ERRATUM TO: THE WIGNER–VILLE DISTRIBUTION ASSOCIATED WITH THE QUATERNION OFFSET LINEAR CANONICAL TRANSFORM

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In [1, Lemma 3.2], the function h should not depend on the variable u , and the lemma must be stated as follows (see [2]):

LEMMA 3.2. *The QOLCT of a signal $f \in L^1(\mathbb{R}^2, \mathbb{H})$ can be reduced to the QFT*

$$(3.1) \quad \mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{f\}(u_1, u_2) = e^{\lambda \left[-\frac{1}{b_1} u_1 (d_1 \tau_1 - b_1 \eta_1) + \frac{d_1}{2b_1} (u_1^2 + \tau_1^2) \right]} \mathcal{F}^{\lambda, \mu} \{h\} \left(\frac{u_1}{b_1}, \frac{u_2}{b_2} \right) \\
\times e^{\mu \left[-\frac{1}{b_2} u_2 (d_2 \tau_2 - b_2 \eta_2) + \frac{d_2}{2b_2} (u_2^2 + \tau_2^2) \right]},$$

with

$$h(t) = \frac{1}{\sqrt{2\pi\lambda b_1}} e^{\lambda \left(\frac{1}{b_1} t_1 \tau_1 + \frac{a_1}{2b_1} t_1^2 \right)} f(t) e^{\mu \left(\frac{1}{b_2} t_2 \tau_2 + \frac{a_2}{2b_2} t_2^2 \right)} \frac{1}{\sqrt{2\pi\mu b_2}}.$$

In [1, Definition 4.1], the definition of the WVD-QOLCT is correct for the case $b_1, b_2 \neq 0$. However, for the three cases $b_1 = 0, b_2 \neq 0$, $b_1 \neq 0, b_2 = 0$ and $b_1 = b_2 = 0$, one should not have the integration symbol, and the definition must be given as follows:

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$$\mathcal{W}_{f,g}^{A_1,A_2}(t,u) = \begin{cases} \int_{\mathbb{R}^2} K_{A_1}^\lambda(s_1, u_1) f\left(t + \frac{s}{2}\right) \bar{g}\left(t - \frac{s}{2}\right) K_{A_2}^\mu(s_2, u_2) ds, & b_1, b_2 \neq 0, \\ \sqrt{d_1} e^{\lambda(\frac{c_1 d_1}{2}(u_1 - \tau_1)^2 + u_1 \tau_1)} f\left(t_1 + \frac{d_1(u_1 - \tau_1)}{2}, t_2 + \frac{s_2}{2}\right) \\ \quad \times \bar{g}\left(t_1 - \frac{d_1(u_1 - \tau_1)}{2}, t_2 - \frac{s_2}{2}\right) K_{A_2}^\mu(s_2, u_2) ds, & b_1 = 0, b_2 \neq 0, \\ \sqrt{d_2} K_{A_1}^\lambda(s_1, u_1) f\left(t_1 + \frac{s_1}{2}, t_2 + \frac{d_2(u_2 - \tau_2)}{2}\right) \\ \quad \times \bar{g}\left(t_1 - \frac{s_1}{2}, t_2 - \frac{d_2(u_2 - \tau_2)}{2}\right) e^{\mu(\frac{c_2 d_2}{2}(u_2 - \tau_2)^2 + u_2 \tau_2)} & b_1 \neq 0, b_2 = 0, \\ \sqrt{d_1 d_2} e^{\lambda(\frac{c_1 d_1}{2}(u_1 - \tau_1)^2 + u_1 \tau_1)} f\left(t_1 + \frac{d_1(u_1 - \tau_1)}{2}, t_2 + \frac{d_2(u_2 - \tau_2)}{2}\right) \\ \quad \times \bar{g}\left(t_1 - \frac{d_1(u_1 - \tau_1)}{2}, t_2 - \frac{d_2(u_2 - \tau_2)}{2}\right) e^{\mu(\frac{c_2 d_2}{2}(u_2 - \tau_2)^2 + u_2 \tau_2)}, & b_1 = b_2 = 0, \end{cases}$$

