the coalitions involved, not all the strata have the same weight within the population of compatible coalitions. This fact should be reflected in the extraction probabilities of each sampling unit.
- Stratifying the sampling units ensures that the population is previously divided into homogeneous subpopulations. Under stratified sampling, simple random sampling with replacement (SRS_{wr}) is performed in each of these subgroups, so that the resulting estimator is obtained as a weighted mean of the corresponding subpopulation estimators. From a statistical point of view, the usage of this methodology ensures a reduction in variability with respect to what is obtained under SRS_{wr} .

We analyse our sampling proposals by providing a collection of statistical results that thoroughly study those properties that the estimators of the Owen value and the Banzhaf–Owen value under stratified sampling respectively satisfy, as well as that provide theoretical bounds of the absolute error incurred in the estimation.

This paper is organized as follows. Section 2 introduces the formal definitions and notations on cooperative game theory. The procedures based on stratified sampling to estimate the Owen value and the Banzhaf–Owen value are described in Sect. 3. In particular, their statistical properties are studied and some theoretical results to bound the error are provided. Section 4 evaluates the performance of the sampling proposals from a computational point of view through the approximation of the above-mentioned coalitional values in three well-known examples where they can be exactly obtained. Some concluding remarks are deferred to Sect. 5. Finally, three appendices are included in an Online Resource section. Appendix A displays the numerical results in the estimation of the Banzhaf–Owen value as an index of the power for the members of the IMF in 2002. Appendix B shows the analogous results corresponding to the Owen value estimation under stratified sampling in the same context. Appendix C details the results of estimating the Owen value by using the proposal based on stratified sampling in a bankruptcy problem.

2 On coalitional values

In this section, we introduce theoretical terminology that is used to build the procedures for estimating coalitional values.

A *transferable utility cooperative game* (or TU-game) is given by a pair (N, v) , with $N = \{1, \dots, n\}$ the set of players and $v : 2^N \rightarrow \mathbb{R}$ a map that assigns to each coalition $S \subseteq N$ a real number $v(S)$ that represents the worth of the cooperation of the members of

S .¹ By convention, $v(\emptyset) = 0$. We denote by $|S|$ the number of elements of S . The class of TU-games with player set N is denoted by G^N .

A relevant issue in cooperative game theory focuses on the definition of procedures for allocating the worth of the cooperation among the involved players. To this aim, coalitional values as the Shapley value can be used. Alternatively, the Banzhaf value is usually used as ranking index due to its non-efficient character. However, the link between them is the fact of its definition in terms of the average of the contributions of a player to the set of coalitions that do not contain it. That is, fixed $i \in N$ and $(N, v) \in G^N$, player i 's *marginal contribution* to coalition $S \subseteq N \setminus \{i\}$ is given by

$$v(S \cup \{i\}) - v(S). \tag{1}$$

The *Shapley value* (Shapley, 1953) is formally defined, for every $i \in N$ and every $(N, v) \in G^N$, as

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S)). \tag{2}$$

Moreover, the *Banzhaf value* (Banzhaf, 1964) is defined for every $i \in N$ and every (N, v) by

$$Bz_i(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S)). \tag{3}$$

However, other approaches can be considered. A *TU-game with a system of a priori unions* is a triple (N, v, P) where (N, v) is a TU-game and $P = \{P_1, \dots, P_m\}$ is a partition of N , i.e., $\cup_{k=1}^m P_k = N$ and $P_k \cap P_l = \emptyset$ for $k \neq l$. P is interpreted as a coalition structure that restricts the cooperation among the players in N . U^N denotes the set of TU-games with a priori unions and with player set N . In what follows, we denote by $P_{(i)}$ the union to which i belongs and $p_i = |P_{(i)}|$. A coalition $T \subseteq N \setminus \{i\}$ is said to be *compatible with partition P* for a player i if $T = \cup_{P_l \in R} P_l \cup S$ for a coalition of unions $R \subseteq P \setminus P_{(i)}$ and a coalition of players $S \subseteq P_{(i)} \setminus \{i\}$.

The *Owen value* (Owen, 1977) is defined, for every $i \in N$ and every $(N, v, P) \in U^N$, as

$$O_i(N, v, P) = \sum_{R \subseteq P \setminus P_{(i)}} \sum_{S \subseteq P_{(i)} \setminus \{i\}} \frac{s!(p_i - s - 1)!r!(m - r - 1)!}{p_i!m!} (v(\cup_{P_l \in R} P_l \cup S \cup \{i\}) - v(\cup_{P_l \in R} P_l \cup S)). \tag{4}$$

Notice that it is considered as an extension of the Shapley value for TU-games with a priori unions since that $O(N, v, P) = Sh(N, v)$ if P contains either one or n a priori unions.

Owen (1982) extends the Banzhaf value to the context of TU-games with a system of a priori unions. The *Banzhaf–Owen value* for every $i \in N$ and every $(N, v, P) \in U^N$ is defined by

$$BzO_i(N, v, P) = \sum_{R \subseteq P \setminus P_{(i)}} \frac{1}{2^{m-1}} \sum_{S \subseteq P_{(i)} \setminus \{i\}} \frac{1}{2^{p_i-1}} (v(\cup_{P_l \in R} P_l \cup S \cup \{i\}) - v(\cup_{P_l \in R} P_l \cup S)) \tag{5}$$

where $P_{(i)} \in P$ such that $i \in P_{(i)}$ and $p_i = |P_{(i)}|$.

¹ In cost settings, a TU-game is known as a cost game and is usually denoted by (N, c) .

From a theoretical point of view, Alonso-Meijide et al. (2007) and Lorenzo-Freire (2017) are examples of references that characterize the above-mentioned coalitional values. In the literature, there are multiple applications of them on real-world problems. However, the main drawback in exactly computing coalitional values as the ones mentioned is computational. The complexity of this calculus increases when the set of players enlarges due to the number of elements to be evaluated increases exponentially.

3 A stratified sampling procedure to estimate coalitional values

In this section, we provide mechanisms to estimate the Owen value and the Banzhaf–Owen value for general games by using stratified sampling techniques. We initially introduce the procedure for their estimation and then, we analyse the properties from a statistical approach.

3.1 Numerical implementation of the sampling algorithm

Here, we propose a sampling methodology to estimate coalitional values, in particular the Owen value and the Banzhaf–Owen value for general TU-games, based on stratified sampling. Take $(N, v, P) \in U^N$ a TU-game with a system of a priori unions and $i \in N$ a fixed player such that $i \in P_{(i)} \in P$.

A stratified sampling procedure (Cochran, 2007) can be applied in this context due to the interpretation of the considered coalitional values of i in Expressions (4) and (5) as being a mean of means. The idea that justifies our proposal is based on the reformulation of the Owen value and the Banzhaf–Owen value as follows, in line with the proposal for the Shapley value estimation for a TU-game in Maleki (2015).

Let $(N, v, P) \in U^N$ be a game with a priori unions and let $i \in N$ be a player such that $i \in P_{(i)} \in P$. First, we consider the set of all possible coalitions distributed into strata. Fixed an index k , with $k \in \{0, 1, \dots, m - 1\}$, and an index h , with $h \in \{0, 1, \dots, p_i - 1\}$, we implicitly determine the strata C_{kh} of all compatible coalitions with k unions in $P \setminus P_{(i)}$ and with h players in $P_{(i)} \setminus \{i\}$.

Consequently, the Owen value allows for an alternative formulation that arises from grouping the addends in Expression (4) according to the cardinality of the evaluated coalitions. So, it can be rewritten as

$$O_i(N, v, P) = \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \left(\sum_{R \subseteq P \setminus P_{(i)}: |R|=k} \sum_{S \subseteq P_{(i)} \setminus \{i\}: |S|=h} \frac{1}{\binom{m-1}{k} \binom{p_i-1}{h}} \left(v \left(\bigcup_{P_l \in R} P_l \cup S \cup \{i\} \right) - v \left(\bigcup_{P_l \in R} P_l \cup S \right) \right) \right) \tag{6}$$

or, alternatively,

$$O_i(N, v, P) = \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} E_i^{kh}, \tag{7}$$

where the expected marginal contribution of player i within strata C_{kh} is denoted by

$$E_i^{kh} = \sum_{R \subseteq P \setminus P_{(i)}: |R|=k} \sum_{S \subseteq P_{(i)} \setminus \{i\}: |S|=h} \frac{1}{\binom{m-1}{k} \binom{p_i-1}{h}} \left(v \left(\bigcup_{P_l \in R} P_l \cup S \cup \{i\} \right) - v \left(\bigcup_{P_l \in R} P_l \cup S \right) \right).$$

Analogously, the Banzhaf–Owen value for (N, v, P) supports the alternative formulation shown below. Thus,

$$BzO_i(N, v, P) = \frac{1}{2^{m-1} 2^{p_i-1}} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \left(\sum_{R \subseteq P \setminus P_{(i)}: |R|=k} \sum_{S \subseteq P_{(i)} \setminus \{i\}: |S|=h} \left(v \left(\bigcup_{P_l \in R} P_l \cup S \cup \{i\} \right) - v \left(\bigcup_{P_l \in R} P_l \cup S \right) \right) \right) \tag{8}$$

or, equivalently, as

$$BzO_i(N, v, P) = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh} E_i^{kh}, \tag{9}$$

where $W_{kh} = \frac{\binom{m-1}{k} \binom{p_i-1}{h}}{2^{m-1} 2^{p_i-1}}$ and E_i^{kh} is the expected marginal contribution of player i within strata C_{kh} .

For a fixed $i \in N$, our procedure is the following. We firstly take with replacement a sample of coalitions of size k composed by unions other than the one to which i belongs. Secondly, for each element of the first sample, we take with replacement a coalition in $P_{(i)} \setminus \{i\}$ of size h . From the formal definition, the Owen value or the Banzhaf–Owen value estimation for i corresponds to different alternatives of averaging the means of the corresponding player i 's marginal contributions in the different strata.

We formally describe the algorithm. Given a TU-game with a priori unions $(N, v, P) \in U^N$, a system of a priori unions $P = \{P_1, \dots, P_m\}$, and an arbitrary player $i \in N$, the steps of the sampling procedure are described below.

