Acta Math. Hungar., **172** (**1**) (2024), 62–79
https://doi.org/10.1007/s10474-024-01405-2
First published online February 28, 2024

ON LOCALLY COMPACT GROUPS OF SMALL TOPOLOGICAL ENTROPY

F. G. RUSSO* and O. WAKA

Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X1, 7701 Rondebosch, Cape Town, South Africa
e-mails: francescog.russo@yahoo.com, wolwethu@gmail.com

(Received March 26, 2023; accepted November 9, 2023)

This paper is dedicated to the memory of Bernhard Banaschewski, whose inspiring seminars influenced the scientific works of the research group in "Topology, Algebra and Dynamical Systems" at the University of Cape Town for decades

Abstract. We discuss the finiteness of the topological entropy of continuous endomorphims for some classes of locally compact groups. Firstly, we focus on the abelian case, imposing the condition of being compactly generated, and note an interesting behaviour of slender groups. Secondly, we remove the condition of being abelian and consider nilpotent periodic locally compact p-groups (p prime), reducing the computations to the case of Sylow p-subgroups. Finally, we investigate locally compact Heisenberg p-groups $\mathbb{H}_n(\mathbb{Q}_p)$ on the field $\mathbb{Q}_p$ of the p-adic rationals with n arbitrary positive integer.

1. Motivations and main results

In the present paper a locally compact group is always assumed to be a topological group whose topology is both Hausdorff and locally compact. Hood [14] formulated a notion of topological entropy involving the well known concept of uniformity for a topological space. His definition applies to a topological groups possessing a left uniformity, since continuous endomorphisms are uniformly continuous (in connection with the given left uniformity). Let us be more formal on Hood's Entropy [14] in the context of what we need to investigate here. For a locally compact group G, we

* Corresponding author.

The authors thank the National Research Foundation of South Africa (South Africa) for the grant ISARP with Ref. No. ITAL22051410615 and for the project "Topology for Tomorrow", but also Gruppo Nazionale per la Fisica Matematica of INdAM (Italy).

Key words and phrases: topological entropy, locally compact group, dynamical system, Sylow p-subgroup, slender group, Heisenberg p-group.

Mathematics Subject Classification: 22A05, 37B40, 54C70.

denote by $\mathcal{CT}(G)$ the collection of all compact neighborhoods of the identity of G, and by μ a left invariant Haar measure on G. For a continuous endomorphism φ of G, an element $V \in \mathcal{CT}(G)$ and an $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$C_n(\varphi, V) = V \cap \varphi^{-1}(V) \cap \cdots \cap \varphi^{-n+1}(V) \in \mathcal{CT}(G) \tag{1.1}$$

defines the *n-th φ-cotrajectory of V*. The *topological entropy* of φ (in the sense of Hood) is

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \sup\Big\{\limsup_{n\to\infty} \Big(\frac{-\log \mu(C_n(\varphi, V))}{n}\Big) \;\Big|\; V \in \mathcal{CT}(G)\Big\}. \tag{1.2}$$

Adler and others [1,3,18] investigated the aforementioned notions, stressing on dynamical properties of topological structures with relations with ergodic theory and mathematical physics. Following [7,8], we may introduce the *topological entropy* of a locally compact group G as

$$\mathsf{E}_{\mathsf{top}}(G) = \big\{ \mathsf{h}_{\mathsf{top}}(\varphi) \mid \varphi \in \mathrm{End}(G)\big\}, \tag{1.3}$$

where $\mathrm{End}(G)$ denotes the ring of continuous endomorphisms of G and $\mathrm{Aut}(G)$ the group of continuous automorphisms of G. Here we investigate the cardinality of (1.3) and relations with structural properties, as made in [3,4,8,17,21,22,24].

Denoting with $\widehat{\mathbb{Q}}$ the topological dual (in the sense of Pontryagin) of the additive group $\mathbb{Q}$ of the rationals, we note that

$$\begin{aligned} &\inf\big\{ \mathsf{E}_{\mathsf{top}}(G) \setminus \{0\} \mid G \text{ is a compact group}\big\} \\ &= \inf\big(\big\{ \mathsf{h}_{\mathsf{top}}(\varphi) \mid \varphi \in \mathrm{Aut}(\widehat{\mathbb{Q}}^n),\ n \in \mathbb{N}\big\} \setminus \{0\}\big) \end{aligned} \tag{1.4}$$

and a formula of Yuzvinski [24] shows that $\mathsf{h}_{\mathsf{top}}(\varphi)$ can be calculated from the solutions of the characteristic polynomial of φ (see [16,24]). Looking at locally compact groups, we also note that $\mathsf{h}_{\mathsf{top}}(\psi)$ is finite for any $\psi \in \mathrm{End}(\mathbb{R})$. Actually, we can do much more: given $t \in \mathsf{E}_{\mathsf{top}}(\mathbb{R}) \setminus \{+\infty\}$ we may construct $\psi \in \mathrm{Aut}(\mathbb{R})$ of $\mathsf{h}_{\mathsf{top}}(\psi) = t$, see [3,23].

