A fuzzy multifactor asset pricing model

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Abstract

This paper introduces a new approach of multifactor asset pricing model estimation. This approach assumes that the monthly returns of financial assets are fuzzy random variables and estimates the multifactor asset pricing model as a fuzzy linear model. The fuzzy random representations allows us to incorporate bias on prices induced by the market microstructure noise and to reflect the intra-period activity in the analysis. The application of fuzzy linear regression enables the uncertainty assessment in an alternative way to confidence interval or hypothesis testing, which is subjected the binding assumption of normal distribution of returns. However, it is well known that the distribution of many asset returns deviates significantly from the normal assumption. We illustrate this estimation in the particular case of the Fama and French's (J Financ Econ 33:3–56, 1993) three factor model. Finally, empirical studies based on Fama and French's portfolios and risk factors, historical dataset highlight the effectiveness of our estimation method and a comparative analysis with the ordinary least square estimation shows its ability to be applied for an optimal decision decision making in the financial market.

Keywords Fuzzy set · Monthly volatility · Asset pricing theory · Multifactor model · Fuzzy linear regression · Weak-BLUE estimator

JEL Classfication G17

1 Introduction

Arbitrage pricing theory (APT) is a general theory of asset pricing in which the expected return of a financial asset can be modeled as a linear function of various factors or market indices. red The sensitivity in each factor is represented by a factor-specific beta coefficient. This theory that can be used to price the asset correctly, differs from the Capital Asset Pricing Model (CAPM) in that it is less restrictive in its assumptions. It allows for an explanatory

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(as opposed to statistical) model of asset returns. It assumes that each investor will hold a unique portfolio with its own particular array of betas, as opposed to the identical "market portfolio". In some ways, the CAPM can be considered a "special case" of the APT in that the securities market line represents a single-factor model of the asset price, where beta is exposed to changes in value of the market. In this sense APT is more satisfactory than the CAPM theory, which relies on both the mean-variance framework and a strong version of equilibrium, which assumes that everyone uses the mean-variance framework. In the APT context, arbitrage consists of trading in two assets with at least one being mispriced. The arbitrageur sells the asset which is relatively too expensive and uses the proceeds to buy one which is relatively too cheap.

Unlike the CAPM, the APT, however, does not reveal the identity of its priced factors - the number and nature of these factors is likely to change over time and between economies. As a result, this issue is essentially empirical in nature. As a practical matter, indices or spot or futures market prices may be used in place of macro-economic factors, which are reported at low frequency (e.g. monthly) and often with significant estimation errors. One deficiencies of the APT is that it fails determine appropriate systematic risk factors.

The identification and estimation of systematic risks affecting assets returns are important questions in financial economics. The approach initiated by Sharpe (1964), Lintner (1965) and Mossin (1966) with the CAPM, identifies the market return as the only relevant systematic risk affecting the asset returns. One of the main criticisms of the CAPM given in Fama and French (1992) is that a single market factor is insufficient to explain expected returns properly. In order to overcome this limit, several multifactor models have been proposed.[1] For more information, the reader may consult Fama and French (1992) and Carhart (1997). In applied econometrics, these models are generally estimated by the Ordinary Least Square (OLS) methods using closing prices of financial asset for returns computation. This estimation approach has been largely criticized over the past decades.

The closing prices of some financial assets used to compute their return are tainted by microstructure noise[2] caused by the imperfections of the trading process. This problem, referred to as error in variables in applied econometrics, tends to lead to the inconsistency of OLS estimation method for multifactor pricing models. Klepper and Leamer (1984) and Leamer (1984), among others, provide evidence of inconsistency of OLS estimator in linear regression with measurement errors in the regressors. Moreover, Cragg (1994) demonstrated that the slope coefficients were biased towards zero and concluded that the measurement error "produces a bias of the opposite sign on the intercept coefficient when the average value of the explanatory variables is positive". The problem of error in variable has been treated in asset pricing by several authors such as Coën and Racicot (2007), Carmichael and Coën (2008) and Coen and Hubner (2009) using different estimation methods.

The impact of the return interval on the systematic risk estimates is another important estimation issue of multifactor models. This question is largely investigated in the literature. Brailsford and Josev (1997), Hawawini (1983), Handa et al. (1989, 1993) report that different beta estimates can be obtained over the same period by changing time step in the computation of the return. Based on empirical fact, Handa et al. (1993) reject the CAPM when monthly returns are used but accept it with yearly returns [whereas Fama (1981, 1990) show that

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[1] We can also cite Chen et al. (1986) that identifies the risk factors as various macroeconomics variables.

[2] As noticed in Ait-Sahalia et al. (2011), these imperfections might be largely divided into three points. The first represents the frictions inherent in the trading process: bid-ask bounces, discreteness of price changes and rounding, trades occurring on different markets or networks, etc. The second point concerns informational effects such as differences in trade sizes or informational content of price changes. The last point encompasses measurement or data recording errors. Therefore, the returns based on these prices contain some imprecisions.

the power of macroeconomic variables in explaining the stock prices increased with time length]. However, the early work of Levhari and Levy (1977) provide evidence that the beta estimates were biased if a shorter time horizon is used in the place of the relevant time horizon. In conclusion, these authors suggest to use the relevant time horizon in the decision making process in order to avoid biasing the beta estimate.

The assumption of linearity of the causal relationship of returns with a set of covariates is usually referred to as risk factors formulated by seminal researchers (Markowitz, Sharpe, Treynor, …) of quantitative finance, has been extensively discussed in the literature in recent years. Bansal and Viswanathan (1993) for example, show that a non-linear Arbitrage Pricing Model outperforms conditional and unconditional linear models, for pricing international equities, bonds and forward currency contracts. We can also cite Chapman (1997) who argues that non-linear pricing kernel in Conditional Capital Asset Pricing Model (CCPAM) is superior over the standard CAPM.

In order to report effectively the relative change of the value of financial assets over a whole period and to decrease the loss of information, it would be advisable to represent the return as a function of all values observed in a thinner time discretization within the period. Such representation may be obtained by the expectation of high frequency returns within a period but we have to summarize all the information to the single first moment of the probability distribution.

In addition, over the last two decades, the ability of humans to store data has passed its ability to analyze them. In financial econometrics, the emergence of high frequency trading provides time series of financial returns available at very fine timestep. However, macroeconomic variables continued to be released at weekly or monthly frequency by public institutions. When estimating a multifactor model with macroeconomic risk factors, econometricians have to synchronize to macroeconomic variables frequency and consequently to ignore some relevant financial high frequency information. The evolution of the financial market ecosystem implies the emergence of new sources of risk that cannot be reflected by the use of weekly or monthly time series in market model estimation. For example, the present and unprecedent coronavirus pandemic crisis has shown that some events (the announcement of a reserach results, treatment or a vaccine ; the appearance of new variants of the disease …etc) can appear and significantly impact the financial markets within a week or months. The market model estimation in COVID-19 context is treated in recent research works. We can cite Diaz et al. (2021) and Liu (2020) among others. The modeling approach that we propose in this article can also be applied in this context because it allows exploiting the information available at high frequency and using monthly time series of macroeconomic variables in the multifactor model estimation. Our representation of the monthly return as a fuzzy random variable reflects the intraperiod market uncertainty.

In this paper, we propose to describe the relative variation of a financial asset through a fuzzy random variable in such a way that its mean value captures different relevant information on the probability distribution of the returns observed in intraperiod. The fuzzy representation of the return of a financial asset has been carried out in the literature by many authors. Bilbao et al. (2006) and Smimou et al. (2009) used fuzzy returns to handle expert's judgments whereas Sadefo et al. (2012) and Moussa et al. (2014b) reflected the imprecision of observed returns by the fuzziness. We can also mention Tanaka and Guo (1999), Parra et al. (2001), Terol et al. (2006), Vercher et al. (2007), Yoshida (2009) and Moussa et al. (2014a), among others. The theory of fuzzy sets has also been applied to other decision-making problems in Economics in order the treat the effects of imprecision and vagueness on each judgment in the decision-making process. For example, we can cite the application of fuzzy random regression by Nureize et al. (2014) for production forecasting. More recently, there are the

works of Dzuche et al. (2021) in Decision Theory and Shiang-Tai and Yueh-Chiang (2021) in Data Analysis. Our approach is distinguished by the use of the intra-month volatility to represent the fuzziness. The final purpose of this paper is to improve the quality of statistical estimation (stability and robustness) of the Arbitrage Pricing Theory (APT), more precisely the multifactor model, by using these fuzzy returns.

