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COLEMAN AUTOMORPHISMS OF FINITE GROUPS WITH SEMIDIHEDRAL SYLOW 2-SUBGROUPS

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Abstract. We study some families of finite groups having inner class-preserving automorphisms. In particular, let G be a finite group and S be a semidihedral Sylow 2-subgroup. Then, in both cases when either Sym(4) is not a homomorphic image of G and $Z(S) < Z(G)$ or G is nilpotent-by-nilpotent, we have that all the Coleman automorphisms of G are inner. As a consequence, these groups satisfy the *normalizer problem*.

1. Introduction

The study of the automorphisms of a finite group that preserve the conjugate classes has long been a subject of great interest. Since the first researches, one of the most studied problems is to find which are the groups whose automorphisms of this type are all inner (see, e.g., [4,6,9,16,19,21]). In this paper we deal with a particular type of class-preserving automorphisms, called *Coleman* automorphisms, using the definition given by Marciniak and Roggenkamp in [18]. Donald B. Coleman, in [5, Theorem 1], showed that the main property of these class-preserving automorphisms, i.e. that of becoming inner if restricted to any Sylow p-subgroup, plays an important role in the study of the normalizer of a group G in the unit group $\mathcal{U}(\mathbb{Z}(G))$ of its integral group ring $\mathbb{Z}G$. Indeed, using Coleman's result and other two results due to Krempa [16, Theorem 3.2] and Jackowski and Marciniak [16, Proposition 2.3], it is possible to prove that if G is a group for which each Coleman automorphism is inner, then G satisfies the *normalizer problem*, i.e.

$$N_{\mathcal{U}(\mathbb{Z}G)}(G) = GZ(\mathcal{U}(\mathbb{Z}G)).$$

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As a reference for knowing this problem and other problems and results related to the study of integral group rings see, e.g., [2]. If $\mathrm{Aut}_{\mathrm{Col}}(G)$ is the group of all Coleman automorphisms of G, here we study the problem to find finite groups G for which $\mathrm{Aut}_{\mathrm{Col}}(G) = \mathrm{Inn}(G)$, which we call $\mathrm{Out}_{\mathrm{Col}}$-problem. In the literature, there are several affirmative answers and counterexamples to this problem (see, e.g., [11,13–15,17,18,22,23]) and also authors which study the characterization of the Coleman automorphism group for some classes of finite groups (see, e.g., [10,12]). In particular, we consider families of finite groups with semidihedral Sylow 2-subgroups. We seem fair to point out that even in [23] the authors dealt with the same problem for groups containing semidihedral Sylow 2-subgroups, considering different hypothesis and thus obtaining different results. In Section 2, starting from a result due to Juriaans, de Miranda and Robério [17, Theorem 2.4], presenting a structure for a counterexample to the $\mathrm{Out}_{\mathrm{Col}}$-problem, we give, with a similar approach, some preliminaries results which we will then use for proving our main results. Finally, in Section 3, we prove our main results in which we give two families of finite groups, with semidihedral Sylow 2-subgroups, satisfying the $\mathrm{Out}_{\mathrm{Col}}$-problem.

2. Preliminary results

Let G be a finite group. Let us denote by $\mathrm{Aut}(G)$ the group of all automorphisms of G and by $\mathrm{Inn}(G)$ the subgroup of $\mathrm{Aut}(G)$ containing all inner automorphisms of G. Let us define the subgroup $\mathrm{Aut}_c(G)$ of $\mathrm{Aut}(G)$ as the set of all class-preserving automorphisms of G, i.e. the automorphisms preserving the conjugacy classes in G. The following definition was introduced by Marciniak and Roggenkamp in [18].

Definition 2.1. A class-preserving automorphism $\varphi \in \mathrm{Aut}_c(G)$ of G is called *Coleman automorphism*, for short C-automorphism, if

(i) $\varphi^2 \in \mathrm{Inn}(G)$;

(ii) for every Sylow p-subgroup of G, we have $\varphi|_S = \mathrm{conj}_g|_S$, for some $g \in G$.

