ORIGINAL RESEARCH

On the facets of stable set polytopes of circular interval graphs

Gianpaolo Oriolo[1] [ORCID] · **Gautier Stauffer[2]**

Abstract

A linear description of the stable set polytope $STAB(G)$ of a quasi-line graph G is given in Eisenbrand et al. (Combinatorica 28(1):45–67, 2008), where the so called Ben Rebea Theorem (Oriolo in Discrete Appl Math 132(3):185–201, 2003) is proved. Such a theorem establishes that, for quasi-line graphs, $STAB(G)$ is completely described by non-negativity constraints, clique inequalities, and clique family inequalities (CFIs). As quasi-line graphs are a superclass of line graphs, Ben Rebea Theorem can be seen as a generalization of Edmonds' characterization of the matching polytope (Edmonds in J Res Natl Bureau Stand B 69:125–130, 1965), showing that the matching polytope can be described by non-negativity constraints, degree constraints and odd-set inequalities. Unfortunately, the description given by the Ben Rebea Theorem is not minimal, i.e., it is not known which are the (non-rank) clique family inequalities that are facet defining for $STAB(G)$. To the contrary, it would be highly desirable to have a minimal description of $STAB(G)$, pairing that of Edmonds and Pulleyblank (in: Berge, Chuadhuri (eds) Hypergraph seminar, pp 214–242, 1974) for the matching polytope. In this paper, we start the investigation of a minimal linear description for the stable set polytope of quasi-line graphs. We focus on circular interval graphs, a subclass of quasi-line graphs that is central in the proof of the Ben Rebea Theorem. For this class of graphs, we move an important step forward, showing some strong sufficient conditions for a CFI to induce a facet of $STAB(G)$. In particular, such conditions come out to be related to the existence of certain proper circulant graphs as subgraphs of G. These results allows us to settle two conjectures on the structure of facet defining inequalities of the stable set polytope of circulant graphs (Pêcher and Wagler in Math Program 105:311–328, 2006) and of (fuzzy) circular graphs (Oriolo and Stauffer in Math Program 115:291–317, 2008), and to slightly refine the Ben Rebea Theorem itself.

Keywords Stable sets · Quasi-line graphs · Polyhedral descriptions · Facets · Circular interval graphs

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1 Introduction

A *stable set* is a set of pairwise non-adjacent vertices of a graph. While finding a maximum size (or weight) stable set is hard in general graphs, the special case of line graphs can be handle in polynomial time through a simple reduction to matching (Edmonds 1965; Lovász and Plummer 1986) (a graph G is *line* if it is the intersection graph of the edges of a graph H—two edges intersect if they share an end—and thus stable sets in G correspond to matchings in H and vice-versa). The matching problem is a standard problem in combinatorial optimization and it has been studied extensively. In particular, there is a good understanding of the polyhedral nature of the problem: the matching polytope of a graph G, i.e., the convex hull of the characteristic vectors of the matchings of G, can be described by non-negativity constraints, degree constraints and odd-set inequalities. Not all degree constraints and odd-set inequalities are needed to describe the matching polytope. In fact the *facets* of this polytope were described by Edmonds and Pulleyblank (1974) : for a connected graph G, a degree constraint $x(\delta(v)) \leq 1$ defines a facet of the matching polytope of G if and only if the degree of v is at least three or it is two and v is not contained in a triangle; while an odd set inequality $x(E(S)) \leq \lfloor \frac{|S|-1}{2} \rfloor$ defines a facet of the matching polytope if and only if $|S| \geq 3$ and $G[S]$ is factor-critical and 2-vertex-connected (factor-critical graphs are graphs H for which $H \setminus v$ has a perfect matching for all $v \in V(H)$).

A graph is *claw-free* if there is no stable set of size three in the neighborhood of any vertex. Claw-free graphs are a superclass of line graphs, and therefore some properties of matching extend naturally to stable sets in claw-free graphs. In particular Berge (1973) proved that a stable set is maximum in a claw-free graph if and only if there does not exist an "augmenting path" (see Berge 1973 for the proper extension of this definition from matching to stable set) and this property was exploited by several authors to devise polynomial time algorithms for the stable set problem (Minty 1980; Sbihi 1980; Nakamura and Tamura 2001) (other types of algorithms also exist, see Lovász and Plummer 1986; Faenza et al. 2014; Nobili and Sassano 2015). Unfortunately, despite considerable efforts and progress (Galluccio et al. 2014a, b), the polyhedral nature of the problem is still not fully understood yet, see Faenza et al. (2011); Oriolo et al. (2011) for detailed discussions and remaining open questions. A subclass of claw-free graphs, namely quasi-line graphs, behaves better polyhedrally. A graph is *quasi-line* if the neighborhood of any vertex can be partitioned into two cliques and these graphs are (strictly) sandwiched between line and claw-free graphs. The *Ben Rebea Theorem* (Eisenbrand et al. 2008) provides a complete description of their stable set polytope, i.e., the convex hull of the characteristic vectors of the stable sets of a graph: non-negativity constraints, *clique* inequalities and the so-called *clique family inequalities* are all one needs to describe this polytope.

Theorem 1 (Ben Rebea Theorem) (Eisenbrand et al. 2008) *The stable set polytope of quasi-line graphs can be described by:*

(i) $x_v \geq 0$*, for all $v \in V$,*
(ii) $x(K) \leq 1$*, for all maximal cliques $K \subseteq V$ and*
(iii) clique family inequalities with parameter $\mathcal{F}$ and p: $|\mathcal{F}| > 2p$, $p \geq 2$ and $gcd(|\mathcal{F}|, p) \neq p$.

Clique family inequalities are defined as follows: given a graph G, a set of cliques $\mathcal{K}$, and an integer $1 \leq p < n = |\mathcal{K}|$, the clique family inequality ($CFI(\mathcal{K}, p)$) is the valid inequality (Oriolo 2003)

$$(p - r) \sum_{v \in S_{\geq p}} x_v + (p - r - 1) \sum_{v \in S_{p-1}} x_v \leq (p - r) \lfloor \frac{n}{p} \rfloor$$

where $r = n \mod p$, $S_{\geq p}$ is the set of vertices contained in at least p cliques of $\mathcal{K}$ and S_{p-1} is the set of vertices contained in exactly $p - 1$ cliques of $\mathcal{K}$. Clique family inequalities are a natural but non-trivial extensions of odd-set inequalities to the stable set setting: on the one hand, they capture odd-set inequalities, as odd set of vertices in a graph H correspond to odd families of cliques in the line graph of H; on the other hand, they capture sophisticated non-rank facet-inducing inequalities that are necessary to describe $STAB(G)$, as shown by examples in Giles and Trotter (1981) and Liebling et al. (2004). (We recall that an inequality of the form $ax \leq b$ is said to be *rank* if a is a 0/1 vector: trivially odd set inequalities are rank).

Not all clique family inequalities define facets and it would be interesting to have an equivalent of Pulleyblank and Edmonds' minimal description of the matching polytope (Edmonds and Pulleyblank 1974) for the stable set polytope of quasi-line graphs. This is a challenging task in particular for non-rank facets, as already interpreting those in Giles and Trotter (1981) and Liebling et al. (2004) is not straightforward. As for rank facets, a positive result is due to Galluccio and Sassano (1997). Their result is for claw-free graphs, but we state it for quasi-line graphs as in this case it simplifies a little bit. Galluccio and Sassano show that, when G is quasi line, each rank facet of $STAB(G)$ can be obtained by sequential lifting of a few *rank-minimal* facets, associated with some relevant subgraphs G' of G (the definitions of circulants will be given shortly, other definitions that are needed in the rest of this section, will be given in § 1.1):

Theorem 2 (Galluccio and Sassano 1997) *Any* rank *facet of a quasi-line graph G can be obtained by sequential lifting of a facet of the form $\sum_{v \in V(G')} x_v \leq \alpha(G')$ with G' an induced subgraph of G and either*

(j) G' *a singleton;*

(jj) G' *a $(\alpha\omega + 1, \omega)$-circulant, for some $\omega \geq 3$ integer (with $\omega = \omega(G')$ and $\alpha = \alpha(G')$) or*

(jjj) G' *the line graph of a minimal 2-vertex-connected factor-critical graph H (minimal means here that H is such that $H \setminus e$ is no longer factor-critical for any $e \in E(H)$).*

Moving to non-rank facets, such a description is not at hand, even if Stauffer (2011) was able to somehow extend Theorem 2 using the notion of simultaneous lifting:

Theorem 3 (Stauffer 2011) *Besides non-negativity constraints, any facet of a quasi-line graph G can be obtained by simultaneous lifting of a facet of the form $\sum_{v \in V(G')} x_v \leq \alpha(G')$ with G' an induced subgraph of G of the form (j) (jj) or (jjj).*

Unfortunately, while Theorem 2 provides a reasonable description of rank facets, Theorem 3 is not fully satisfactory, as simultaneous lifting is not as easy to interpret as sequential lifting.

In the paper, we therefore focus on the study of non-rank facets and we concentrate on a particular class of quasi-line graphs with circular structure, named *circular interval graphs*, that were already central in the proof of the Ben Rebea Theorem. Circular interval graphs generalize circulant graphs, whose definition we now recall:

Definition 4 A graph $G(V, E)$ is an (n, p)-circulant or *antiweb* (Trotter 1975), for $n, p \in \mathbb{N}$ with $n \geq 2$ and $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$, if the vertices in V can be labeled $0, \ldots, n - 1$ so that $E = \{(i, j) : i, j \in V \text{ with } |i - j| \leq p - 1 \mod n\}$ (i.e., i is adjacent to $\{i - p + 1, \ldots, i - 1, i + 1, \ldots, i + p - 1\}$ where sums are taken modulo n).

Observe that $\sum_{i \in V} x_i \leq \alpha(C(n, p)) = \lfloor \frac{n}{p} \rfloor$ is a valid inequality for $STAB(C(n, p))$ and it induces a facet if and only if p does not divide n (see for instance Wagler 2004).

Definition 5 A graph $G = (V, E)$ is a *circular interval graph* (CIG) if there is an injective mapping Φ from V to the unit circle $\mathcal{C}$ and a set of intervals $\mathcal{I}$ of $\mathcal{C}$, none including another, such that $(u, v) \in E \iff \exists I \in \mathcal{I} : \phi(u), \phi(v) \in I$. We call $(\phi, \mathcal{I})$ a representation of G. *Fuzzy circular interval graphs* (FCIG) are a slight generalization of circular interval graphs and will be formally defined in § 1.1

We now motivate our interest in circular interval graphs. It is known that the facets of $STAB(G)$ are rank when G is a quasi-line graph that is not a *fuzzy circular interval graph* (see Eisenbrand et al. 2008). Now, for FCIG, Eisenbrand and al. have observed (cf. Lemma 5 in Eisenbrand et al. (2008) and Lemma 25 in the following) that a facet inducing inequality of the stable set polytope of a FCIG is also a facet inducing inequality of the stable set polytope of a suitable CIG: this lemma can be used to extend most polyhedral results from circular interval graphs to fuzzy circular interval graphs. Therefore, providing a minimal linear description of the stable set polytope for CIGs is a crucial step to providing a minimal linear description of the stable set polytope of FCIGs and quasi-line graphs.

