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A BOUNDED BELOW, NONCONTRACTIBLE, ACYCLIC COMPLEX OF PROJECTIVE MODULES

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Abstract. We construct examples of bounded below, noncontractible, acyclic complexes of finitely generated projective modules over some rings S, as well as bounded above, noncontractible, acyclic complexes of injective modules. The rings S are certain rings of infinite matrices with entries in the rings of commutative polynomials or formal power series in infinitely many variables. In the world of comodules or contramodules over coalgebras over fields, similar examples exist over the cocommutative symmetric coalgebra of an infinite-dimensional vector space. A simpler, universal example of a bounded below, noncontractible, acyclic complex of free modules with one generator, communicated to the author by Canonaco, is included at the end of the paper.

Introduction

Bounded above acyclic complexes of projective objects are contractible. So are bounded below acyclic complexes of injective objects. On the other hand, there is an easy, thematic example of a doubly unbounded, acyclic, noncontractible complex of finitely generated projective-injective modules over the algebra of dual numbers $R = k[\varepsilon]/(\varepsilon^2)$ (over any field k):

$$\cdots \longrightarrow R \xrightarrow{\varepsilon*} R \xrightarrow{\varepsilon*} R \longrightarrow \cdots . \tag{1}$$

We refer to [9, Prologue], [10, Sections 7.4–7.5] and the references therein for a discussion of the role of the complex (1) in the context of derived Koszul duality and derived categories of the second kind.

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Do there exist bounded below, noncontractible, acyclic complexes of projective modules; and if so, under what rings? Dual-analogously, are there any bounded above, noncontractible, acyclic complexes of injective modules? These questions were posed, in the context of potential applications to the Finitistic Dimension Conjecture, in the recent preprint of Shaul [16]. According to [16, Theorem 5.1], nonexistence of such complexes of projective/injective modules over a two-sided Noetherian ring S with a dualizing complex would imply finiteness of the finitistic dimensions of S.

The aim of the present paper is to show that, over certain rather big rings S, such complexes do exist. The examples of rings S which we obtain are certainly noncommutative and non-Noetherian. The more explicit ones among them are rings of column-finite or row/column-zero-convergent infinite matrices with entries in the rings of commutative polynomials or formal power series in infinitely many variables.

On the other hand, in the world of coalgebras over fields, we demonstrate examples of bounded above, acyclic, noncontractible complexes of injective comodules and bounded below, acyclic, noncontractible complexes of projective contramodules over certain *cocommutative* coalgebras dual to algebras of formal power series in infinitely many variables. These examples go back to [6, Section 0.2.7], where they were very briefly discussed in the context of semi-infinite homological algebra and derived comodule-contramodule correspondence.

Almost all the examples presented in this paper are based on one idea, namely, that of the dual Koszul complex of the ring of polynomials in infinitely many variables. A straightforward realization of this idea is possible in the worlds of comodules and contramodules, but we need an additional trick with a passage to infinite matrices in order to produce examples of complexes of projective/injective modules. The only exception is the (much simpler) *universal* example, communicated to the author by A. Canonaco. We reproduce it at the end of the paper in Example 8.4.

The approach to the Finitistic Dimension Conjecture developed in [15,16] goes back to Rickard's paper [14], where it was shown that if the injective modules over a finite-dimensional algebra generate its unbounded derived category as a triangulated category with coproducts, then the finitistic dimension is finite. A counterexample in [14, Theorem 3.5] shows that for the ring of commutative polynomials in infinitely many variables, the generation property fails. Our examples in this paper follow in the footsteps of [6, Section 0.2.7] and [14, Theorem 3.5]. We also provide some details of the claims in [6, Section 0.2.7] which were skipped in the book [6].

1. Projective, flat, and injective bounded acyclicity problems

The general convention in this paper is that complexes are presumed to be cohomologically graded, so the differential raises the degree. A complex

$C^\bullet = (C^n,\, d_n\colon C^n \to C^{n+1})$ is called *bounded above* if $C^n = 0$ for $n \gg 0$, and $C^\bullet$ is *bounded below* if $C^n = 0$ for $n \ll 0$. In this notation, it is a standard fact that every bounded above acyclic complex of projective modules/objects (in an abelian or exact category) is contractible, and every bounded below acyclic complex of injective modules/objects is contractible. When we occasionally consider homologically graded complexes, we use the notation with lower indices, $P_\bullet = (P_n,\, d_n\colon P_n \to P_{n-1})$.

Let S be an associative ring. The two "wrong-sided bounded projective/injective acyclicity problems" posed in [16, Theorem 5.1(4–5)] are:

- Is every bounded above acyclic complex of injective S-modules contractible?
- Is every bounded below acyclic complex of projective S-modules contractible?

In addition to the above two, we would like to ask a similar question about flat S-modules. Here one has to be careful: even a two-sided bounded acyclic complex of flat modules need not be contractible. However, such a complex is always *pure acyclic*, or in other words, has flat modules of cocycles. Thus we ask:

- Is every bounded below acyclic complex of flat S-modules pure acyclic?

Given a ring S and a left S-module M, the *character module* $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a right S-module. The following lemma is well-known.

LEMMA 1.1. *A left S-module F is flat if and only if the right S-module F^+ is injective.*

The next proposition explains the connection between the injective, flat, and projective wrong-sided bounded acyclicity questions, and shows that presenting a counterexample to the "projective" question is enough.

PROPOSITION 1.2. *Given a ring S, consider the following three properties*:

(1) *Every bounded above acyclic complex of injective right S-modules is contractible.*

(2) *Every bounded below acyclic complex of flat left S-modules is pure acyclic.*

(3) *Every bounded below acyclic complex of projective left S-modules is contractible.*

Then the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) *hold.*

PROOF. (1) $\Longrightarrow$ (2). Let $F^\bullet = (0 \to F^0 \to F^1 \to F^2 \to \cdots)$ be a bounded below acyclic complex of flat left S-modules. Then, by the direct implication of Lemma 1.1, $F^{\bullet,+} = (\cdots \to F^{2,+} \to F^{1,+} \to F^{0,+} \to 0)$ is a bounded above acyclic complex of injective right S-modules. A complex of injective modules is contractible if and only if its modules of cocycles are injective. If this is the case for the complex $F^{\bullet,+}$, then the inverse implication of Lemma 1.1

tells that the modules of cocycles of the complex $F^\bullet$ are flat; so $F^\bullet$ is a pure acyclic complex of flat modules.

(2) $\Longrightarrow$ (3). By Neeman's theorem [5, Theorem 8.6(iii) $\Rightarrow$ (i)], any pure acyclic complex of projective modules is contractible. (Cf. [16, proof of Theorem A.7].) $\square$

2. The injective construction of acyclic complex of projectives

Let k be a field, $(x_\alpha)_{\alpha\in A}$ be an infinite set of variables, and $R = k[x_\alpha : \alpha\in A]$ be the commutative ring of polynomials in the variables x_α over k. Endow the one-dimensional vector space k over k with the R-module structure by the obvious rule: all the elements $x_\alpha \in R$ act by zero in k.

Theorem 2.1 (Rickard [14]). *For any injective R-module J and all integers $n \ge 0$, one has* $\mathrm{Ext}^n_R(J,k) = 0$.

Proof. For a countably infinite set of variables x_α, this is formulated and proved in [14, Theorem 3.5]. The general case of a possibly uncountable index set A is similar. One represents A as the union of its finite subsets $B \subset A$, so the ring R the direct limit of the related polynomial rings R_B in finitely many variables, considers the direct limit of finite Koszul complex indexed by the finite subsets $B \subset A$, etc. (Cf. the proof of Theorem 3.1 below for some further details.) $\square$

Let A be an additive category and $M \in \mathsf{A}$ be an object. Then we denote by $\mathsf{add}(M)$ the full subcategory in A formed by the direct summands of finite direct sums of copies of M. The following lemma is a straightforward category-theoretic generalization of a well-known module-theoretic observation going back to Dress [3].

Lemma 2.2. *Let A be an idempotent-complete additive category and $M \in \mathsf{A}$ be an object.*

(a) *Let $S = \mathrm{Hom}_{\mathsf{A}}(M,M)^{\mathrm{op}}$ be the opposite ring to the endomorphism ring of the object $M \in \mathsf{A}$; so the ring S acts on the object M on the right. Then the covariant functor $\mathrm{Hom}_{\mathsf{A}}(M,-)\colon \mathsf{A} \to S\text{–}\mathsf{Mod}$ restricts to an equivalence of additive categories*

$$\mathrm{Hom}_{\mathsf{A}}(M,-) : \mathsf{add}(M) \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$$

between the full subcategory $\mathsf{add}(M) \subset \mathsf{A}$ and the full subcategory of finitely generated projective left S-modules $S\text{–}\mathsf{mod}_{\mathsf{proj}}$ in the category of left S-modules $S\text{–}\mathsf{Mod}$.