Zeng and Keane (2005) have initiated a similar approach by estimating a linear best covering fuzzy function. However, this possibilistic modeling method which can be considered as a precursor of the fuzzy linear regression has potential limitations. The estimation strongly depends on learning dataset (Bardossy 1990; Bardossy et al. 1990), and the issue of forecasting has to be addressed (Savic and Pedrycz 1991). The model is extremely sensitive to outliers and it may tend to become multicolinear as more independent variables are collected (Kim et al. 1996).

The remainder of this paper is organized as follows. The Sects. 2 and 4 are successively devoted to a brief review of the basics concepts of fuzzy set theory and to the presentation of the Arbitrage Pricing Theory (APT). Section 3 introduces the process of the fuzzy representation of an asset return. The Sect. 5 is assigned to the presentation of fuzzy multifactor model. Section 6 gives several numerical examples based on Fama and French's dataset. A discussion is made in Sect. 7 and some conclusions are finally listed in Sect. 8.

2 Preliminaries

Before proceeding to a formal presentation of fuzzy multifactor model, we first review briefly three of the basics concepts of fuzzy set theory: fuzzy sets, fuzzy numbers and fuzzy random variable. For a more detailed presentation of fuzzy set theory, see Zimmermann (2001).

2.1 Fuzzy sets and fuzzy numbers

Let $\mathbf{X}$ be a crisp set. A fuzzy subset A of $\mathbf{X}$ is defined by its membership function $\mu_A : \mathbf{X} \to [0, 1]$ which associates each element x of $\mathbf{X}$ with its membership degree $\mu_A(x)$ (Zadeh 1965). The degree of membership of an element x to a fuzzy set A is equal to 0 (respectively 1) if we want to express with certainty that x does not belong (respectively belongs) to A.

The crisp set of elements that belong to the fuzzy set A at least to the degree α is called the α-cut or α-level set and defined by:

$$A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\}. \tag{1}$$

A_0 is the closure[3] of the support of A. Recall from Shapiro (2009) that the support of A is the set of all x such that $\mu_A(x) > 0$.

Fuzzy numbers have some properties, examples of which are the notions of "around ten percent" and "close to zero". Dubois and Prade (1980, p. 26) characterizes the fuzzy numbers as follows:

Definition 2.1 A fuzzy subset A of $\mathbb{R}$ with membership $\mu_A : \mathbb{R} \to [0, 1]$ is called fuzzy number if

1. A is normal, i.e. $\exists x_0 \in \mathbb{R} : \mu_A(x_0) = 1$;
2. A is fuzzy convex, i.e.

$$\forall x_1, \ x_2 \in \mathbb{R} \ : \ \mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, \ \forall \lambda \in [0, 1];$$

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[3] The closure of the support of A is the smallest closed interval containing the support of A (Shapiro (2009))

3. μ_A is upper semi continuous[4];
4. $supp(A)$ is bounded.

Definition 2.2 (Zimmermann ([1996](#), p. 64)) A LR-fuzzy number, denoted by $\tilde{A} = \langle l, c, r \rangle_{LR}$, where $c \in \mathbb{R}^+$ is called central value, and $l \in \mathbb{R}_+^*$ and $r \in \mathbb{R}_+^*$ is the left and the right spread, respectively, is characterized by a membership function of the form

$$\mu_A(x) = \begin{cases} L\left(\frac{c-x}{l}\right) & \text{if } c - l \le x \le c, \\ R\left(\frac{x-c}{r}\right) & \text{if } r + c \ge x \ge c, \\ 0 & \text{else where .} \end{cases} \tag{2}$$

$L : \mathbb{R}^+ \to [0, 1]$, $R : \mathbb{R}^+ \to [0, 1]$ are fixed left-continuous and non-increasing functions with $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$. L and R are called the left and the right shape functions respectively. If right and left spreads are equal and $L := R$, the LR-fuzzy number is said to be a symmetric fuzzy number and denoted $\tilde{A} = (c, \Delta)$. Δ is the spread equal to $l = r$.

Without loss of generality, we limit the present study to triangular fuzzy [5] characterized by the shape functions $R(x) := L(x) := \max\{1 - x, 0\}$.

Using Zadeh's extension principle (Zadeh [1965](#)), which is a rule providing a general method to extend a function $f : \mathbb{R}^k \to \mathbb{R}$ to the set of fuzzy numbers, we can define binary operator such as addition, subtraction, multiplication for two fuzzy numbers. When $k = 2$, this method defines the membership function of the result as follows

$$\mu_{\tilde{A}_1 \circ \tilde{A}_2}(z) = \sup_{(x_1, x_2) \in \tilde{A}_1 \times \tilde{A}_2} \{\min\left(\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2)\right) \mid x_1 \circ x_2 = z\}, \tag{3}$$

where $\circ$ is the binary operator.

2.2 Fuzzy random variables

Different approaches of the concept of fuzzy random variables (FRV) have been developed in the literature since the 70's. The most often cited being introduced by Kwakernaak ([1978](#)) and enhanced by Kruse and Meyer ([1987](#)), and the one by Puri and Ralescu ([1985](#), [1986](#)). An extensive discussion on these two approaches is given by Shapiro ([2009](#)). For the purpose of this study, we adopt the concept of FRVs of Puri and Ralescu ([1986](#)).

We first recall the following definitions:

Definition 2.3 (Körner [1997](#)) A fuzzy set $\tilde{A}$ is called a normal convex fuzzy subset of $\mathbb{R}$ if $\tilde{A}$ is normal, the α-cuts of $\tilde{A}$ are convex and compact and the support of $\tilde{A}$ is compact.

Definition 2.4 (Gil et al. [2006](#)) A convex compact random set is a Borel-measurable mappings with the Borel σ-field generated by the topology associated with the Hausdorff metric on $\mathcal{F}_c(\mathbb{R})$.

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[4] Semi-continuity is a weak form of continuity. Intuitively, a function f is called upper semi-continuous at point x_0 if the function's values for arguments near x_0 are either close to $f(x_0)$ or less than $f(x_0)$

[5] This assumption is also made in many articles such as Koissi and Shapiro ([2006](#)), Andrés-Sànchez ([2007](#)) and BerryStölzle et al. ([2010](#)), among others.

Let $\mathcal{F}_c(\mathbb{R})$ denote the set of all normal convex fuzzy subsets of $\mathbb{R}$ and $(\Omega, \mathcal{A}, P)$[6] a probability space.

Recall that if $\tilde{f} : \omega \rightarrow \mathcal{F}(\mathbb{R})$ is a fuzzy set-valued function then $\tilde{f}_\alpha$ is a set-valued function for all $\alpha \in [0, 1]$, where $\tilde{f}_\alpha(\omega) = \{x \in \mathbb{R} | \tilde{f}_\alpha(\omega)(x) \geq \alpha\}$. The fuzzy set-valued function $\tilde{f}$ is said measurable[7] if and only if $\tilde{f}_\alpha$ is (set-valued) measurable for all $\alpha \in [0, 1]$.

Kwakernaak ([1978](#)) have introduced a FRV by

Definition 2.5 The fuzzy valued function $\mathcal{X} : \Omega \rightarrow \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable if $\mathcal{X}$ is measurable.

Puri and Ralescu ([1986](#)) have defined a FRV as follows

Definition 2.6 The mapping $\mathcal{X} : \Omega \rightarrow \mathcal{F}_c(\mathbb{R})$ is said to be a FRV on $\mathbb{R}$ if for any $\alpha \in [0, 1]$, the α-cut is a convex compact random set.