1. The population of the sampling process is the set of compatible coalitions with P for i .
2. The parameter under study is $O_i(N, v, P)$ and $BzO_i(N, v, P)$, that is, the Owen value and Banzhaf–Owen value for player i .
3. The characteristic to study in each sampling unit corresponds to player i 's marginal contribution for each coalition T that is a compatible coalition with P for i . Then, if we consider $T \subseteq N \setminus \{i\}$ in terms of the $R = \{P_k \mid P_k \subseteq T\}$ and $S = T \cap P_{(i)}$, we have

$$x(R, S)_i = v \left(\bigcup_{P_l \in R} P_l \cup S \cup \{i\} \right) - v \left(\bigcup_{P_l \in R} P_l \cup S \right).$$

4. The sampling procedure is as follows. Formally, we take with replacement in each strata C_{kh} a sample $S_{kh} = \{(R_1, S_1), \dots, (R_{\ell_{kh}}, S_{\ell_{kh}})\}$ of ℓ_{kh} pair of coalitions (R_j, S_j) such that $R_j \subseteq P \setminus P_{(i)}$, with $|R_j| = k$, and $S_j \subseteq P_{(i)} \setminus \{i\}$, with $|S_j| = h$, and for all $j \in \{1, \dots, \ell_{kh}\}$.

As a consequence of this sampling procedure, we obtain a sample of ℓ_{kh} compatible coalitions, where each element takes the form $\bigcup_{P_l \in R_j} P_l \cup S_j$ for $j = 1, \dots, \ell_{kh}$.

5. The estimations of O_i and BzO_i by stratified sampling are obtained as the mean of the marginal contributions over the samples in two different ways. We denote by \bar{E}_i^{kh} the estimation of E_i^{kh} . This approximation is obtained as the mean of the marginal contributions

over the sample \mathcal{S}_{kh} , i.e.

$$\bar{E}_i^{kh} = \frac{1}{\ell_{kh}} \sum_{t=1}^{\ell_{kh}} x(R_t, S_t)_i, \text{ for all } k \in \{0, \dots, m-1\} \text{ and for all } h \in \{0, \dots, p_i-1\}.$$

5.1. Thus, the estimation of O_i under stratified sampling, denoted by \bar{O}_i^{st} , is given by

$$\bar{O}_i^{st} = \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \bar{E}_i^{kh}. \tag{10}$$

5.2. Analogously, the estimation of BzO_i under stratified sampling, denoted by \overline{BzO}_i^{st} , is given by

$$\overline{BzO}_i^{st} = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \left(W_{kh} \bar{E}_i^{kh} \right) \tag{11}$$

where $W_{kh} = \frac{\binom{m-1}{k} \binom{p_i-1}{h}}{2^{m-1} 2^{p_i-1}}$ is the weight of strata C_{kh} in the population and $\ell = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \ell_{kh}$ denotes the size of the sample used for estimating BzO_i .

The pseudocode of the algorithm for estimating the Owen value and the Banzhaf–Owen value using stratified sampling is as follows.

Procedure 3.1 Let $(N, v, P) \in U^N$ be a TU-game with a priori unions and take $i \in N$.

Initialize $\bar{O}_i^{kh} = 0$ and $\overline{BzO}_i^{kh} = 0$ for all $k \in \{0, \dots, m-1\}$ and for all $h \in \{0, \dots, p_i-1\}$.

Take $\ell_{kh} > 0$ for all $k \in \{0, \dots, m-1\}$ and for all $h \in \{0, \dots, p_i-1\}$ and $\ell = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \ell_{kh}$.

Do $k = 0$.

while $k < m - 1$ **do**

 Do $h = 0$.

while $h < p_i - 1$ **do**

 Do $j = 0$.

while $j < \ell_{kh}$ **do**

 Do $j = j + 1$.

 Take $R_j \subseteq P \setminus P_{(i)}$ of size k and $S_j \subseteq P_{(i)} \setminus \{i\}$ of size h with replacement.

 Calculate $x(R_j, S_j)_i = v(\cup_{P_l \in R_j} P_l \cup S_j \cup \{i\}) - v(\cup_{P_l \in R_j} P_l \cup S_j)$.

 Do $\bar{E}_i^{kh} = \bar{E}_i^{kh} + \frac{1}{\ell_{kh}} x(R_j, S_j)_i$.

end while

 Do $h = h + 1$.

end while

 Do $k = k + 1$.

end while

Finally, $\bar{O}_i^{st} = \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \bar{E}_i^{kh}$ and $\overline{BzO}_i^{st} = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh} \bar{E}_i^{kh}$.

When applying this procedure for all $i \in N$, the vectors $\bar{O}^{st} = (\bar{O}_1^{st}, \dots, \bar{O}_n^{st})$ and $\overline{BzO}^{st} = (\overline{BzO}_1^{st}, \dots, \overline{BzO}_n^{st})$ correspond to the estimation of the Owen value and the Banzhaf–Owen value for the game with a system of a priori unions (N, v, P) , respectively.

Before the theoretical analysis from a statistical approach, we introduce a brief game-theoretical remark. The estimated Owen value is not efficient under stratification in the sense that $\sum_{i=1}^n \bar{O}_i^{st}$ is not necessarily equal to $v(N)$. Following the ideas of Castro et al. (2017), an efficient estimator for the Owen value can be directly obtained as

$$\bar{O}_i^{eff} = \bar{O}_i^{st} + \left(v(N) - \sum_{i=1}^n \bar{O}_i^{st} \right) \frac{\bar{O}_i^{st}}{\sum_{i=1}^n \bar{O}_i^{st}}, \text{ for each } i \in N.$$

The first problem that arises when using a stratified sampling procedure consists of deciding how to divide the total amount of samples among the different strata. This step is called the allocation procedure in sampling. Implicitly, it determines in which strata the sampling methodology under consideration has to make a large effort. For instance, Neyman (1934) establishes a specific procedure for determining the size of the sample that minimizes the variance of the estimators under stratified sampling and, as a consequence, the error in the estimation. However, the main drawback when using this mechanism in large-scale problems is the computational difficulty arisen from its obtaining.

Fixed $i \in N$, take $\ell > 0$ as the total amount of samples that we extract from the population under stratified sampling. Below, we distinguish two scenarios.

- The simplest allocation procedure consists of assigning the same number of samples from each strata. This procedure is named as the *uniform allocation procedure* under stratified sampling. Its usage is only recommended for those cases in which no information is available about variability of units within strata. Thus, if $\lceil \cdot \rceil$ denotes the ceiling function,

$$\ell_{kh} = \left\lceil \frac{\ell}{m \cdot p_i} \right\rceil$$

for each strata C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. Under this approach, we have to extract the same amount of samples per strata.

- As the strata sizes are not necessarily equal, a *proportional allocation procedure* is used to maintain a steady sampling fraction throughout the population. The total sample size, ℓ , should be allocated to the strata proportionally to their sizes. Formally, take a strata of compatible coalitions C_{kh} for a given $k \in \{0, \dots, m - 1\}$ and a given $h \in \{0, \dots, p_i - 1\}$. The total amount of compatible coalitions for i formed by k unions in $P \setminus P_{(i)}$ and h players in $P_{(i)} \setminus \{i\}$ is $\binom{m-1}{k} \binom{p_i-1}{h}$. Thus, the *proportional allocation procedure* assigns to each strata C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$, a total amount of coalitions to be extracted from it equal to

$$\ell_{kh} = \lceil \ell \cdot W_{kh} \rceil,$$

where $W_{kh} = \frac{\binom{m-1}{k} \binom{p_i-1}{h}}{2^{m-1} 2^{p_i-1}}$ is the weight of the strata. Notice that this approach assigns more effort for those larger strata.

In what follows, we will analyse the statistical properties of the proposed estimators in estimating the Owen value and the Banzhaf–Owen value. For each of them, we will further distinguish two scenarios according to the sampling allocation procedure used.

3.2 The estimator of the Owen value

In this section, we focus on analysing the properties of the estimator of the Owen value for player i , O_i^{st} , from a statistical perspective. In this section, we only include the most relevant theoretical results.

By the interpretation of the estimator as a mean of means, we firstly analyse the properties of \bar{E}_i^{kh} . We have that $\bar{E}_i^{kh} = \frac{1}{\ell_{kh}} \sum_{k=1}^{\ell_{kh}} x(R_j, S_j)_i$ is the unbiased estimator of E_i^{kh} , that is, the theoretical mean of player i 's marginal contributions using all compatible coalitions associated to strata C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. Thus, $\mathbb{E}(\bar{E}_i^{kh}) = E_i^{kh}$. In terms of its variance, it holds that $Var(\bar{E}_i^{kh}) = \frac{\theta_{kh}^2}{\ell_{kh}}$, where θ_{kh}^2 denotes the variance of the marginal contributions in the strata with respect to its theoretical mean E_i^{kh} given by $\theta_{kh}^2 = \frac{1}{\binom{m-1}{k} \binom{p_i-1}{h}} \sum_{R \subseteq P \setminus P(i): |R|=k} \sum_{S \subseteq P(i) \setminus \{i\}: |S|=h} (x(R, S)_i - E_i^{kh})^2$.

First, according to the definition of this estimator O_i^{st} , we know that it is unbiased since $\mathbb{E}(\bar{O}_i^{st}) = O_i$, where $\mathbb{E}(\cdot)$ is the mean operator. In terms of the variability, this estimator satisfies $Var(\bar{O}_i^{st}) = \mathbb{E}((\bar{O}_i^{st} - O_i)^2)$. Formally,

$$Var(\bar{O}_i^{st}) = \frac{1}{m^2 \cdot p_i^2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} Var(\bar{E}_i^{kh}) = \frac{1}{m^2 \cdot p_i^2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \frac{\theta_{kh}^2}{\ell_{kh}}$$

According to the procedure for allocating the sample among the considered strata by using stratified sampling, we distinguish two possibilities.