Even though we will not be able to provide a minimal description of $STAB(G)$ for CIGs, we will however provide strong necessary conditions for clique family inequalities to be facet-defining. Our main result is Theorem 6 below, showing that circulant subgraphs are "responsible" for the non-trivial facets of $STAB(G)$, when G is a CIG. Consider a CIG C whose vertices are numbered clockwise. Then, for each $i \in V(C)$, $N(i) = \{n_l(i), n_l(i) + 1, \ldots, i - 1, i + 1, \ldots, n_r(i) - 1, n_r(i)\}$ (sums are taken again modulo n), for some values $n_l(i), n_r(i)$. Let $Q_i = \{n_l(i), \ldots, i - 1, i\}$ and for a set $I \subseteq V(C)$ let $\mathcal{Q}(I) = \{Q_i, \text{ for all } i \in I\}$. Now suppose that the set $I \subseteq V(C)$ induces an (n', p')-circulant on C, then the *clique family inequality associated with I* is that with parameters $\mathcal{Q}(I)$ and p'.

Theorem 6 *Let C be a circular interval graph and $(\phi, \mathcal{I})$ a representation for it. The stable set polytope of C can be characterized by: non-negativity inequalities, clique inequalities, the inequality $\sum_{v \in V(C)} x_v \leq \alpha(C)$ and clique family inequalities associated with sets $I \subset V(W)$ that induce (n, p)-circulants on W, with n and p are relatively prime and such that $\alpha(I) < \alpha(W)$.*

Theorem 6 is indeed a generalization (and a sharpening) to the stable set polytope of CIGs of a conjecture of Pêcher and Wagler for circulant graphs (Pêcher and Wagler 2006), that we therefore settle.

While Theorem 6 gives a description of the stable set polytope of CIGs that is finer than those given by Theorem 1 and Theorem 3, it does not provide necessary and sufficient conditions for clique family inequalities to be facet-defining. We will however provide strong necessary conditions for clique family inequalities to be facet-defining for CIGs, and the proof of Theorem 6 builds upon these conditions. Remarkably, these conditions and Theorem 6 itself will also allow us to sharpen and settle another conjecture from the literature, due to Oriolo and Stauffer (2008), on the stable set polytope of FCIGs. Finally, we will also be able to sharpen Theorem 1, by showing that we can restrict to pairs $(\mathcal{F}, p)$ such that $|\mathcal{F}|$ and p are relatively prime.

The paper is organized as follows. In Sect. 2, we recall important elements from the proof in Eisenbrand et al. (2008) and we restate and sharpen some of the results. In Sect .3 and 4, we exploit our refinements to prove different properties of non-rank facets that will already lead us to sharpen Ben Rebea Theorem. Then in Sect. 5, we use those necessary conditions to prove Theorem 6 and the above conjectures.

1.1 Definitions and properties

A *stable set* (resp. *clique*) is a set of pairwise non-adjacent (resp. adjacent) vertices of a graph. The *stability number* (resp. *clique number*) of a graph G is the size of a maximum stable set (resp. clique) in G and is denoted $\alpha(G)$ (resp. $\omega(G)$). Given a graph $G = (V, E)$ and $W \subseteq V$, $G[W]$ represents the subgraph of G induced by the vertices of W. $G[W]$ is said to be α-*maximal* if $\alpha(G[W \cup \{v\}]) > \alpha(G[W])$ for every $v \notin W$.

The *stable set polytope* $STAB(G)$ of a graph G is the convex hull of the characteristic vectors of the stable sets of G. *Facets* of $STAB(G)$ are inequalities of the form $ax \leq b$ which are satisfied by the characteristic vectors of any stable set of G and that are necessary to describe the polytope, i.e., there are $|V(G)|$ affinely independent stable sets, or roots, satisfying $ax = b$: stable sets whose characteristic vectors satisfy a valid inequality $ax \leq b$ at equality are called *roots* of the inequality. $STAB(G)$ is down-monotone and, apart from non-negativity constraints (sometimes call *trivial facets*), all facets of $STAB(G)$ are of the form $ax \leq b$ with $a, b \geq 0$. A (non-trivial) facet is said to be *rank* if a is a 0/1 vector and *non-rank* otherwise.

Let $G(V, E)$ be a graph and let $F' : \sum_{v \in V'} a_v x_v \leq b$ be a facet of $STAB(G[V'])$, for some $V' \subset V$: the facets of $STAB(G)$ of the form $F : \sum_{v \in V'} a_v x_v + \sum_{v \in V \setminus V'} a'_v x_v \leq b$ (with a'_v not necessarily integer) are called *simultaneous liftings* of F', see Zemel (1978). Moreover, if there exists an ordering $v_1, \ldots, v_K$ of the vertices in $V \setminus V'$ such that, for each $k = 1..K$, the inequality $\sum_{v \in V'} a_v x_v + \sum_{v \in \{v_1, \ldots, v_k\}} a'_v x_v \leq b$ is a facet of $STAB(G[V' \cup \{v_1, \ldots, v_k\}])$, then F is a *sequential liftings* of F' (i.e., it can be obtained from F' by "lifting" one coefficient at a time), see Padberg (1973).

Fuzzy circular interval graphs are a slight generalization of circular interval graphs where Φ is not necessarily injective, no two interval of $\mathcal{I}$ shares an endpoint and the vertices mapped at the two extremities of an interval might be adjacent or not. More formally:

Definition 7 A graph $G(V, E)$ is a fuzzy circular interval (FCIG) if the following conditions hold.

(i) There is a map Φ from V to a circle $\mathcal{C}$.
(ii) There is a set of intervals $\mathcal{I}$ of $\mathcal{C}$, none including another, such that no point of $\mathcal{C}$ is an endpoint of more than one interval so that:

    (a) If two vertices u and v are adjacent, then $\Phi(u)$ and $\Phi(v)$ belong to a common interval.
    (b) If two vertices u and v belong to a same interval, which is not an interval with distinct endpoints $\Phi(u)$ and $\Phi(v)$, then they are adjacent.

In this case, we also say that the pair $(\Phi, \mathcal{I})$ gives a *fuzzy representation* of G.

In other words, in a FCIG, adjacencies are completely described by the pair $(\Phi, \mathcal{I})$, except for vertices u and v such that $\mathcal{I}$ contains an interval with endpoints $\Phi(u)$ and $\Phi(v)$. For these vertices adjacency is fuzzy. If $[p, q]$ is an interval of $\mathcal{I}$ such that $\Phi^{-1}(p)$ and $\Phi^{-1}(q)$ are both non-empty, then we call the cliques $(\Phi^{-1}(p), \Phi^{-1}(q))$ a *fuzzy pair*. Here $\Phi^{-1}(p)$ denotes the clique $\{v \in V \mid \Phi(v) = p\}$.

We close this section by defining clique-circulants.

Definition 8 A quasi-line graph $G(V, E)$ is a (n, p)-*clique-circulant* if (i) there exist a partition of V into n non-empty cliques $Q_1, \ldots, Q_n$ and an integer p, with $n \geq 2p \geq 4$, such that $\Lambda(Q_i) \supseteq Q_{i-p+1} \cup \cdots \cup Q_{i-1} \cup Q_{i+1} \cup \cdots \cup Q_{i+p-1}$, for $i = 1, \ldots, n$ and (ii) there exists an induced (n, p)-circulant on vertex set $W = \{v_1, \ldots, v_n\} \subseteq V$ with $v_i \in Q_i$ for $i = 1, \ldots, n$. (for $W \subseteq V$, $\Lambda(W) := \{v \in V : (u, v) \in E, \forall u \in W\}$, $\Lambda(W)$ represent the strong neighborhood of W).
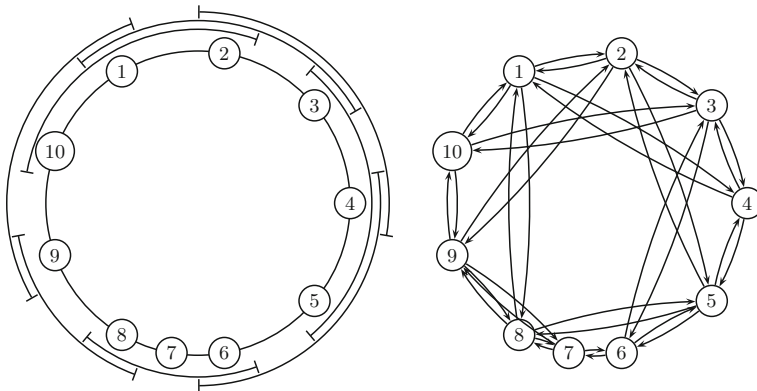
2 Warm up

Let $D = (V, \mathcal{A})$ (resp. $G(V, E)$) be a directed (resp. undirected) graph, a *cyclic order* on V is a linear ordering $v_1, \ldots, v_n$ of the vertices of V with the additional relation that v_1 follows v_n. Given a circular interval graph C and a representation $(\phi, \mathcal{I})$, we can define a cyclic order by listing the vertices clockwise. In the following, we will suppose that the vertices are labeled $1, \ldots, n$ with respect to this ordering. Moreover given a vertex i we will often refer to $i - 1$ as the *predecessor*, or left neighbor, of i and $i + 1$ as the *successor*, or right neighbor, of i. The *closest predecessor* of i having a given property will be the vertex $i - k$ having the required property with $k > 0$ as small as possible and analogously for the *closest successor*. Also we can suppose that for all $v \in V(C)$, $\phi(v)$ is the angular polar coordinate of v. We will sometimes denote $[a, b]$ the set of points of the unit circle with polar coordinate θ such that $b \leq \theta \leq a$. (NB: indices are taken modulo n and inequalities on θ are taken modulo 2π).

In the following, when we consider a CIG C we always assume that we are given a representation $(\phi, \mathcal{I})$. So let C be a CIG. Following Eisenbrand et al. (2008), we can associate an auxiliary graph to C. Indeed, any interval of $\mathcal{I}$ corresponds to a clique in C and we denote the family of cliques stemming from intervals by $\mathcal{K}_\mathcal{I}$. Now let $A \in \{0, 1\}^{m.n}$ be the clique vertex incidence matrix of $\mathcal{K}_\mathcal{I}$ and $V(C)$ ($m = |\mathcal{K}_\mathcal{I}|$); note that A has the circular one property. The characteristic vectors of stable sets of G are integer solutions in $\{x \in \mathbb{R}^n : Ax \leq 1, x \geq 0\}$. Introducing the totally unimodular (and invertible) transformation

$$
T = \begin{pmatrix} \overset{1}{-1} & 1 & & & \\ & -1 & 1 & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & -1 & 1 \end{pmatrix},
$$

we can rewrite the set $\{x \in \mathbb{Z}^n : Ax \leq 1, x \geq 0\}$ as $\{x = Ty : y \in \mathbb{Z}^n, \left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right) Ty \leq \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\}$. The matrix $\left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right) T$ is obtained from $\left(\begin{smallmatrix} A \\ -I \end{smallmatrix}\right)$ by substituting each column (but the last one) by the current column minus the next and it can thus be easily seen to be "almost" a network matrix, i.e., it is of the form $(N|v)$ where N^t is a network matrix minus one row (see Eisenbrand et al. 2008 for more details). We then define the auxiliary graph $Aux(C) = (U, \mathcal{A})$ associated with C by considering the arc-node incidence matrix $M = \begin{pmatrix} N & w \\ -N & -w \end{pmatrix}$ where w is the negative sum of the columns of N.