3 Fuzzy representation of returns

Fuzzy random variables (FRV) were introduced and defined by Kwakarnaak (1978), Puri and Ralescu ([1986](#)) as a well-formalized model for fuzzy set-valued random elements. Since these definitions, numerous studies in probability theory have been developed to analyze the properties of this new class of random variables (cf. Gil et al. [2006](#) for an overview). For the last three decades, there has been numerous work in this area, we refer the reader to Puri and Ralescu ([1985](#), [1986](#)), Klement et al. ([1986](#)) and Colubi et al. ([2002](#)) for work related to the formalization of the measurability. As for the work realted to laws of large numbers which strengthen the suitability of the fuzzy mean, the reader may consult Colubi et al. ([1999](#)) Molchanov ([1999](#)) and Proske and Puri ([2002](#)) (Pas de trace de cet article en reference). The authors in Körner ([2000](#)), Montenegro et al. ([2004](#)), González-Rodríguez et al. ([2006](#)), Ramos-Guajardo et al. ([2010](#)) develop the hypothesis testing. Despite the existence of this complete mathematical analysis framework, the application of these theoretical results is still quite limited because of the difficulties that we have met to observe and measure FRVs in practice. Hence the necessity of building methods to provide fuzzy representation of observations, which are often crisp. A solution was proposed by González-Rodríguez et al. ([2006](#)) who introduced a family of fuzzy representation of random variables. Each of the representations transforms a crisp random variable into a fuzzy random variables whose means capture different relevant information on the probabilistic distribution of the original real-valued random variable. However the application of this method requires a priori assumptions about the distribution of the real variable and the shape of the membership function of the fuzzy random variable. This double assumption may lead to a significant bias of information. There also exists other seminal ways to characterize fuzziness of a fuzzy variable crisply observed. Their review is given in Dubois and Prade ([1980](#), pp. 255–264). This characterization generally consists in the estimation of the membership function. As pointed out by Ross ([1995](#), pp. 179–180), the assignment of the membership function can be intuitive or based on algorithms or logical operations. An example of such membership function assignment for a financial risk factor is given by Smimou et al. ([2008](#)) in order to incorporate the experts' judgments in the financial returns measure. In addition, Koissi

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[6] Where Ω is the set of all possible outcomes described by the probability space, $\mathcal{A}$ is σ-fields of subsets of Ω, and the function P defined on $\mathcal{A}$ is a probability measure.

[7] The function f from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ is said to be measurable if and only if $f^{-1}(F) \in \mathcal{F}_1 \ \forall \ F \in \mathcal{F}_2$

and Shapiro (2006, p.291)[8] specified that a crisp data can be fuzzified by adding a number $\pm\Delta$ to each value, where Δ is chosen small compared to the center value. Following these two studies, we make a fuzzy representation of financial returns in the aim at combining two sources of information. We associate the closing prices based return usually used in the analysis, to a statistic summary of information observed in intraperiod at a higher frequency. For this purpose, we add positive numbers to the observed returns. For each time period $[t, t + 1[$, the number Δ is chosen as the scaled volatility observed within this period. The central value of the obtained fuzzy return is the closing prices based return and its spread is the scaled volatility. The intraperiod volatility scaling is necessary in order to standardize orders of magnitude of the central value and the spread of the fuzzy return. We proceed as follows:

The asset price time series is initially partitioned into sub-groups according to periods (months in our case). On each period, successive observed returns are calculated and the corresponding empirical probabilistic distribution are taken into consideration. The first two moments (expected value and variance) are computed for each probabilistic distribution (i.e. month). The monthly return is then represented as a symmetric LR-fuzzy number with central value the mean and the spread is the standard deviation. In this paper, we use a triangular shape function for the fuzzy returns. Recall that this assumption is also made in many articles such as Koissi and Shapiro (2006), Andrés-Sànchez (2007) and BerryStölzle et al. (2010), among others.

The fuzzy representation process can be summarized as in the following procedure.

We denote by t the sub-period $[t, t + 1]$.

Procedure 3.1 .

> **Step 1:** *Partition the price time series in sub-groups $P_t = \{P_{t+\frac{i}{n}}, i = 0, \ldots, n.\}$ with size $n + 1$ each one corresponding to a period t. The sample size $(n + 1) \geq 2$ has to be sufficiently large.[9]*
> *For each period t*
> **Step 2:** *Compute the return over the period $R_t = \frac{P_{t+1} - P_t}{P_t}$*
> **Step 3:** *Compute the returns within the period $R_{t,i} = \frac{P_{t+\frac{i+1}{n}} - P_{t+\frac{i}{n}}}{P_{t+\frac{i}{n}}}$, $i \in \{0, \ldots, n-1\}$*
> **Step 4:** *Estimate empirically the variance $\hat{\sigma}_t^2$ of $R_{t,i}$ as*
>
> $$\hat{\sigma}_t^2 = \frac{1}{n} \sum_{i=0}^{n-1} \left(R_{t,i} - \hat{\mu}_t \right)^2 \tag{4}$$
>
> *where*
>
> $$\hat{\mu}_t = \frac{1}{n} \sum_{i=0}^{n-1} R_{t,i} \tag{5}$$
>
> **Step 5:** *Scale the intraperiod volatility by $\Delta_t = \sqrt{n}\hat{\sigma}_t$*
> **Step 6:** *Fit the membership function of the symmetric LR-fuzzy return with central value R_t and spread Δ_t*

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[8] Koissi and Shapiro (2006) also precise that the choice Δ might be arbitrary (Chang and Ayyub (2001,p.192), randomly generated (Diamond (1988, p.152) or resulting from fuzzy regression Chang and Ayyub (2001).

[9] In this study, we use daily closing prices time series which are partitioned into monthly periods, hence $n = 20$.

In the previous procedure, the use of the intra-period standard deviation as volatility to express imprecision can be econometrically explained. Standard deviation by definition is a statistical indicator often used to define the range of possible values of a random variable. Moreover, within the framework of financial econometrics dealing with returns tainted with imprecision, Ait-Sahalia et al. (2010) justify the use of intraday volatility in order to reflect the imprecision on returns based on closing prices. It is important to remark that the covariance does not appear in the procedure because in a first stage we aim at dealing the precision of each asset independently.

Finally, we have the following statement:

Proposition 1 *If returns successively observed are assumed to real random variables, the symmetric LR-fuzzy set constructed as in the Procedure 3.1 is a fuzzy random variable in the Kwakaarnak sense.*

Proof Let $\tilde{R}_i = \langle R_i, \Delta_i \rangle$ constructed as in the procedure 3.1. Recall that $R_i = \frac{P_n^i - P_1^i}{P_1^i}$ and $\Delta_i = \sqrt{n-1}\hat{\sigma}_i$.

The α-cuts of the fuzzy return are denoted $\tilde{R}_{i\alpha} = [R_{i\alpha}^d, R_{i\alpha}^u] \; \forall \alpha \in (0,1]$.

$$\tilde{R}_{i\alpha}^d = \mu_i - \Delta_i L^{-1}(\alpha), \; R_{i\alpha}^u = \mu_i + \Delta_i L^{-1}(\alpha) \; \forall \alpha \in (0,1] \tag{6}$$

and

$$R_{i0}^d = \mu_i - \Delta_i, \; R_{i\alpha}^u = \mu_i + \Delta_i \tag{7}$$

Remark that L^{-1} exists because L is a continuous decreasing (cf. Definition 2.2) function hence it is a bijection.

Since R_i and $\sqrt{n-1}\hat{\sigma}_i$ are random variables, relations (6) and (7) define α-cuts endpoints as random variables $\forall \alpha \in [0,1]$. Therefore, as a consequence of the Proposition 1, $\tilde{R}_i$ is a fuzzy random variable. □

4 Arbitrage pricing theory

Even if the Capital Asset Pricing Model (CAPM) (Lintner 1965; Sharpe 1964) proposed the first quantification of the tradeoff between risk and expected return, a number of studies have presented evidence rejecting its validity. These limitations can be explained by the numerous specific assumptions of the model on investors' beliefs. In order to overcome the CAPM's weakness, Ross (1976) and Roll and Ross (1980) proposed the Arbitrage Pricing Theory (APT) which starts with specific assumptions on the distribution of asset returns and relies on approximate arbitrage arguments.