- First, if the uniform allocation procedure is used, $\ell_{kh} = \lceil \frac{\ell}{m \cdot p_i} \rceil$ for each C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. Thus, we have that

$$Var(\bar{O}_i^{st}) = \frac{1}{m^2 \cdot p_i^2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \frac{\theta_{kh}^2}{\lceil \frac{\ell}{m \cdot p_i} \rceil}$$

- Otherwise, under the proportional allocation procedure, $\ell_{kh} = \lceil \ell \cdot W_{kh} \rceil$ for each C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$,

$$Var(\bar{O}_i^{st}) = \frac{1}{m^2 \cdot p_i^2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \frac{\theta_{kh}^2}{\lceil \ell W_{kh} \rceil}$$

Then, the mean squared error (MSE) of \bar{O}_i^{st} is

$$MSE(\bar{O}_i^{st}) = (\mathbb{E}(\bar{O}_i^{st}) - O_i)^2 + Var(\bar{O}_i^{st})$$

By the unbiased character of the estimator, $MSE(\bar{O}_i^{st}) = Var(\bar{O}_i^{st})$. In terms of the variance of \bar{O}_i^{st} , we have ensured that its value is zero when the sample size is equal to the population size. Then, the estimator is consistent since that

$$\lim_{\ell \rightarrow 2^{p_i-1} 2^{m-1}} MSE(\bar{O}_i^{st}) = 0$$

It is important to remark that the exact value of $Var(\bar{O}_i^{st})$ is usually unknown. Thus, its value can be estimated by

$$\widehat{Var}(\bar{O}_i^{st}) = \frac{1}{m^2 \cdot p_i^2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \frac{\hat{\theta}_{kh}^2}{\ell_{kh}}, \tag{12}$$

being $\hat{\theta}_{kh}^2$ the unbiased estimation of the variance θ_{kh}^2 on the sample S_{kh} , whose expression is

$$\hat{\theta}_{kh}^2 = \frac{1}{\ell_{kh} - 1} \sum_{j=1}^{\ell_{kh}} (x(R_j, S_j)_i - \bar{E}_i^{kh})^2. \tag{13}$$

3.2.1 Error analysis of the Owen value estimation

A fundamental issue in the problem we are dealing with focuses on bounding the error in the estimation, that is, the difference between the approximated value and the exact one. This error is often not possible to be measured and thus, a probabilistic bound on this is theoretically provided instead. In words, we have that the approximation is at a distance greater than ε of the real value with a probability α as maximum. Formally, this is equivalent to

$$\mathbb{P}(|\bar{O}_i^{st} - O_i| \geq \varepsilon) \leq \alpha, \text{ with } \varepsilon > 0 \text{ and } \alpha \in (0, 1].$$

Thus, the estimated value usually becomes a good approximation of the real one, in this case of the Owen value and under stratified sampling, when sampling sizes sufficiently enlarge.

Moreover, the availability of information on the marginal contributions for each player $i \in N$ results fundamental for bounding the error in the estimation. From a mathematical point of view, *Hoeffding's inequality* (Hoeffding, 1963) results useful. It states that if $\sum_{j=1}^k X_j$ denotes the sum of k observations X_1, \dots, X_k drawn with replacement, with $a_j \leq X_j \leq b_j$ for all $j \in \{1, \dots, k\}$, then

$$\mathbb{P}(|\sum_{j=1}^k X_j - \mathbb{E}(\sum_{j=1}^k X_j)| \geq t) \leq 2 \exp\left(\frac{-2t^2}{\sum_{j=1}^k (b_j - a_j)^2}\right), \text{ for all } t \geq 0. \tag{14}$$

Firstly, we provide a general result for bounding the error in the Owen value estimation based on Hoeffding's inequality. Notice that the resulting bound coincides with that one obtained for the case of simple random sampling with replacement, as in Bachrach et al. (2010).

Proposition 3.2 *Let $(N, v, P) \in U^N$ be a TU-game with a priori unions. Take $\varepsilon > 0$, $\alpha \in (0, 1]$ and denote the range of the marginal contributions for (N, v, P) by*

$$w_{i,kh} = \max_{(R,S),(R',S') \in C_{kh}} (x(R, S)_i - x(R', S')_i),$$

for all $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$ and by w_i , the maximum of these values. Then,

$$\ell \geq \frac{\ln(2/\alpha)w_i^2}{2\varepsilon^2} \text{ implies that } \mathbb{P}(|\bar{O}_i^{st} - O_i| \geq \varepsilon) \leq \alpha.$$

Proof Clearly, since that $\bar{O}_i^{st} = \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \left(\frac{1}{\ell_{kh}} \sum_{t=1}^{\ell_{kh}} x(R_t, S_t)_i \right)$ for a sample, it is easy to check that

$$\begin{aligned} \mathbb{P} \left(|\bar{O}_i^{st} - O_i| \geq \varepsilon \right) &= \mathbb{P} \left(|\bar{O}_i^{st} - \mathbb{E}(\bar{O}_i^{st})| \geq \varepsilon \right) \\ &= \mathbb{P} \left(\left| \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \left(\frac{1}{\ell_{kh}} \sum_{t=1}^{\ell_{kh}} x(R_t, S_t)_i \right) - \mathbb{E} \left(\frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \left(\frac{1}{\ell_{kh}} \sum_{t=1}^{\ell_{kh}} x(R_t, S_t)_i \right) \right) \right| \geq \varepsilon \right) \\ &= \mathbb{P} \left(\left| \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sum_{t=1}^{\ell_{kh}} \frac{x(R_t, S_t)_i}{m p_i \ell_{kh}} - \mathbb{E} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sum_{t=1}^{\ell_{kh}} \frac{x(R_t, S_t)_i}{m p_i \ell_{kh}} \right) \right| \geq \varepsilon \right). \end{aligned} \tag{15}$$

Thus, we can distinguish two possibilities according to the allocation procedure used in each case.

- First, if we use the uniform allocation procedure, $\ell_{kh} = \lceil \frac{\ell}{m \cdot p_i} \rceil$ for each $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. As approximation, we take $\ell_{kh} = \frac{\ell}{m \cdot p_i}$. Thus, the final expression in (15) can be written as:

$$\mathbb{P} \left(\left| \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sum_{t=1}^{\ell_{kh}} x(R_t, S_t)_i - \mathbb{E} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sum_{t=1}^{\ell_{kh}} x(R_t, S_t)_i \right) \right| \geq \varepsilon \ell \right) \tag{16}$$

- Secondly, under the proportional allocation procedure, $\ell_{kh} = \lceil \ell \cdot W_{kh} \rceil$ for each $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. We can naturally consider that $\ell_{kh} \approx \ell \cdot W_{kh}$. Consequently, the final expression in (15) can be written as:

$$\mathbb{P} \left(\left| \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sum_{t=1}^{\ell_{kh}} \frac{x(R_t, S_t)_i}{m p_i W_{kh}} - \mathbb{E} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sum_{t=1}^{\ell_{kh}} \frac{x(R_t, S_t)_i}{m p_i W_{kh}} \right) \right| \geq \varepsilon \ell \right) \tag{17}$$

It is possible to check that the quantities in (16) and (17) are upper-bounded by $2 \exp \left(\frac{-2\varepsilon^2 \ell}{w_i^2} \right)$,

being $w_i = \max \{w_{i,kh}\}$. Taking $\alpha > 0$ such that $2 \exp \left(\frac{-2\varepsilon^2 \ell}{w_i^2} \right) \leq \alpha$, we conclude the proof. \square

Secondly, we provide a general result for bounding the error in the estimation of E_i^{kh} , with $k \in \{0, 1, \dots, m - 1\}$ and $h \in \{0, 1, \dots, p_i - 1\}$, based on the range of the marginal contributions. Before, we introduce a statement based on Hoeffding’s concentration inequality in order to bound the error in the estimation of E_i^{kh} for C_{kh} when using the range of marginal contributions.

Proposition 3.3 *Let $(N, v, P) \in U^N$ be a TU-game with a priori unions. Take $\varepsilon > 0$ and $\alpha \in (0, 1]$. Then,*

$$\ell_{kh} \geq \frac{\ln(2/\alpha) w_{i,kh}^2}{2\varepsilon^2} \text{ implies that } \mathbb{P}(|\bar{E}_i^{kh} - E_i^{kh}| \geq \varepsilon) \leq \alpha,$$

being $w_{i,kh}$ the range of the marginal contributions of player i for (N, v, P) .

Proof Clearly, since that $\bar{E}_i^{kh} = \frac{1}{\ell_{kh}} \sum_{(R,S) \in \mathcal{S}_{kh}} x(R, S)_i$ for a sample of ℓ_{kh} elements,

$$\begin{aligned} \mathbb{P}(|\bar{E}_i^{kh} - E_i^{kh}| \geq \varepsilon) &= \mathbb{P}(|\bar{E}_i^{kh} - \mathbb{E}(\bar{E}_i^{kh})| \geq \varepsilon) \\ &= \mathbb{P}\left(\left| \sum_{(R,S) \in \mathcal{S}_{kh}} x(R, S)_i - \mathbb{E}\left(\sum_{(R,S) \in \mathcal{S}_{kh}} x(R, S)_i\right)\right| \geq \varepsilon \ell_{kh}\right). \end{aligned}$$

Applying Hoeffding’s inequality (14), it holds that

$$\mathbb{P}\left(\left| \sum_{(R,S) \in \mathcal{S}_{kh}} x(R, S)_i - \mathbb{E}\left(\sum_{(R,S) \in \mathcal{S}_{kh}} x(R, S)_i\right)\right| \geq \varepsilon \ell_{kh}\right) \leq 2 \exp\left(\frac{-2\varepsilon^2 \ell_{kh}}{w_{i,kh}^2}\right) \leq \alpha,$$

concluding the proof. □

Using the formulation of the Owen value in Expression (7), the total estimation error of O_i^{st} under stratified sampling can be bounded as follows

$$\begin{aligned} |O_i - \bar{O}_i^{st}| &\leq \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} |E_i^{kh} - \bar{E}_i^{kh}| \\ &\leq \frac{1}{m \cdot p_i} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{\ln(2/\alpha) w_{kh}^2}{2 \ell_{kh}}} = \frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2}} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{w_{i,kh}^2}{\ell_{kh}}} \end{aligned} \tag{18}$$

where the second inequality holds by applying Hoeffding’s inequality for bounding the estimation error of E_i^{kh} on each strata C_{kh} .

Thus, we provide the following result that generalizes the analysis of the error in this setting.

Proposition 3.4 *Let $(N, v, P) \in U^N$ be a TU-game with a priori unions. Take $\varepsilon > 0$, $\alpha \in (0, 1]$.*

- *If the uniform allocation procedure is used,*

$$\ell \geq \frac{1}{\varepsilon^2} \frac{\ln(2/\alpha)}{2 \cdot m \cdot p_i} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} w_{i,kh}\right)^2, \text{ ensures that } |O_i - \bar{O}_i^{st}| \leq \varepsilon.$$

- *If the proportional allocation procedure is used*

$$\ell \geq \frac{1}{\varepsilon^2} \frac{\ln(2/\alpha)}{2 \cdot m^2 \cdot p_i^2} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \frac{w_{i,kh}}{\sqrt{W_{kh}}}\right)^2, \text{ ensures that } |O_i - \bar{O}_i^{st}| \leq \varepsilon.$$

Proof In view of the inequality in (18), it holds that

$$|O_i - \bar{O}_i^{st}| \leq \frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2}} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{w_{i,kh}^2}{\ell_{kh}}}$$

Thus, if we distinguish by the allocation procedure used under stratification, we have the following.