(b) *Let $S = \mathrm{Hom}_{\mathsf{A}}(M,M)$ be the endomorphism ring of the object $M \in \mathsf{A}$; so the ring S acts on the object M on the left. Then the contravariant func-*

tor $\mathrm{Hom}_{\mathsf{A}}(-,M)\colon \mathsf{A}^{\mathsf{op}}\to S\text{–}\mathsf{Mod}$ *restricts to an anti-equivalence of additive categories*

$$\mathrm{Hom}_{\mathsf{A}}(-,M)\colon \mathsf{add}(M)^{\mathsf{op}}\simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$$

between the full subcategory $\mathsf{add}(M)\subset\mathsf{A}$ *and the full subcategory of finitely generated projective left* S*-modules* $S\text{–}\mathsf{mod}_{\mathsf{proj}}\subset S\text{–}\mathsf{Mod}$.

The following corollary sums up the "injective coresolution construction of a bounded below acyclic complex of projective modules".

Corollary 2.3. *Let* $R=k[x_\alpha:\alpha\in A]$ *be the ring of polynomials in infinitely many variables over a field* k, *and let*

$$0\longrightarrow k\longrightarrow J^0\longrightarrow J^1\longrightarrow J^2\longrightarrow\cdots \tag{2}$$

be an injective coresolution of the one-dimensional R*-module* k. *Let* J *be an injective* R*-module such that the* R*-module* J^n *is a direct summand of* J *for all* $n\ge 0$. *Let*

$$0\longrightarrow \mathrm{Hom}_R(J,J^0)\longrightarrow \mathrm{Hom}_R(J,J^1)\longrightarrow \mathrm{Hom}_R(J,J^2)\longrightarrow\cdots \tag{3}$$

be the complex obtained by applying the functor $\mathrm{Hom}_R(J,-)$ *to the truncated coresolution* (2). *Then* (3) *is a bounded below, noncontractible, acyclic complex of finitely generated projective left modules over the ring* $S=\mathrm{Hom}_R(J,J)^{\mathrm{op}}$.

Proof. The complex (3) is acyclic by Theorem 2.1. The left S-module $\mathrm{Hom}_R(J,J^n)$ is a direct summand of the left S-module $\mathrm{Hom}_R(J,J)=S$ for every $n\ge 0$, since the R-module J^n is a direct summand of J. So (3) is even a complex of cyclic projective left S-modules (i.e., projective S-modules with one generator).

It remains to explain why the complex of S-modules (3) is not contractible. For this purpose, one observes that, given a full subcategory B in an additive category A, a complex $C^\bullet$ in B is contractible in A if and only if it is contractible in B. Indeed, any contracting homotopy for $C^\bullet$ as a complex in A would be a collection of morphisms in A between objects from B, which means a collection of morphisms in B.

By Lemma 2.2(a) (for $\mathsf{A}=R\text{–}\mathsf{Mod}$ and $M=J$), the functor $\mathrm{Hom}_R(J,-)$ is an equivalence of categories $\mathsf{add}(J)\simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$. The truncated coresolution (2),

$$0\longrightarrow J^0\longrightarrow J^1\longrightarrow J^2\longrightarrow\cdots$$

is a noncontractible (since nonacyclic) complex in $R\text{–}\mathsf{Mod}$ with the terms belonging to $\mathsf{add}(J)$, so it is a noncontractible complex in $\mathsf{add}(J)$. Applying the equivalence of additive categories $\mathsf{add}(J)\simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$, we obtain a noncontractible complex (3) in $S\text{–}\mathsf{mod}_{\mathsf{proj}}$, which is consequently also noncontractible in $S\text{–}\mathsf{Mod}$. It is important for this argument that the functor $\mathrm{Hom}_R(J,-)\colon \mathsf{add}(J)\to S\text{–}\mathsf{Mod}$ is fully faithful. $\square$

3. Dual Rickard's acyclicity theorem

The aim of this section is to prove the following dual version of Rickard's theorem [14, Theorem 3.5].

THEOREM 3.1. *Let $R=k[x_\alpha:\alpha\in A]$ be the ring of polynomials in infinitely many variables over a field k. As above, we endow the one-dimensional k-vector space k with the obvious R-module structure. Then, for any flat R-module P and all integers $n\ge 0$, one has $\mathrm{Ext}^n_R(k,P)=0$.*

PROOF. For every $\alpha\in A$, consider the two-term Koszul complex of free R-modules with one generator

$$\cdots\longrightarrow 0\longrightarrow R\xrightarrow{x_\alpha *} R\longrightarrow 0\longrightarrow\cdots \tag{4}$$

situated in the cohomological degrees -1 and 0. For every finite subset of indices $B\subset A$, denote by $K^B_\bullet(R)$ the tensor product, taken over the ring R, of the complexes (4) with $\alpha\in B$. As a finite subset $B\subset A$ varies, the complexes $K^B_\bullet(R)$ form an inductive system, indexed by the poset of all finite subsets $B\subset A$ ordered by inclusion.

Put $K_\bullet(R)=\varinjlim_{B\subset A}K^B_\bullet(R)$. Then $K_\bullet(R)$ is a bounded above complex of free R-modules. One has $K_n(R)=0$ for $n<0$, $K_0(R)=R$, and $K_n(R)$ is a free R-module with a set of generators of the cardinality equal to the cardinality of the set A for all $n>0$. (More invariantly, $K_n(R)$ is the free R-module spanned by the set of all subsets in A of the finite cardinality n).

For any finite subset $B\subset A$, the complex $K^B_\bullet(A)$ is a finite resolution of the R-module $R/\sum_{\alpha\in B}x_\alpha R$ by finitely generated free R-modules. Passing to the direct limit, one can easily see that $K_\bullet(R)$ is a free R-module resolution of the one-dimensional R-module $k=R/\sum_{\alpha\in A}x_\alpha R$.

The following three lemmas are straightforward or standard.

LEMMA 3.2. *Let R be an associative ring, Ξ be a directed poset, and $(F_\xi)_{\xi\in\Xi}$ be an inductive system of projective R-modules whose direct limit $F=\varinjlim_{\xi\in\Xi}F_\xi$ is also a projective R-module. Let P be an arbitrary R-module. Then the higher derived inverse limit functors vanish on the projective system $\mathrm{Hom}_R(F_\xi,P)_{\xi\in\Xi}$,*

$$\varprojlim_{\xi\in\Xi}\mathrm{Hom}_R(F_\xi,P)=\mathrm{Hom}_R(F,P)$$

and

$$\varprojlim\nolimits^i_{\xi\in\Xi}\mathrm{Hom}_R(F_\xi,P)=0\quad\textit{for all } i\ge 1.$$

Proof. Dropping the condition that the R-module $F = \varinjlim_{\xi\in\Xi} F_\xi$ is projective (but keeping the conditions that the R-modules F_ξ are projective), one would have $\varprojlim^i_{\xi\in\Xi} \operatorname{Hom}_R(F_\xi, P) = \operatorname{Ext}^i_R(F, P)$ for every $i \ge 0$. □

Lemma 3.3. *Let A be an infinite set and $(V_B)_{B\subset A}$ be a projective system of abelian groups, indexed by the poset of all finite subsets $B \subset A$ ordered by inclusion. Assume that there exists an integer $n \ge 0$ such that $V_B = 0$ whenever the cardinality of B exceeds n. Then the whole derived inverse limit functor vanishes on the projective system $(V_B)_{B\subset A}$,*

$$\varprojlim\nolimits^i_{B\subset A} V_B = 0 \quad \textit{for all } i \ge 0.$$

Proof. This is a special case of the assertion that the derived functors of inverse limit are preserved by the passage to a cofinal subsystem. This can be deduced from fact that the derived inverse limits vanish on so-called weakly flabby (faiblement flasque) projective systems [4, Théorème 1.8]. A stronger result that the derived inverse limit (in an abelian category with exact product functors) only depends on the pro-object represented by the given projective system can be found in [13, Corollary 7.3.7]. □

Lemma 3.4. *Let Ξ be a directed poset and $(C^\bullet_\xi)_{\xi\in\Xi}$ be a projective system of complexes of abelian groups with $C^n_\xi = 0$ for $n < 0$. Then there are two spectral sequences ${}'E^{pq}_r$ and ${}''E^{pq}_r$, starting from the pages*

$$'E^{pq}_2 = \varprojlim\nolimits^p H^q(C^\bullet_\xi), \quad p,\, q \ge 0,$$

$$''E^{pq}_1 = \varprojlim\nolimits^q C^p_\xi, \quad p,\, q \ge 0,$$

with the differentials ${}'d^{pq}_r\colon {}'E^{p,q}_r \to {}'E^{p+r,q-r+1}_r$ and ${}''d^{pq}_r\colon {}''E^{p,q}_r \to {}''E^{p+r,q-r+1}_r$, converging to the associated graded groups to two different filtrations ${}'F^pE^n$ and ${}''F^pE^n$ on the same graded abelian group E^n, $n = p + q$.