The basic assumption of the APT is that returns of an asset i are generated by a linear factor model called the multifactor model and defined as

$$R_{it} = \alpha_i + \sum_{k=1}^{m} \beta_{ki} F_{kt} + \epsilon_{it}, \; i = 1, \ldots, n, \; \forall t, \tag{8}$$

where α_i is the expected value of the asset i returns R_i, β_{ki} are the sensitivities of R_i to m common factors F_k which are generally with zero mean and ϵ_{it} is a white noise. That is equivalent to

$$Var(\epsilon_{it}) = \sigma_{\epsilon_i}^2, \quad \forall t,$$

$$\mathbb{E}(F_{kt}) = \mathbb{E}(\epsilon_{it}) = 0, \quad \forall\, i = 1, \ldots, n;\ \forall\, k,$$
$$Cov(\epsilon_i, F_k) = 0, \forall\, i = 1, \ldots, n, \quad \forall\, k = 1, \ldots, m,$$
$$Cov(\epsilon_i, \epsilon_j) = 0, \quad \forall i \neq j,$$
$$Cov(\epsilon_{it}, \epsilon_{jt'}) = 0, \quad \forall i \neq j,\ \forall t \neq t^{'}.$$

In order to simplify the analysis, common factors are assumed uncorrelated.

Under the assumption of mutual independence of common factors, the variance of the return of asset i can be decomposed as in (9)

$$\sigma_i^2 = \sigma_{\epsilon_i}^2 + \sum_{k=1}^{m} \beta_{ki}^2 \sigma_{F_k}^2 \tag{9}$$

and the covariance between returns of assets i and j is

$$\sigma_{ij}^2 = \sum_{k=1}^{m} \beta_{ki} \beta_{kj} \sigma_{F_k}^2 \ \forall\, i,\ \forall\, i \neq j. \tag{10}$$

Recall that the particular corresponding to $m = 1$ is the Sharpe's market line if the factor F_1 represents the market portfolio.

In practice, three types of factor models are available for studying asset returns.

- The statistical and econometric approach initiated by Roll and Ross (1980) based on factor analysis and principal component analysis; this approach treats the common factors as latent or unobservable variables which explain a great proportion of the variance of returns time series.
- The second approach uses macroeconomic variables such as growth rate of GDP, interest rates, inflation rates, unemployment rate? to describe the common behavior of asset returns (Cf. Chen et al. 1986).
- The third approach is the fundamental factor models that use firm or asset specific attributes such as firm size, book and market values, and industrial classification to construct common factors (Fama and French 1992; Grinold and Khan 2000).

The introduction of APT has permitted to overcome some limitations of the CAPM but the difficulties of its statistical estimation still remain. These difficulties which reduce the scope of this model, can be summarized to the following points:

1. The multifactor model is statistically estimated using a learning dataset formed by past observations of returns and of common factors. The estimated betas do not inform about the actual asset returns sensitivity to common factors when the training sample contains old information. For this reason, it is tempting to estimate the beta using recent information, often more relevant. However, such an initiative creates another problem because the limited size of training sample gives significant weight to each observation.
2. The subsequent model checking reveals generally an autocorrelation, a heteroscedasticity or a non-normality of errors. That implies the invalidity of statistical hypothesis testing and the estimation of uncertainty given by the confidence intervals.

In the next section, we propose to revisit the statistical estimation of the APT assuming that returns are fuzzy-set valued. This modeling approach aims at using the representation of the return over a period, containing more information of the variability within this period instead of the single closing prices based return.

5 Fuzzy multifactor model

In this section, using the fuzzy representation of the return of Sect. 3, we introduce the fuzzy multifactor model. This model defines as fuzzy linear regression model, expresses the asset sensitivities to the risk factors as fuzzy numbers. The section describes this fuzzy multifactor model and presents a method for its statistical estimation.

5.1 Model's description

The fuzzy multifactor model is presented as follows:

$$\tilde{R}_{it} = \tilde{\alpha}_i + \sum_{k=1}^{m} \tilde{\beta}_{ik} F_{kt} + \tilde{\epsilon}_{it}, \ k = 1, \ldots, m; \ \forall \, t, \tag{11}$$

where F_k are n common factors to all assets of the market and the error term $\tilde{\epsilon}_{it} = (\epsilon_{it}^R, \epsilon_{it}^\Delta)$ is a bivariate random vector independent and identically distributed with zero mean and constant variance-covariance matrix. That is equivalent to

$$\mathbb{E}(\tilde{\epsilon}_{it}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \forall \, t, \ \forall \, i,$$

$$\mathbb{E}(\tilde{\epsilon}_{it} \tilde{\epsilon}_{it}^t) = \begin{pmatrix} \sigma_R^2 & 0 \\ 0 & \sigma_\Delta^2 \end{pmatrix}, \ \forall \, t, \ \forall \, i,$$

$$\mathbb{E}(\tilde{\epsilon}_{it} \tilde{\epsilon}_{is}^t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \forall \, t \neq s, \ \forall \, i,$$

where $Var(\epsilon_{it}^R) = \sigma_R^2$ and $Var(\epsilon_{it}^\Delta) = \sigma_\Delta^2$.

5.2 Model estimation

In order to give a possible continuation to our study, we propose to use an estimator which has certain properties such as linearity, unbiasedness and minimum variance. The property of linearity will potentially allow determining the probability distribution of the estimator and hence construct hypothesis tests. The unbiasedness ensures that the deviation of the parameter estimate from the true value is zero on average and minimality of variance attests the best accuracy of the estimate.

The random spreads and central values of the fuzzy returns are assumed to homoscedastic, non-autocorrelated, i.e. their variance-covariance matrix are $\Sigma_R = \sigma_R I_T$ et $\Sigma_\Delta = \sigma_\Delta I_T$ respectively with $\sigma_R = \sqrt{Var(R_i)}$, $\sigma_\Delta = \sqrt{Var(\Delta_i)}$ and I_T unit matrix of size T. The weak-BLUE[10] estimator obtained by minimizing the error variance subject to linearity and unbiasedness (Näther 2006; for more details) is given by

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[10] The term Weak-BLUE is used by Näther (1997) to specify that only the estimator of central values of parameters is the BLUE (best linear unbiased estimator)

Theorem 1 (Näther 2001) *If the matrix $\boldsymbol{F}^t\boldsymbol{F}$ is regular, the weak-BLUE of parameters of the model 11 is*

$$
\begin{pmatrix} \tilde{\alpha}_i^* \\ \tilde{\beta}_{i1}^* \\ \tilde{\beta}_{i2}^* \\ \tilde{\beta}_{i3}^* \end{pmatrix} = (\boldsymbol{F}^t\boldsymbol{F})^{-1}\boldsymbol{F}^t\,\tilde{R}_i,
\tag{12}
$$

where $\boldsymbol{F} = \begin{pmatrix} 1 & F_{11} & F_{21} & F_{31} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & F_{1T} & F_{2T} & F_{3T} \end{pmatrix}$ *and* $\tilde{R}_i = \left(\tilde{R}_{i1}, \ldots, \tilde{R}_{iT}\right)$ *with* $\boldsymbol{F}^t$ *the transposed of* $\boldsymbol{F}$.

6 Empirical studies

In this section, we derive and compare the asset pricing according to the crisp and the fuzzy multifactor model. We focus on the model of Fama and French (1993) using three common factors to the risky assets of the market. The first part describes the data set used in this empirical analysis. The second part presents the multifactor model of Fama and French (1993). The third part compares systematic risks estimated with the crisp and the fuzzy approaches.