- By using the uniform procedure, we have $\ell_{kh} = \lceil \frac{\ell}{m \cdot p_i} \rceil$ for each strata C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. Besides, it satisfies that $\ell_{kh} \geq \frac{\ell}{m \cdot p_i}$. Thus, we only have to take $\varepsilon > 0$ such that

$$\frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{w_{i,kh}^2}{\ell_{kh}}}} \leq \frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{w_{i,kh}^2}{\ell/(m \cdot p_i)}}} \leq \varepsilon.$$

- If we use the proportional procedure, we have $\ell_{kh} = \lceil \ell W_{kh} \rceil$ for each strata C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. Besides, it holds that $\ell_{kh} \geq \ell W_{kh}$. Then, it satisfies that

$$\begin{aligned} \frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{w_{i,kh}^2}{\ell_{kh}}}} &\leq \frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} w_{i,kh} \min \left\{ \frac{1}{\sqrt{\ell W_{kh}}}, 1 \right\}} \\ &\leq \frac{1}{m \cdot p_i} \sqrt{\frac{\ln(2/\alpha)}{2} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{\frac{w_{i,kh}^2}{\ell W_{kh}}}} \leq \varepsilon \end{aligned} \tag{19}$$

for a given $\varepsilon > 0$.

This concludes the proof. □

3.3 The estimator of the Banzhaf–Owen value

Here, we analyse the statistical properties of the estimator of the Banzhaf–Owen value for player i , BzO_i^{st} . The study that we do is analogous to the one in the previous section. We refer to it when it is necessary.

From the previous section, it is easy to check that the estimator \overline{BzO}^{st} is unbiased since that $\mathbb{E}(\overline{BzO}_i^{st}) = BzO_i$ for all $i \in N$. The variance under stratification is given by $\text{Var}(\overline{BzO}_i^{st}) = \mathbb{E}((\overline{BzO}_i^{st} - BzO_i)^2)$ and, more explicitly, by

$$\text{Var}(\overline{BzO}_i^{st}) = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh}^2 \text{Var}(\overline{E}_i^{kh}) = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh}^2 \frac{\theta_{kh}^2}{\ell_{kh}}.$$

Again, this expression takes a different form depending on the procedure considered for allocating the sample among the strata under stratified sampling.

- When using the uniform allocation procedure, as $\ell_{kh} = \lceil \frac{\ell}{m \cdot p_i} \rceil$ for each C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$, we have that

$$\text{Var}(\overline{BzO}_i^{st}) = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh}^2 \frac{\theta_{kh}^2}{\lceil \frac{\ell}{m \cdot p_i} \rceil}.$$

- If we use a proportional allocation procedure, $\ell_{kh} = \lceil \ell \cdot W_{kh} \rceil$ for each C_{kh} , with $k \in \{0, \dots, m - 1\}$ and $h \in \{0, \dots, p_i - 1\}$. Hence,

$$\text{Var}(\overline{BzO}_i^{st}) = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh}^2 \frac{\theta_{kh}^2}{\lceil \ell W_{kh} \rceil}.$$

In this case, the mean squared error (MSE) of \overline{BzO}_i^{st} is

$$MSE(\overline{BzO}_i^{st}) = (\mathbb{E}(\overline{BzO}_i^{st}) - BzO_i)^2 + \text{Var}(\overline{BzO}_i^{st}).$$

By the null bias of the estimator, it holds that $MSE(\overline{BzO}_i^{st}) = \text{Var}(\overline{BzO}_i^{st})$. Moreover, according to the expressions of the variance of \overline{BzO}_i^{st} , it satisfies that its value reduces when the sample size tends to the population size. In this sense, the estimator is also consistent since that $\lim_{\ell \rightarrow 2^{p_i-1}2^{m-1}} MSE(\overline{BzO}_i^{st}) = 0$.

For those situation in which the exact value of $\text{Var}(\overline{BzO}_i^{st})$ is unknown, the estimation of its value is given by

$$\widehat{\text{Var}}(\overline{BzO}_i^{st}) = \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh}^2 \frac{\hat{\theta}_{kh}^2}{\ell_{kh}}, \tag{20}$$

being $\hat{\theta}_{kh}^2$ the unbiased estimation of the variance θ_{kh}^2 given in Expression (13).

3.3.1 Error analysis of the Banzhaf–Owen value estimation

Again, a fundamental issue in estimation consists in managing the accuracy of the estimator using the size of the sample. Also in this context, fixed $\varepsilon > 0$, it is desirable that \overline{BzO}_i^{st} is within a distance of ε from BzO_i . Formally, the problem can be formulated in terms of a confidence interval for BzO_i , that is

$$\mathbb{P}(|\overline{BzO}_i^{st} - BzO_i| \geq \varepsilon) \leq \alpha, \text{ with } \varepsilon > 0 \text{ and } \alpha \in (0, 1].$$

Therefore, fixed an accuracy of ε and a confidence level of $1 - \alpha$, which is the required sampling size to ensure this? Next we include some results that can be helpful to solve this question. For this purpose, we initially follow Proposition 3.2.

Proposition 3.5 *Let $(N, v, P) \in U^N$ be a TU-game with a priori unions. Take $\varepsilon > 0$ and $\alpha \in (0, 1]$. Then,*

$$\ell \geq \frac{\ln(2/\alpha)w_i^2}{2\varepsilon^2} \text{ implies that } \mathbb{P}(|\overline{BzO}_i^{st} - BzO_i| \geq \varepsilon) \leq \alpha,$$

being w_i the maximum of the range of the marginal contributions for (N, v, P) .

Proof This proof follows the scheme considered in Proposition 3.2. □

Following Proposition 3.3, we bound the error in estimating BzO_i^{st} because

$$\begin{aligned} |BzO_i - \overline{BzO}_i^{st}| &\leq \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh} |E_i^{kh} - \overline{E}_i^{kh}| \\ &\leq \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh} \sqrt{\frac{\ln(2/\alpha)w_{kh}^2}{2\ell_{kh}}} = \sqrt{\frac{\ln(2/\alpha)}{2}} \sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh} \sqrt{\frac{w_{i,kh}^2}{\ell_{kh}}}. \end{aligned} \tag{21}$$

The second inequality is satisfied due to the use of Hoeffding’s inequality for bounding the estimation error of BzO_i^{kh} on the strata C_{kh} .

Next result allows an analysis of the absolute error in the Banzhaf–Owen value estimation following Proposition 3.4.

Proposition 3.6 *Let $(N, v, P) \in U^N$ be a TU-game with a priori unions. Take $\varepsilon > 0$, $\alpha \in (0, 1]$.*

- *If the uniform allocation procedure is applied,*

$$\ell \geq \frac{1}{\varepsilon^2} \frac{\ln(2/\alpha)}{2} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} W_{kh} w_{i,kh} \right)^2, \text{ ensures that } |BzO_i - \overline{BzO}_i^{st}| \leq \varepsilon.$$

- *If the proportional allocation procedure is used*

$$\ell \geq \frac{1}{\varepsilon^2} \frac{\ln(2/\alpha)}{2} \left(\sum_{k=0}^{m-1} \sum_{h=0}^{p_i-1} \sqrt{W_{kh} w_{i,kh}} \right)^2, \text{ ensures that } |BzO_i - \overline{BzO}_i^{st}| \leq \varepsilon.$$

Proof This proof follows the scheme considered in Proposition 3.4. □

4 Empirical analysis

We illustrate the performance of our sampling procedures for some well-known examples in literature where the exact coalitional values can be exactly determined as a mechanism of distributing the worth of the cooperation. In these situations, the computation of the Banzhaf–Owen value and the Owen value is not an easy task when using the general formulations because the set of agents is sufficiently large. However, there exist polynomial expressions for both coalitional values in the considered examples, in such a way that it is possible to compare the simulation results with the exact ones.

The results included in this section have been performed by computing the proposed sampling methods into the statistical software R (R Core Team, 2022) on a personal computer with an Intel(R) Core(TM) i5-7400 and 8 GB of memory and a single 3.00GHz CPU processor. In particular, the usage of *sampling* package in R software makes easier to get coalitions under simple random sampling with and without replacement.

4.1 A weighted majority game: the IMF in 2002

The first example we consider is the weighted majority game analysed in Alonso-Mejide and Bowles (2005). In this context, coalitional values are usually considered to analyse the distribution of the players’ power.

A *weighted majority game* models those voting situations in which the involved agents have different weights. Moreover, there exists a quota $q > 0$ that imposes the majority in the voting and a collection of non-negative weights h_1, \dots, h_n , with h_i associated to each $i \in N$. Formally, a *weighted majority game* is formally given by a *simple game* (N, v) such that, for each $S \subseteq N$,

$$v(S) = 1, \text{ if } \sum_{i \in S} h_i - q \geq 0, \text{ and } v(S) = 0, \text{ otherwise.}$$

For instance, Alonso-Mejide and Bowles (2005) used the Banzhaf–Owen value to know the distribution of the power of the countries belonging to the Board of Governors of the International Monetary Fund (IMF). Each country has associated a weight given by its voting right and they are organized into constituencies that induce a partition of the countries. The

Table 1 Sample sizes for the weighted majority game by using Proposition 3.5

	$\varepsilon = 0.00100$	$\varepsilon = 0.00075$	$\varepsilon = 0.00050$
$\alpha = 0.1$	1,497,866	2,662,873	5,991,465
$\alpha = 0.05$	1,844,440	3,279,004	7,377,759
$\alpha = 0.01$	2,649,159	4,709,615	10,596,635

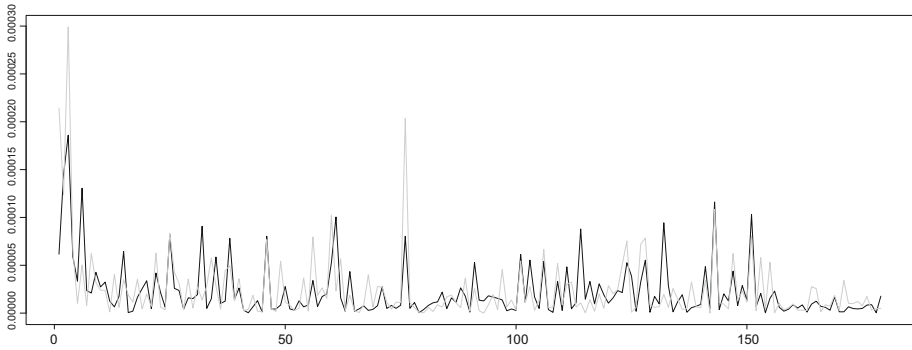


Fig. 1 Absolute error in estimating the Banzhaf–Owen value for the 179 players using stratified sampling (in black) and *SRSwr* (in gray) with $q = 50\%$

modelling of a voting situation of the Board of Governors as a TU-game with a priori unions is due to the fact of that passing a law depends on the required quota for the majority.