We shall denote by $\mathrm{Aut}_{\mathrm{Col}}(G)$ the group of the C-automorphisms of G.

Remark 1. By definition, $\mathrm{Aut}_{\mathrm{Col}}(G) \geq \mathrm{Inn}(G)$ and C-automorphisms of odd order are inner and so, taking a suitable odd power, we can assume that such automorphisms have order a power of 2. Moreover, since $|\mathrm{Aut}_{\mathrm{Col}}(G)|$ is divisible only by the prime numbers dividing $|G|$, we can suppose that G has order even. Indeed, suppose that $\varphi \in \mathrm{Aut}_{\mathrm{Col}}(G)$ has order a prime p such that $p \nmid |G|$ and let $H = \{g \in G \mid \varphi(g) = g\}$. Then p does not divide the cardinality of any conjugate class C in G, hence $H \cap C \neq \emptyset$, and so $H = G$, i.e. $\varphi = \mathrm{Id}$. For another definition of Coleman automorphisms for which it is possible consider such automorphisms with a p-power order, also with $p \neq 2$, see for example [14,15].

Now we give some results which we will exploit, in Section 3, for presenting two families of groups G, with semidihedral Sylow 2-subgroups, for which each C-automorphism is inner, or, in other words, $\mathrm{Aut}_{\mathrm{Col}}(G) = \mathrm{Inn}(G)$, which is equivalent to say

$$\mathrm{Out}_{\mathrm{Col}}(G) = \mathrm{Aut}_{\mathrm{Col}}(G)/\,\mathrm{Inn}(G) = \{\mathrm{Id}\}.$$

For the sake of simplicity, we will say that a group G *satisfies the* $\mathrm{Out}_{\mathrm{Col}}$-*problem* if $\mathrm{Out}_{\mathrm{Col}}(G) = \{\mathrm{Id}\}$.

We start from the following result giving a structure of a certain class of groups which is a minimal counterexample to the $\mathrm{Out}_{\mathrm{Col}}$-problem. As usual, we denote by $O_p(G)$ the maximal normal p-subgroup of G and by $O_{p'}(G)$ the maximal normal subgroup of G whose order is coprime to p.

THEOREM 2.2 [17, Theorem 2.4]. *Let G be a finite solvable group which is a minimal counterexample to the* $\mathrm{Out}_{\mathrm{Col}}$-*problem. Suppose that $Z(S) < Z(G)$, for S a Sylow 2-subgroup of G. Then G has a non-inner C-automorphism φ and contains a normal subgroup H such that*:

(i) *φ induces a derivation $\rho\colon G \to H$ such that $\rho(g) = \varphi(g)g^{-1} \in H$, for every $g \in G$*;

(ii) *there exists a Sylow 2-subgroup S on which φ acts as the identity*;

(iii) *the Fitting subgroup* $\mathrm{Fit}(G)$ *of G is $H \times F_1$, $H = O_{2'}(\mathrm{Fit}(G))$ is minimal normal in G and $F_1 \neq \{1\}$.*

(iv) *$\varphi|_H = \mathrm{conj}_{x_0} = \mathrm{conj}_{x_0^g}$, for some $x_0 \in S$ and for every $g \in G$*;

(v) *$\varphi^2 = \mathrm{Id}$ and $\varphi(h) = x_0^{-1}hx_0 = h^{-1}$, for every $h \in H$.*

From now on we will consider G a finite group with semidihedral Sylow 2-subgroups. Let us recall that a semidihedral group S of order 2^n is a group with the following presentation

$$\left\langle a, b \mid a^{2^{n-1}} = b^2 = 1,\ bab = a^{2^{n-2}-1} \right\rangle,$$

or, in other words, is the semidirect product $C_{2^{n-1}} \rtimes C_2$, where C_2 acts on $C_{2^{n-1}}$ by $x \mapsto x^{2^{n-2}-1}$. In particular, all the elements in $S \setminus \langle a \rangle$ have order 2 or 4. Moreover, we have the following useful properties (see [1, Lemma 1, p. 9]):