The set of arcs $\mathcal{A}$ partitions in two classes $\mathcal{A}_L$ and $\mathcal{A}_R$: $\mathcal{A}_L$ are the arcs with arc-node incidence matrix $(N|w)$ and $\mathcal{A}_R$ are the arcs with arc-node incidence matrix $(-N| - w)$. The arcs $\mathcal{A}_R$ are simply the reverse of the arcs $\mathcal{A}_L$. In turn, $\mathcal{A}_L$ consists of two sets of arcs $\mathcal{S}_L$ and $\mathcal{T}_L$, where $\mathcal{S}_L$ is the set of arcs stemming from clique inequalities and $\mathcal{T}_L$ are the arcs stemming from the lower bounds $x \geq 0$. Likewise $\mathcal{A}_R$ can be partitioned into $\mathcal{S}_R$ and $\mathcal{T}_R$. In other words, if we look at the arc-node incidence matrix M, we can assume that the rows of M appear in the order $\mathcal{S}_L, \mathcal{T}_L, \mathcal{S}_R, \mathcal{T}_R$. A clique $\{i, i + 1, \ldots, i + p\}$ of C generates the arcs $(i - 1, i + p) \in \mathcal{S}_L$ and $(i + p, i - 1) \in \mathcal{S}_R$ and a lower bound $-x_i \leq 0$ generates the two arcs $(i, i - 1) \in \mathcal{T}_L$ and $(i - 1, i) \in \mathcal{T}_R$. We call *clique arcs* the arcs of $\mathcal{S}_L \cup \mathcal{S}_R$ and *non-negativity arcs* the arcs of $\mathcal{T}_L \cup \mathcal{T}_R$.

It is easy to build $Aux(C)$ from a representation of C and conversely, it is easy to give a representation of C associated with the auxiliary graph $Aux(C)$. In particular, there is a one to one correspondence between the vertices U and $V(C)$. Therefore, we will often abuse notations and denote by $V(C)$ the vertices of the auxiliary graph (Fig. 1).

Fig. 1 A circular interval graph C and its corresponding auxiliary graph $Aux(C)$. For instance : arc $(4, 3) \in \mathcal{T}_L$, arc $(3, 4) \in \mathcal{T}_R$, arc $(6, 3) \in \mathcal{S}_R$ and arc $(3, 6) \in \mathcal{S}_L$

Eisenbrand et al. (2008) have proved that each facet-inducing inequality $cx \leq \delta$ of $STAB(C)$, which is not induced by an inequality of the system $Ax \leq \mathbf{1}$, $x \geq 0$, *"is associated with"* at least one simple (directed) cycle Γ of the auxiliary graph $Aux(C)$. Indeed this is the essence of the following Theorem which is basically Theorem 7 in Eisenbrand et al. (2008) restated in our notations (where we exhibit the values of certain parameters that can be read directly from the proof).

We stick to the notation in Eisenbrand et al. (2008) and we denote cycles in $Aux(C)$ by $\Gamma = (f_L, f_R)$ where f_L (resp. f_R) is the characteristic vector of Γ over $\mathcal{A}_L$ (resp. over $\mathcal{A}_R$). Abusing notation, we will often write $f_L(x_v \geq 0)$ (resp. $f_R(x_v \geq 0)$, $f_L(K), f_R(K)$) instead of $f_L(e)$ for e being the non-negativity arc corresponding to $x_v \geq 0$ or $x(K) \leq 1$.

Theorem 9 (Eisenbrand et al. 2008) *For each facet-inducing inequality $cx \leq \delta$ of $STAB(C)$, which is not induced by an inequality of the system $Ax \leq \mathbf{1}$, $x \geq 0$, there exist an integer $1 \leq \beta \leq \alpha(C)$ and a vector (f_L, f_R), which is the incidence vector of a simple cycle Γ of the directed graph $Aux(C)$, a negative integer $f_{R,0} = c(n) - f_R v$ and a positive integer $f_{L,0} = c(n) - f_L v$ (with $c(n) = f_L d_L - f_R d_R$, $d_L = d - \beta v$, $d_R = d - (\beta+1)v$, $d = (\mathbf{1}\, \mathbf{0})^t$) and such that $c\, x \leq \delta$ is derived from the systems*

$$
\begin{array}{lll}
\mathbf{1}x \leq \beta & \quad & -\mathbf{1}x \leq -(\beta + 1) \\
Ax \leq 1 & and & Ax \leq 1 \\
-x \leq 0. & & -x \leq 0,
\end{array}
\tag{1}
$$

with the weights $f_{L,0}$, f_L and $|f_{R,0}|$, f_R respectively. Moreover, we can find $|V(C)|$ affinely independent stable sets of size β and $\beta + 1$ satisfying $cx = \delta$.

(The last statement of the theorem follows from simple convexity arguments.) Eisenbrand et al. have also proved that we can in fact restrict our attention to simple directed cycles with at least one arc from $\mathcal{S}_R$ and without arcs from $\mathcal{S}_L$ (Lemma 9 and 10 in Eisenbrand et al. 2008).

Remark 10 From now on we indeed assume that there is no arc from $\mathcal{S}_L$ in $Aux(C)$, i.e., $Aux(C) = (U, \mathcal{T}_L \cup \mathcal{S}_R \cup \mathcal{T}_R)$. Moreover, in our drawings, we will usually represent non-negativity arcs by undirected edges, while we will only specify the orientation when one of the two arcs is used in a simple cycle.

Fig. 2 A cycle and the different types of vertices. The winding number of the cycle is 2
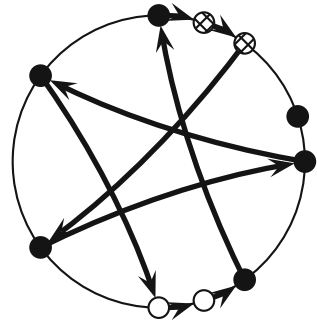
3 Exploiting the auxiliary graph *Aux*(Γ)

We start with a concept introduce by Sebò (2004). Given a cyclic order $1, \ldots, n$ of the vertices of a directed graph $D(U, \mathcal{A})$, one can define the *winding* of a directed cycle Γ as follows. We define the *length* of an arc $(i, i + k)$, for some $n \geq i \geq 1, n - 1 \geq k \geq 1$, as $l((i, i+k)) = k$ (sums of indices are taken modulo n). The winding of Γ is the integer $ind(\Gamma) = \frac{\sum_{e \in \Gamma} l(e)}{|V(G)|}$. We would like to generalize this notion of winding allowing some "backward steps". We therefore let $\mathcal{A}_F \cup \mathcal{A}_B$ be a partition of $\mathcal{A}$ into two classes: $\mathcal{A}_F$ the forward arcs and $\mathcal{A}_B$ the backward arcs. Now we can define the length of an arc $(i, i + k)$, for some $n \geq i \geq 1$, $n - 1 \geq k \geq 1$, to be $\mathbf{l}((i, i + k)) = k$ if it is forward and $\mathbf{l}((i, i + k)) = k - n$ if it is backward. Hence, the length of an arc $(i, i - 1)$ will be $n - 1$ if $(i, i - 1) \in \mathcal{A}_F$ and -1 if $(i, i-1) \in \mathcal{A}_B$. We now define the *(net) winding* to be the integer $\mathbf{ind}_{\mathcal{A}_F, \mathcal{A}_B}(\Gamma) = \frac{\sum_{e \in \Gamma} \mathbf{l}(e)}{n}$.

As we already pointed out, we can restrict our attention to simple, directed cycles of $Aux(C)$ with at least one arc from $\mathcal{S}_R$. For such cycles, we will consider the (net) winding with respect to a counter-clockwise cyclic ordering of the vertices and with respect to the sets of forward arcs $\mathcal{A}_F = \mathcal{S}_R \cup \mathcal{T}_L$ and backward arcs $\mathcal{A}_B = \mathcal{T}_R$, and we define $p(\Gamma) = \mathbf{ind}_{\mathcal{A}_F, \mathcal{A}_B}(\Gamma)$. It is convenient to introduce the following definition:

Definition 11 Let $\Gamma = (f_L, f_R)$ be a simple, directed cycle in $Aux(C)$ with at least one arc from $\mathcal{S}_R$. We denote by $\mathcal{F}(\Gamma)$ be the family of cliques corresponding to the clique arcs of Γ and $p(\Gamma) = \mathbf{ind}_{\mathcal{S}_R \cup \mathcal{T}_L, \mathcal{T}_R}$, its winding number (again w.r.t. a counter-clockwise ordering of the vertices). We also let $\beta(\Gamma) = \lfloor \frac{|\mathcal{F}(\Gamma)|}{p(\Gamma)} \rfloor$ and $r(\Gamma) = |\mathcal{F}(\Gamma)| - p(\Gamma) \cdot \beta(\Gamma)$. We finally partition the vertices of $V(C)$ into three classes with respect to Γ:

- Circles $\equiv S_\circ(\Gamma) := \{v : f_L(x_v \geq 0) = 1\}$,
- Bullets $\equiv S_\bullet(\Gamma) := \{v : f_L(x_v \geq 0) = f_R(x_v \geq 0) = 0\}$ and
- Crosses $\equiv S_\otimes(\Gamma) := \{v : f_R(x_v \geq 0) = 1\}$.

Figure 2 illustrates the previous definitions. Note that, if v is a vertex of circle or cross type, then v belongs to the cycle because $f_L(x_v \geq 0) = 1$ or $f_R(x_v \geq 0) = 1$. Also, because the cycle is simple and contains at least one clique arc, a sequence of circle vertices on the cycle must start and end with a bullet. For the same reason, a sequence of cross vertices must start and end with a bullet. Regarding the vertices of type bullet, some are in the cycle, some are not.

The next theorem is a refinement of Theorem 9. For the sake of completeness we give a complete proof.

Theorem 12 *Let C be a CIG. Then any facet of $STAB(C)$, which is not $x \geq 0$ or a clique inequality, is the clique family inequality $CFI(\Gamma)$ for some cycle $\Gamma = (f_L, f_R)$ of $Aux(C)$*

such that $|\mathcal{F}(\Gamma)| > 2p(\Gamma) \geq 4$ and $|\mathcal{F}(\Gamma)|\ mod\ p(\Gamma) \neq 0$. Moreover, there exist $|V(C)|$ affinely independent roots for $CFI(\Gamma)$ of size $\beta(\Gamma)$ and $\beta(\Gamma) + 1$.

Proof Theorem 9 implies that any facet $cx \leq \delta$ which is not induced by $Ax \leq \mathbf{1}$, $x \geq 0$ is a nonnegative integer combination of the system on the left in (1) with nonnegative weights $f_{L,0}$, f_L. As already mentioned, Lemma 9 in Eisenbrand et al. (2008) implies that f_L can be chosen such that the only nonzero $(+1)$ entries of f_L are corresponding to lower bounds $-x(v) \leq 0$. Therefore $cx \leq \delta$ is of the form

$$a \sum_{v \in T} x(v) + (a - 1) \sum_{v \notin T} x(v) \leq a\,\beta, \tag{2}$$

with $a = f_{L,0}$ and T set to those variables, whose lower bound inequality does not appear in the derivation: in other words, if $v \notin T$, then $f_L(x(v) \geq 0) = 1$.