6.1 The three factor model of Fama and French (1993)

Fama and French (1993) proposed a three-factor model to capture the expected risk premium anomalies. This model states that the risk premium of a risky portfolio is linearly explained by three factors: the excess return of the market portfolio, the difference between the return on a portfolio covering small-size stocks and the return on a portfolio covering large-size stocks, SMB (small minus big); and the difference between the return on a portfolio of high-book-to-markets stocks and the return on a portfolio of low-book-to-market stocks, HML (high minus low). This relationship is expressed in this following linear regression model

$$
R_{p,t} - r_{f,t} = a_p + b_p\left(R_{p,t} - r_{f,t}\right) + s_p SMB_t + h_p HML_t + \epsilon_{p,t},
\tag{13}
$$

where $\left(R_{p,t} - r_{f,t}\right)$ and $\left(R_{m,t} - r_{f,t}\right)$ are respectively the risk premiums of the portfolio p and the market m at time t; SMB_t and HML_t are the other two common factors previously described at time t; $\epsilon_{p,t}$ is a error term assumed to be zero mean and uncorrelated with all other variables. The slope coefficients b_p, s_p and h_p in this time series regression are the sensitivities of the portfolio p to the common risk factors.

As the particular case of the fuzzy multifactor presented in Sect. 5, we introduce the fuzzy three factor model of Fama and French (1993). The portfolio's risk premium at time t is assumed to be a fuzzy number $\tilde{r}_{p,t}$. Consequently, the fuzzy three factor model is rewritten as follows

$$
\tilde{R}_{p,t} - r_{f,t} = \tilde{a}_p + \tilde{b}_p\left(\tilde{R}_{m,t} - r_{f,t}\right) + \tilde{s}_p SMB_t + \tilde{h}_p HML_t + \tilde{\epsilon}_{p,t}.
\tag{14}
$$

$\tilde{\epsilon}_{p,t}$ is the fuzzy error term verifying the conditions presented in 12.

The slope coefficients $\tilde{b}_p$, $\tilde{s}_p$ and $\tilde{h}_p$ in the fuzzy linear regression model (14) are the fuzzy sensitivities of the portfolio p to the common risk factors. These parameters are estimated using the method presented in Sect. 5.

6.2 The used data set

For the empirical analysis, we use the dataset of 25 portfolios and the three risk factors formed by Fama and French and available on their web site.[11] This database contains equal-weighted returns for the intersections of 5 size markets equity (ME) and 5 book-to-market equity (BE/ME) portfolios. These two authors provide on their website historical research data on the risk factors used in the different approaches of their model. These constantly updated and archived research dataset constitute a benchmark for empirical analysis of researchers like us who focus on Multifactor Asset Pricing.

We use the daily time series of the portfolios real returns in order to construct the monthly fuzzy returns following the fuzzification methods presented in Sect. 3. The time series of the three commons factors are directly given in monthly time step.

The portfolios composition and statistic summaries of the monthly returns (in percentage) are given in Table 1. The spreads of the average fuzzy returns are also presented in the table. Recall that the central values of these average fuzzy returns coincide exactly with the closing prices based returns. The average returns of portfolios are positive and homogeneous (in the range 0.4-1.47). This homogeneity is also observed with standard deviations. These standard deviations are very close to the average spreads.

6.3 Analysis of systematic risks estimation

The estimates of the real and fuzzy sensitivities to the risk factors are presented respectively in Tables 3 and 4. These tables report the estimates and the Student's t-statistic test of the nullity for the crisp sensitivities. In the fuzzy case, the central values and the spreads of the sensitivities and their ratios are exposed.

Table 3 is devoted to the results of the crisp sensitivities. For each risk factor, the estimates and the Student's t-statistic are exposed. The estimates of the sensitivities are homogeneous for all portfolios except for the first sensitivity (a_p) of Portfolio 1 . The box plot of b_p depicted in Fig. 1 enhances this outlier. This outlier is excluded from the rest of the analysis.

Table of results 4 is devoted to fuzzy sensitivities. Their central values coincide exactly with the crisp sensitivity previously described. The supports of the fuzzy sensitivities are expressed as the closed intervals around the central values whose radius are the spreads. These supports are presented in Fig. 2. The lengths of these intervals express the variability and uncertainty related to common risk factors and inform about the significance of the sensitivities. We observe that the spreads of the portfolios fuzzy sensitivities to the first risk factor (the market risk premium) is higher than the spreads of the other two.

In order to compare the two approaches of the three multifactor model estimation, we compare the expressions of uncertainty. In the MCO approach, the uncertainty of estimation is expressed via the Student test of sensitivities nullity. The Student's t-statistic is a ratio between the estimate and its standard deviation; hence it is a relative quantity. It reflects the significance of the sensitivity estimated. Since the spreads of the fuzzy sensitivity expresses its uncertainty, we define in a similar way, a measure equivalent to the Student's t-statistic

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Table 1 Compositions and statistic summaries of 25 portfolios monthly returns

Portfolio	ME	BE/ME	Mean returns	SD	Mean spreads
Portfolio 1	1	1	0.5776	7.7722	7.2523
Portfolio 2	1	2	0.9269	7.4198	6.9187
Portfolio 3	1	3	1.0554	6.8617	7.0003
Portfolio 4	1	4	0.8700	6.3803	6.9876
Portfolio 5	1	5	1.1798	7.4204	6.9344
Portfolio 6	2	1	1.2022	7.2672	6.8909
Portfolio 7	2	2	1.1500	7.1413	7.0156
Portfolio 8	2	3	1.2943	7.4952	7.3721
Portfolio 9	2	4	0.8358	7.5218	7.3132
Portfolio 10	2	5	1.2576	9.2266	8.5299
Portfolio 11	3	1	1.0175	7.0660	6.6778
Portfolio 12	3	2	1.2198	7.2191	6.3020
Portfolio 13	3	3	1.1412	6.9739	6.6027
Portfolio 14	3	4	1.0019	7.2513	6.7498
Portfolio 15	3	5	1.4684	7.9945	7.1975
Portfolio 16	4	1	1.0937	6.2958	5.9858
Portfolio 17	4	2	0.7563	6.7787	6.1814
Portfolio 18	4	3	0.6902	7.4908	6.7810
Portfolio 19	4	4	0.7808	6.9451	6.6237
Portfolio 20	4	5	0.9308	8.5933	7.2640
Portfolio 21	5	1	0.6477	5.6889	5.0309
Portfolio 22	5	2	0.7350	5.8926	5.4653
Portfolio 23	5	3	0.5091	6.3580	6.1498
Portfolio 24	5	4	0.3861	6.6527	6.4670
Portfolio 25	5	5	0.8941	7.0508	7.6568
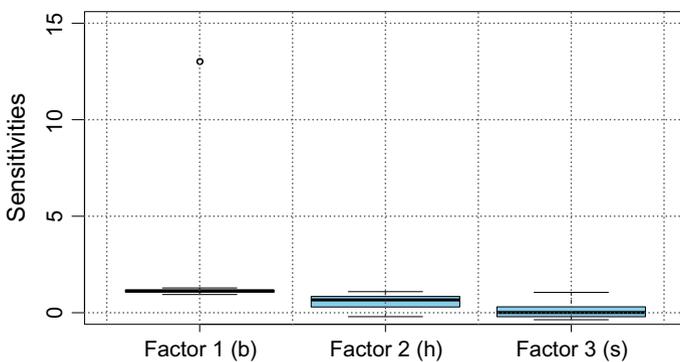
Fig. 1 Box plots of portfolios sensitivities to the three common risk factors

Table 2 Correlations between the fuzzy and crisp uncertainty measures

Factor	Factor 1 (b)	Factor 2 (s)	Factor 3 (h)
Corr. Coef. Spreads and t-statistics	− 0.6763	0.4219	− 0.5765
Corr. Coef. Ratios and t-statistics	0.5734	0.2675	0.4558

which assesses the relevance of the corresponding risk factor in explaining the portfolio return. This measure is introduced as a ratio between the central value and the spread of the fuzzy sensitivities. The obtained results are presented in Columns 4, 7 and 10 of Table 4. The correlation coefficients between these two measures are presented in Table 2. Portfolios 1 above is presented as an outlier, is excluded from this calculation. The correlation coefficients between the spreads and t-statistics are higher than those between ratios and t-statistics for the three risk factors. Factor 1 (market risk premium) presents the highest values in the two case whereas Factor 2 has the lowest values. The correlation coefficients related to Factor 1 are absolutely superior to 0.57. The two expression of the uncertainty generated by Factor 1 are significantly correlated. In the case of Factor 3 ("high minus low" factor), this correlation exists in a lower proportion. This correlation is not valid for Factor 2.