(i) $Z(S) = C_2$ and $S/Z(S)$ is dihedral $D_{2^{n-2}}$ of order 2^{n-1};

(ii) S possesses precisely three maximal subgroups, respectively cyclic, generalized quaternion, and dihedral, and so, more in general, the subgroups of S are cyclic, generalized quaternion, dihedral and Klein groups contained in dihedrial subgroups;

(iii) if D is a dihedral subgroup of S of order at least 8, then the maximal cyclic subgroup of D is contained in the maximal cyclic subgroup $C_{2^{n-1}}$ of S.

Notice that, by definition, n has to be bigger or equal to 4, or, in other words, S has order at least 16.

LEMMA 2.3. *Let G be a finite solvable group which is a minimal counterexample to the* $\mathrm{Out_{Col}}$*-problem, containing a semidihedral Sylow* 2*-subgroup S such that $Z(S) < Z(G)$. Let H be the normal subgroup and x_0 the element of S of Theorem* 2.2*. Then there exist $s_1 \in S$ of order four and $h_1 \in H \setminus \{1\}$ such that $[x_0, s_1] \neq 1 = [h_1, s_1]$.*

PROOF. By Theorem 2.2, $x_0 \notin Z(G) > Z(S)$ and, since S is a semidihedral group, it is possible to choose $y \in S \setminus \{\langle x_0 \rangle\}$ of order 4. We can notice that $[x_0, y] \neq 1$. If we suppose $C_H(y) \neq \{1\}$, then we can set $s_1 = y$ and so $[h_1, s_1] = 1$, for any $h_1 \in C_H(s_1) \setminus \{1\}$. Let us suppose that $C_H(y) = \{1\}$. In this case, for any positive integer k, we have that $x_0^k y$ trivially acts on H. Since $[x_0, x_0^k y] = [x_0, y] \neq 1$, taking k such that $x_0^k y$ has order 4 and setting $s_1 = x_0^k y$, we also have $[s_1, h_1] = 1$, for any $h_1 \in H \setminus \{1\}$. □

COROLLARY 2.4. *With the same notation and the same hypotheses of Lemma* 2.3*, we have that a conjugate of s_1 belongs to the maximal cyclic subgroup of S and there exists a* 2*-element s_0 of order at least* 8*, inverting h_1 and commuting with s_1.*

PROOF. Let us define $f = h_1 s_1$ and let φ be the C-automorphism of Theorem 2.2. Since $\varphi(f) = \varphi(h_1)\varphi(s_1) = h_1^{-1} s_1$ is conjugate to f by a 2-element s_0, we get

$$h_1^{-1} s_1 = \varphi(f) = (s_0^{-1} h_1 s_0)(s_0^{-1} s_1 s_0).$$

By $[h_1, s_1] = 1$ and the uniqueness of decomposition of $\varphi(f)$, it follows that s_0 inverts h_1 and commutes with s_1. Moreover, since S is a semidihedrial group and $[s_0, h_1] \neq 1$ but $[h_1, s_1] = [s_0, s_1] = 1$, then $\mathrm{ord}(s_0) > \mathrm{ord}(s_1) = 4$; hence $\mathrm{ord}(s_0) \geq 8$ and so s_1 belongs to $\langle s_0 \rangle$ and, again because of S is a semidihedrial group, $\langle s_0 \rangle$ is contained in a conjugate of the maximal cyclic subgroup of S. □