From the same Theorem 9 , we know also that $cx \leq \delta$ can be derived from the system

$$\begin{aligned} -\mathbf{1}x &\leq -(\beta + 1) \\ Ax &\leq \mathbf{1} \\ -x &\leq 0, \end{aligned} \tag{3}$$

with weights $|f_{R,0}|$, f_R, where $f_{R,0}$ is a negative integer while f_R is a 0-1 vector. Observe that each vertex v has to belong to a root of size β or size $\beta + 1$, otherwise the facet would be induced by $x(v) \geq 0$ (since we know there exist $|V(C)|$ affinely independent roots of size β or $\beta + 1$, see Theorem 9). Moreover, the multiplier of f_R associated with a lower bound $-x(v) \leq 0$ must be 0 if v belongs to a root of size $\beta + 1$ (indeed a root of size $\beta + 1$ being tight for $-\mathbf{1}x \leq -(\beta+1)$, it must be tight for all inequalities used with positive multipliers in the right of (1)): in other words, if $v \in T$ and v belongs to a root of size $\beta + 1$, then $f_R(x(v) \geq 0) = 0$; and if $v \in T$ and v does not belong to a root of size $\beta + 1$, then $f_R(x(v) \geq 0)$ is either 0 or 1. Finally, observe that by definition $\mathcal{F}(\Gamma) = \{K \in \mathcal{K}_I \mid f_R(K) \neq 0\}$. We have therefore:

$$\begin{aligned} -|f_{R,0}| + |\{K \in \mathcal{F}(\Gamma) \mid v \in K\}| &= a - 1 \quad \forall v \notin T \\ -|f_{R,0}| + |\{K \in \mathcal{F}(\Gamma) \mid v \in K\}| &= a \qquad \forall v \in T, v \text{ is in a root of size } \beta + 1 \\ -|f_{R,0}| + |\{K \in \mathcal{F}(\Gamma) \mid v \in K\}| &\geq a \qquad \forall v \in T, v \text{ is not in a root of size } \beta + 1 \\ -|f_{R,0}|(\beta + 1) + |\mathcal{F}(\Gamma)| &= a\beta \end{aligned}$$

If we let $p := a + |f_{R,0}|$, then any vertex not in T belongs to exactly $p - 1$ cliques from $\mathcal{F}(\Gamma)$, while each vertex in T belongs to at least p cliques from $\mathcal{F}(\Gamma)$. Moreover, since $|\mathcal{F}(\Gamma)| = (a + |f_{R,0}|)\beta + |f_{R,0}|$, we have that $|\mathcal{F}(\Gamma)|\ mod\ p = |f_{R,0}| \neq 0$. Therefore, the inequality $cx \leq \delta \equiv a \sum_{v \in T} x(v) + (a - 1) \sum_{v \notin T} x(v) \leq a\,\beta$, is the clique family inequality associated with $\mathcal{F}(\Gamma)$ and p.

We now show that $p = p(\Gamma)$, i.e., $cx \leq \delta$ is indeed equivalent to $CFI(\Gamma)$. From Theorem 9, we have $a = f_{L,0}$, $f_{L,0} = c(n) - f_L \cdot v$ and $f_{R,0} = c(n) - f_R \cdot v$ for some $c(n)$, and $f_{R,0}$ is a negative integer so $|f_{R,0}| = -f_{R,0}$. It follows that $p = (f_R - f_L) \cdot v$. We indeed have that $p = p(\Gamma)$, as $(f_R - f_L) \cdot v = p(\Gamma)$: see Claim 1 in Lemma 14. It also follows that $\beta = \beta(\Gamma)$ and therefore there exist $|V(C)|$ affinely independent roots for $CFI(\Gamma)$ of size $\beta(\Gamma)$ and $\beta(\Gamma) + 1$.

We finally observe that, since $a \geq 1$ and $|f_{R,0}| \geq 1$, it follows that $p(\Gamma) \geq 2$. Moreover, if $|\mathcal{F}(\Gamma)| = 3$, then the clique family inequality is a clique inequality (as the right hand side is 1). Thus for a non-clique inequality, we have $|\mathcal{F}(\Gamma)| > 2p(\Gamma) \geq 4$ and thus we have at least five clique arcs. $\qquad\square$

Theorem 12 motivates the following:

Definition 13 A directed cycle of $Aux(C)$ that is simple, such that $|\mathcal{F}(\Gamma)| > 2p(\Gamma) \geq 4$ and $|\mathcal{F}(\Gamma)| \bmod p(\Gamma) \neq 0$ is called *good*.

We will now show a "counterpart" to Theorem 12. Namely we show that we can associate with *every* simple directed cycle of $Aux(C)$ with at least one arc from $\mathcal{S}_R$ a clique family inequality.

Lemma 14 *Let $\Gamma = (f_L, f_R)$ a simple, directed cycle in $Aux(C)$ with at least one arc from $\mathcal{S}_R$.*

(i) *the vertices of $S_\otimes(\Gamma)$ (resp. $S_\bullet(\Gamma)$, $S_\circ(\Gamma)$) are covered by exactly $p(\Gamma) + 1$ (resp. $p(\Gamma)$, $p(\Gamma) - 1$) cliques of $\mathcal{F}(\Gamma)$;*
(ii) *the clique family inequality $CFI(\mathcal{F}(\Gamma), p(\Gamma))$, which we simply denote by $CFI(\Gamma)$, is then valid: $(p(\Gamma) - r(\Gamma)) \sum_{v \in S_\bullet(\Gamma) \cup S_\otimes(\Gamma)} x_v + (p(\Gamma) - r(\Gamma) - 1) \sum_{v \in S_\circ(\Gamma)} x_v \leq (p(\Gamma) - r(\Gamma))\beta(\Gamma)$;*
(iii) *$CFI(\Gamma)$ can be also derived from the systems:*

$$
\begin{array}{lll}
\mathbf{1}x \leq \beta(\Gamma) & & -\mathbf{1}x \leq -(\beta(\Gamma) + 1) \\
Ax \leq 1 & and & Ax \leq 1 \\
-x \leq 0. & & -x \leq 0,
\end{array}
\tag{4}
$$

with the weights $p(\Gamma) - r(\Gamma)$, f_L and $r(\Gamma)$, f_R respectively.

Proof We start with the following:

Claim For each $u \in C$, $\sum_{K:u \in K} f_R(K) + f_L(x_u \geq 0) - f_R(x_u \geq 0) = p(\Gamma)$ and, in its turn, $p(\Gamma) = (f_R - f_L) \cdot v$.

Observe that $(f_R - f_L)N = [f_R|f_L]\binom{N}{-N} = [0 \ldots 0]$, since Γ is a a simple cycle of $Aux(C)$ and $\begin{pmatrix} N & w \\ -N & -w \end{pmatrix}$ is the arc-node incidence matrix of $Aux(C)$. Also observe that $T^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$. Then we have: $(f_R - f_L)\binom{A}{-I} = (f_R - f_L)[N|v]T^{-1} = [0 \ldots 0|q]T^{-1} = [q \ldots q]$, where we let $q := (f_R - f_L) \cdot v$. Thus for all $u \in V(C)$, we have $q = |\{K : u \in K \text{ and } f_R(K) > 0\}| + f_L(x_u \geq 0) - f_R(x_u \geq 0) = \sum_{K:u \in K} f_R(K) + f_L(x_u \geq 0) - f_R(x_u \geq 0)$. We are left with showing that q is indeed equal to $p(\Gamma)$. By definition, $p(\Gamma) = \frac{\sum_{e \in \Gamma} \mathbf{l}(e)}{|V(C)|}$ but $\sum_{e \in \Gamma} \mathbf{l}(e) = \sum_{K:f_R(K)>0} |K| + \sum_u f_L(x_u \geq 0) - \sum_u f_R(x_u \geq 0) = \sum_u (|K : u \in K \text{ and } f_R(K) > 0| + f_L(x_u \geq 0) - f_R(x_u \geq 0)) = |V(C)| \cdot q$ and thus $p(\Gamma) = q$. ■

(i) Since Γ is simple, it follows that, for each $u \in V$, it has at most one of $f_L(x_u \geq 0)$ or $f_R(x_u \geq 0)$ equal to one. The statement then follows from the claim and Definition 11; (ii) it follows from the definition of clique family inequalities (Oriolo 2003); (iii) this can be easily checked by the reader. □

Now one may wonder if there is a one-to-one correspondence between the good cycles of $Aux(C)$ and the facet-defining inequalities of $STAB(C)$. Unfortunately, that is not true. First of all, there might in general be *several* good cycles "producing" the same clique family inequality. Consider the CIG of Fig. 1 together with the corresponding auxiliary graph. The two cycles of Fig. 3 gives exactly the same clique family (facet-defining) inequality, i.e.,
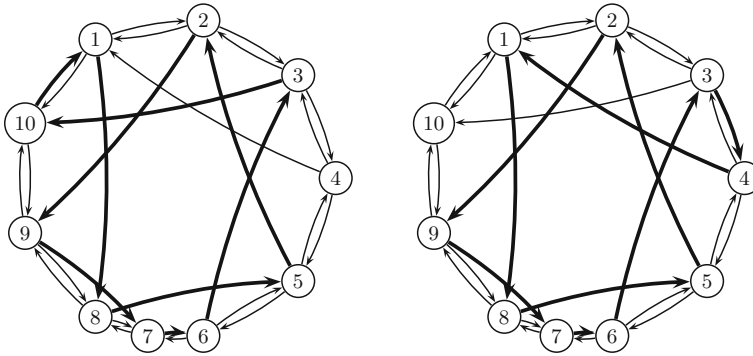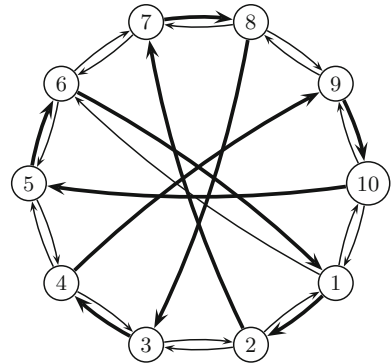
Fig. 3 Two cycles giving the same facet

Fig. 4 A good cycle "producing" a valid inequality $\sum_v x_v \leq 2$ that is not a facet: in fact the arc $(1, 6)$ and the arc $(6, 1)$ correspond to the cliques $K_1 = \{7, 8, 9, 10, 1\}$ and $K_2 = \{2, 3, 4, 5, 6\}$ respectively: therefore the inequality $\sum_v x_v \leq 2$ is the sum of the inequalities $\sum_{v \in K_1} x_v \leq 1$ and $\sum_{v \in K_2} x_v \leq 1$

$\sum_{i \neq 7} x_i \leq 3$. Note that since each cycle Γ univocally determines the family of cliques $\mathcal{F}(\Gamma)$ that example shows that indeed two different families of cliques may generate the same clique family inequality.

Even worse, it is not true in general that the clique family inequality associated with a good cycle of $Aux(C)$ is facet-defining for $STAB(C)$. We give an example in Fig. 4.

4 The circulant structure of good cycles in *Aux*(*C*)

In this section, we will refine Theorem 12 by showing that we may restrict our attention to some good cycles of $Aux(C)$ exhibiting a nice circulant structure. We start with a few definitions.

Definition 15 Given a circular interval graph $C = (V, E)$ with representation $(\phi, \mathcal{I})$ and a good cycle Γ in $Aux(C)$, we let $C[\mathcal{F}(\Gamma)] = (V, E')$ be the circular interval graph with representation $(\phi, \mathcal{I}_{\mathcal{F}})$ where $\mathcal{I}_{\mathcal{F}} \subseteq \mathcal{I}$ is the subset of intervals corresponding to the cliques in $\mathcal{F}(\Gamma)$, i.e., we keep only adjacencies defined by $\mathcal{I}_{\mathcal{F}}$.