In [1, Remark 4.2], and in all the rest of the paper, the function h must be noted $h_{f,g,t}(s)$, that is, the variable t is fixed. Thus relation (4.1) will be in the form

$$\mathcal{W}_{f,g}^{A_1,A_2}(t,u) = \mathcal{O}_{A_1,A_2}^{\lambda,\mu}\{h_{f,g,t}\}(u).$$

In [1, Lemma 4.3], the function $k_{f,g}$ must not depend on the variable u on which $\mathcal{F}^{\lambda,\mu}$ depends. The correction of this lemma is as follows:

LEMMA 4.3. *We have*

$$(4.2) \quad \mathcal{W}_{f,g}^{A_1,A_2}(t,u) = e^{\lambda\left[-\frac{1}{b_1}u_1(d_1\tau_1 - b_1\eta_1) + \frac{d_1}{2b_1}(u_1^2 + \tau_1^2)\right]} \mathcal{F}^{\lambda,\mu}\{k_{f,g,t}\}\left(\frac{u_1}{b_1}, \frac{u_2}{b_2}\right) \\ \times e^{\mu\left[-\frac{1}{b_2}u_2(d_2\tau_2 - b_2\eta_2) + \frac{d_2}{2b_2}(u_2^2 + \tau_2^2)\right]},$$

where

$$k_{f,g,t}(s) = \frac{1}{\sqrt{2\pi\lambda b_1}} e^{\lambda\left(\frac{1}{b_1}s_1\tau_1 + \frac{a_1}{2b_1}s_1^2\right)} h_{f,g,t}(s) e^{\mu\left(\frac{1}{b_2}s_2\tau_2 + \frac{a_2}{2b_2}s_2^2\right)} \frac{1}{\sqrt{2\pi\mu b_2}}.$$

In the statement of [1, Theorem 4.9], the function $f(t + \frac{s}{2})\bar{g}(t - \frac{s}{2})$ should be written $f(t + \frac{s+k}{2})\bar{g}(t - \frac{s+k}{2})$ and the theorem can be stated in an improved form as follows:

THEOREM 4.9. *Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$. Then we have*

$$\sum_{(k_1, k_2) \in \mathbb{Z}^2} e^{\frac{i}{b_1}k_1\tau_1 + i\frac{a_1}{2b_1}k_1^2} f\left(t + \frac{k}{2}\right) \bar{g}\left(t - \frac{k}{2}\right) e^{\frac{j}{b_2}k_2\tau_2 + j\frac{a_2}{2b_2}k_2^2}$$

$$\begin{aligned}
&= \sqrt{2\pi i b_1} \left[\sum_{(k_1, k_2) \in \mathbb{Z}^2} e^{2\pi i k_1 (d_1 \tau_1 - b_1 \eta_1) - i \frac{d_1}{2b_1} (4\pi^2 b_1^2 k_1^2 + \tau_1^2)} \right. \\
&\quad \times \mathcal{W}_{f,g}^{A_1, A_2}(t, (2\pi b_1 k_1, 2\pi b_2 k_2)) \\
&\quad \left. \times e^{2\pi j k_2 s_2} e^{2\pi j k_2 (d_2 \tau_2 - b_2 \eta_2) - j \frac{d_2}{2b_2} (4\pi^2 b_2^2 k_2^2 + \tau_2^2)} \right] \sqrt{2\pi j b_2}.
\end{aligned}$$

PROOF. Let $\omega_{f,g,t}(s) = e^{i \frac{1}{b_1} s_1 \tau_1 + i \frac{a_1}{2b_1} s_1^2} f(t + \frac{s}{2}) \overline{g}(t - \frac{s}{2}) e^{j \frac{1}{b_2} s_2 \tau_2 + j \frac{a_2}{2b_2} s_2^2}$.

As $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$, by Hölder's inequality we have $\omega_{f,g,t} \in L^1(\mathbb{R}^2, \mathbb{H})$. Then by Proposition 4.8 one has

$$\begin{aligned}
&\sum_{(k_1, k_2) \in \mathbb{Z}^2} \omega_{f,g,t}(s_1 + k_1, s_2 + k_2) \\
&= \sum_{(k_1, k_2) \in \mathbb{Z}^2} e^{2\pi i k_1 s_1} \mathcal{F}^{i,j} \{\omega_{f,g,t}\}(2\pi k_1, 2\pi k_2) e^{2\pi j k_2 s_2}.
\end{aligned}$$