LEMMA 2.5. *Let G be a finite group with a semidihedral Sylow* 2*-subgroup S. If $Z(S) < Z(G)$ and G is a counterexample of minimal order to the* $\mathrm{Out_{Col}}$*-problem, then either $G = S \ltimes O_{2'}(G)$ or $G/N = \mathrm{Sym}(4)$, where $N = Z(S) \times O_{2'}(G)$. In particular, if $G = S \ltimes O_{2'}(G)$, then the element s_1 in S of Lemma* 2.3 *is contained in the maximal cyclic subgroup of S and the element x_0 in S of Theorem* 2.2 *has order* 2 *or* 4*.*

PROOF. By hypothesis, $Z(S) < Z(G)$ and so we can define the quotient $H = G/Z(S)$. Since S is semidihedral, then H contains the Sylow 2-subgroup $S/Z(S) = D_{2^{n-2}}$, with $n \geq 4$, and so it contains a dihedral Sylow 2-subgroup of order at least 8 and hence the order of H is a multiple of 8. Notice that in [8, Theorem 2], an immediate corollary of the classification of Gorenstein and Walter of the finite groups with dihedral Sylow 2-subgroups (in the case when the group is simple), the hypothesis is satisfied when the group has order $4m$, with m odd, and in this case, the simple group can only be $\mathrm{PSL}(2,q)$, for $q>3$. Hence, in our case, since $|H/O_{2'}(H)|=8m$, for some positive integer m, then we have $H/O_{2'}(H) \neq \mathrm{PSL}(2,q)$, for $q>3$. Moreover, $H/O_{2'}(H)$ cannot be a subgroup K of $\mathrm{P\Gamma L}(2,q)$ containing as subgroup $\mathrm{PSL}(2,q)$, for $q>3$. Indeed, if by contradiction $G/(Z(S)\times O_{2'}(G))=H/O_{2'}(H)=K$, then

$$|G/(Z(S)\times O_{2'}(G))|=k(q^2-1)q,$$

for some positive integer k. Therefore, there exists a subgroup A of G such that $A/(Z(S)\times O_{2'}(G))=\mathrm{PSL}(2,q)$, which is simple, since $\mathrm{PSL}(2,q)$ is simple for $q>3$. Moreover, notice that $|G|=2hk(q^2-1)q$ and $|A|=h(q^2-1)q$, where $|O_{2'}(G)|=h$ and recalling that $|Z(S)|=2$ and

$$|\mathrm{PSL}(2,q)|=\frac{1}{2}(q^2-1)q,$$

and so the index of A in G is $2k$. Hence a Sylow 2-subgroup S' of A is a proper subgroup of S, and so S' can be either cyclic or generalized quaternion or dihedral. It is well known that a Sylow 2-subgroup of a simple group cannot be cyclic, and in [3], the authors proved that it cannot even be generalized quaternion. Moreover S' cannot even be dihedral since, for order reasons, $A\neq \mathrm{PSL}(2,q)$, and so we have a contradiction. Therefore, by the Gorenstein-Walter classification we have either $G/O_{2'}(G)\cong S$ or $H/O_{2'}(H) \cong \mathrm{PGL}(2,3)=\mathrm{Sym}(4)$. Consequently, applying the Schur–Zassenhaus Theorem (see, e.g., [20, Theorem 9.1.2]), we have either $G=S\ltimes O_{2'}(G)$ or $G/(Z(S)\times O_{2'}(G))=\mathrm{Sym}(4)$.

Let us now suppose that $G=S\ltimes O_{2'}(G)$. Notice that, by the Feit-Thompson Theorem [7], $O_{2'}(G)$ is solvable, and also S is it, being is a 2-group, so G is solvable and we can apply Corollary 2.4. Hence there exists $s_1\in S$ such that, for some $g\in G$, s_1^g is contained in the maximal cyclic subgroup $\langle a\rangle$ of S. If $\pi: G\to S\cong G/O_{2'}(G)$ is the natural projection with kernel $O_{2'}(G)$, then π fixes S, and so applying π on s_1^g, it is possible to choose $g\in S$ such that $s_1\in\langle a\rangle$. □

Proposition 2.6. *Let G be a finite group whose Sylow 2-subgroup are semidihedral, G is a counterexample of minimal order to the* $\mathrm{Out}_{\mathrm{Col}}$*-problem and there exists a normal nilpotent subgroup N of G such that a Sylow 2-subgroup of G/N is normal. Then any $\varphi\in\mathrm{Aut}_{\mathrm{Col}}(G)$ can be modified, modulo* $\mathrm{Inn}(G)$, *in a such way that* $\varphi|_{O_{2'}(N)}=\mathrm{Id}$.