The Board of Governors of the IMF in January 2002 is composed of 179 members. All information about the Executive Board for IMF members is available at the website www.imf.org. At that time, countries are organized into 24 constituencies. Thus, (N, v, P) denotes the weighted majority game with a priori unions that models the problem. Appendix A of Online Resource provides the list of the country’s voting rights (column 2 of the first three tables). The different constituencies are separated by dashed lines.

4.1.1 Analysis of the Banzhaf–Owen value estimation

Firstly, we evaluate the performance of our sampling proposal with the proportional allocation procedure for estimating the Banzhaf–Owen value in this example. Appendix A of Online Resource also depicts the Banzhaf–Owen value by using $q = 50\%$, $q = 70\%$ and $q = 85\%$ (columns 3, 6 and 9, respectively).

Table 1 initially depicts the required sample sizes by directly using Hoeffding’s inequality. We consider the usual values of α as well as some bearable values for the absolute error in this setting. For determining these amounts, Proposition 3.5 is required.

Table 2 additionally displays the required sampling sizes to guarantee a maximum absolute error of ε with a probability of at least $1 - \alpha$ for the different constituency lengths. These sampling sizes are obtained from the expression in Proposition 3.6 by taking $w_i = 1$ for all $i \in N$. Besides, the obtained sampling sizes are always smaller than the population size.

We compare the results obtained under stratified sampling with those obtained under simple random sampling with replacement (*SRSwr*). To this aim, we use the ideas of Bachrach et al. (2010), in line with the one suggested by Laruelle and Valenciano (2004) that ensures that the Banzhaf–Owen value for (N, v, P) is the Banzhaf value for an alternative TU-game. Thus, we estimate the Banzhaf–Owen value for each player $i \in N$ using both methodologies

Table 2 Sample sizes for the weighted majority game by using Proposition 3.6

p_i	$\varepsilon = 0.001$			$\varepsilon = 0.005$			$\varepsilon = 0.010$		
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
	1	17,790,211	21,906,478	31,464,154	711,609	876,260	1,258,567	177,903	219,065
4	66,393,970	81,756,088	117,425,822	2,655,759	3,270,244	4,697,033	663,940	817,561	1,174,259
6	91,038,794	112,103,188	161,013,194	3,641,552	4,484,128	6,440,528	910,388	1,121,032	1,610,132
7	101,588,449	125,093,802	179,671,543	4,063,538	5,003,753	7,186,862	1,015,885	1,250,939	1,796,716
8	111,224,532	136,959,465	196,714,130	4,448,982	5,478,379	7,868,566	1,112,246	1,369,595	1,967,142
9	120,106,281	147,896,258	212,422,585	4,804,252	5,915,851	8,496,904	1,201,063	1,478,963	2,124,226
10	128,361,875	158,062,016	227,023,608	5,134,475	6,322,481	9,080,945	1,283,619	1,580,621	2,270,237
12	143,382,810	176,558,468	253,589,961	5,735,313	7,062,339	10,143,599	1,433,829	1,765,585	2,535,900
13	150,294,583	185,069,475	265,814,273	6,011,784	7,402,779	10,632,571	1,502,946	1,850,695	2,658,143
14	156,881,039	193,179,894	277,463,224	6,275,242	7,727,196	11,098,529	1,568,811	1,931,799	2,774,633
21	196,614,644	242,106,989	347,736,943	7,864,586	9,684,280	13,909,478	1,966,147	2,421,070	3,477,370
23	206,520,039	254,304,276	365,255,840	8,260,802	10,172,172	14,610,234	2,065,201	2,543,043	3,652,559

Table 3 Theoretical errors for the weighted majority game by using Proposition 3.6

p_i	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
1	0.00133	0.00148	0.00177
4	0.00258	0.00286	0.00343
6	0.00302	0.00335	0.00401
7	0.00319	0.00354	0.00424
8	0.00333	0.00370	0.00443
9	0.00347	0.00385	0.00461
10	0.00358	0.00397	0.00476
12	0.00379	0.00420	0.00503
13	0.00388	0.00430	0.00515
14	0.00396	0.00439	0.00527
21	0.00443	0.00492	0.00589
23	0.00454	0.00504	0.00604

Table 4 Theoretical errors for the weighted majority game by using Proposition 3.5

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Theoretical error	3.8702×10^{-4}	4.2947×10^{-4}	5.1470×10^{-4}

with a sample size equal to $\ell = 10^7$ with a proportional allocation procedure under stratified sampling. In practice, we only take a small size of the population of compatible coalitions with P for each member in each constituency to estimate the Banzhaf–Owen value.

Figure 1 displays the graphical comparison of the absolute errors obtained under the sampling methodologies mentioned above for the set of players in the weighted majority game for the case $q = 50\%$. Appendix A of Online Resource numerically details the results obtained for all countries, including the absolute errors as well as the estimation of the associated variances obtained under both methods, stratified sampling and SRS_{wr} . In this example, we can conclude that both approaches provide correct approximations of the Banzhaf–Owen value. Besides, we check that stratified sampling usually reduces the incurred absolute errors and the estimations of the variances with respect to those obtained under simple random sampling with replacement. Appendix A of Online Resource also includes a detailed analysis of the absolute errors and the estimated variances for the cases $q = 70\%$ and $q = 85\%$, for which analogous conclusions can be extracted. In view of the values displayed in Table 3, we can hypothesize that the theoretical bounds of the error given by the expression on the right side of the last inequality in (21) are very conservative in practice.

However, it is noteworthy that Hoeffding's inequality also provides values of the theoretical error for several values of α with $\ell = 10^7$ (see by using Proposition 3.5). They are depicted in Table 4. In general, they are lesser than those ones given by Proposition 3.6.

Notice that, for the specific case of weighted majority games, those bounds induced by Proposition 3.5 coincide with those given by Bachrach et al. (2010) when simple random sampling with replacement on coalitions is assumed for the Banzhaf values estimation. Besides, Saavedra-Nieves and Fiestras-Janeiro (2021) establish those conditions ensuring that Serfling's inequality, the analogous probabilistic bound to Hoeffding's inequality in the case of using simple random sampling without replacement (SRS_{wor}), provide smaller sampling sizes (not always). In this example, the theoretical errors in Table 4, ensured by Hoeffding's

inequality, are quite similar to those ensured by Serfling's inequality when no replacement in sampling is assumed ($\varepsilon = 3.8657 \cdot 10^{-4}$, $4.2897 \cdot 10^{-4}$ and $5.1410 \cdot 10^{-4}$, for $\alpha = 0.1, 0.05$ and 0.01 , respectively). See more details in Saavedra-Nieves and Fiestras-Janeiro (2021).

We do a simulation study to check how the above-described sampling procedures perform. We choose Austria and estimate 1000 times its Banzhaf–Owen value for several quotas ($q = 50\%$, 70% and 85%) using the above-described sampling methods with a sample size equal to $\ell = 10^7$.

From Table 5, we check that the theoretical bounds provided by Hoeffding's inequality in Table 4 are conservative on this example. Additionally, we depict the minimum, maximum and mean absolute errors observed in these 1000 estimations as well as the results of the estimated variances and processing times (in seconds) under both proposals. By comparing the observed values with respect to the original ones, we observe that stratified sampling ensures generally better results than SRS_{wr} (except the case $q = 70\%$, where the difference is negligible regarding to the bias). However, the most relevant fact is in the reduction of the estimated variance, that stratified sampling theoretically ensures, in the three cases considered. This leads to an increase in the processing time required to obtain it with respect to simple random sampling with replacement. Appendix A.2 of Online Resource completes the study of the distribution of the absolute errors.

As mentioned above, a theoretical comparison of the bounds of the error in the Banzhaf–Owen value estimation is made in Saavedra-Nieves and Fiestras-Janeiro (2021). However, due to its multiple applications in practice, an empirical comparison of sampling methodologies is also of interest. Table 5 also includes the results associated with the estimation of the Banzhaf–Owen value for Austria when assuming simple random sampling without replacement, SRS_{wor} , when $\ell = 10^7$. They are extracted from Saavedra-Nieves and Fiestras-Janeiro (2021). In terms of bias, SRS_{wor} only improves on previous results in the case $q = 50\%$. However, in terms of estimated variance, the stratified version of sampling always provides the best results as expected.

4.1.2 Analysis of the Owen value estimation

Secondly, we check how the proposal based on stratified sampling performs in the estimation of the Owen value also in the example of the International Monetary Fund. The considered voting rights are again the ones given in Appendix A of Online Resource.

Table 6 depicts the required sampling sizes to ensure a maximum absolute error of ε with a probability of at least $1 - \alpha$ for the different constituency. These sampling sizes are obtained accordingly the corresponding expression in Proposition 3.2, with $w_i = 1$ for all $i \in N$. However, it is worth to mention that the used bound provides sampling sizes are, in some cases, larger than the population size (they are highlighted in bold). These correspond to joints of relatively small size and for very small values of ε . Proposition 3.4 provides worse results than those given directly by Hoeffding's inequality in Table 1. Note that, although those values were obtained for the case of estimating the Banzhaf–Owen value, they are also valid for their usage in this different context.

In this section we estimate the Owen value for each player $i \in N$ using stratified sampling by using the proportional allocation procedure of the sample with $\ell = 10^7$. We are only extracting a small portion of the population of compatible coalitions with P for each player in N .