We shall first recall the characteristics of the common risk factors. Factor 1 (market risk premium) and Factor 3 (difference between the return on a portfolio of high-book-to-markets stocks and the return on a portfolio of low-book-to-market stocks) are directly constructed from observed dataset whereas Factor 2 ("small minus big" factor) comes from companies and firms classification. Hence, the correlation between the crisp and fuzzy approaches only exists with Factors defined from observed dataset. This result is one of the main empirical finding of our study.

7 Discussions

The empirical analysis carried out in the previous section, allowed us to make a comparative study of the evaluation of uncertainty by fuzzy linear regression and that performed in a more traditional way by testing for the nullity of parameters in a regression model. This analysis is made in the framework of Arbitrage Pricing Theory. We observed that there are concordance and consistency between the two modeling approaches. Indeed, a significant correlation is measured between the Student test statistic and the width of the spreads of the fuzzy parameters.

It should be remembered that our fuzzy set theory modeling approach for revisiting the APT is motivated by the three following reasons:

- associating intra-period information to the unique closing prices-based returns for more robustness.
- reflecting the imprecision and vagueness generated by market microstructure noises on the returns measurement and then Asset Pricing Model.
- evaluating the significance of the parameters (sensitivity of portfolios to factors) without being subjected to the residual normality assumption which is essential for the validity of the Student test. Let us remember that this assumption is rarely verified on financial returns.

The first point was addressed using intra-period volatility as a measure of uncertainty as justified by some empirical findings such as Ait-Sahalia et al. (2011). The second point is

dealt with by using a fuzzy random variable to represent the precision and vagueness of profitability. Finally, for the third point, the fuzzy linear regression has been implemented thus allowing the evaluation of the parameters significance through the spreads of the fuzzy sets without having to worry about the normality of the model residuals.

From an Economic institution and point of view, the fuzzy APT that we propose can be seen as an extension of its classic version under Knightian uncertainty. Recall that Knightian uncertainty applies to situations where we cannot observe precisely random risk factors in order to measure accurate odds. In these situations, uncertainty has two sources: randomness and imprecision (vagueness). This asset pricing under Knightian uncertainty can be extended to performance evaluation or portfolio allocation in a future studies.

8 Conclusion

In view of completing the information given by the monthly returns computed from the closing prices and to decrease the loss of information caused by the edge discretization in time, we proposed in this paper to associate the closing prices based return with the monthly volatility for a representation through a fuzzy set. The monthly volatility is computed using intra-month returns observed at a daily frequency. These fuzzy returns are used to reformulate the Fama and French (1993) as fuzzy linear regression model. The portfolios sensitivities to the three risk factors are then defined as fuzzy number. Finally, in an empirical study based on Fama and French dataset, we highlighted that the uncertainty expressed by the spreads of the fuzzy sensitivities are correlated with the significance of their real versions, especially for the sensitivities related to the market risk premium and the "high minus low" factor. The proposed fuzzy sensitivities allow associating information at different frequency for the decision making.

Tables of results and figures

See Fig. 2 and Tables 3, 4.
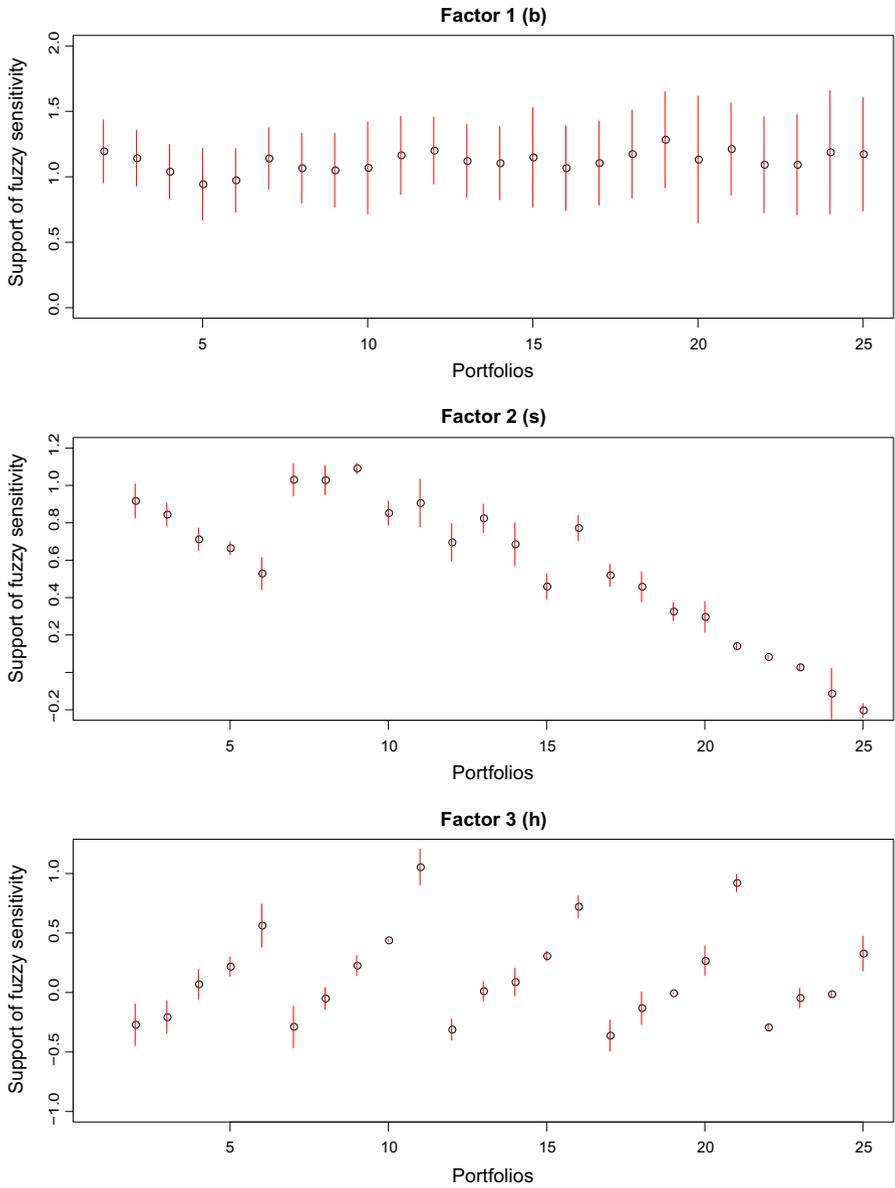
Fig. 2 Supports of the fuzzy sensitivities for Portfolios 2-25

Table 3 Estimates of portfolio's sensitivities to the risk factors

Portfolio	b_p		s_p		h_p	
	Estimate	T-Stat	Estimate	T-Stat	Estimate	T-Stat
Portfolio 1	13.0165	3.8190	− 18.4049	0.5393	− 5.9268	4.8516
Portfolio 2	1.1952	24.0268	0.9168	3.3795	− 0.2709	378.8211
Portfolio 3	1.1422	23.0497	0.8439	2.2179	− 0.2076	389.4348
Portfolio 4	1.0389	26.5668	0.7114	0.3592	0.0688	514.4615
Portfolio 5	0.9433	37.4104	0.6641	5.4441	0.2172	690.5202
Portfolio 6	0.9730	8.5679	0.5288	10.9735	0.5624	238.2237
Portfolio 7	1.1404	93.3360	1.0304	11.7278	− 0.2884	1096.2263
Portfolio 8	1.0657	99.8684	1.0283	0.3674	− 0.0508	1065.7125
Portfolio 9	1.0488	121.2391	1.0910	7.3003	0.2259	1171.1354
Portfolio 10	1.0681	86.3804	0.8520	29.6044	0.4381	1301.7810
Portfolio 11	1.1644	60.2912	0.9057	90.1989	1.0534	959.0198
Portfolio 12	1.2001	40.2850	0.6954	13.6358	− 0.3118	1072.3398
Portfolio 13	1.1205	54.4019	0.8236	0.0144	0.0110	930.6483
Portfolio 14	1.1041	53.6230	0.6854	1.2664	0.0879	1232.1262
Portfolio 15	1.1480	19.9939	0.4591	11.0846	0.3060	960.7525
Portfolio 16	1.0665	42.2033	0.7720	42.9580	0.7206	746.3746
Portfolio 17	1.1047	22.9802	0.5195	20.2646	− 0.3624	932.2234
Portfolio 18	1.1732	20.3857	0.4581	2.6860	− 0.1311	1132.3208
Portfolio 19	1.2830	5.5189	0.3252	0.0028	− 0.0060	671.1790
Portfolio 20	1.1314	7.7592	0.2966	7.3589	0.2661	771.6076
Portfolio 21	1.2122	1.8330	0.1403	32.6437	0.9200	382.8939
Portfolio 22	1.0922	0.3284	0.0821	23.5553	− 0.2935	1419.5753
Portfolio 23	1.0918	0.0663	0.0269	0.5061	− 0.0467	1304.6553
Portfolio 24	1.1876	1.3574	− 0.1134	0.0319	− 0.0145	974.3040
Portfolio 25	1.17256	2.3138	− 0.2034	14.5699	0.3273	942.4382