Applying (4.2) leads to

$$\begin{aligned}
&\sum_{(k_1, k_2) \in \mathbb{Z}^2} e^{i \frac{1}{b_1} (s_1 + k_1) \tau_1 + i \frac{a_1}{2b_1} (s_1 + k_1)^2} \\
&\quad \times f\left(t + \frac{s + k}{2}\right) \overline{g}\left(t - \frac{s + k}{2}\right) e^{j \frac{1}{b_2} (s_2 + k_2) \tau_2 + j \frac{a_2}{2b_2} (s_2 + k_2)^2} \\
&= \sqrt{2\pi i b_1} \sum_{(k_1, k_2) \in \mathbb{Z}^2} e^{2\pi i k_1 s_1} e^{i[2\pi k_1 (d_1 \tau_1 - b_1 \eta_1) - \frac{d_1}{2b_1} (4\pi^2 b_1^2 k_1^2 + \tau_1^2)]} \\
&\quad \times \mathcal{W}_{f,g}^{A_1, A_2}(t, (2\pi b_1 k_1, 2\pi b_2 k_2)) e^{j[2\pi k_2 (d_2 \tau_2 - b_2 \eta_2) - \frac{d_2}{2b_2} (4\pi^2 b_2^2 k_2^2 + \tau_2^2)]} e^{2\pi j k_2 s_2} \sqrt{2\pi j b_2}.
\end{aligned}$$

Therefore, we get the result by evaluating $(s_1, s_2) = (0, 0)$. $\square$

[1, Lemma 4.12] must be corrected as follows:

LEMMA 4.12. Let $A_l = \begin{bmatrix} a_l & b_l \\ c_l & d_l \end{bmatrix} \begin{smallmatrix} \tau_l \\ \eta_l \end{smallmatrix}$ and $B_l = \begin{pmatrix} a_l & b_l \\ c_l & d_l \end{pmatrix}$ with $a_l d_l - b_l c_l = 1$ for $l = 1, 2$. For $f \in L^1(\mathbb{R}^2, \mathbb{H})$, we have the relation

$$\begin{aligned}
(4.14) \quad &\mathcal{O}_{A_1, A_2}^{\lambda, \mu} \{f(t)\}(u) = e^{\lambda(-2u_1(d_1 \tau_1 - b_1 \eta_1) + d_1 \tau_1^2) \frac{1}{2b_1}} \\
&\times \mathcal{L}_{B_1, B_2} \left\{ e^{\lambda \frac{\tau_1}{b_1} t_1} f(t) e^{\mu \frac{\tau_2}{b_2} t_2} \right\}(u) e^{\mu(-2u_2(d_2 \tau_2 - b_2 \eta_2) + d_2 \tau_2^2) \frac{1}{2b_2}}.
\end{aligned}$$

The proof of [1, Theorem 4.11] starts as follows:

By equation (4.14) we have

$$\begin{aligned} \left(\int_{\mathbb{R}^2} |\mathcal{W}_{f,g}^{A_1,A_2}(t,u)|_Q^p du \right)^{\frac{1}{p}} &= \left(\int_{\mathbb{R}^2} |\mathcal{O}_{A_1,A_2}^{\lambda,\mu} \{h_{f,g,t}(s)\}(u)|_Q^p du \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^2} |\mathcal{L}_{B_1,B_2}^{\lambda,\mu} \{e^{\lambda \frac{\tau_1}{b_1} s_1} h_{f,g,t}(s) e^{\mu \frac{\tau_2}{b_2} s_2}\}(u)|_Q^p du \right)^{\frac{1}{p}} \\ &\leq \frac{|b_1 b_2|^{\frac{-1}{2} + \frac{1}{p}}}{2\pi} \underbrace{\left(\int_{\mathbb{R}^2} \left| e^{\lambda \frac{\tau_1}{b_1} s_1} f\left(t + \frac{s}{2}\right) \overline{g}\left(t - \frac{s}{2}\right) e^{\mu \frac{\tau_2}{b_2} s_2} \right|_Q^q ds \right)^{\frac{1}{q}}}_{= \left(\int_{\mathbb{R}^2} |f(t + \frac{s}{2}) \overline{g}(t - \frac{s}{2})|_Q^q ds \right)^{\frac{1}{q}}}. \end{aligned}$$

The rest of the proof is valid.

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References

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