Let $Aux(C[\mathcal{F}(\Gamma)])$ be the auxiliary graph associated with $C[\mathcal{F}(\Gamma)]$. Note that $Aux(C[\mathcal{F}(\Gamma)])$ is a spanning subgraph of $Aux(C)$, namely the subgraph with vertex set $V(C)$ and arc set $\mathcal{T}_L \cup \mathcal{T}_R \cup (\mathcal{S}_R \cap \Gamma)$. Let us call *essential* a bullet vertex that belongs to Γ. We will now prove that the essential vertices induce a circulant graph in $C[\mathcal{F}(\Gamma)]$.

Lemma 16 *Let C be a circular interval graph and let Γ be a good cycle of $Aux(C)$. The essential vertices of Γ induce an (n, p)-circulant in $C[\mathcal{F}(\Gamma)]$, with $n = |\mathcal{F}(\Gamma)|$ and $p = p(\Gamma)$. Moreover n and p are relatively prime.*

Proof Let C' be the subgraph of $C[\mathcal{F}(\Gamma)]$ induced by the essential vertices. By construction, C' is a CIG and we can then associate the auxiliary graph $Aux(C')$ with it. It is straightforward to check that the vertices and the clique arcs of $Aux(C')$ form a good cycle Γ' of $Aux(C')$. In fact, each clique arc of Γ corresponds to a clique arc of $Aux(C')$ and viceversa: a clique arc of Γ ending in a circle vertex v will correspond to a clique arc of $Aux(C')$ ending in the closest essential predecessor of v; a clique arc of Γ starting from a cross vertex v will correspond to a clique arc of $Aux(C')$ starting from the closest essential predecessor of v; the other clique arcs of Γ will stay unchanged. It follows that, for each vertex in $Aux(C')$, there is exactly one clique arc of $Aux(C')$ leaving from that vertex and exactly one (other) clique arc entering that vertex. Therefore, the vertices and the clique arcs of $Aux(C')$ form a cycle Γ' with $|V(C')| = |V(Aux(C'))| = |F(\Gamma')| = |F(\Gamma)| = n$.

Let us number, clockwise, $1, \ldots, n$ the vertices of C' and let $K_1, \ldots, K_n$ be the n cliques of $\mathcal{F}(\Gamma')$, with K_i being the clique corresponding to the clique arc of $Aux(C')$ leaving from vertex i, with $1 \leq i \leq n$. Note that K_i takes consecutive elements of $V(C')$, say $i, i-1, \ldots, i-l(i)+1$ with $1 \leq l(i) \leq n$. We claim that, for each i, $l(i) = p(\Gamma)$, i.e., each clique takes $p(\Gamma)$ vertices. From Lemma 14, each vertex of C' belongs to exactly p cliques of $\mathcal{F}(\Gamma)$ and, by construction, to exactly p cliques of $\mathcal{F}(\Gamma')$. Moreover, there is no inclusion among cliques in $\mathcal{F}(\Gamma)$ and, again by construction, there is no inclusion among cliques in $\mathcal{F}(\Gamma')$. The vertex $i - l(i) + 1$ is then covered by the cliques K_i, K_{i-1},..., $K_{i-l(i)+1}$ and only by these. But because it is covered exactly p times, it follows that $l(i) = p$ and thus K_i takes the p consecutive elements $i, i-1, \ldots, i-p+1$ (for each i). As a consequence, C' is an (n, p)-circulant. Note that because we are dealing with simple cycles, we can also conclude that n and p are relatively prime. Indeed if we follow Γ' starting from n, we will visit the vertices $n, n-p, n-2p,\ldots,n$ in this sequence. But in order to visit vertex 1, there should exist $\mu \in \mathbb{Z}$, $\mu < 0$ such that $n + \mu p = 1 (mod \ n)$, i.e., there should exist $\lambda, \mu \in \mathbb{Z}$ such that $\lambda n + \mu p = 1$. By Bézout's theorem, this is equivalent to n and p being relatively prime. $\qquad \square$

Remark 17 Ben Rebea Theorem, i.e. Theorem 1, can be sharpened by restricting to pairs $(\mathcal{F}, p)$ such that $|\mathcal{F}|$ and p are relatively prime. Therefore, the stable set polytope of quasi-line graphs can be described by: (i) $x_v \geq 0$, for all $v \in V$; (ii) $x(K) \leq 1$, for all maximal cliques $K \subseteq V$; (iii) clique family inequalities with parameter $\mathcal{F}$ and p: $|\mathcal{F}| > 2p$, $p \geq 2$, $|\mathcal{F}|$ and p relatively prime.

It is important to notice that the essential vertices of a good cycle Γ do not necessarily induce an (n, p)-circulant in the original graph C, even if $|\mathcal{F}(\Gamma)|$ and $p(\Gamma)$ are relatively prime. One can study the examples of Fig. 3 to see this. Indeed the essential vertices of the first cycle induce a $(7, 2)$-circulant in C whereas this is not the case for the essential vertices of the second cycle. The example of Fig. 5 illustrates another, more complicated, situation where the essential vertices of the cycle Γ on the left induce a $(8, 3)$-circulant in $C[\mathcal{F}(\Gamma)]$, but there is no subgraph of C that is an $(8, 3)$-circulant. However, we can find another good cycle Γ' (on the right) whose essential vertices induce a $(5, 2)$-circulant in C and that yields the same clique family inequality as Γ $(\sum_{v \neq o} x_v \leq 2)$.

This is the essence of the following lemma. We prove that if a good cycle does not induce an (n, p)-circulant in the original graph, we can find another (n', p')-cycle with $\lfloor \frac{n'}{p'} \rfloor = \lfloor \frac{n}{p} \rfloor$
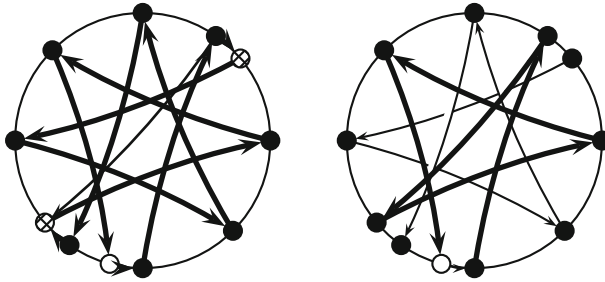
Fig. 5 Left: a good cycle Γ whose essential vertices induce a (8, 3)-circulant in $C[\mathcal{F}(\Gamma)]$, even though there is no (8, 3)-circulant in C. Right: another cycle Γ' that induce a (5, 2) both in $C[\mathcal{F}(\Gamma')]$ and C and is such that $CFI(\Gamma) = CFI(\Gamma')$

representing the same facet and having an induced (n', p')-circulant structure. We postpone the technical proof of this lemma to the "Appendix".

Lemma 18 *Let $C = (V, E)$ be a circular interval graph. Let F be a facet of $STAB(C)$ different from a non-negativity constraint, a clique inequality and $\sum_{v \in V} x_v \leq \alpha(C)$. There always exists a good cycle Γ of $Aux(C)$ such that F coincides with $CFI(\Gamma)$ and the essential vertices of Γ induce an (n, p)-circulant graph in C, with $n = |\mathcal{F}(\Gamma)|$ and $p = p(\Gamma)$ relatively prime.*

5 Circulant graphs for the stable set polytope of CIGs

We are now ready to prove the main result of the paper showing that circulant subgraphs are "responsible" for the non-trivial facets of the stable set polytope of CIGs. We will exploit this result to prove two conjectures from the literature due to Pêcher and Wagler (2006) and Oriolo and Stauffer (2008) respectively. It is indeed convenient to start with the former conjecture, on the stable set polytope of circulant graphs.

Let W be an (n, p)-circulant graph with $V(W) = \{1, 2, \ldots, n\}$. For each $i \in V(W)$, let Q_i be the clique $\{i - p + 1, i - p + 2, \ldots, i\}$ (modulo n as usual) and for a set $I \subseteq V(W)$ let $\mathcal{Q}(I) = \{Q_i, \text{ for all } i \in I\}$. Now suppose that the set $I \subseteq V(W)$ induces an (n', p')-circulant on W. The *clique family inequality associated with I* is that with parameters $\mathcal{Q}(I)$ and p'. Pêcher and Wagler (2006) conjectured the following:

Conjecture 19 *(Pêcher and Wagler 2006) The stable set polytope of an (n, p)-circulant W can be characterized by non-negativity inequalities, clique inequalities, the inequality $\sum_{v \in V(W)} x_v \leq \alpha(W)$ and clique family inequalities associated with sets $I \subset V(W)$ that induce (n', p')-circulants on W, with $n' \bmod p' \neq 0$ and $\alpha(I) < \alpha(W)$.*

We will prove the above conjecture on the superclass of CIGs. We only need to suitably redefine the cliques Q_i, the rest will be unchanged. So let C be a CIG and $(\phi, \mathcal{I})$ a representation with $V(G) = \{1, \ldots, n\}$, where the vertices are numbered clockwise. For any vertex $i \in V(C)$, $N(i) = \{n_l(i), n_l(i) + 1, \ldots, i - 1, i + 1, \ldots, n_r(i) - 1, n_r(i)\}$, for some values $n_l(i), n_r(i)$. So we now let $Q_i = \{n_l(i), \ldots, i - 1, i\}$.

Theorem 20 *Let C be a circular interval graph and $(\phi, \mathcal{I})$ a representation for it. The stable set polytope of C can be characterized by: non-negativity inequalities, clique inequalities, the inequality $\sum_{v \in V(C)} x_v \leq \alpha(C)$ and clique family inequalities associated with sets $I \subset V(W)$ that induce (n, p)-circulants on W, with n and p relatively prime and $\alpha(I) < \alpha(W)$.*

Proof Lemma 18 implies that, if a facet-inducing inequality of C does not belong to one of the three first classes of inequalities, then it is the clique family inequality associated with some good cycle Γ such that the essential vertices of Γ induce an (n, p)-circulant W of C, with $n = |\mathcal{F}(\Gamma)|$, $p = p(\Gamma)$ and n and p are relatively prime. Now if Γ does not contain any cross vertex, then $CFI(\Gamma)$ coincides with the clique family inequalities associated with $V(W)$. Otherwise let j be a cross vertex such that Γ contains a clique arc leaving from j; note that the clique corresponding to that arc is exactly the clique Q_j and $Q_j \in \mathcal{F}(\Gamma)$. We replace the clique Q_j by the clique Q_i, where i is the closest (bullet) predecessor of v in W: let $\mathcal{F}' := (\mathcal{F}(\Gamma) \setminus Q_j) \cup Q_i$. The clique family inequality associated with $\mathcal{F}'$ and p dominates $CFI(\Gamma)$: indeed, all the predecessors of j before i are crosses and, by Lemma 14, they were covered by $p + 1$ cliques of $\mathcal{F}(\Gamma)$ and thus they are still covered by p clique of $\mathcal{F}'$; moreover, all the other vertices are either unaffected by this change or covered one more time. If we now iterate this procedure on each cross vertex of Γ that is the head of a clique arc, we end up with a clique family inequality, that is exactly the the clique family inequalities associated with $V(W)$, that dominates $CFI(\Gamma)$.