Proof. These are called “the two hypercohomology spectral sequences” (for the derived functor of inverse limit); cf. [2, Section XVII.3]. The groups E^n are the cohomology groups of the complex obtained by applying the derived functor of inverse limit to the whole complex of projective systems $(C^\bullet_\xi)$. □

Now we can finish the proof of the theorem. By the definition, we have $\operatorname{Ext}^n_R(k, P) = H^n \operatorname{Hom}_R(K_\bullet(R), P)$. The complex $\operatorname{Hom}_R(K_\bullet(R), P)$ is the inverse limit

$$\operatorname{Hom}_R(K_\bullet(R), P) = \varprojlim\nolimits_{B\subset A} \operatorname{Hom}_R(K^B_\bullet(R), P).$$

For every $n \ge 0$, Lemma 3.2 (with the poset Ξ of all finite subsets $B \subset A$, finitely generated free R-modules F_B, and an infinitely generated free R-module F) tells that $\varprojlim^i_{B\subset A} \operatorname{Hom}_R(K^B_n(R), P) = 0$ for all $i \ge 1$.

On the other hand, the complex $\mathrm{Hom}_R(K^B_\bullet(R), P)$ has its only nonzero cohomology module situated in the cohomological degree n equal to the cardinality of B (as P is a flat module over the ring $R_B = k[x_\alpha : \alpha \in B]$). By Lemma 3.3, we have $\varprojlim^i_{B\subset A} H^n \mathrm{Hom}_R(K^B_\bullet(R), P) = 0$ for all $i \geq 0$ and $n \geq 0$.

In the context of Lemma 3.4, put $C^\bullet_B = \mathrm{Hom}_R(K^B_\bullet(R), P)$. Then we know that ${}'E_2^{pq} = \varprojlim^p_{B\subset A} H^q(C^\bullet_B) = 0$ for all p, $q \geq 0$, and ${}''E_1^{pq} = \varprojlim^q_{B\subset A} C^p_B = 0$ for all $q \geq 1$. Thus $E^n = 0$ and

$$H^n(\varprojlim_{B\subset A} \mathrm{Hom}_R(K^B_\bullet(R), P)) = {}''E_2^{n,0} = 0$$

for all $n \geq 0$. $\square$

4. The projective construction of acyclic complex of projectives

Now we are ready to present the "projective resolution construction of a bounded below acyclic complex of projective modules".

Corollary 4.1. *Let $R = k[x_\alpha : \alpha \in A]$ be the ring of polynomials in infinitely many variables over a field k, and let*

$$0 \longleftarrow k \longleftarrow P_0 \longleftarrow P_1 \longrightarrow P_2 \longleftarrow \cdots \tag{5}$$

be a projective resolution of the one-dimensional R-module k. Let P be a projective R-module such that the R-module P_n is a direct summand of P for all $n \geq 0$. Let

$$0 \longrightarrow \mathrm{Hom}_R(P_0, P) \longrightarrow \mathrm{Hom}_R(P_1, P) \longrightarrow \mathrm{Hom}_R(P_2, P) \longrightarrow \cdots \tag{6}$$

be the complex obtained by applying the contravariant functor $\mathrm{Hom}_R(-, P)$ *to the truncated resolution* (5). *Then* (6) *is a bounded below, noncontractible, acyclic complex of finitely generated projective left modules over the ring $S =$* $\mathrm{Hom}_R(P, P)$.

Proof. The complex (6) is acyclic by Theorem 3.1. The left S-module $\mathrm{Hom}_R(P_n, P)$ is a direct summand of the left S-module $\mathrm{Hom}_R(P, P) = S$ for every $n \geq 0$, since the R-module P_n is a direct summand of P. So (6) is even a complex of cyclic projective left S-modules.

The proof of the assertion that the complex of S-modules (6) is not contractible is similar to the argument in the proof of Corollary 2.3. By Lemma 2.2(b) (for $\mathsf{A} = R\text{–}\mathsf{Mod}$ and $M = P$), the functor $\mathrm{Hom}_R(-, P)$ is an anti-equivalence of categories $\mathsf{add}(P)^{\mathsf{op}} \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$. The truncated resolution (5),

$$0 \longleftarrow P_0 \longleftarrow P_1 \longrightarrow P_2 \longleftarrow \cdots$$

is a noncontractible (since nonacyclic) complex in R–Mod with the terms belonging to $\mathsf{add}(P)$, so it is a noncontractible complex in $\mathsf{add}(P)$. Applying the anti-equivalence of additive categories $\mathsf{add}(P)^{\mathsf{op}} \simeq S$–$\mathsf{mod}_{\mathsf{proj}}$, we obtain a noncontractible complex (6) in S–$\mathsf{mod}_{\mathsf{proj}}$, which is consequently also noncontractible in S–Mod. It is important for this argument that the contravariant functor $\mathrm{Hom}_R(-,P)\colon \mathsf{add}(P)^{\mathsf{op}} \longrightarrow S$–$\mathsf{Mod}$ is fully faithful. $\square$

5. Brief preliminaries on coalgebras

In this section and the next two, we consider comodules and contramodules over coassociative, counital coalgebras $\mathcal{C}$ over a field k. We refer to the book [17] and the survey papers [7, Section 1], [10, Sections 3 and 8] for background material on coalgebras, comodules, and contramodules.

For any coalgebra $\mathcal{C}$, there are locally finite Grothendieck abelian categories of left and right $\mathcal{C}$-comodules $\mathcal{C}$–Comod and Comod–$\mathcal{C}$, and a locally presentable abelian category of left $\mathcal{C}$-contramodules $\mathcal{C}$–Contra. There are enough injective objects in $\mathcal{C}$–Comod, and they are precisely the direct summands of the *cofree* left $\mathcal{C}$-comodules $\mathcal{C}\otimes_k V$ (where V ranges over the k-vector spaces). Dual-analogously, there are enough projective objects in $\mathcal{C}$–Contra, and they are precisely the direct summands of the *free* left $\mathcal{C}$-contramodules $\mathrm{Hom}_k(\mathcal{C},V)$ (where $V\in k$–Vect).

The additive categories of injective left $\mathcal{C}$-comodules and projective left $\mathcal{C}$-contramodules are naturally equivalent,

$$\Psi_{\mathcal{C}}\colon \mathcal{C}\text{–}\mathsf{Comod}_{\mathsf{inj}} \simeq \mathcal{C}\text{–}\mathsf{Contra}_{\mathsf{proj}} \;:\!\Phi_{\mathcal{C}}. \tag{7}$$

The equivalence is provided by the restrictions of the adjoint functors

$$\Psi_{\mathcal{C}}\colon \mathcal{C}\text{–}\mathsf{comod} \leftrightarrows \mathcal{C}\text{–}\mathsf{Contra} \;:\!\Phi_{\mathcal{C}},$$

the functor $\Phi_{\mathcal{C}}$ being the left adjoint and $\Psi_{\mathcal{C}}$ the right adjoint. The functors $\Psi_{\mathcal{C}}$ and $\Phi_{\mathcal{C}}$ are constructed as

$$\Psi_{\mathcal{C}}(\mathcal{M}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C},\mathcal{M}) \quad\text{and}\quad \Phi_{\mathcal{C}}(\mathfrak{P}) = \mathcal{C}\odot_{\mathcal{C}}\mathfrak{P}$$

for all $\mathcal{M}\in\mathcal{C}$–$\mathsf{Comod}$ and $\mathfrak{P}\in\mathcal{C}$–$\mathsf{Contra}$. Here $\odot_{\mathcal{C}}\colon \mathsf{Comod}$–$\mathcal{C}\times\mathcal{C}$–$\mathsf{Contra}$ $\to k$–Vect is the functor of *contratensor product* over a coalgebra $\mathcal{C}$, while $\mathrm{Hom}_{\mathcal{C}}$ denotes the Hom functor in the comodule category $\mathcal{C}$–Comod. The equivalence of additive categories (7) is called the (*underived*) *comodule-contramodule correspondence*. We refer to [7, Sections 1.2 and 3.1], [10, Sections 8.6–8.7], or [6, Sections 0.2.6 and 5.1] for a more detailed discussion.