Table 4 Estimates of portfolio's fuzzy sensitivities to the risk factors

Portfolio	$\tilde{b}_p$			$\tilde{s}_p$			$\tilde{h}_p$		
	Estimate	Spread	Ratio	Estimate	Spread	Ratio	Estimate	Spread	Ratio
Portfolio 1	13.0166	0.2681	48.5598	−18.4050	0.0769	239.3771	−5.9268	0.1928	30.7482
Portfolio 2	1.1952	0.2408	4.9638	0.9169	0.0908	10.0996	−0.2709	0.1748	1.5495
Portfolio 3	1.1422	0.2125	5.3742	0.8439	0.0622	13.5716	−0.2077	0.1375	1.5109
Portfolio 4	1.0390	0.2079	4.9979	0.7115	0.0610	11.6616	0.0688	0.1264	0.5443
Portfolio 5	0.9433	0.2739	3.4437	0.6641	0.0352	18.8545	0.2172	0.0826	2.6283
Portfolio 6	0.9730	0.2436	3.9942	0.5288	0.0860	6.1523	0.5624	0.1822	3.0876
Portfolio 7	1.1405	0.2360	4.8330	1.0305	0.0871	11.8243	−0.2884	0.1760	1.6388
Portfolio 8	1.0658	0.2668	3.9951	1.0283	0.0775	13.2742	−0.0509	0.0907	0.5604
Portfolio 9	1.0489	0.2833	3.7027	1.0910	0.0306	35.7055	0.2260	0.0846	2.6710
Portfolio 10	1.0682	0.3518	3.0365	0.8520	0.0645	13.2144	0.4381	0.0076	57.4371
Portfolio 11	1.1644	0.2983	3.9029	0.9058	0.1276	7.0986	1.0534	0.1515	6.9513
Portfolio 12	1.2001	0.2557	4.6929	0.6954	0.1001	6.9452	−0.3119	0.0894	3.4879
Portfolio 13	1.1206	0.2793	4.0113	0.8237	0.0765	10.7691	0.0110	0.0829	0.1332
Portfolio 14	1.1041	0.2805	3.9359	0.6855	0.1149	5.9666	0.0880	0.1162	0.7572
Portfolio 15	1.1481	0.3790	3.0296	0.4591	0.0689	6.6599	0.3061	0.0403	7.6023
Portfolio 16	1.0665	0.3230	3.3019	0.7721	0.0687	11.2349	0.7207	0.0928	7.7656
Portfolio 17	1.1048	0.3210	3.4413	0.5195	0.0590	8.8051	−0.3624	0.1309	2.7690
Portfolio 18	1.1733	0.3368	3.4833	0.4581	0.0806	5.6857	−0.1312	0.1379	0.9511
Portfolio 19	1.2830	0.3677	3.4893	0.3253	0.0483	6.7414	−0.0061	0.0038	1.5763
Portfolio 20	1.1314	0.4860	2.3279	0.2967	0.0829	3.5770	0.2661	0.1249	2.1313
Portfolio 21	1.2123	0.3527	3.4367	0.1403	0.0129	10.8384	0.9200	0.0734	12.5388
Portfolio 22	1.0922	0.3672	2.9745	0.0822	0.0081	10.1205	−0.2936	0.0288	10.1777
Portfolio 23	1.0918	0.3838	2.8447	0.0270	0.0187	1.4398	−0.0468	0.0814	0.5742
Portfolio 24	1.1877	0.4707	2.5234	−0.1135	0.1353	0.8386	−0.0145	0.0251	0.5784
Portfolio 25	1.1726	0.4340	2.7019	−0.2035	0.0373	5.4481	0.3273	0.1461	2.2398

References

Ait-Sahalia, Y., Mykland, P. A., & Zhang, L. (2011). Ultra high frequency volatility estimation with dependent microstructure noise. *Journal of Econometrics, 160,* 190–203.

Andrés-Sànchez, J. de., (2007). Claim reserving with fuzzy regression and Taylor's geometric separation method. *Insurance: Mathematics and Economics, 40*, 145–163.

Bansal, R., & Viswanathan, S. (1993). No arbitrage and arbitrage pricing: A new approach. *Journal of Finance, 48*, 1231–1262.

Bardossy, A. (1990). Note on fuzzy regression. *Fuzzy Sets and Systems, 67,* 65–75.

Bardossy, A., Bogardi, I., & Duckstein, L. (1990). Fuzzy regression in hydrology. *Water Resources Research, 26,* 1497–1508.

Bilbao, A., Arenas, M., Jiménez, M., Gladish, B. P., & Rodríguez, M. V. (2006). An extension of Sharpe's single-index model: Portfolio selection with expert betas. *Journal of the Operational Research Society, 57*(12), 1442–1451.

BerryStölzle, T. R., Koissi, M.-C., & Shapiro, A. F. (2010). Detecting fuzzy relationships in regression models: The case of insurer solvency surveillance in Germany. *Insurance: Mathematics and Economics, 46,* 554–567.

Brailsford, T. J., & Josev, T. (1997). The impact of the return interval on the estimation of systematic risk. *Pacific-Basin Finance Journal, 5*(3), 357–376.

Carhart, M. M. (1997). On persistence in mutual fund performance. *The Journal of Finance, 52,* 57–82.

Carmichael, B., & Coën, A. (2008). Asset pricing models with errors-in-variables. *Journal of Empirical Finance, 15*(4), 778–788.

Chapman D. A. (1997). Approximating the asset pricing kernel. *Journal of Finance*, (*52*), 1383–1410.

Chen, N.-F., Roll, R., & Ross, A. (1986). Time-varying beta: A boundedly rational equilibrium approach. *The Journal of Business, 59*(3), 383–404.

Coën, A., & Racicot, F.-E. (2007). Capital asset pricing models revisited: Evidence from errors in variables. *Economics Letters, 95*(3), 443–450.

Coen, A., & Hubner, G. (2009), Risk and performance estimation in hedge funds revisited: Evidence from errors in variables. *Journal of Empirical Finance*, *16*(1), 112–125.

Cragg, J. G. (1994). Making good inferences from bad data. *Canadian Journal of Economics, 27,* 776–800.

Colubi,A., López–Diiaz, M., Domiinguez–Menchero, J. S., & Gil, M.A.(1999).AGeneralized strong law of large numbers. probability Theory and Related Fields, **114**(3), 401–417.

Colubi, C., Fernández–García, C., & Gil, M. A. (2002). Simulation of random fuzzy variables: An empirical approach to statistical/probabilistic studies with fuzzy experimental data. *IEEE Transactions on Fuzzy Systems*, *10*(3), 384–390.

Diaz, V., Ibrushi, D., & Zhao, J. (2021). Reconsidering systematic factors during the Covid-19 pandemic—The rising importance of ESG. *Finance Research Letters, 38,* 101870.

Dubois, D., & Prade, H. (1980). *Theory and application, fuzzy sets and systems*. Academic Publishers.