We are left with showing that $\alpha(W) < \alpha(G)$. Suppose the contrary: then, since W is an (n, p)-circulant and $n \bmod p \neq 0$, the inequality $\sum_{v \in V(W)} x_v \leq \alpha(W)$ is facet-inducing for $STAB(G[W])$. Now recall the following fact (a proof can be found in Oriolo 2003): If $G = (V, E)$ is a quasi-line graph and Q a subset of vertices, the inequality $\sum_{j \in Q} x_j \leq \alpha(G[Q])$ is facet-inducing for $STAB(G)$ if and only if $\sum_{j \in Q} x_j \leq \alpha(G[Q])$ is facet-inducing for $STAB(G[Q])$ and $G[Q]$ is α-maximal. In this case, since $\alpha(W) = \alpha(G)$, it follows that the inequality $\sum_{v \in V(G)} x_v \leq \alpha(G)$ is facet-inducing for $STAB(G)$. But then the latter inequality would dominate $CFI(\Gamma)$. Now because $CFI(\Gamma)$ is facet-inducing, $CFI(\Gamma)$ coincides with $\sum_{v \in V(G)} x_v \leq \alpha(G)$, this is a contradiction. □

The conjecture by Pêcher and Wagler follows from Theorem 6.

Corollary 21 *Conjecture* 19 *holds true. Moreover, it can be strengthened by claiming that* n' *and* p' *are relatively prime and such that* $n' > 2p' \geq 4$.

5.1 The conjecture by Oriolo and Stauffer for the SSP of FCIG

We devote this section to the solution of another conjecture, on the stable set polytope of fuzzy circular interval graphs (FCIGs), due to Oriolo and Stauffer (2008). FCIGs are a slight generalization of CIGs (see § 1.1 for a formal definition). As we already discussed, FCIGs are a quite relevant subclass of quasi-line graphs because all facets are rank when the quasi-line graph is not fuzzy circular interval (see Eisenbrand et al. 2008).

Oriolo and Stauffer (2008) gave a description of the rank facets of FCIGs, that is more detailed than the one provided by Theorem 2. They showed that all rank facets are inequalities associated with clique-circulant (see § 1.1 for the definition).

Theorem 22 (Oriolo and Stauffer 2008) *Let* $G = (V, E)$ *be a fuzzy circular interval graph. An inequality* $\sum_{v \in V(G')} x_v \leq \alpha(G')$ *is a facet of* $STAB(G)$ *if and only if* G' *is either a maximal clique or an* α-maximal (n, p)-clique-circulant with $n \bmod p \neq 0$.

They also conjectured that clique-circulants are at the heart of non-rank facets too. In order to present their conjecture, we need a couple of definitions. First, an induced (n, p)-clique-circulant $\overline{C}$ of a FCIG G is *maximal* if for each vertex $v \in N(\overline{C})$, $\overline{C} \cup v$ is no more an (n, p)-clique-circulant. Oriolo and Stauffer proved that, if G is a FCIG and $\overline{C}$ an induced (n, p)-clique-circulant of G, then a simple clique family inequality can be associated with $\overline{C}$:

Theorem 23 (Oriolo and Stauffer 2008) *Let $G = (V, E)$ be a a fuzzy circular interval graph and $\overline{C}$ a maximal (n, p)-clique-circulant graph. The following inequality, called the* clique family inequality associated with $\overline{C}$, *is valid for $STAB(G)$:*

$$(p - r) \cdot \sum_{v \in V(\overline{C})} x_v + (p - r - 1) \cdot \sum_{v \in N(\overline{C})} x_v \leq (p - r) \cdot \left\lfloor \frac{n}{p} \right\rfloor. \tag{5}$$

Conjecture 24 (Oriolo and Stauffer 2008) *Each facet of the stable set polytope of a FCIG G is: either a non-negativity constraint; or a maximal clique inequality; or a clique family inequality (5) associated with a maximal (n, p)-clique-circulant $\overline{C}$ such that $n \mod p \neq 0$.*

It is easy to see that for rank facets Conjecture 24 reduces to Theorem 22; therefore to settle the conjecture we just need to deal with non-rank facets. We first deal with the case where G is indeed a CIG: therefore, as usual we replace G by C. So assume that C is a CIG and let $(\phi, \mathcal{I})$ be a representation for it. We know from Lemma 18 that any non-rank facet-inducing inequality can be associated with a good cycle Γ of the auxiliary graph $Aux(C)$ with $n(\Gamma)$ and $p(\Gamma)$ relatively prime and such that the graph induced by the essential vertices is an $(n(\Gamma), p(\Gamma))$-circulant (in the following, for shortness, we let $n = n(\Gamma)$ and $p = p(\Gamma)$). Now let $v_1, \ldots, v_n$ be the essential vertices of Γ numbered clockwise. Let us define:

$$Q_i = \{v \in S_\otimes(\Gamma) : \phi(v) \in (\phi(v_i), \phi(v_{i+1}))\} \cup \{v \in S_\bullet(\Gamma) :$$
$$\phi(v) \in (\phi(v_{i-1}), \phi(v_i)]\} \text{ for } i = 1, \ldots, n.$$

Observe that $\{Q_1, \ldots, Q_n\}$ defines a partition of the vertices in $S_\otimes(\Gamma) \cup S_\bullet(\Gamma)$. We claim that $\overline{C} = C[\bigcup_{1,\ldots,n} Q_i]$ is a clique-circulant. First, $W = \{v_1, \ldots, v_n\}$ defines an (n, p)-circulant such that $v_i \in Q_i$ for all $i = 1, \ldots, n$. Then it may easily be checked that $\Lambda(Q_i) \supseteq Q_{i-p+1} \cup \ldots \cup Q_{i-1} \cup Q_{i+1} \cup \ldots \cup Q_{i+p-1}$, for $i = 1, \ldots, n$. $\overline{C}$ is thus an (n, p)-clique-circulant. Now observe that in order to show that the clique family inequality associated with Γ is the same as the clique family inequality associated with $\overline{C}$, it is enough to show that $N(\overline{C}) = S_\circ(\Gamma)$. That is trivial, as $V(\overline{C}) = \bigcup_{1,\ldots,n} Q_i = S_\otimes(\Gamma) \cup S_\bullet(\Gamma)$ and $S_\circ(\Gamma) = N(S_\otimes(\Gamma) \cup S_\bullet(\Gamma))$. In order to prove the conjecture for FCIGs, it thus remains to prove that $\overline{C}$ is maximal. If not, then there would exist a vertex v such that $S_\otimes(\Gamma) \cup S_\bullet(\Gamma) \cup \{v\}$ is also an (n, p)-clique-circulant. But then, since $N(S_\otimes(\Gamma) \cup S_\bullet(\Gamma) \cup \{v\}) \supseteq N(S_\otimes(\Gamma) \cup S_\bullet(\Gamma)) \setminus \{v\}$, the clique family inequality associated with this new (n, p)-clique-circulant would dominate the inequality associated with $\overline{C}$, which is facet-inducing, a contradiction.

Now it is not difficult to extend the result from CIGs to FCIGs. We need the following result from Eisenbrand et al. (2008). Given a graph $G(V, E)$ and a facet-inducing inequality $F : ax \leq b$ of $STAB(G)$, an edge e is said to be *F-critical* if there is a stable set of $G \setminus e$ (i.e., the graph with vertex set V and edge set $E \setminus \{e\}$) that violates the inequality.

Lemma 25 (Eisenbrand et al. 2008) *Let F be a facet of $STAB(G)$, where G is a fuzzy circular interval graph. Then F is also a facet of $STAB(G')$, where G' is a circular interval graph and is obtained from G by removing non-F-critical edges between fuzzy pairs of cliques (see Def. in § 1.1).*

So let G be a FCIG and consider a non-rank facet F of $STAB(G)$. Let G' be the CUG obtained from G by removing non-F-critical edges between fuzzy pairs. From the discussion above, we know that F is a clique family inequality associated with a maximal (n, p)-clique-circulant $\overline{C}$ of G'. We therefore let $\{Q_i, i = 1, \ldots, n\}$ be the partition of the vertices of $\overline{C}$ and $W = \{v_1, \ldots, v_n\}$ the vertices of the (n, p)-circulant, with $v_i \in Q_i$, defined as above, that obey Definition 8.

We need to show that $\overline{C}$ is a maximal (n, p)-clique-circulant also in G. However, it will be enough to show that $\overline{C}$ is (n, p)-clique-circulant in G, as maximality follows then by the same arguments as above for the case of CIGs. Because $\overline{C}$ is a maximal (n, p)-clique-circulant in G' and when moving from G' to G we only add edges, in order to show that $\overline{C}$ is an (n, p)-clique-circulant also in G it is enough to prove that W is still an (n, p)-circulant in G (or that we can anyhow choose vertices $w_i' \in Q_i$, i=1,...,n, so that $G[\{w_i', i = 1, \ldots, n\}]$ is (n, p)-circulant of G). If there are no F-critical edges between vertices of $\overline{C}$, we are clearly fine. So assume that there is a fuzzy pair of cliques (P_1, P_2) and two vertices $u \in P_1 \cap V(W)$ and $v \in P_2 \cap V(W)$ for which we removed the non F-critical edge $uv \in E(G)$. Now observe that u and v have maximum coefficient in the inequality F and therefore all other edges between vertices of P_1 and vertices of P_2 are non F-critical too and so in G' there are no edges between vertices of P_1 and vertices P_2. It follows that the vertices in P_1 (resp. P_2) are thus copies in G' and that they are all in $\overline{C}$ (they should get the same coefficient in the facet) and, in particular, we may assume without loss of generality that P_1 belong to some set Q_i, with $i \in \{1, \ldots, n\}$ and P_2 belong to some set Q_j, with $j \in \{1, \ldots, n\}$. Now recall that we can also assume (see e.g. Lemma 1 in Eisenbrand et al. 2008) that, without loss of generality, that there exist $u' \in P_1$ and $v' \in P_2$ that are not adjacent in G. Substituting u' for u and v' for v in W, we create in G' another (n, p)-circulant graph W' (fuzzy pairs are *homogeneous* pairs of cliques i.e., two vertices in P_1—resp. P_2—have the same neighborhood outside P_2—resp. P_1) that together with the sets Q_i obey Definition 8. If we repeat this argument for each fuzzy pair of cliques (we can treat each fuzzy pair independently), we end up with another (n, p)-circulant graph W'' that is an (n, p)-circulant in G and is such that W'' together with the sets Q_i obey Definition 8. The result follows.

Theorem 26 *Conjecture 24 holds true. Moreover, it can be strengthened by claiming that n and p are relatively prime and such that $n > 2p \geq 4$.*

6 A concluding remark

We would like to add a final remark. Even though not every clique family inequality associated with a good cycle Γ induces a facet (the nature of $S_\circ(\Gamma)$ and its interplay with $S_\bullet(\Gamma) \cup S_\otimes(\Gamma)$ is crucial), it is quite easy to use our findings to define sufficient conditions for a cycle to be facet-defining when restricting to $C[\mathcal{F}(\Gamma)]$ (in fact we can also easily remove cross vertices if we are only interested in sufficient conditions). When the circle vertices are organized in a "regular" manner, linear independence of the roots of size $\beta(\Gamma) + 1$ can easily be ensured. The following Fig. 6 gives examples.