In fact, we are only interested in one special kind of coalgebras, namely, the *symmetric coalgebra* $\mathcal{S}ym(U)$ of a k-vector space U. To define the symmetric coalgebra, consider the *tensor coalgebra* $\mathcal{T}en(U) = \bigoplus_{n=0}^{\infty} U^{\otimes n}$, as defined, e. g., in [10, Section 2.3] (where the notation is slightly different). The tensor coalgebra is the cofree conilpotent coalgebra cospanned by U [10, Remark 3.2]; it is also naturally graded. The symmetric coalgebra is simplest defined as the graded subcoalgebra in $\mathcal{T}en(U)$ whose grading components $\mathcal{S}ym_n(U) \subset \mathcal{T}en_n(U) = U^{\otimes n}$ are the subspaces of symmetric tensors $\mathcal{S}ym_n(U) \subset U^{\otimes n}$ in the tensor powers of the vector space U. So the whole symmetric coalgebra is $\mathcal{S}ym(U) = \bigoplus_{n=0}^{\infty} \mathcal{S}ym_n(U) = k \oplus U \oplus \mathcal{S}ym_2(U) \oplus \cdots$.

Following the discussion in [7, Section 1.3–1.4] or [10, Section 8.3], coalgebras $\mathcal{C}$ can be described (and in fact, defined) in terms of their vector space dual algebras $\mathcal{C}^* = \operatorname{Hom}_k(\mathcal{C}, k)$, which carry natural linearly compact (= pseudocompact) topologies. In particular, if U is a finite-dimensional k-vector space with a basis $x_1^*, \ldots, x_m^*$, then the dual algebra $\mathcal{S}ym(U)^*$ to the symmetric coalgebra $\mathcal{S}ym(U)$ is the topological algebra of formal Taylor power series $\mathcal{S}ym(U)^* = k[[x_1, \ldots, x_m]]$.

Generally speaking, for an infinite-dimensional k-vector space W, one has $\mathcal{S}ym(W) = \varinjlim_{U \subset W} \mathcal{S}ym(U)$ and $\mathcal{S}ym(W)^* = \varprojlim_{U \subset W} \mathcal{S}ym(U)^*$, where U ranges over the finite-dimensional vector subspaces of W. So, if $\{x_\alpha^* : \alpha \in A\}$ is a k-vector space basis of W, indexed by some set A, then $\mathcal{S}ym(W)^* = \varprojlim_{B \subset A} k[[x_\alpha : \alpha \in B]]$, where B ranges over the finite subsets of A. Here, given two finite subsets $B' \subset B'' \subset A$, the transition map $k[[x_\alpha : \alpha \in B'']] \to k[[x_\alpha : \alpha \in B']]$ in the projective system takes x_α to x_α for all $\alpha \in B'$ and x_β to 0 for all $\beta \in B'' \setminus B'$. Such rings $\mathcal{S}ym(W)^* = \varprojlim_{B \subset A} k[[x_\alpha : \alpha \in B]]$ are the "commutative rings of formal power series in infinitely many variables" that we are interested in.

6. Comodule and contramodule acyclicity theorems

As above, we denote by W an infinite-dimensional k-vector space with a basis $\{x_\alpha^* : \alpha \in A\}$ indexed by a set A. Given a finite set B, we let $\widehat{R}_B = k[[x_\alpha : \alpha \in B]]$ be the (topological) ring of commutative formal Taylor power series in finitely many variables indexed by B. Furthermore, we put $\widehat{R} = \varprojlim_{B \subset A} \widehat{R}_B$ (with the transition maps described in the previous section). So, denoting by $U_B \subset W$ the finite-dimensional vector subspace spanned by $\{x_\alpha^* : \alpha \in B\}$, we have $\widehat{R}_B = \mathcal{S}ym(U_B)^*$ and $\widehat{R} = \varprojlim_{B \subset A} k[[x_\alpha : \alpha \in B]] = \mathcal{S}ym(W)^*$. Let us also introduce the notation $\mathcal{C}_B = \mathcal{S}ym(U_B)$ and $\mathcal{C} = \mathcal{S}ym(W)$ for the symmetric coalgebras.

As in the proof of Theorem 3.1, we start with considering the two-term Koszul complex of free $\widehat{R}_B$-modules with one generator

$$\cdots \longrightarrow 0 \longrightarrow \widehat{R}_B \xrightarrow{x_\alpha *} \widehat{R}_B \longrightarrow 0 \longrightarrow \cdots \tag{8}$$

situated in the cohomological degrees -1 and 0 (where $\alpha \in B$). Denote by $K^B_\bullet(\widehat{R}_B)$ the tensor product, taken over the ring $\widehat{R}_B$, of the complexes (8). As the elements $\{x_\alpha : \alpha \in B\}$ form a regular sequence in the formal power series ring $\widehat{R}_B$, the complex $K^B_\bullet(\widehat{R}_B)$ is a finite resolution of the one-dimensional $\widehat{R}_B$-module $k = \widehat{R}_B / \sum_{\alpha \in B} x_\alpha \widehat{R}_B$ by finitely generated free $\widehat{R}_B$-modules.

The (augmented) Koszul complex $K^B_\bullet(\widehat{R}_B) \to k$ is a complex of linearly compact topological k-vector spaces; so it can be obtained by applying the vector space dualization functor $\mathrm{Hom}_k(-,k)$ to a certain complex of discrete vector spaces. The latter complex has the form

$$(9)\qquad 0 \longrightarrow k \longrightarrow \mathcal{S}ym(U_B) \longrightarrow \mathcal{S}ym(U_B) \otimes_k U_B$$
$$\longrightarrow \mathcal{S}ym(U_B) \otimes_k \Lambda^2(U_B) \longrightarrow \cdots \longrightarrow \mathcal{S}ym(U_B) \otimes_k \Lambda^m(U_B) \longrightarrow 0,$$

where $m = \dim U_B$ and $\Lambda^n(V)$, $n \geq 0$, denotes the exterior powers of a vector space V. The complex (9) is an injective/cofree $\mathcal{C}_B$-comodule coresolution of the trivial one-dimensional comodule k over the conilpotent coalgebra $\mathcal{C}_B = \mathcal{S}ym(U_B)$.

Passing to the direct limit of the finite complexes (9) over all the finite subsets $B \subset A$, we obtain a bounded below complex

$$(10)\qquad 0 \longrightarrow k \longrightarrow \mathcal{S}ym(W) \longrightarrow \mathcal{S}ym(W) \otimes_k W$$
$$\longrightarrow \mathcal{S}ym(W) \otimes_k \Lambda^2(W) \longrightarrow \cdots \longrightarrow \mathcal{S}ym(W) \otimes_k \Lambda^n(W) \longrightarrow \cdots$$

The complex (10) is an injective/cofree $\mathcal{C}$-comodule coresolution of the trivial one-dimensional comodule k over the conilpotent coalgebra $\mathcal{C} = \mathcal{S}ym(W)$.

One can easily check that the coresolutions (9) and (10) are well-defined and functorial for any k-vector spaces U (in place of U_B) and W, and do not depend on the choice of any bases in the vector spaces. In fact, the differential $\mathcal{S}ym(W) \otimes_k \Lambda^n(W) \to \mathcal{S}ym(W) \otimes_k \Lambda^{n+1}(W)$ can be constructed as the composition $\mathcal{S}ym(W) \otimes_k \Lambda^n(W) \to \mathcal{S}ym(W) \otimes_k W \otimes_k \Lambda^n(W) \to \mathcal{S}ym(W) \otimes_k \Lambda^{n+1}(W)$ of the map induced by the comultiplication map $\mathcal{S}ym(W) \to \mathcal{S}ym(W) \otimes_k W$ and the map induced by the multiplication map $W \otimes_k \Lambda^n(W) \to \Lambda^{n+1}(W)$.

Applying the vector space dualization functor $\mathrm{Hom}_k(-,k)$ to the complex (10), we obtain a bounded above complex

$$(11)\qquad 0 \longleftarrow k \longleftarrow \mathrm{Hom}_k(\mathcal{C},k) \longleftarrow \mathrm{Hom}_k(\mathcal{C},W^*)$$
$$\longleftarrow \mathrm{Hom}_k(\mathcal{C},\Lambda^2(W)^*) \longleftarrow \cdots \longleftarrow \mathrm{Hom}_k(\mathcal{C},\Lambda^n(W)^*) \longleftarrow \cdots$$

The complex (11) is a projective/free $\mathcal{C}$-contramodule resolution of the trivial one-dimensional $\mathcal{C}$-contramodule k.