Proof. Suppose that G is a minimal counterexample to the $\mathrm{Out}_{\mathrm{Col}}$-problem. Let us denote $O_{2'}(N)$ by O. Since N is nilpotent, also O is it. So we can consider O as a product of its Sylow p-subgroups O_p. If the Frattini subgroup $\Phi(O)$ is non-trivial, then $\Phi(O_{\bar p}) \neq \{1\}$, for some prime $\bar p$ dividing the order of O. Since G is a minimal counterexample, then $G/\Phi(O_{\bar p})$ satisfies the $\mathrm{Out}_{\mathrm{Col}}$-problem. Hence, modulo $\mathrm{Inn}(G)$, we can define, modifying φ, a derivation $\rho\colon G \to \Phi(O_{\bar p})$ such that $\rho(g) = \varphi(g)g^{-1} \in \Phi(O_{\bar p})$, for every $g \in G$. Choose a prime divisor q of the order of O and notice that φ fixes O_q. If $q \neq \bar p$, then, for each $g \in O_q$, we have $\rho(g) = 1$ and so $\varphi|_{O_q} = \mathrm{Id}$. If $q = \bar p$, then we have $O_{\bar p} = C_{O_{\bar p}}(\varphi)[O_{\bar p}, \varphi] \subset C_{O_{\bar p}}(\varphi)\Phi(O_{\bar p})$; but we also have $C_{O_{\bar p}}(\varphi)\Phi(O_{\bar p}) \subset O_{\bar p}$. Hence $O_{\bar p} = \langle C_{O_{\bar p}}(\varphi), \Phi(O_{\bar p})\rangle$, and so, by definition of Frattini group, $O_{\bar p} = C_{O_{\bar p}}(\varphi)$. Therefore $\varphi|_{O_p} = \mathrm{Id}$, for every p dividing $|O|$, and so $\varphi|_O = \mathrm{Id}$.