Observe also that because we can focus on affinely independent roots of size $\beta(\Gamma)$ and $\beta(\Gamma) + 1$, and because the graph induced by the vertices with maximum coefficient must induce a rank facet, "facetness" of the clique family inequality associated with a good cycle only depends on the existence of $|S_\circ(\Gamma)|$ roots of size $\beta(\Gamma) + 1$ whose restriction to $S_\circ(\Gamma)$ are affinely independent. Observe that each root of size $\beta(\Gamma) + 1$ picks exactly $p(\Gamma) - r(\Gamma)$ vertices in $S_\circ$ and thus there should be some nice structure involved there too. We believe that, for non-rank inequalities associated with good cycles whose essential vertices induce a circulant, it is possible to build an auxiliary graph G' on the circle vertices with a nice "circular" structure and with the property that (1) $\alpha(G') = p(\Gamma) - r(\Gamma)$ and (2) $x(V(G')) \leq \alpha(G')$ is a facet if and only if $CFI(\Gamma)$ is facet-defining. Because understanding rank facets is easier, such a relation, together with our result, would certainly give a pretty good description of the non-rank facets. Unfortunately, although we have preliminary results in this direction
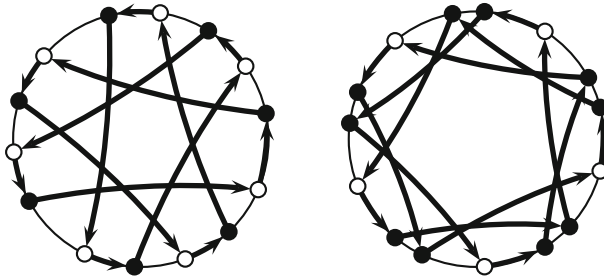
Fig. 6 Two facet-defining cycles : **a** $\sum_{v \in \circ} x_v + 2 \sum_{v \in \bullet} x_v \leq 4$, **b** $\sum_{v \in \circ} x_v + 2 \sum_{v \in \bullet} x_v \leq 6$

(e.g. such graphs are easy to build for the two examples above, and in this case they are even circular interval graphs), we do not have a clear picture of the right construction yet. However, we believe that this might be useful for further investigations and this is why we bring it to the attention of the reader.

N.B. Our techniques and theorems could be extended naturally to the set covering problem in circulant matrices (see Bianchi et al. 2017 *): in fact, row family inequalities are the counterpart of clique family inequalities in this setting and the stable set polytope of circulant graphs and the set covering polytope of circulant matrices bear strong resemblance.*

A Appendix

A.1 Proof of Lemma 18

Before going to the proof of Lemma 18, we discuss some properties of facet inducing clique family inequality associated with good cycles Γ. We first recall Lemma 3.2 in Stauffer (2011). (Note that we subtly changed the statement in the original lemma by substituting "Let $F : \sum a_v x_v \leq b$ be a non-clique facet of $STAB(G)$" for "Let $F : \sum a_v x_v \leq b$ be a facet of $STAB(G)$, that is not a clique in $\mathcal{K}_{\mathcal{I}}$". However the proof does not change.)

Lemma 27 (Stauffer 2011) *Let* $G = (V, E)$ *be a circular interval graph. Let* $F : \sum a_v x_v \leq b$ *be a facet of* $STAB(G)$, *that is not a clique in* $\mathcal{K}_{\mathcal{I}}$, *and* β *the smallest cardinality of a root of* F. *Then for all* $Q \in \mathcal{K}_{\mathcal{I}}$, *there exists a root* S *of* F *with* $|S| = \beta$ *such that* $Q \cap S = \emptyset$.

Lemma 28 *Let* C *be a circular interval graph and* $\Gamma = (f_L, f_R)$ *a good cycle of* $Aux(C)$ *such that the clique family inequality* $CFI(\Gamma)$ *is a facet of* $STAB(C)$. *There exists has a set of* $|V(C)|$ *affinely independent roots for* $CFI(\Gamma)$ *of size* $\beta(\Gamma)$ *and* $\beta(\Gamma) + 1$. *Moreover:*

Fig. 7 Situation where we do not have an (n, p)-circulant
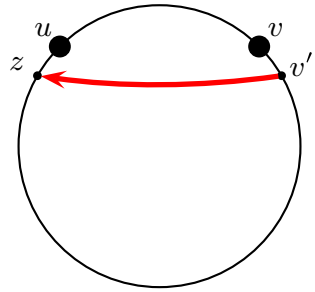
- *For any clique Q in $\mathcal{K}_{\mathcal{I}}$, there is a root of size $\beta(\Gamma)$ that does not intersect Q.*
- *Each root of $\mathcal{S}$ of size $\beta(\Gamma) + 1$ intersect each clique in $\mathcal{F}(\Gamma)$.*
- *No root of $\mathcal{S}$ of size $\beta(\Gamma) + 1$ contains a vertex of $S_{\otimes}(\Gamma)$.*

Proof Since Γ is facet-producing and good, it follows that $CFI(\Gamma)$ is not an inequality of the system $Ax \leq 1$, $x \geq 0$, and therefore, from Theorem 12, there exist a good cycle Γ' of $Aux(C)$ and $|V(C)|$ affinely independent roots of size $\beta(\Gamma')$ and $\beta(\Gamma') + 1$ for the corresponding clique family inequality $CFI(\Gamma')$ such that $CFI(\Gamma') = CFI(\Gamma)$. Note that, in general, Γ' needs not to be equal to Γ; however, since $\beta(\Gamma') = \beta(\Gamma)$, the above roots trivially form a set $|V(C)|$ affinely independent roots of size $\beta(\Gamma)$ and $\beta(\Gamma) + 1$ for $CFI(\Gamma)$ too.

Observe now that trivially the minimum size of a root of $CFI(\Gamma)$ is equal to $\beta(\Gamma)$; the first statement of the lemma follows then from Lemma 27. We now show that the second and the third statements follow from Lemma 14. The inequality $CFI(\Gamma)$ can be indeed derived from the system $-\mathbf{1}x \leq -(\beta(\Gamma) + 1)$, $Ax \leq 1$, $-x \leq 0$ with multipliers $r(\Gamma)$ and f_R. Consider now a root of $CFI(\Gamma)$ of size $\beta(\Gamma) + 1$; since the root is tight for $-\mathbf{1}x \leq -(\beta(\Gamma) + 1)$, it follows that it is also tight for all other inequalities with a non-zero multiplier f_R. In particular, each clique in $\mathcal{F}(\Gamma)$ must be tight for all roots of size $\beta(\Gamma) + 1$. This proves the second statement. Finally observe that the multiplier $f_R(x_v \geq 0)$ must be 0 if v belongs to a root of size $\beta(\Gamma) + 1$. Since, by definition, $f_R(x_v \geq 0) = 1$ for the vertices in $S_{\otimes}(\Gamma)$, the third statement follows. □

We can now prove Lemma 18. From Theorem 12 and Lemma 16, there exists a cycle Γ of $Aux(C)$ such that F is of the form $(p - r)x(T) + (p - r - 1)x(\bar{T}) \leq (p - r)\beta$, with $T = S_{\bullet}(\Gamma) \cup S_{\otimes}(\Gamma)$ and $\bar{T} = S_{\circ}(\Gamma)$, where $r = n \mod p$, $n = |\mathcal{F}(\Gamma)|$, $p = p(\Gamma)$ and $\beta = \beta(\Gamma)$. Moreover, the essential vertices of Γ induce an (n, p)-circulant W in $C[\mathcal{F}(\Gamma)]$ and n and p are relatively prime. We are in a configuration as depicted in Fig. 2. Let us choose Γ among all cycles representing the facet with the property that $r = r(\Gamma)$ is minimal.

Suppose that W does not induce an (n, p)-circulant in C. Let $v_1, \ldots, v_n$ be the vertices of W in clockwise order. Then without loss of generality there exist two vertices u, v of W such that $v = v_i$, $u = v_{i-p}$ and u and v are adjacent in the original graph C. Therefore, there exists an arc (v', z) in $Aux(C)$ corresponding to a clique containing both u and v: we will call this clique and its corresponding arc the *red clique* and the *red arc*. Note that u must differ from z (otherwise u is not covered by the red clique) but v' and v can coincide. We are in the situation represented in Fig. 7.

Since $v = v_i$ is adjacent to v_{i-p+1} in $C[\mathcal{F}(\Gamma)]$, there must exist another arc a' in $Aux(C)$ that induces that adjacency. We are indeed in one of the situations depicted in Fig. 8: a' leaves from a successor w of v' and the red clique starts at a node in $[\phi(v), \phi(w))$: that is because
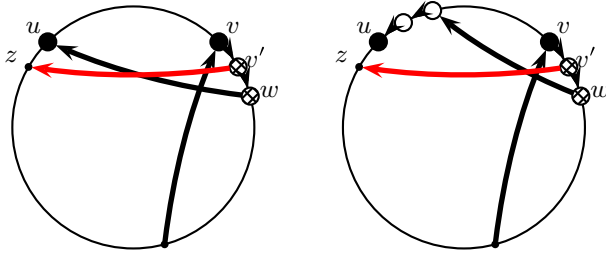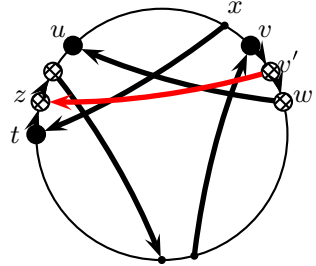
Fig. 8 Situation in the original graph

Fig. 9 Situation where z is on the cycle

otherwise the red clique would dominate a'. In particular, w has to be a cross vertex, as otherwise W is not an (n, p)-circulant in $C[\mathcal{F}(\Gamma)]$; therefore v has no direct circle successor on the circle and so there is another arc of Γ entering into v. Note that Fig. 8 is for illustration purposes: there might be more cross vertices between v and w; again v and v' may coincide. As it will be clear in the following, we do not need to distinguish between the two different situations of Fig. 8.

We will show that the situations depicted in Fig. 8 are not possible.

We first show that z cannot belong to Γ.

Suppose z belongs to the cycle Γ. We claim that z cannot be of type bullet or circle. In fact, if so and since z belongs to the cycle Γ, there must exist another arc a'' of Γ that that ends either in z or in a direct circle successor of z. Note that a'' has to start at a predecessor of v, otherwise the clique corresponding to a'' would contain both u and v. But in this case, then this clique would be dominated by the red clique. For the same reason, there cannot be any other essential or circle vertex between u and z. We are thus in the situation of Fig. 9, where $t = v_{i-p-1}$ (note again that Fig. 9 is for illustration purposes and there might be more cross vertices between t and z and between z and u; moreover the arc leaving from w might end up in some circle successor of u, see Fig. 8).

We claim that in this case $r \geq 2$. We know that $r > 0$ so assume that r is indeed equal to 1. We know form Lemma 28 that there is a root of size β that misses the red clique. As no root of size β can include a circle vertex, there should exist a root of size β whose vertices are cross and bullet vertices in $(\phi(v'), \phi(z)]$. Now recall that there are exactly p vertices of W in the interval $(\phi(t), \phi(v))$. Therefore, if $r = 1$ and we move along the cycle Γ from z, after β -1 clique arcs we must arrive at v. But then this set of $\beta - 1$ cliques covers every non-circle vertex in $(\phi(v'), \phi(z)]$. Hence it is not possible to have a stable set of size β in $(\phi(v'), \phi(z)]$, which is a contradiction.

Therefore, from now on, we assume that $r \geq 2$. The red arc defines a "shortcut" in the cycle Γ so that it is possible to define a new cycle Γ' using this shortcut. More formally,

consider the path P' on Γ from v' to z: we define the cycle Γ' by replacing P' with the arc (v', z). We will compare the clique family inequalities associated with Γ and Γ', but first we take a detour to analyze more in detail Γ'.