Applying the functor $\Phi_{\mathcal{C}} = \mathcal{C} \odot_{\mathcal{C}} -$ to the truncated $\mathcal{C}$-contramodule resolution (11), we obtain a bounded above complex of injective/cofree $\mathcal{C}$-comodules

$$
\begin{aligned}
(12) \qquad & 0 \longleftarrow \mathcal{C} \longleftarrow \mathcal{C} \otimes_k W^* \longleftarrow \mathcal{C} \otimes_k \Lambda^2(W)^* \\
& \longleftarrow \cdots \longleftarrow \mathcal{C} \otimes_k \Lambda^n(W)^* \longleftarrow \cdots
\end{aligned}
$$

Applying the functor $\Psi_{\mathcal{C}} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}, -)$ to the truncated $\mathcal{C}$-comodule coresolution (10), we obtain a bounded below complex of projective/free $\mathcal{C}$-contramodules

$$
\begin{aligned}
(13) \qquad & 0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}, W) \\
& \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \Lambda^2(W)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(\mathcal{C}, \Lambda^n(W)) \longrightarrow \cdots
\end{aligned}
$$

THEOREM 6.1. *For any infinite-dimensional k-vector space W, the complex of cofree comodules* (12) *is acyclic* (*i.e., its cohomology spaces vanish in all the degrees*).

PROOF. This was stated in [6, Section 0.2.7] (as a part of introductory/preliminary material for the book). The proof is not difficult.

The complex (12) is the direct limit of its subcomplexes

$$
\begin{aligned}
(14) \qquad & 0 \longleftarrow \mathcal{C}_B \longleftarrow \mathcal{C}_B \otimes_k W^* \longleftarrow \mathcal{C}_B \otimes_k \Lambda^2(W)^* \\
& \longleftarrow \cdots \longleftarrow \mathcal{C}_B \otimes_k \Lambda^n(W)^* \longleftarrow \cdots
\end{aligned}
$$

taken over the directed poset of all finite subsets $B \subset A$. The complex (14), which is a complex of comodules over the subcoalgebra $\mathcal{C}_B = \mathcal{S}ym(U_B)$ of the coalgebra $\mathcal{C} = \mathcal{S}ym(W)$, can be obtained by applying the cotensor product functor $\mathcal{C}_B \,\square_{\mathcal{C}} -$ to the complex (12) (see [7, Sections 2.5–2.6] or [6, Section 0.2.1 or 1.2.1]).

The complex (14) is *not* acyclic, but its cohomology spaces gradually vanish as the size of the finite subset $B \subset A$ grows. Indeed, applying the vector space dualization functor $\mathrm{Hom}_k(-, k)$ to the finite complex (9), we obtain a finite Koszul complex that was denoted above by $K^B_\bullet(\widehat{R}_B)$. It has the form

$$
\begin{aligned}
(15) \qquad & 0 \longleftarrow k \longleftarrow \mathrm{Hom}_k(\mathcal{C}_B, k) \longleftarrow \mathrm{Hom}_k(\mathcal{C}, U_B^*) \\
& \longleftarrow \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^2(U_B)^*) \longleftarrow \cdots \longleftarrow \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^m(U_B)^*) \longleftarrow 0
\end{aligned}
$$

and can be viewed as a projective/free $\mathcal{C}_B$-contramodule resolution of the trivial one-dimensional $\mathcal{C}_B$-contramodule k.

Applying the functor $\Phi_{\mathcal{C}_B} = \mathcal{C}_B \odot_{\mathcal{C}_B} -$ to the truncated $\mathcal{C}_B$-contramodule resolution (15), we obtain a finite complex of injective/cofree $\mathcal{C}_B$-comodules

$$0 \longleftarrow \mathcal{C}_B \longleftarrow \mathcal{C}_B \otimes_k U_B^* \longleftarrow \mathcal{C}_B \otimes_k \Lambda^2(U_B)^* \longleftarrow \cdots \longleftarrow \mathcal{C}_B \otimes_k \Lambda^m(U_B)^* \longleftarrow 0. \tag{16}$$

The only cohomology space of the complex (16) is the one-dimensional k-vector space $\Lambda^m(U_B)^*$ situated in the cohomological degree $-m$, i.e., at the rightmost term.

Consider the direct sum decomposition $W = U_B \oplus V_B$, where $V_B \subset W$ is the subspace with the basis $\{x_\alpha^* : \alpha \in A \setminus B\}$. Consider the graded dual vector space to the exterior algebra $\bigoplus_{n=0}^{\infty} \Lambda^n(V_B)$, and view it as a complex

$$0 \longleftarrow k \xleftarrow{0} V_B^* \xleftarrow{0} \Lambda^2(V_B)^* \xleftarrow{0} \cdots \xleftarrow{0} \Lambda^n(V_B)^* \xleftarrow{0} \cdots \tag{17}$$

with zero differential. Then the complex (14) is the tensor product, taken over the field k, of the complexes (16) and (17). Accordingly, the cohomology spaces of the complex (14) are concentrated in the cohomological degrees $\le -m$, where m is the cardinality of the set B.

As the size of the subset $B \subset A$ grows, the cohomology of the complex (14) move away and disappear at the cohomological degree $-\infty$. So the direct limit (12) of the complexes (14) is acyclic. $\square$

THEOREM 6.2. *For any infinite-dimensional k-vector space W, the complex of free contramodules* (13) *is acyclic* (*i.e., its cohomology spaces vanish in all the degrees*).

PROOF. This was also stated in [6, Section 0.2.7]. The proof is only slightly more complicated than the proof of the previous theorem, in that one needs to deal with inverse limits. However, we have done all the preparatory work already.

The complex (13) is the inverse limit of its quotient complexes

$$0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, W) \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^2(W)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^n(W)) \longrightarrow \cdots \tag{18}$$

taken over the directed poset of all finite subsets $B \subset A$. The complex (18), which is a complex of contramodules over the subcoalgebra $\mathcal{C}_B \subset \mathcal{C}$, can be obtained by applying the Cohom functor $\mathrm{Cohom}_{\mathcal{C}}(\mathcal{C}_B, -)$ to the complex (13) (see [7, Sections 2.5–2.6] or [6, Section 0.2.4 or 3.2.1]).

Similarly to the previous proof, the complex (18) is *not* acyclic, but its cohomology spaces gradually vanish as the size of the finite subset $B \subset A$ grows. For the sake of completeness of the exposition, let us start with applying the functor $\Psi_{\mathcal{C}_B} = \mathrm{Hom}_{\mathcal{C}_B}(\mathcal{C}_B, -)$ to the truncated $\mathcal{C}_B$-comodule coresolution (9). We obtain a finite complex of projective/free $\mathcal{C}_B$-contramodules

$$(19) \qquad 0 \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, k) \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, U_B) \\ \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^2(U_B)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^m(U_B)) \longrightarrow 0.$$

The only cohomology space of the complex (19) is the one-dimensional k-vector space $\Lambda^m(U_B)$ situated in the cohomological degree m, i. e, at the rightmost term. In fact, the complex of contramodules (19) can be obtained by applying the vector space dualization functor $\mathrm{Hom}_k(-, k)$ to the complex of comodules (16).

Consider the exterior algebra $\bigoplus_{n=0}^{\infty} \Lambda^n(V_B)$, where as in the previous proof $W = U_B \oplus V_B$, and view it as a complex

$$(20) \qquad 0 \longrightarrow k \xrightarrow{0} V_B \xrightarrow{0} \Lambda^2(V_B) \xrightarrow{0} \cdots \xrightarrow{0} \Lambda^n(V_B) \xrightarrow{0} \cdots$$

with zero differential. Then the complex (18) is the complex of k-vector space morphisms, $\mathrm{Hom}_k(-,-)$, from the complex (16) into the complex (20). Accordingly, the cohomology spaces of the complex (18) are concentrated in the cohomological degrees $\geq m$.

The rest of the argument proceeds along the lines of the proof of Theorem 3.1, based on Lemmas 3.2–3.4. As mentioned above, the complex (13) is the inverse limit of the complexes (18) taken over the directed poset Ξ of all finite subsets $B \subset A$ with respect to inclusion. At every cohomological degree $n \geq 0$, Lemma 3.2 (for $R = k$, $F_B = \mathcal{C}_B$, and $P = \Lambda^n(W)$) tells that $\varprojlim^i_{B \subset A} \mathrm{Hom}_k(\mathcal{C}_B, \Lambda^n(W)) = 0$ for all $i \geq 1$.