Dzuche, J., Deffo Tassak, C., Sadefo, J., & Fono, L. A. (2021). On two dominances of fuzzy variables based on a parametrized fuzzy measures and application to portfolio selection with fuzzy returns. *Annals of Operations Research, 301*(6), 355–368.

Fama, E. F. (1990). Stock returns, expected returns, and real activity. *The Journal of Finance*, *45*(4), 1089–1108.

Fama, E., & French, K. R. (1992). The cross-section of expected stock returns. *Journal of Finance, 47,* 427–465.

Fama, E. F. (1981). Stock returns, real activity, inflation, and money. *The American Economic Review*, *71*(4), 545–565.

Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics, 33,* 3–56.

González-Rodríguez, G., Colubi, A., & Gil, M. Á. (2006). A fuzzy representation of random variables: An operational tool in exploratory analysis and hypothesis testing. *Computational Statistics & Data Analysis, 51,* 163–176.

Gil, M. Á., López-Díaz, M., & Ralescu, D. A. (2006). Overview on the development of fuzzy random variables. *Fuzzy Sets and Systems, 157,* 2546–2557.

Grinold, R. C., & Khan, R. H. (2000). *Active portfolio management: A quantitative approach for producing superior returns and controlling risk* (2nd ed.). Mc Graw-Hill.

Handa, P., Kothari, S. P., & Wasley, C. (1989). The relation between the return interval and betas: Implications for the size effect. *Journal of Financial Economics, 23,* 79–100.

Handa, P., Kothari, S. P., & Wasley, C. (1993). Sensitivity of multivariate tests of the capital asset pricing to the return interval measurement. *Journal of Finance, 48,* 15–43.

Hawawini, G. (1983). Why beta shifts as the return interval changes. *Financial Analysts Journal, 39*(3), 73–77.

Kim, K. J., Moskowitz, H., & Koksalan, M. (1996). Fuzzy versus statistical linear regression. *European Journal of Operational Research, 92*(2), 417–434.

Klement, E. P., Puri, M. L., Ralescu, D. A. (1986). Limit theorems for fuzzy random variables. *Proc Royal Soc. London-Series A 19,* 171–182.

Klepper, S., & Leamer, E. E. (1984). Consistent sets of estimates for regressions with errors in all variables. *Econometrica, 52*(1), 163–184.

Koissi, M.-C., & Shapiro, A. F. (2006). Fuzzy formulation of the Lee–Carter model for mortality forecasting. *Insurance: Mathematics and Economics, 39,* 287–309.

Körner, R. (2000). An asymptotic a-test for the expectation of random fuzzy *variables. Journal of Statistical Planning and Inference, 83,* 331–346.

Körner, R. (1997). On the variance of fuzzy random variable. *Fuzzy Sets and Systems, 92,* 83–93.

Kruse, R., & Meyer, K. D. (1987). *Statistics with Vague Data*. Dordrecht: D. Reidel Publishing Company.

Kwakernaak, H. (1978). Fuzzy random variables definitions and theorems. *Information Sciences, 15,* 1–29.

Leamer, E. E. (1984). Errors in variables in linear systems. *Econometrica, 55*(4), 893–909.

Levhari, D., & Levy, H. (1977). The capital asset pricing model and the investment horizon. *The Review of Economics and Statistics, 59*(1), 92–104.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, *47*(1), 13–37.

Liu, S. (2020). Analysis of COVID-19 on service industry based on Fama and French five-factor model. In *Management science informatization and economic innovation development conference*.

Molchanov, I. (1999). On strong laws of large numbers for random upper semicontinuous functions. *Journal of Mathematical Analysis and Applications*, *235,* 349–355.

Montenegro, M., Colubi, A., Casals, M. R., & Gil, M. A. (2004). Asymptotic and Bootstrap techniques for testing the expected value of a fuzzy random variable. *Metrika, 59,* 31–49.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica, 34*(4), 768–783.

Moussa, M., Sadefo, K. J., Shapiro, Arnold F., & Terraza, M. (2014a). CAPM with fuzzy returns and hypothesis testing. *Insurance: Mathematics and Economics, 55,* 40–57.

Moussa, M., Sadefo, K. J., & Terraza, M. (2014b). Fuzzy value-at-risk and expected shortfall for portfolios with heavy-tailed returns. *Economic Modelling, 39,* 247–256.

Näther, W. (1997). Linear statistical inference for random fuzzy data. *Statistics, 29,* 221–240.

Näther, W. (2001). Random fuzzy variables of second order and applications to statistical inference. *Information Sciences, 133,* 69–88.

Näther, W. (2006). Regression with fuzzy random variable. *Computational Statistical Data Analysis, 51,* 235–252.

Nureize, A., Watada, J., & Wang, S. (2014). Fuzzy random regression based multi-attribute evaluation and its application to oil palm fruit grading. *Annals of Operations Research, 2019,* 299–315.

Parra, M. A., Terol, A. B., & Uria, M. V. R. (2001). A fuzzy goal programming approach to portfolio selection. *European Journal of Operational Research, 133,* 287–297.

Proske, F. N., & Puri, M. L. (2002). Central Limit Theorem for Banach Space valued fuzzy random variables. *Proceedings of the American Mathematical Society, 130,* 1493–1501.

Puri, M. L., & Ralescu, D. A. (1985). The concept of normality for fuzzy random variables. *Annals of Probability, 11,* 1373–1379.

Puri, M. L., & Ralescu, D. A. (1986). Fuzzy random variables. *Journal of Mathematical Analysis and Applications, 114,* 409–422.

Ramos-Guajardo, A. B., Colubi, A., González-Rodríguez, G., & Gil, M. A. (2010). One-sample tests for a generalized Fréchet variance of a fuzzy random variable. *Metrika, 71,* 185–202.

Roll, R., & Ross, S. (1980). An empirical investigation of the arbitrage pricing theory. *The Journal of Finance, 35,* 1073–1103.

Ross, I. T. J. (1995). *Fuzzy logic with engineering applications*. New York: McGraw-Hill.

Ross, S. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory, 13*(3), 341–360.

Sadefo, K. J., Mbairadjim Moussa, A., & Terraza, M. (2012). Fuzzy risk adjusted performance measures: Application to hedge funds. *Insurance: Mathematics and Economics, 51,* 702–712.

Savic, D. A., & Pedrycz, W. (1991). Evaluation of fuzzy linear regression models. *Fuzzy Sets and Systems, 39*(1), 51–63.

Shapiro, A. F. (2009). Fuzzy random variable. *Insurance: Mathematics and Economics, 44,* 307–314.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance, 19*(3), 425–442.

Shiang-Tai, L., & Yueh-Chiang, L. (2021). Fuzzy measures for fuzzy cross efficiency in data envelopment analysis. *Annals of Operations Research, 301*(6), 369–414.

Smimou, K., Bector, C. R., & Jacoby, G. (2008). Portfolio selection subject to experts' judgments. *International Review of Financial Analysis, 17,* 1036–1054.

Tanaka, H., & Guo, P. (1999). Portfolio selection based on upper and lower exponential possibility distributions. *European Journal of Operational Research, 114,* 115–126.

Terol, A. B., Gladish, B. P., Parra, M. A., & Uria, M. V. R. (2006). Fuzzy compromise programming for portfolio selection. *Applied Mathematics and Computation, 173,* 251–264.

Vercher, E., Bermúdez, J. D., & Segura, J. V. (2007). Fuzzy portfolio optimization under downside risk measures. *Fuzzy Sets and Systems, 158,* 769–782.

Yoshida, Y. (2009). An estimation model of value-at-risk portfolio under uncertainty. *Fuzzy Sets and Systems*, *160*, 3250–3262.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8,* 338–353.

Zeng, X. J., & Keane, J. A. (2005). Approximation capabilities of hierarchical fuzzy systems. *IEEE Transactions on Fuzzy Systems*, *13*(5), 659–672.

Zimmermann, H.-J. (1996). *Fuzzy Set Theory and its Applications* (3rd Edn.). Boston.

Zimmermann, H.-J. (2001). *Fuzzy set theory and its applications*. Kluwer Academic Publishers.