We start dealing with $|\mathcal{F}(\Gamma')|$ and $p(\Gamma')$. The path P' takes some number $n_0 + 1$ of cliques of Γ, with $n > n_0 \geq 1$. Now "project" the path P' on the circulant W. Since there are exactly p vertices of W in the interval $(\phi(t), \phi(v))$, and n and p are relatively prime, it follows that $(n_0 + 1)p \bmod n = (p + 1)$, i.e., $n_0 p \bmod n = 1$. Since $n_0 p \bmod n = 1$, there exists p_0 such that $n_0 p = p_0 n + 1$, with $1 \leq p_0 < p$, and p_0 can be indeed interpreted as the "winding number" of the path P'; therefore the winding number of the cycle Γ' is equal to $p - p_0$. It follows that Γ' is a cycle with $|\mathcal{F}(\Gamma')| = n - n_0 > 0$ and $p(\Gamma') = p - p_0 > 0$, with $n > n_0 \geq 1$, $p > p_0 \geq 1$ and $n_0 p = p_0 n + 1$. In the following we let $n' = n - n_0$ and $p' = p - p_0$.

We now show that $\beta(\Gamma') = \lfloor \frac{n'}{p'} \rfloor = \beta(\Gamma)$ (we denote $\beta(\Gamma)$ simply by β from now on). In order to prove that, we define $\bar{r} = n' - \beta p'$ and show that $\bar{r}$ is positive and smaller than p'. The key is showing that $rp' - p\bar{r} = 1$: in fact, by definition, $\bar{r} = n' - \frac{n-r}{p}p'$, therefore $rp' - p\bar{r} = np' - n'p = (n - n')p - (p - p')n = n_0 p - p_0 n = 1$. Now since $rp' - p\bar{r} = 1$, $r \geq 2$ and $p' \geq 1$, it follows that $\bar{r}$ is positive. Moreover $\bar{r} < p'$, since otherwise $1 = rp' - p\bar{r} \leq rp' - pp' = p'(r - p) < 0$.

We are now ready to compare the clique family inequalities associated with Γ and Γ'. Let $T' = S_\bullet(\Gamma') \cup S_\otimes(\Gamma')$ and $\bar{T}' = S_\circ(\Gamma')$. The inequality associated with Γ' is then $F' : (p' - r')x(T') + (p' - r' - 1)x(\bar{T}') \leq (p' - r')\beta$. Recall that the clique family inequality associated with Γ is $F : (p - r)x(T) + (p - r - 1)x(\bar{T}) \leq (p - r)\beta$, with $T = S_\bullet(\Gamma) \cup S_\otimes(\Gamma)$ and $\bar{T} = S_\circ(\Gamma)$. We will show that indeed $F \equiv F'$ by showing that there exists a family $\mathcal{Q}$ of $|V(C)|$ affinely independent roots for F that are also roots of F'.

First recall that we can assume that each stable set in $\mathcal{Q}$ has size either β or $\beta + 1$ by Lemma 28. Then observe that, by construction, $T \subseteq T'$ since by shortcutting Γ, we can only move some vertices of $\bar{T} = S_\circ(\Gamma)$ to $T' = S_\bullet(\Gamma') \cup S_\otimes(\Gamma')$. Therefore each stable set in $\mathcal{Q}$ of size β is also tight for F'. We are left with showing that also each stable set of $\mathcal{Q}$ of size $\beta + 1$ is tight for F': we let $\mathcal{Q}^{\beta+1}$ be the sub-family of stable sets of $\mathcal{Q}$ of size $\beta + 1$. In order to prove that, it will be enough to show that each stable set in $\mathcal{Q}^{\beta+1}$ meets each clique that is used in the right disjunction of F' (recall that F' is associated with the disjunction $(\sum_{v \in V} x_v \leq \beta) \vee (\sum_{v \in V} x_v \geq \beta + 1)$, cfr. Lemma 14), i.e., the cliques corresponding to arcs in Γ'. Note that the only arc that belongs to Γ' and not to Γ is the red arc. Therefore, as from Lemma 28 each stable set in $\mathcal{Q}^{\beta+1}$ intersects each clique in $\mathcal{F}(\Gamma)$, it follows that each stable set in $\mathcal{Q}^{\beta+1}$ also intersects each clique in $\mathcal{F}(\Gamma')$, but possibly for the red clique. But we now show that it holds for the red clique too. Observe that the clique K corresponding to the arc (x, t) (cfr. Fig. 9) is in $\mathcal{F}(\Gamma)$ and thus each stable set in $\mathcal{Q}^{\beta+1}$ intersects K. Note also that, again from Lemma 28, no cross vertex belong to a stable set in $\mathcal{Q}^{\beta+1}$. Therefore, since the only vertices that belong to K but not to the red clique are crosses, it follows that also the red clique intersects each stable set in $\mathcal{Q}^{\beta+1}$.

Hence $F' \equiv F$ and so Γ' is another cycle representing F. We have $r(\Gamma') = n' - \beta p' = n - n_0 - \beta p + \beta p_0 = r - r_0$ with $r_0 = n_0 - \beta p_0$. We now show that $r_0 p - r p_0 = 1$. In fact, $r_0 p - r p_0 = (n_0 - \beta p_0)p - (n - \beta p)p_0 = n_0 p - n p_0 = 1$. It follows, from $r \geq 2$ and $p_0 \geq 1$, that $r_0 > 0$, which is a contradiction since we chose Γ among all cycles representing F so that $r(\Gamma)$ is minimal. So from now on we assume that z does not belong to Γ.

We will now prove that z cannot be outside Γ as well.

In this case, trivially z is a bullet vertex. Recall from Fig. 8 that we are now in one of the two possible situations depicted in Fig. 10. Also observe that, by the same arguments used
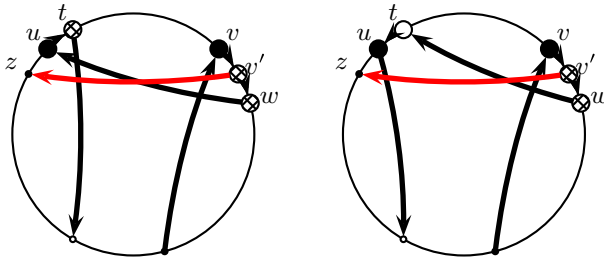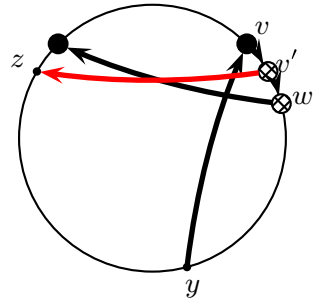
Fig. 10 z does not belong to Γ

Fig. 11 The cycle Γ and the current red arc

above, there is no essential vertex between z and u, and in fact each vertex between z and u has to be a bullet non-essential vertex.

In both cases we can modify Γ slightly to produce a non weaker valid inequality. Indeed, in the former case, we can replace the path $(v', \ldots, w, u, \ldots, t)$—there might be more cross vertices between v and w, as well more cross vertices between u and t—by the path $(v', z, \ldots, u, \ldots, t)$—there might be more bullet vertices between z and u. In the latter case, we can replace the path $(v', \ldots, w, t, \ldots, u)$—there might be more cross vertices between v and w, as well more circle vertices between t and u—by the path $(v', z, \ldots, u)$—there might be more bullet vertices between z and u. Let us call Γ' this new cycle. It is easy to see that we get a (strictly) stronger clique family inequality in the latter case (as all circle vertices between t and u become bullet vertices) proving in addition that this configuration is impossible, and that we get the same inequality in the former case. Note that even if we change Γ, we keep $r(\Gamma') = r(\Gamma)$ minimal. Observe also that, with respect to Γ', all vertices on the circle after v' up to w become bullet non-essential vertices, all vertices after z up to u on the circle become cross vertices and z becomes essential.

We are now wondering if Γ' satisfies the condition of the lemma. If not, this means that there is another clique that plays for Γ' the same role that the red clique played for Γ and therefore "kills" the circulant structure. But since Γ' has $r(\Gamma')$ minimal, we can use the same arguments as above and conclude that the head of the arc corresponding to this clique does not belong to Γ'. We can therefore define another cycle Γ'' by applying the transformation just discussed to Γ', and in case iterate. If this process does not end, this is because we cycle (indeed there are a finite number of cycles we can visit).

So let us suppose that using this procedure, we cycle. Let Γ be a cycle we have visited twice and let us choose a red clique associated with Γ. We are in the situation of Fig. 11.

Observe that when going from Γ to Γ' the number of direct cross successors of v decreased and that this number can only increases again if, for some (current) cycle $\tilde{\Gamma}$, v plays the role

Fig. 12 The cycle Γ and the successor of y in $\tilde{\Gamma}$ (in blue)
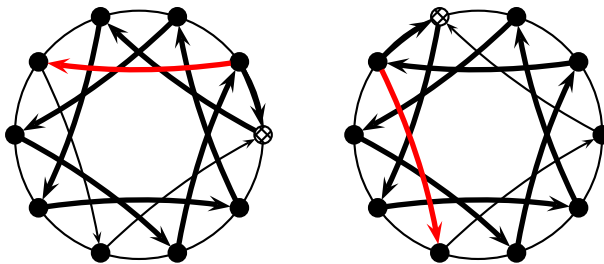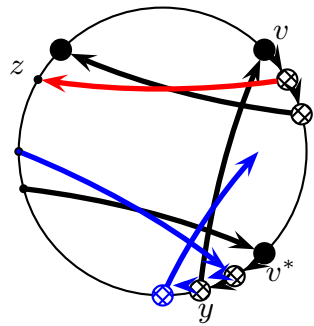

Fig. 13 (10, 3)-circulant and a cycle with (7, 2)-structure

that z played for Γ. Therefore, the clique that corresponds to the arc leaving from y (cfr. Fig. 11) will play for $\tilde{\Gamma}$ the same role that the red clique played for Γ. Note that y is either a cross or an essential vertex for Γ. However, we now show that, if y is an essential vertex for Γ, we can rule that it is preceded by a circle vertex in Γ. First, note that y plays for $\tilde{\Gamma}$ the same role that v' played for Γ and therefore y will have a cross successor. Then observe that our procedure does not ever modify the set of circle vertices, therefore the successor of y cannot be a circle for Γ. Therefore we are in the situation depicted in Fig. 12, where v^* represents the closest essential predecessor of y (if y is a bullet, $v^* = y$).

But again, since (y, v) plays the role of the red clique for $\tilde{\Gamma}$, it follows that y, with respect to $\tilde{\Gamma}$, has a cross successor. Now we can apply the same argumentation as above, replacing the red arc by (y, v), v by v^* etc. We will again show that, with respect to Γ, the tail of the clique arc ending in v^* is either a bullet vertex not preceded by a circle vertex or a cross vertex. Iterating this argumentation, we will visit all the bullet vertices of Γ, since n and p are relatively prime (see Lemma 16), and we will thus prove that no bullet vertex is preceded by a circle vertex in Γ. Hence it implies that there are no circle vertices for Γ. But this means that $\bar{T} = \emptyset$ and this is a contradiction with F not being of the form $\sum_{v \in V} x_v \leq \beta$. $hfill\square$

The example of Fig. 13 illustrates the fact that the last part of the proof does not hold for cycles associated with facets of type $\sum_{v \in V} x_v \leq \beta$. In fact, we have a (10, 3)-circulant but the cycle at hand (with $r(\Gamma)$ minimal) is associated with a (7, 2)-circulant. Successively applying the procedure yields after 11 steps the initial cycle. It is clear that there is no (7, 2)-circulant. Nevertheless, we believe that Lemma 18 could be extended with some care to rank facets of full support. We would have to change of course the pair (n, p) in this case as illustrated by this example.

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