Let us now suppose that $\Phi(O) = \{1\}$. By a well-known property of the Frattini subgroup, we have that O is a direct product of elementary abelian groups. Let us write $O = O_p \times O_1$, where O_p is a Sylow p-subgroup of O and O_1 is the complement of O_p in O. We necessarily have that $O \neq O_p$, otherwise, since $\varphi \in \mathrm{Aut}_{\mathrm{Col}}(G)$, we have $\varphi|_O = \mathrm{conj}_g|_O$, for some $g \in G$, a contradiction. We modify again φ in a such way that we can define $\rho\colon G \to O_p$ such that $\rho(g) = \varphi(g)g^{-1} \in O_p$, for every $g \in G$. Therefore, since φ is a 2-element, we have that if $g \in O$, then $\varphi(g) = s^{-1}gs$, where s is a 2-element. Let us fix a Sylow 2-subgroup such that $\varphi|_S = \mathrm{Id}$. By hypothesis G/N has unique Sylow 2-subgroup, and so, modulo N, s belongs to S, or, in other words, there exist $n \in N$ and $x \in S$ such that $s = nx$. In particular, since O is abelian, we can conjugate g by a 2-element s and choose $s \in S$. In other words, we proved that $g \in O$ is conjugate to $\varphi(g)$ by an element of S. Let us define $H = \langle \varphi|_O, \mathrm{conj}_g|_O \mid g \in S\rangle$, which is a 2-group, since $\varphi|_S = \mathrm{Id}$ and φ is a 2-element. Since O is a direct product of elementary abelian subgroups, then it can be written as a direct product of irreducible H-modulos $M_1, \ldots,$ M_k. Let us consider $m = m_1 \cdots m_k$, where $1 \neq m_i \in M_i$, for every $1 \leq i \leq k$. As mentioned before, we can choose $s \in S$ such that $\varphi(m) = s^{-1}ms$, and so $\psi = \varphi \cdot \mathrm{conj}_{s^{-1}} \in C_H(m)$. Since S is semidihedral, then $Z(S) = \langle \hat s\rangle = C_2$, for some $\hat s \in S$ of order 2, and, being $O = O_p \times O_1$, $m = zw$ with $z \in O_p$ and $w \in O_1$. Since $\rho(g) \in O_p$ for each $g \in G$ and, O_p and O_1 are normal in G, we have that $\varphi|_{O_1} = \mathrm{Id}$. By definition, ψ fixes m and $(\mathrm{ord}(z), \mathrm{ord}(w)) = 1$, and so we have that $sw = ws$. If φ fixes also z, then again $sz = zs$, and so we have that the sets $\{t \in M_i \mid t \in O_p, \varphi(t) = t\}$ are H-invariant and so they coincide with M_i, since M_i is irreducible. Hence φ fixes also O_p, and so O. Let us now suppose that $s \notin C_H(z)$. In particular, we can suppose that $s \neq 1$ and, since we have chosen s in the maximal cyclic subgroup of S and S is semidihedral, we have $Z(S) = \langle \hat s\rangle \leq \langle s\rangle < C_H(w)$. Therefore, the sets $\{t \in M_i \mid t \in O_1, [\hat s, t] = 1\}$ are H-invariant and so again they coincide with M_i. This implies that $[\hat s, O_1] = 1$. Finally, it is enough to prove that, if $s \notin C_H(z)$, then $\varphi|_{O_p} = \mathrm{conj}_{\hat s}|_{O_p}$. Since G is a minimal counterexample to the $\mathrm{Out}_{\mathrm{Col}}$-problem, then $\overline{G} = G/O_1$ satisfies $\mathrm{Out}_{\mathrm{Col}}(\overline{G}) = \{\mathrm{Id}\}$, and so, in partucilar, there exists a 2-element $\overline{g} \in \overline{G}$ such that $\overline{\varphi} = \mathrm{conj}_{\overline{g}}$, where $\overline{\varphi}$ is a C-automorphism of $\overline{G}$ induced by φ. Since, by hypothesis, $\overline{\varphi}|_{\overline{S}} = \mathrm{Id}$, where $\overline{S}$ is a Sylow 2-subgroup of $\overline{G}$, we have that $\overline{g} \in Z(\overline{S})$. Because of O_1 has order odd, then $S \cap O_1 = \{1\}$, and so we can choose $g = \hat{s}$. Since $\rho(g) \in O_p$, we obtain that $\varphi(g) = \mathrm{conj}_{\hat{s}}(g)$, for each $g \in O_p$. Since we already proved that φ and $\mathrm{conj}_{\hat{s}}$ act as the identity on O_1, we obtain that $\varphi|_O = \mathrm{conj}_{\hat{s}}|_O$. Finally, also when $\Phi(O) = \{1\}$, we can modify φ, modulo $\mathrm{Inn}(G)$, such that $\varphi|_O = \mathrm{Id}$. □

For a proof of a more general version of Proposition 2.6, see [15, Corollary 3]

3. Main results

In this section we prove our main results. In particular we exhibit two families of finite groups, with semidihedral Sylow 2-subgroups, satisfying the $\mathrm{Out_{Col}}$-problem. We will use the same notation of Theorem 2.2 and Lemma 2.3.