Denote by $C^{\bullet}_B$ the complex (18). By Lemma 3.3, we have $\varprojlim^i_{B \subset A} H^n(C^{\bullet}_B) = 0$ for all $i \geq 0$ and $n \geq 0$. Now in the context of Lemma 3.4 we have ${}'E_2^{pq} = 0$ for all p, $q \geq 0$, and ${}''E_1^{pq} = 0$ for all $q \geq 1$. Therefore, $E^n = 0$ and $H^n(\varprojlim_{B \subset A} C^{\bullet}_B) = {}''E_2^{n,0} = 0$ for all $n \geq 0$. $\square$

7. Two contramodule constructions of acyclic complexes of projectives

We have essentially already constructed the promised bounded above, noncontractible, acyclic complex of injective comodules and bounded below, noncontractible, acyclic complex of projective contramodules over the cocommutative coalgebra $\mathcal{C} = \mathcal{S}ym(W)$. Let us state this as a corollary.

COROLLARY 7.1. *Let W be an infinite-dimensional vector space over a field k and $\mathcal{C} = \mathcal{S}ym(W)$ be the symmetric coalgebra. Then*

(a) *the complex* (12) *is a bounded above, noncontractible, acyclic complex of injective comodules over $\mathcal{C}$;*

(b) *the complex* (13) *is a bounded below, noncontractible, acyclic complex of projective contramodules over* $\mathcal{C}$.

Proof. Part (a): the complex (12) is acyclic by Theorem 6.1. It remains to explain why the complex of $\mathcal{C}$-comodules (12) is not contractible.

The truncated resolution (11),

$$0 \longleftarrow \mathrm{Hom}_k(\mathcal{C}, k) \longleftarrow \mathrm{Hom}_k(\mathcal{C}, W^*)$$
$$\longleftarrow \mathrm{Hom}_k(\mathcal{C}, \Lambda^2(W)^*) \longleftarrow \cdots \longleftarrow \mathrm{Hom}_k(\mathcal{C}, \Lambda^n(W)^*) \longleftarrow \cdots$$

is a noncontractible (since nonacyclic) complex in the abelian category $\mathcal{C}$–Contra with the terms belonging to the full subcategory of projective objects $\mathcal{C}$–$\mathsf{Contra}_{\mathsf{proj}}$, so it is a noncontractible complex in $\mathcal{C}$–$\mathsf{Contra}_{\mathsf{proj}}$. Applying the equivalence of additive categories $\Phi_{\mathcal{C}}\colon \mathcal{C}$–$\mathsf{Contra}_{\mathsf{proj}} \simeq \mathcal{C}$–$\mathsf{Comod}_{\mathsf{inj}}$ (7), we obtain a noncontractible complex (12) in $\mathcal{C}$–$\mathsf{Comod}_{\mathsf{inj}}$, which is consequently also noncontractible in $\mathcal{C}$–Comod.

Part (b): the complex (13) is acyclic by Theorem 6.2. It remains to explain why the complex of $\mathcal{C}$-contramodules (13) is not contractible.

The truncated coresolution (10),

$$0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{C} \otimes_k W \longrightarrow \mathcal{C} \otimes_k \Lambda^2(W) \longrightarrow \cdots \longrightarrow \mathcal{C} \otimes_k \Lambda^n(W) \longrightarrow \cdots$$

is a noncontractible (since nonacyclic) complex in the abelian category $\mathcal{C}$–Comod with the terms belonging to the full subcategory of injective objects $\mathcal{C}$–$\mathsf{Comod}_{\mathsf{inj}}$, so it is a noncontractible complex in $\mathcal{C}$–$\mathsf{Comod}_{\mathsf{inj}}$. Applying the equivalence of additive categories $\Psi_{\mathcal{C}}\colon \mathcal{C}$–$\mathsf{Comod}_{\mathsf{inj}} \simeq \mathcal{C}$–$\mathsf{Contra}_{\mathsf{proj}}$ (7), we obtain a noncontractible complex (13) in $\mathcal{C}$–$\mathsf{Contra}_{\mathsf{proj}}$, which is consequently also noncontractible in $\mathcal{C}$–Contra. $\square$

Now let us present the two contramodule constructions of bounded below, noncontractible, acyclic complexes of projective modules. Recall the notation $\mathrm{Hom}_{\mathcal{C}}(-,-)$ for the Hom spaces in the category $\mathcal{C}$–Comod. The notation $\mathrm{Hom}^{\mathcal{C}}(-,-)$ stands for the Hom spaces in the category $\mathcal{C}$–Contra.

Corollary 7.2. *Let W be an infinite-dimensional vector space over a field k and $\mathcal{C} = \mathcal{S}ym(W)$ be the symmetric coalgebra. Let*

$$0 \longrightarrow k \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \cdots$$

be a notation for the injective coresolution (10) *of the trivial one-dimensional $\mathcal{C}$-comodule k. Denote by $\mathcal{J}$ the cofree $\mathcal{C}$-comodule $\mathcal{C} \otimes_k W$ cospanned by W. Let*

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J}^0) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J}^1) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J}^2) \longrightarrow \cdots \tag{21}$$

be the complex obtained by applying the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, -)$ *to the truncated coresolution* (10). *Then* (21) *is a bounded below, noncontractible,* *acyclic complex of finitely generated projective left modules over the ring* $S = \mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J})^{\mathrm{op}}$.

PROOF. For any $\mathcal{C}$-comodule $\mathcal{M}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C} \otimes_k W,\ \mathcal{M}) = \mathrm{Hom}_k(W, \mathrm{Hom}_k(\mathcal{C}, \mathcal{M})) = \mathrm{Hom}_k(W, \Psi_{\mathcal{C}}(\mathcal{M})).$$

Thus the complex (21) can be obtained by applying the vector space Hom functor $\mathrm{Hom}_k(W, -)$ to the complex (13), and it follows from Theorem 6.2 that the complex (21) is acyclic.

Furthermore, by construction, the $\mathcal{C}$-comodule $\mathcal{J}^0 = \mathcal{C}$ is a direct summand of $\mathcal{J}$, while the $\mathcal{C}$-comodules $\mathcal{J}^n = \mathcal{C} \otimes_k \Lambda^n(W)$ are isomorphic to $\mathcal{J}$ for $n \geq 1$. Hence the left S-module $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J}^0)$ is a direct summand of $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J}) = S$, and the left S-modules $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, \mathcal{J}^n)$ are isomorphic to S for $n \geq 1$. So (21) is even a complex of cyclic projective left S-modules.

The assertion that the complex of S-modules (21) is not contractible is provable similarly to the argument in the proof of Corollary 2.3. By Lemma 2.2(a) (for $\mathsf{A} = \mathcal{C}\text{–}\mathsf{Comod}$ and $M = \mathcal{J}$), the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, -)$ is an equivalence of categories $\mathsf{add}(\mathcal{J}) \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$. The truncated coresolution (10),

$$0 \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \mathcal{J}^2 \longrightarrow \cdots$$

is a noncontractible (since nonacyclic) complex in $\mathcal{C}\text{–}\mathsf{Comod}$ with the terms belonging to $\mathsf{add}(\mathcal{J})$, so it is a noncontractible complex in $\mathsf{add}(\mathcal{J})$. Applying the equivalence of additive categories $\mathsf{add}(\mathcal{J}) \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$, we obtain a noncontractible complex (21) in $S\text{–}\mathsf{mod}_{\mathsf{proj}}$, which is consequently also noncontractible in $S\text{–}\mathsf{Mod}$. It is important for this argument that the functor $\mathrm{Hom}_{\mathcal{C}}(\mathcal{J}, -)\colon \mathsf{add}(\mathcal{J}) \to S\text{–}\mathsf{Mod}$ is fully faithful.

Alternatively, put $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{C}, W) \in \mathcal{C}\text{–}\mathsf{Contra}_{\mathsf{proj}}$. Then the co-contra correspondence (7) restricts to an equivalence of additive categories $\mathsf{add}(\mathcal{J}) \simeq \mathsf{add}(\mathfrak{P})$ taking $\mathcal{J}$ to $\mathfrak{P}$. Hence the ring S can be alternatively described as $S = \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{P})^{\mathrm{op}}$. The complex of left S-modules (21) can be constructed by applying the functor $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, -)$ to the complex of $\mathcal{C}$-contramodules (13), whose terms belong to $\mathsf{add}(\mathfrak{P})$.