Figure 2 graphically describes the absolute errors obtained under stratified sampling for the set of players in the weighted majority game for the case $q = 50\%$. Appendix B of Online Resource includes the numerical results obtained for the set of members. More specifically,

Table 5 Basic statistics for 1000 estimations of the Banzhaf–Owen value for Austria

$q = 50\%$	Absolute error	Minimum	Average	Maximum
	<i>Strat. sampling</i>	1.572×10^{-9}	3.975×10^{-5}	1.460×10^{-4}
	<i>SRS_{Swr}</i>	1.037×10^{-7}	4.020×10^{-5}	1.925×10^{-4}
	<i>SRS_{Swor}</i>	3.700×10^{-9}	3.902×10^{-5}	1.445×10^{-4}
	Estimated variance	Minimum	Average	Maximum
	<i>Strat. sampling</i>	2.384×10^{-9}	2.397×10^{-9}	2.411×10^{-9}
	<i>SRS_{Swr}</i>	2.402×10^{-9}	2.420×10^{-9}	2.437×10^{-9}
	<i>SRS_{Swor}</i>	2.402×10^{-9}	2.415×10^{-9}	2.428×10^{-9}
	Processing time	Minimum	Average	Maximum
	<i>Strat. sampling</i>	223.146	238.163	318.315
	<i>SRS_{Swr}</i>	133.277	139.522	152.954
	<i>SRS_{Swor}</i>	206.470	229.189	249.270
$q = 70\%$	Absolute error	Minimum	Average	Maximum
	<i>Strat. sampling</i>	2.596×10^{-8}	2.266×10^{-5}	9.653×10^{-5}
	<i>SRS_{Swr}</i>	2.800×10^{-8}	2.251×10^{-5}	9.593×10^{-5}
	<i>SRS_{Swor}</i>	2.800×10^{-8}	2.298×10^{-5}	1.234×10^{-4}
	Estimated variance	Minimum	Average	Maximum
	<i>Strat. sampling</i>	7.880×10^{-10}	7.966×10^{-10}	8.035×10^{-10}
	<i>SRS_{Swr}</i>	8.232×10^{-10}	8.327×10^{-10}	8.402×10^{-10}
	<i>SRS_{Swor}</i>	8.212×10^{-10}	8.306×10^{-10}	8.428×10^{-10}
	Processing time	Minimum	Average	Maximum
	<i>Strat. sampling</i>	218.174	246.340	313.948
	<i>SRS_{Swr}</i>	132.296	143.296	154.965
	<i>SRS_{Swor}</i>	210.500	225.600	241.900
$q = 85\%$	Absolute error	Minimum	Average	Maximum
	<i>Strat. sampling</i>	1.934×10^{-9}	2.961×10^{-6}	1.258×10^{-5}
	<i>SRS_{Swr}</i>	7.000×10^{-9}	3.122×10^{-6}	1.321×10^{-5}
	<i>SRS_{Swor}</i>	7.000×10^{-9}	3.081×10^{-6}	1.449×10^{-5}
	Estimated variance	Minimum	Average	Maximum
	<i>Strat. sampling</i>	1.352×10^{-11}	1.457×10^{-11}	1.551×10^{-11}
	<i>SRS_{Swr}</i>	1.449×10^{-11}	1.564×10^{-11}	1.696×10^{-11}
	<i>SRS_{Swor}</i>	1.416×10^{-11}	1.640×10^{-11}	8.288×10^{-10}
	Processing time	Minimum	Average	Maximum
	<i>Strat. sampling</i>	224.294	243.331	307.395
	<i>SRS_{Swr}</i>	128.472	137.557	143.877
	<i>SRS_{Swor}</i>	223.700	234.200	281.800

Table 6 Sample sizes for estimating the Owen value for the weighted majority game under stratified sampling

p_i	$\varepsilon = 0.025$			$\varepsilon = 0.050$			$\varepsilon = 0.100$		
	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
1	1.2941×10^8	1.2941×10^8	1.2941×10^8	22,524,140	22,524,140	22,524,140	8,087,825	8,087,825	8,087,825
4	1.7254×10^8	1.7254×10^8	1.7254×10^8	30,032,186	30,032,186	30,032,186	10,783,767	10,783,767	10,783,767
6	2.9907×10^8	2.9907×10^8	2.9907×10^8	$52,055,790$	$52,055,790$	$52,055,790$	18,691,862	18,691,862	18,691,862
7	4.2537×10^8	4.2537×10^8	4.2537×10^8	$74,038,532$	$74,038,532$	$74,038,532$	$26,585,285$	$26,585,285$	$26,585,285$
8	6.3100×10^8	6.3100×10^8	6.3100×10^8	$109,831,995$	$109,831,995$	$109,831,995$	$39,437,774$	$39,437,774$	$39,437,774$
9	9.6988×10^8	9.6988×10^8	9.6988×10^8	$168,815,844$	$168,815,844$	$168,815,844$	$60,617,318$	$60,617,318$	$60,617,318$
10	1.5354×10^9	1.5354×10^9	1.5354×10^9	$267,257,854$	$267,257,854$	$267,257,854$	$95,965,248$	$95,965,248$	$95,965,248$
12	4.1238×10^9	4.1238×10^9	4.1238×10^9	$717,790,914$	$717,790,914$	$717,790,914$	$257,739,790$	$257,739,790$	$257,739,790$
13	6.9430×10^9	6.9430×10^9	6.9430×10^9	$1,208,486,733$	$1,208,486,733$	$1,208,486,733$	$433,935,720$	$433,935,720$	$433,935,720$
14	1.1856×10^{10}	1.1856×10^{10}	1.1856×10^{10}	$2,063,583,348$	$2,063,583,348$	$2,063,583,348$	$740,978,367$	$740,978,367$	$740,978,367$
21	6.5013×10^{11}	6.5013×10^{11}	6.5013×10^{11}	$113,160,609,341$	$113,160,609,341$	$113,160,609,341$	$40,632,990,936$	$40,632,990,936$	$40,632,990,936$
23	2.1559×10^{12}	2.1559×10^{12}	2.1559×10^{12}	$375,261,479,217$	$375,261,479,217$	$375,261,479,217$	$134,746,502,093$	$134,746,502,093$	$134,746,502,093$

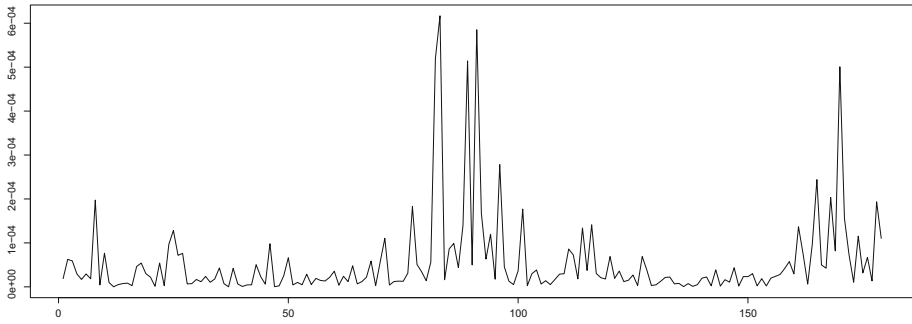


Fig. 2 Absolute error in estimating the Owen value for the 179 players using stratified sampling with $q = 50\%$

Table 7 Theoretical errors for the weighted majority game by using Proposition 3.4

p_i	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
1	0.12337	0.13690	0.16407
4	0.16882	0.18733	0.22451
6	0.19888	0.22069	0.26449
7	0.20838	0.23123	0.27712
8	0.22197	0.24632	0.29520
9	0.23849	0.26465	0.31717
10	0.24932	0.27666	0.33157
12	0.27978	0.31047	0.37208
13	0.29394	0.32618	0.39091
14	0.30939	0.34332	0.41146
21	0.43887	0.48700	0.58365
23	0.46769	0.51899	0.62198

we include the absolute errors and the estimation of the associated variances. In view of these results, we can conclude that this new approach provides a correct approximation of the Owen value. Appendix B of Online Resource also details the numerical results for the cases $q = 70\%$ and $q = 85\%$. Again, we can hypothesize that the theoretical bounds of the error, given by the theoretical bounds of the error given by the right side of the first inequality in (19), are very conservative in view of the values depicted in Table 7. Even so, we use this second alternative since that Proposition 3.2 provides extremely large theoretical bounds of the error.

Again, we can use as theoretical errors in the Owen value estimation those amounts that were obtained for the weighted majority game by means of the Hoeffding’s inequality (see Table 4). Those amounts seem more reasonable than the ones in Table 7.

We complete this analysis by doing a new simulation study to compare which of the two methods proposed is the best in terms of approximating the Owen value. We take Austria and we estimate 1000 times its Owen value for several quotas ($q = 50\%$, 70% and 85%) using stratified sampling with a sample size equal to $\ell = 10^7$. If we use those theoretical errors in Table 4, we check that for the case of Austria ($p_i = 10$) the obtained absolute errors in the simulation study are smaller than the theoretical ones (except for the case $q = 85\%$).

Table 8 Basic statistics for 1000 estimations of the Owen value for Austria

		Minimum	Average	Maximum
$q = 50\%$	Absolute error	1.224×10^{-7}	4.998×10^{-5}	2.334×10^{-4}
	Estimated variance	3.265×10^{-9}	3.782×10^{-9}	4.340×10^{-9}
	Processing time	219.896	238.082	292.914
$q = 70\%$	Absolute error	2.665×10^{-8}	2.827×10^{-4}	1.387×10^{-3}
	Estimated variance	1.327×10^{-8}	1.458×10^{-7}	1.098×10^{-6}
	Processing time	217.079	228.782	242.273
$q = 85\%$	Absolute error	9.277×10^{-7}	9.146×10^{-4}	6.063×10^{-3}
	Estimated variance	1.119×10^{-7}	8.691×10^{-7}	2.899×10^{-7}
	Processing time	219.429	231.147	243.612

From Table 8, it is easy to check that the theoretical bounds adequately perform also under this approach. A statistical summary of the absolute errors, the estimated variances and the processing times in these 1000 estimations under stratified sampling is also included. Appendix B.1 of Online Resource displays the graphical analysis of the distribution of the absolute errors.

4.2 A bankruptcy game

Along this section, we estimate the Owen value for a *bankruptcy game* with a priori unions (N, v, P) . This example is partially extracted from the real bankruptcy problem considered in Saavedra-Nieves and Saavedra-Nieves (2020).