THEOREM 3.1. *Let G be a finite group and S be a Sylow 2-subgroup. Suppose that S is semidihedral and $Z(S) < Z(G)$. If* $\mathrm{Sym}(4)$ *is not a homomorphic image of G, then* $\mathrm{Out_{Col}}(G) = \{\mathrm{Id}\}$. *In particular this is the case when* $|S| > 16$.

PROOF. Suppose that G is a minimal counterexample to the $\mathrm{Out_{Col}}$-problem. By Corollary 2.4, we have the

- $s_1 \in \langle s_0 \rangle$,
- s_1 and a conjugate s_0^g of s_0, for some $g \in G$, are contained in the maximal cyclic subgroup of S, and
- the order of x_0 is 2 or 4.

First, suppose the order of x_0 is 4. The element $s_2 = x_0 s_0^g$ belongs to S, has order 4 and commutes with h_1^g, indeed, considering Theorem 2.2(v) and Corollary 2.4, we have

$$s_0^{-g} x_0^{-1} h_1^g x_0 s_0^g = (s_0^{-1} h_1^{-1} s_0)^g = h_1^g$$

Since, by Theorem 2.2(ii)–(v), $(h_1^g)^{-1} s_2 = \varphi(h_1^g s_2)$ is conjugate to $h_1^g s_2$, then there exists a 2-element f commuting with s_2 and inverting h_1^g. Hence the order of f is at least 8 and, by Lemma 2.5, we have that s_2 belongs to the maximal cyclic subgroup of S, and so also x_0 is contained in the maximal cyclic subgroup of S. This implies that x_0 has to commute with s_1 and that is a contradiction. In the case when x_0 has order 2, we can proceed in a similar way, considering $s_3 = x_0^2 s_0^g$. Finally, we obtain the last claim of our theorem, applying again Lemma 2.5. □

THEOREM 3.2. *Let G be a nilpotent-by-nilpotent finite group whose Sylow 2-subgroups are semidihedral. Then* $\mathrm{Out}_{\mathrm{Col}}(G) = \{\mathrm{Id}\}$.

PROOF. Suppose that G is a minimal counterexample to the $\mathrm{Out}_{\mathrm{Col}}$-problem. Let $F = \mathrm{Fit}(G)$ be the Fitting subgroup of G and $\varphi \in \mathrm{Aut}_{\mathrm{Col}}(G)$ be non-inner. Let N be a normal nilpotent subgroup of G such that G/N is nilpotent, and so also $(G/N)/(F/N) = G/F$ is nilpotent. By Proposition 2.6, we can suppose that φ acts as the identity on $O_{2'}(F)$. If $F = O_{2'}(F)$, then φ is a C-automorphism which acts as the identity on F; hence, if $g \in G$ and $f \in F$, we have

$$g^{-1}fg = \varphi(g^{-1}fg) = \varphi(g^{-1})f\varphi(g).$$

Therefore $\varphi(g)g^{-1}$ centralizes F. Since G is solvable, being nilpotent-by-nilpotent, we have that F contains its centralizing, and so $\rho(g) = \varphi(g)g^{-1} \in Z(F)$. Let $m = \exp(F)$, then $\varphi^m(g) = \rho^m(g)g = g$; this means that φ has odd order, since m is odd, and so we obtain a contradiction, recalling that if a C-automorphism has odd order then it is inner. Suppose now that $|F|$ is even. Then the intersection $O_2(G)$ of all the semidihedral Sylow 2-subgroups is non-trivial and so it contains a cyclic group C_2 of order 2, which is characteristic in $O_2(G)$. This implies that C_2 is normal, and hence central, in each semidihedral Sylow 2-subgroup S and thus $C_2 = Z(S)$. Now, since it is characteristic in $O_2(G)$, it is normal, and hence central, in G, and so we can conclude that $Z(S) < Z(G)$. Finally, since G is nilpotent-by-nilpotent, $\mathrm{Sym}(4)$ can not be a homomorphic image of G, and so the claim follows from Theorem 3.1. $\square$

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