Then the noncontractibility argument can be based on Corollary 7.1(b) and the fact that the functor $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, -)$ is an equivalence of categories $\mathsf{add}(\mathfrak{P}) \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$ (by Lemma 2.2(a) for $\mathsf{A} = \mathcal{C}\text{–}\mathsf{Contra}$ and $M = \mathfrak{P}$). Once again, it is important for this argument that the functor $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, -)\colon \mathsf{add}(\mathfrak{P}) \to S\text{–}\mathsf{Mod}$ is fully faithful. In fact, the whole functor $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, -)\colon \mathcal{C}\text{–}\mathsf{Contra} \to S\text{–}\mathsf{Mod}$ is fully faithful (on the whole abelian category $\mathcal{C}\text{–}\mathsf{Contra}$) by [11, Theorem 6.10]. The latter conclusion is based on the observations that $\mathfrak{P}$ is the coproduct of $\dim W$ copies of the projective generator $\mathcal{C}^* = \mathrm{Hom}_k(\mathcal{C}, k)$ of the abelian category $\mathcal{C}\text{–}\mathsf{Contra}$, and $\mathcal{C}^*$ is abstractly κ-small in $\mathcal{C}\text{–}\mathsf{Contra}$ for $\mathcal{C} = \mathcal{S}ym(W)$ if κ is the successor cardinality of $\dim W$. $\square$

Corollary 7.3. *Let W be an infinite-dimensional vector space over a field k and $\mathcal{C} = \mathcal{S}ym(W)$ be the symmetric coalgebra. Let*

$$0 \longleftarrow k \longleftarrow \mathfrak{P}_0 \longleftarrow \mathfrak{P}_1 \longleftarrow \mathfrak{P}_2 \longleftarrow \cdots$$

be a notation for the projective resolution (11) *of the trivial one-dimensional $\mathcal{C}$-contramodule k. Denote by $\mathfrak{P}$ the free $\mathcal{C}$-contramodule* $\mathrm{Hom}_k(\mathcal{C}, W^*)$ *spanned by the vector space* $W^* = \mathrm{Hom}_k(W, k)$. *Let*

$$0 \longrightarrow \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_0, \mathfrak{P}) \longrightarrow \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_1, \mathfrak{P}) \longrightarrow \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_2, \mathfrak{P}) \longrightarrow \cdots \tag{22}$$

be the complex obtained by applying the contravariant functor $\mathrm{Hom}^{\mathcal{C}}(-, \mathfrak{P})$ *to the truncated resolution* (11). *Then* (22) *is a bounded below, noncontractible, acyclic complex of finitely generated projective left modules over the ring* $S = \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{P})$.

Proof. For any right $\mathcal{C}$-comodule $\mathcal{N}$, any left $\mathcal{C}$-contramodule $\mathfrak{Q}$, and any k-vector space V, there is a natural isomorphism of k-vector spaces

$$\mathrm{Hom}^{\mathcal{C}}(\mathfrak{Q}, \mathrm{Hom}_k(\mathcal{N}, V)) \simeq \mathrm{Hom}_k(\mathcal{N} \odot_{\mathcal{C}} \mathfrak{Q},\ V)$$

[7, Section 3.1], [10, Section 8.6], or [6, Sections 0.2.6 and 5.1.1]. In particular, we have natural isomorphisms

$$\mathrm{Hom}^{\mathcal{C}}(\mathfrak{Q}, \mathfrak{P}) = \mathrm{Hom}_k(\mathcal{C} \odot_{\mathcal{C}} \mathfrak{Q},\ W^*) = \mathrm{Hom}_k(\Phi_{\mathcal{C}}(\mathfrak{Q}), W^*).$$

Thus the complex (22) can be obtained by applying the contravariant vector space Hom functor $\mathrm{Hom}_k(-, W^*)$ to the complex (12), and it follows from Theorem 6.1 that the complex (22) is acyclic.

Furthermore, by construction, the $\mathcal{C}$-comodule $\mathfrak{P}_0 = \mathcal{C}^*$ is is a direct summand of $\mathfrak{P}$, while the $\mathcal{C}$-contramodules $\mathfrak{P}_n = \mathrm{Hom}_k(\mathcal{C}, \Lambda^n(W)^*)$ are isomorphic to $\mathfrak{P}$ for $n \geq 1$. Hence the left S-module $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_0, \mathfrak{P})$ is a direct summand of $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}, \mathfrak{P}) = S$, and the left S-modules $\mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_n, \mathfrak{P})$ are isomorphic to S for $n \geq 1$. So (22) is even a complex of cyclic projective left S-modules.

The assertion that the complex of S-modules (22) is not contractible is provable similarly to the argument in the proof of Corollary 4.1. By Lemma 2.2(b) (for $\mathsf{A} = \mathcal{C}\text{–}\mathsf{Contra}$ and $M = \mathfrak{P}$), the functor $\mathrm{Hom}^{\mathcal{C}}(-, \mathfrak{P})$ is an anti-equivalence of categories $\mathsf{add}(\mathfrak{P})^{\mathsf{op}} \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$. The truncated resolution (11),

$$0 \longleftarrow \mathfrak{P}_0 \longleftarrow \mathfrak{P}_1 \longleftarrow \mathfrak{P}_2 \longleftarrow \cdots$$

is a noncontractible (since nonacyclic) complex in $\mathcal{C}\text{–}\mathsf{Contra}$ with the terms belonging to $\mathsf{add}(\mathfrak{P})$, so it is a noncontractible complex in $\mathsf{add}(\mathfrak{P})$. Applying the anti-equivalence of additive categories $\mathsf{add}(\mathfrak{P})^{\mathsf{op}} \simeq S\text{–}\mathsf{mod}_{\mathsf{proj}}$, we obtain a noncontractible complex (22), which is consequently also noncontractible in S–Mod. It is important for this argument that the contravariant functor $\mathrm{Hom}^{\mathcal{C}}(-,\mathfrak{P})\colon \mathsf{add}(\mathfrak{P})^{\mathsf{op}} \to S$–$\mathsf{Mod}$ is fully faithful. $\square$

8. Summary of the examples obtained

Now we can summarize our constructions as follows.

Conclusion 8.1. *There exists an associative ring S for which*

(a) *there is a bounded above acyclic complex of injective right S-modules that is not contractible*;

(b) *there is a bounded below acyclic complex of flat left S-modules that is not pure acyclic*;

(c) *there is a bounded below acyclic complex of* (*finitely generated*) *projective left S-modules that is not contractible.*

Proof. Proposition 1.2 tells that any ring S satisfying (c) also satisfies (a) and (b). Various examples of associative rings S satisfying (c) are provided by Corollaries 2.3, 4.1, 7.2, and 7.3. $\square$

What can one say about the rings S appearing in Corollaries 2.3, 4.1, 7.2, and 7.3? First of all, *none* of them is commutative (while we have cocommutative coalgebra examples in Corollary 7.1).

Let us denote the respective versions of the ring S by $S_{2.3}$, $S_{4.1}$, $S_{7.2}$, and $S_{7.3}$. While the ring $S_{2.3}$ (from Corollary 2.3) appears to be complicated and hard to visualize, the rings $S_{4.1}$, $S_{7.2}$, and $S_{7.3}$ can be described rather explicitly.

In the context of Corollary 4.1, it makes sense to choose the infinite Koszul complex $K_\bullet(R) = \varinjlim_{B\subset A} K_\bullet^B(R)$ to play the role of the projective resolution $P_\bullet$ (5) of the R-module k. In this case, one can take P to be the free R-module with A generators, $P = \bigoplus_{\alpha\in A} R$. Then the R-module $P_0 = R$ is a direct summand of P, while the R-module P_n is isomorphic to P for $n \ge 1$, so the assumption of the corollary is satisfied. The resulting ring $S_{4.1} = \mathrm{Hom}_R(P,P)$ is the ring of infinite, column-finite $A\times A$ matrices with entries from the commutative polynomial ring $R = k[x_\alpha : \alpha\in A]$ in infinitely many variables.

In the context of Corollaries 7.2 and 7.3, it makes sense to introduce the notation $\mathcal{J}_{7.2}$ for the cofree comodule $\mathcal{J} = \mathcal{C}\otimes_k W$ appearing in Corollary 7.2 and the notation $\mathfrak{P}_{7.2}$ for the free contramodule $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{C},W)$ mentioned in the discussion in its proof. Then the notation $\mathfrak{P}_{7.3}$ can be used for the bigger free contramodule $\mathfrak{P} = \mathrm{Hom}_k(\mathcal{C},W^*)$ from Corollary 7.3, and we can also denote by $\mathcal{J}_{7.3}$ the corresponding cofree comodule $\mathcal{J} = \mathcal{C}\otimes_k W^*$.