A bankruptcy problem can be also analysed from a game theoretic approach. In this setting, a *bankruptcy game* (O'Neill, 1982) is said to be a TU-game (N, v) associated to each $(N, c, E) \in B^N$ and given, for each $S \subseteq N$, by

$$v(S) = \max \left\{ 0, E - \sum_{i \notin S} c_i \right\}. \quad (22)$$

Saavedra-Nieves and Saavedra-Nieves (2020) refer to the milk problem arisen in Galicia (Spain) after the suppression of the European milk quotas in April 2015. The usage of bankruptcy rules in this setting is justified as mechanisms of determining new systems of quotas (cf. Gallástegui, Iñarra and Prellezo 2002). A main assumption for describing this situation in terms of a bankruptcy problem is that the maximum of tons of milk in 2014–2015 imposed for Galicia is reduced. This fact directly determines the needing of establishing a new distribution of the milk production for the councils. As innovation, it seems reasonable to additionally evaluate the effect in the new quotas of the organization of the Galician councils into counties. They are territorial units higher than councils and that are lower than the province. It is important to highlight the key role of Galicia's regional articulation in establishing new legislation. According to this, the bankruptcy rules that consider a priori unions (see Borm, Carpenste, Casas-Méndez, and Hendrickx 2005) can be applied for the mentioned purpose and, among others, the Owen value. As the task is mainly focused on comparing the estimations with the exact Owen value, we have only restricted our analysis to the province of A Coruña (Galicia) by reducing the aggregate milk quota for the province in 2015 by $\rho = 40\%$. The information about the individual milk production of the councils

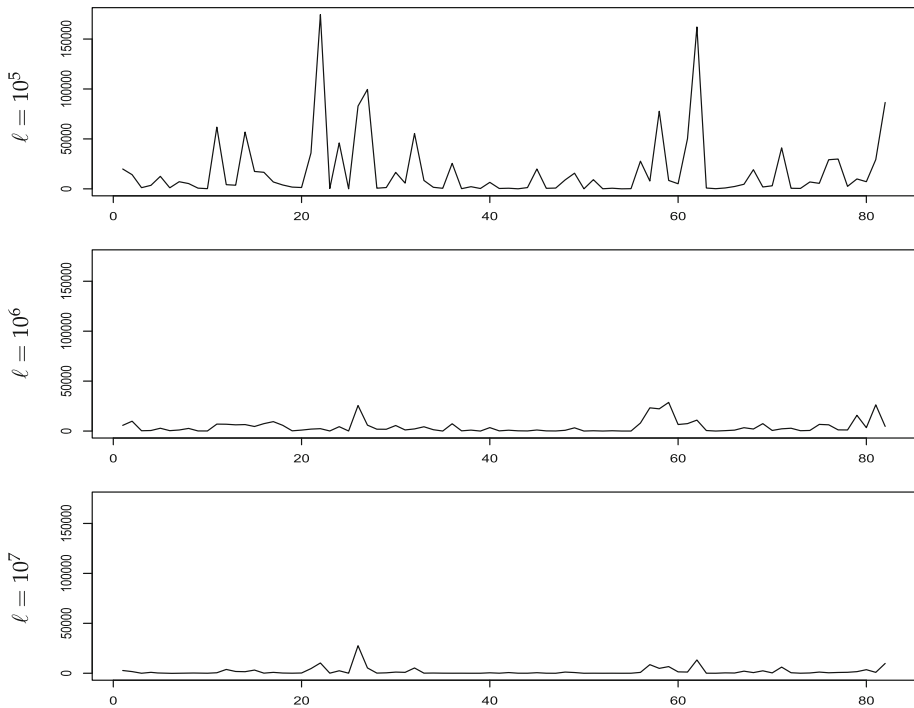


Fig. 3 Absolute error for the 82 councils of A Coruña by using $\ell = 10^5$, $\ell = 10^6$ and $\ell = 10^7$

of A Coruña in April 2015 as well as their organization in counties, that induces the required partition P , is described in the first column of Table C.10 in the Online Resource section. We also include in the fifth column the exact Owen value that was obtained by using the characterization of this coalitional value given in Lorenzo-Freire (2017). Notice that the corresponding game with a priori unions has associated a total amount of 82 players and, hence, the Owen value is a vector of dimension 82.

Firstly, we estimate the Owen value for the associated bankruptcy game and that it is depicted in Table C.10 in Appendix C of the Online Resource section. Figure 3 graphically describes the absolute errors obtained under stratified sampling for the set of players in the bankruptcy game, with $\ell = 10^5$, $\ell = 10^6$ and $\ell = 10^7$. Appendix C of Online Resource also completes the numerical results with the estimated variances obtained for the set of involved players. It is obvious that both measures reduce when sampling size increases. We can naturally conclude a correct performance of our sampling proposal since it is easy to check that the theoretical errors given by Proposition 3.2 are conservative enough (Table 9).

Finally, we do a small simulation study to evaluate how stratified sampling performs to approximate the Owen value. We take Cabana de Bergantiños and we estimate 1000 times its Owen value with a sample size equal to $\ell = 10^5$.

In view of these results, we can ensure that the theoretical bounds of the error obtained from the right side of the first inequality in (19) are also very conservative in bankruptcy settings. See, for instance, the results given in Table 10 for several values of α .

Table 9 Basic statistics for 1000 estimations of the Owen value for Cabana de Bergantiños

	Minimum	Average	Maximum
Absolute error	41.42	12425.36	73110.17
Estimated variance	1.262×10^8	2.530×10^8	1.827×10^9
Processing time	1.540	2.053	4.520

Table 10 Theoretical errors for the bankruptcy game by using Proposition 3.5

	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Theoretical error	36802.91	40839.26	48944.02

4.3 An airport game

This example corresponds to the estimation of the Owen value for the called *airport game* with a priori unions (N, c, P) initially studied in Vázquez-Brage et al. (1997). It allows the establishment of the fees for the planes operating in an airport. We use this example since that it admits a polynomial expression of the exact Owen value and thus, the comparison can be done.

Below, we briefly remind the airport games following the ideas of Littlechild and Owen (1973). The elements that characterize an airport game are the following. We denote by \mathcal{T} the set of types of planes operating in an airport in a fixed period. Thus, N_τ is the set of movements operated by planes of type $\tau \in \mathcal{T}$ and N the set of all movements, being $N = \cup_{\tau \in \mathcal{T}} N_\tau$. Besides, c_τ is the cost of a runway that is suitable for planes of type τ in that period, satisfying that

$$0 \leq c_1 \leq c_2 \leq \dots \leq c_{|\mathcal{T}|}.$$

Formally, the airport games is defined, for every $S \subseteq N$, by

$$c(S) = \max\{c_\tau : S \cap N_\tau \neq \emptyset\},$$

in such way that $c(S)$ is the cost of a runway that is used by all the movements in S . Thus, the definition of a fee for each movement is based on the allocation of $c(N)$ among all the movements.

For this purpose, Vázquez-Brage et al. (1997) model this situation by using cost games with a priori unions (N, c, P) , being P the partition in N induced by the airlines to which the planes belong, and the Owen value. The appropriate use of this coalitional value is because the airport industry is, in practice, organized through a system of airlines. Vázquez-Brage et al. (1997) specifies a formula for obtaining the Owen value with less computational complexity in this class of games. Its usage is illustrated on the real example that corresponds to the situation given by the movements of planes in the first three months of 1993 at Lavacolla, the airport of Santiago de Compostela (Spain). The information about these movements as well as the further elements that characterize the airport game are described in Table 11. It depicts the partition P induced by the airlines in which the movements are organized, the numbers of movements per airline, the types of planes with runway costs, and the Owen value. The corresponding cost game with a priori unions has 1258 players and, hence, the Owen value is a vector in \mathbb{R}^{1258} . By the property of symmetry of the Owen value, its computation reduces to 25 different components (one per type in each airline).

Table 11 Airlines using Lavacolla in the first three months of 1993 and the Owen value

Airline	Movements	Types	Costs	Owen value	Theoretical error	\overline{O}_i^{st}
Air Europa	36	B-757(3)	32,496	8.110	30044.9622	10.422
	172	B-737(5)	39,494	11.183	36515.1322	9.718
Aviaco	12	DC-9(4)	34,265	151.093	5275.6132	124.554
Britannia	6	B-737(5)	39,494	369.224	3594.6030	371.497
British airways	2	B-757(3)	32,496	843.378	1335.6328	863.932
Condor Flugdienst	2	B-757(3)	32,496	843.378	1335.6328	855.244
Caledonian airways	2	B-757(3)	32,496	843.378	1335.6328	830.662
Eurobelgian airlines	2	B-737(5)	39,494	1107.673	1623.2607	1092.225
Futura	32	B-737(5)	39,494	69.230	17494.6815	63.168
Gestair executive set	2	CESSNA(1)	8120	176.522	333.7438	176.522
Iberia	452	DC-9(4)	34,265	2.037	37221.3161	1.760
	438	B-727(6)	44,850	9.070	48719.5689	12.340
Air charter	2	B-737(5)	39,494	1107.673	1623.2607	1122.612
Corse Air	4	B-737(5)	39,494	553.836	2560.4507	558.168
Air UK Leisure	2	B-737(5)	39,494	1107.673	1623.2607	1096.867
Ibertrans	2	CESSNA(1)	8120	176.522	333.7438	176.522
LTE	36	B-757(3)	32,496	46.854	15803.6980	39.246
Mac aviation	6	LEARJET-25(2)	15,134	120.367	1377.4427	124.074
Monarch airlines ltd	2	B-737(5)	39,494	1107.673	1623.2607	1103.305
Sobelair	6	B-737(5)	39,494	369.224	3594.6030	352.580
Trabajos Aéreos	2	CESSNA(1)	8120	176.522	333.7438	176.522
Tea Basel LTD	2	B-737(5)	39,494	1107.673	1623.2607	1114.205
Oleohidráulica Balear SA	4	CESSNA(1)	8120	88.261	526.4308	88.261
Viasa	30	DC-10(7)	50,000	334.778	21062.6664	349.594
Spanair	2	B-737(5)	39,494	1107.673	1623.2607	1110.996

Let us estimate the Owen value using our sampling procedure. It is worth to mention that airport games are concave and thus, by Proposition 3.4, the minimum sampling sizes to ensure that the absolute error is smaller than or equal to ε , with $\varepsilon > 0$ with probability at least $1 - \alpha$ can be exactly determined. This numerical example illustrates the main drawbacks of using stratified sampling in this setting. We take a sample of size $\ell = 10^8$. The second last column of Table 11 depicts the theoretical maximum absolute errors when using this sample size with $\alpha = 0.1$. As the bound of the error depends on the number of strata associated with each $i \in N$, we obtain theoretical errors large enough in those cases with a large number of movements. See, for instance, the case of Airlines 1 and 10.