The ring $S_{7.2} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{J}_{7.2},\mathcal{J}_{7.2})^{\mathrm{op}} = \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_{7.2},\mathfrak{P}_{7.2})^{\mathrm{op}}$ is the ring of infinite, row-zero-convergent $A\times A$ matrices with entries from the topological commutative formal power series ring $\widehat{R} = \mathcal{C}^* = \varprojlim_{B\subset A} k[[x_\alpha : \alpha\in B]]$

in infinitely many variables. Such rings of row-zero-convergent matrices were discussed in the papers [11, Example 7.10] and [12, Section 5].

Let D denote the indexing set of a basis $\{y_\delta\colon \delta \in D\}$ in the k-vector space W^*. The cardinality $|D|$ of the set D is equal to $|k|^{|A|}$, where $|k|$ is the cardinality of the field k and $|A|$ is the cardinality of the set A. Then the ring $S_{7.3} = \mathrm{Hom}_{\mathcal{C}}(\mathcal{J}_{7.3}, \mathcal{J}_{7.3}) = \mathrm{Hom}^{\mathcal{C}}(\mathfrak{P}_{7.3}, \mathfrak{P}_{7.3})$ is the ring of infinite, column-zero-convergent $D \times D$ matrices with entries from the topological commutative formal power series ring $\widehat{R}$ in infinitely many variables indexed by A.

Endowed with its natural topology, the ring $S_{7.2}$ becomes a complete, separated right linear topological ring (i.e., a topological ring with a base of neighborhoods of zero formed by open right ideals). Such topological rings were discussed in the papers [8,11,12]. Moreover, the ring $S_{7.2}$ is *topologically left perfect* in the sense of [12, Section 14] (as one can see from the discussion in [11, Section 7.3] or [12, Section 5] together with [8, Example 12.3]). Similarly, the ring $S_{7.3}$ is a complete, separated, left linear topological ring which is, moreover, topologically right perfect. In other words, the ring $S_{7.3}$ is the endomorphism ring of a module with perfect decomposition [12, Section 10], while $S_{7.2}$ is the opposite ring to the endomorphism ring of such a module.

Viewed as abstract rings, both the rings $S_{7.2}$ and $S_{7.3}$ are *semiregular* in the sense of [1, Section 4]; see the discussion in [12, Remark 10.5]. (The semiregularity is a left-right symmetric property.) The ring $S_{4.1}$, on the other hand, has vanishing Jacobson radical.

The ring $S_{7.3}$ differs from the opposite ring to $S_{7.2}$ for the only reason that the cardinality of the set D is larger than that of the set A. One can employ the bigger cofree comodule $\mathcal{J}_{7.3} = \mathcal{C} \otimes_k W^*$ in lieu of the cofree comodule $\mathcal{J}_{7.2} = \mathcal{C} \otimes_k W$ in the construction of Corollary 7.2 (while leaving the rest of the construction unchanged). This will produce a pair of opposite rings S and S^{op}, both of them semiregular, both of them satisfying all the claims of Conclusion 8.1.

REMARK 8.2. *None* of the rings $S_{4.1}$, $S_{7.2}$, and $S_{7.3}$ is Noetherian on either side. Indeed, consider the ring $T = \mathrm{Hom}_k(k^{(E)}, k^{(E)})$ of infinite, column-finite $E \times E$ matrices with entries from the field k (where E is a set and $k^{(E)}$ is the k-vector space with a basis indexed by E). Then T is a quotient ring of $S_{4.1}$ (for $E = A$) and of $S_{7.3}$ (for $E = D$), while the opposite ring to T is a quotient ring of $S_{7.2}$ (for $E = A$). The ring T is von Neumann regular, so it cannot be left or right Noetherian (as any one-sided Noetherian von Neumann regular ring is semisimple Artinian).

REMARK 8.3. A simpler construction of rings and complexes of modules satisfying Conclusion 8.1 than the one discussed above exists (see Example 8.4 below). But the following naïve attempt at constructing an example for Conclusion 8.1(b) fails.

The commutative ring $R = k[x_\alpha : \alpha \in A]$ of polynomials in infinitely many variables over a field k is not Noetherian, but it is coherent. Hence the class of flat R-modules is closed under infinite products in $R\text{–}\mathsf{Mod}$, and it follows that the R-module $\mathrm{Hom}_R(Q, P)$ is flat for any projective R-module Q and flat R-module P. Thus the complex (6) from Corollary 4.1 is a bounded below, acyclic complex of flat R-modules. One does not even need the R-modules P_n to be direct summands of P for this claim to hold; it suffices to take $P = R$.

However, this is *not* an example for Conclusion 8.1(b), because the complex of R-modules (6) is actually pure acyclic (for any flat R-module P). Indeed, it suffices to show that, for any finitely presented R-module M, applying the functor $M \otimes_R -$ preserves acyclicity of the complex (6). Denote the complex (6) by $F^\bullet$.

Any finitely presented module over the ring of polynomials R in infinitely many variables has a finite projective resolution $G_\bullet$ by finitely generated projective R-modules. Since $F^\bullet$ is a complex of flat R-modules and $G_\bullet$ is a finite resolution, the complexes $M \otimes_R F^\bullet$ and $G_\bullet \otimes_R F^\bullet$ are quasi-isomorphic. Finally, viewed as an object of the homotopy category of complexes of R-modules $\mathsf{K}(R\text{–}\mathsf{Mod})$, the complex $G_\bullet \otimes_R F^\bullet$ belongs to the thick subcategory spanned by the complex $F^\bullet$ (since the complex $G_\bullet$ belongs to the thick subcategory spanned by the one-term complex of R-modules R). As the complex $F^\bullet$ is acyclic, so is the complex $G_\bullet \otimes_R F^\bullet$.

EXAMPLE 8.4. The following example has a different nature than all the previous examples in this paper. It was communicated to the author by A. Canonaco and is reproduced here with his kind permission.

Suppose that we have a bounded below complex of *free modules with one generator* over a ring S. Obviously, such a complex of (left) modules has the form

$$0 \longrightarrow S \xrightarrow{*z_0} S \xrightarrow{*z_1} S \xrightarrow{*z_2} S \longrightarrow \cdots, \tag{23}$$

where $z_0, z_1, z_2, \ldots$ is some sequence of elements in S. For the sequence of maps (23) to be a complex, the equation $z_n z_{n+1} = 0$ has to be satisfied in S for all integers $n \geq 0$.

Now let k be a field and S_{uni} be the k-algebra generated by a sequence of elements $x_0, x_1, x_2, \ldots$ with the imposed relations $x_n x_{n+1} = 0$ for all $n \geq 0$, and no other relations. Then one can easily see that any element $s \in S_{\mathrm{uni}}$ satisfying the equation $sx_0 = 0$ vanishes, while any element $s \in S_{\mathrm{uni}}$ satisfying $sx_{n+1} = 0$ with $n \geq 0$ has the form $s = tx_n$ for some $t \in S_{\mathrm{uni}}$. It suffices to represent s as a k-linear combination of monomials in the variables x_n, $n \geq 0$, etc. In other words, the bounded below complex of free S_{uni}-modules

$$0 \longrightarrow S_{\mathrm{uni}} \xrightarrow{*x_0} S_{\mathrm{uni}} \xrightarrow{*x_1} S_{\mathrm{uni}} \xrightarrow{*x_2} S_{\mathrm{uni}} \longrightarrow \cdots \tag{24}$$

is acyclic. On the other hand, if k is endowed with the right S_{uni}-module structure in which all the elements x_n act by zero in k, then applying the functor $k \otimes_{S_{\mathrm{uni}}} -$ to the complex (24) produces a nonacyclic complex with zero differential. So the complex (24) is *not* contractible.

The bounded below complex of free S_{uni}-modules with one generator (24) is *universal* in the following sense. Let S be an associative k-algebra and $C^\bullet$ be a complex of free S-modules with one generator such that $C^i = 0$ for $i < 0$. Then there exists a k-algebra homomorphism $f\colon S_{\mathrm{uni}} \to S$ such that the complex $C^\bullet$ is obtained by applying the functor of extension of scalars $S \otimes_{S_{\mathrm{uni}}} -$ to the complex (24). Indeed, the complex $C^\bullet$ has the form (23) for some elements $z_n \in S$, $n \geq 0$ satisfying the equations $z_n z_{n+1} = 0$, and it remains to let $f\colon S_{\mathrm{uni}} \to S$ be the homomorphism taking x_n to z_n for every $n \geq 0$.

While the example in Example 8.4 is certainly simpler (to construct and prove its properties) than the examples in Corollaries 2.3, 4.1, 7.2, and 7.3, *no* example of a bounded below, noncontractible, acyclic complex of projective modules (or of a bounded above, noncontractible, acyclic complex of injective modules) can be *too* simple. The results of [16, Appendix A] demonstrate this.

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