Table 11 also includes the estimations obtained by means of our methodology in the last column. It is easy to see that, in general, these estimations are quite far from the exact Owen value. Moreover, this effect is accentuated in those components associated with airlines with many movements. Furthermore, this methodology worsens the results obtained for the estimation of the Owen value also in this example by using simple random sampling with replacement (see Saavedra-Nieves et al., 2018). The fact of that the most of the players within the unions are symmetrical and the large size of the strata may justify this effect. In this way,

only for those compatible coalitions with less costly movements than the one associated with i , the corresponding marginal contribution is not equal to zero.

5 Concluding remarks

Multiple real-world situations can be modelled by using cooperative game theory in a wide variety of contexts. The objective common to all of them is to share the joint costs of cooperation between all the players involved and, to face it, coalitional values are usually used as tool. In particular, we focus on those situations in which cooperation among players is restricted by the existence of a structure of a priori unions. Notice that there exists a list of methods that avoid the use of general expressions in some contexts. However, computing the coalitional values becomes a hard task for the most of TU-games since that the complexity increases with the enlarging of the set of players. This drawback arises in the most of real applications and, for this reason, sampling techniques are introduced as an alternative to their approximation.

According to the reformulations of the expressions of the Owen value and the Banzhaf–Owen value, we have provided two procedures based on stratified sampling to estimate the above-mentioned coalitional values for general TU-games with a priori unions.

1. We have introduced a sampling procedure to estimate the Owen value for general TU-games based on stratified sampling.
2. Analogously, we have innovatively extended these methodologies to approach the Banzhaf–Owen value for general TU-games with a priori unions also based on stratified sampling.

The proposals based on stratified sampling have been studied and theoretically analysed from a statistical approach, and their performance was evaluated on well-known examples in literature where the Banzhaf–Owen value and the Owen value can be exactly computed. In particular, we compare the distribution of the power of the countries belonging to the Board of Governors of the International Monetary Fund (IMF) in 2002, for which the Banzhaf–Owen value and the Owen value were able to be exactly obtained. We also estimate the Owen value in a bankruptcy problem for which the exact allocation is obtained after a strong computational effort. Moreover, we repeat this study on an airport game for which the Owen value take a polynomial expression. Although both estimators provides good approximations for weighted majority games, we check that the methodology based on stratified sampling for estimating the Owen value provides worse results than, for example, the methodology proposed in Saavedra-Nieves et al. (2018) in those airport games with a large amount of symmetrical players. This phenomenon would also affect the estimation of the Shapley value in line with that suggested by Castro et al. (2017) or Maleki (2015). Hence, we can conclude that the class of TU-games on which the methodologies are applied is decisive in terms of being affected by this effect. From a statistical point of view, the establishment of new theoretical bounds of the incurred error that improve the ones hereby provided could be useful when analysing the quality of the resulting estimations.

Supplementary Information The online version contains supplementary material available at <https://doi.org/10.1007/s10479-022-05044-0>.

Acknowledgements The author acknowledges the financial support of *Ministerio de Economía y Competitividad* of the Spanish government under grants MTM2017-87197-C3-2-P and PID2021-124030NB-C32, and of Xunta de Galicia through the ERDF (*Grupos de Referencia Competitiva*) ED431C 2021/24. The author also thanks the computational resources of the Centro de Supercomputación de Galicia (CESGA).

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Algaba, E., Bilbao, J. M., & Fernández, J. R. (2007). The distribution of power in the European constitution. *European Journal of Operational Research*, 176(3), 1752–1766.
- Algaba, E., Bilbao, J. M., Fernández-García, J. R., & López, J. J. (2003). Computing power indices in weighted multiple majority games. *Mathematical Social Sciences*, 46(1), 63–80.
- Algaba, E., Fragnelli, V., & Sánchez-Soriano, J. (2019). *Handbook of the Shapley value*. CRC Press.
- Alonso-Meijide, J. M., Bilbao, J. M., Casas-Méndez, B., & Fernández, J. R. (2009). Weighted multiple majority games with unions: Generating functions and applications to the European union. *European Journal of Operational Research*, 198(2), 530–544.
- Alonso-Meijide, J. M., & Bowles, C. (2005). Generating functions for coalitional power indices: An application to the IMF. *Annals of Operations Research*, 137(1), 21–44.
- Alonso-Meijide, J. M., Carreras, F., Fiestras-Janeiro, M. G., & Owen, G. (2007). A comparative axiomatic characterization of the Banzhaf–Owen coalitional value. *Decision Support Systems*, 43(3), 701–712.
- Bachrach, Y., Markakis, E., Resnick, E., Procaccia, A. D., Rosenschein, J. S., & Saberi, A. (2010). Approximating power indices: Theoretical and empirical analysis. *Autonomous Agents and Multi-Agent Systems*, 20(2), 105–122.
- Banzhaf, J. F. (1964). Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19, 317.
- Benati, S., López-Blázquez, F., & Puerto, J. (2019). A stochastic approach to approximate values in cooperative games. *European Journal of Operational Research*, 279(1), 93–106.
- Borm, P., Carpentier, L., Casas-Méndez, B., & Hendrickx, R. (2005). The constrained equal awards rule for bankruptcy problems with a priori unions. *Annals of Operations Research*, 137(1), 211–227.
- Castro, J., Gómez, D., Molina, E., & Tejada, J. (2017). Improving polynomial estimation of the Shapley value by stratified random sampling with optimum allocation. *Computers & Operations Research*, 82, 180–188.
- Castro, J., Gómez, D., & Tejada, J. (2009). Polynomial calculation of the Shapley value based on sampling. *Computers & Operations Research*, 36(5), 1726–1730.
- Cochran, W. G. (2007). *Sampling techniques*. JohnWiley & Sons.
- Costa, J. (2016). A polynomial expression for the Owen value in the maintenance cost game. *Optimization*, 65(4), 797–809.
- Deng, X., & Papadimitriou, C. H. (1994). On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19(2), 257–266.
- Fernández-García, F., & Puerto-Albandoz, J. (2006). *Teoría de Juegos Multiobjetivo*. Sevilla: Imagraf Impresores SA.
- Fragnelli, V., García-Jurado, I., Norde, H., Patrone, F., and Tijs, S. (2000). How to share railways infrastructure costs? In *Game practice: contributions from applied game theory* (pp. 91–101). Springer.
- Gallástegui, M. C., Iñarra, E., & Prellezo, R. (2002). Bankruptcy of fishing resources: The northern European anglerfish fishery. *Marine Resource Economics*, 17(4), 291–307.
- Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American statistical association*, 58(301), 13–30.
- Laruelle, A., & Valenciano, F. (2004). On the meaning of Owen–Banzhaf coalitional value in voting situations. *Theory and Decision*, 56(1), 113–123.

- Leech, D. (2002). Voting power in the governance of the international monetary fund. *Annals of Operations Research*, 109(1), 375–397.
- Leech, D. (2003). Computing power indices for large voting games. *Management Science*, 49(6), 831–837.
- Littlechild, S. C., & Owen, G. (1973). A simple expression for the Shapley value in a special case. *Management Science*, 20(3), 370–372.
- Lorenzo-Freire, S. (2017). New characterizations of the Owen and Banzhaf-Owen values using the intracoalitional balanced contributions property. *TOP*, 25(3), 579–600.
- Lucchetti, R., Moretti, S., Patrone, F., & Radrizzani, P. (2010). The Shapley and Banzhaf values in microarray games. *Computers & Operations Research*, 37(8), 1406–1412.
- Maleki, S. (2015). Addressing the computational issues of the Shapley value with applications in the smart grid (Unpublished doctoral dissertation). University of Southampton.
- Mann, I., & Shapley, L. S. (1960). *Values of large games, IV: Evaluating the electoral college by Monte Carlo techniques*. The RAND Corporation.
- Moretti, S., Patrone, F., & Bonassi, S. (2007). The class of microarray games and the relevance index for genes. *TOP*, 15(2), 256–280.
- Neyman, J. (1934). On the two different aspects of the representative method: The method of stratified sampling and the method of purposive selection. *Journal of the Royal Statistical Society*, 97(4), 558–625.
- O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 2(4), 345–371.
- Owen, G. (1972). Multilinear extensions of games. *Management Science*, 18(5–part–2), 64–79.
- Owen, G. (1975). Multilinear extensions and the Banzhaf value. *Naval research logistics quarterly*, 22(4), 741–750.
- Owen, G. (1977). Values of games with a priori unions. In *Mathematical economics and game theory* (pp. 76–88). Springer.
- Owen, G. (1982). Modification of the Banzhaf-Coleman index for games with a priori unions. In *Power, voting, and voting power* (pp. 232–238). Springer.
- R Core Team. (2022). R: A language and environment for statistical computing [Computer software manual]. Vienna, Austria. Retrieved from <https://www.R-project.org/>
- Saavedra-Nieves, A., & Fiestras-Janeiro, M. G. (2021). Sampling methods to estimate the Banzhaf-Owen value. *Annals of Operations Research*, 301(1), 199–223.
- Saavedra-Nieves, A., García-Jurado, I., & Fiestras-Janeiro, M. G. (2018). Estimation of the Owen value based on sampling. In E. Gil, E. Gil, J. Gil, & M. Á. Gil (Eds.), *The mathematics of the uncertain: A tribute to Pedro Gil* (pp. 347–356). Springer.
- Saavedra-Nieves, A., & Saavedra-Nieves, P. (2020). On systems of quotas from bankruptcy perspective: The sampling estimation of the random arrival rule. *European Journal of Operational Research*, 285(2), 655–669.
- Shapley, L. S. (1953). A value for n-person games. *Contributions to the Theory of Games*, 2(28), 307–317.
- Vázquez-Brage, M., van den Nouweland, A., & García-Jurado, I. (1997). Owen's coalitional value and aircraft landing fees. *Mathematical Social Sciences*, 34(3), 273–286.