Following [7,8,10,22], we introduce (for a locally compact group G)

$$\mathfrak{E}_0 = \{G \mid \mathsf{E}_{\mathsf{top}}(G) = \{0\}\} \quad \text{and} \quad \mathfrak{E}_{<\infty} = \{G \mid \mathsf{E}_{\mathsf{top}}(G) = [0, +\infty)\} \tag{1.5}$$

and note that there are results which describe the abelian cases in $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$. The characterization of groups in $\mathfrak{E}_0$ can indicate the presence of structural theorems. For instance, finite abelian groups are in $\mathfrak{E}_0$ and have a decomposition in direct product. On the other hand, very little is known in the nonabelian case in $\mathfrak{E}_{<\infty}$ and $\mathfrak{E}_0$.

Following [12, Definition 2.2] and denoting by $\mathbb{P}$ the set of all primes, an element g of a locally compact group G is called *p-element*, if the sequence g^{p^k} with $k \in \mathbb{N}$ tends to the identity element in G. A locally compact group G is called *p-group*, if G coincides with

$$G_p = \{\, g \in G \mid g \text{ is a } p\text{-element}\} = \Big\{ g \in G \mid \lim_{k\to\infty} g^{p^k} = 1 \Big\}. \tag{1.6}$$

A maximal p-subgroup of a locally compact group G is called *p-Sylow subgroup* of G. Note that G_p turns out to be a closed subgroup by [12, Lemma 2.6], when G is totally disconnected. Following [12,13], we denote by G_0 the connected component of the identity and say that G is *compactly covered*, if for an arbitrary $x \in G$ we can always find a compact subgroup C of G such that $x \in C$. From [12, p. 5], a *compact element* of G is an element $g \in G$ such that $\overline{\langle g \rangle}$ is compact and the set

$$\mathrm{comp}(G) = \{\, g \in G \mid g \text{ is a compact element}\} \tag{1.7}$$

is described in [12, Proposition 1.3, Lemma 1.6]. For instance, $G = \mathrm{comp}(G)$ when G is locally compact abelian, but in general $\mathrm{comp}(G)$ is just a subset of G, not necessarily a subgroup. Note that $\mathrm{comp}(G)$ is denoted by $B(G)$ in [2,7,8]; similarly, G_0 by $c(G)$. Following [12, Proposition 1.3], we call *periodic* those locally compact groups G such that $G_0 = 1$ and $\overline{\langle g \rangle}$ is compact for all $g \in G$. Of course, periodic locally compact groups are totally disconnected, so their Sylow p-subgroups are closed and $\mathrm{comp}(G) = G$ by [12, Lemma 1.6].

A locally compact group G is *topologically finitely generated*, if there exists a finite subset X of G such that $G = \overline{\langle X \rangle}$. In particular, a locally compact p-group G has *?nite p-rank*, if

$$\mathrm{rank}_p(G) = \max \{\, \mathrm{rank}_p(H) \mid H \text{ closed subgroup of } G\} \tag{1.8}$$

is a positive integer, where also the following quantities are positive integers

$$\mathrm{rank}_p(H) = \min \{\, |Y| \mid Y \subseteq H \text{ and } \overline{\langle Y \rangle} = H \}. \tag{1.9}$$

For compact p-groups, see also [19, §2.4]. Following [12,13], a locally compact group G is *compactly generated* if there exists a compact set C such that $G = \langle C \rangle$. It is possible to provide examples of periodic locally compact groups, which are not compactly generated. It is also possible to provide examples which show that “topologically finitely generated groups” and “compactly generated groups” are two different notions.

THEOREM 1.1 [13, Theorem 7.57]. *Every compactly generated locally compact abelian group is isomorphic to a direct sum $\mathbb{R}^d \oplus \mathbb{Z}^m \oplus K$ for a compact abelian group K and two nonnegative integers d, m.*

We are going to focus on specific classes of locally compact abelian groups and check whether the topological entropy of their continuous endomorphisms is finite or not; results of the type of Theorem 1.1 are fundamental for this scope. Denote the cartesian sum of countably many copies of $\mathbb{Z}$ by $\mathbb{Z}^{\mathbb{N}} = \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{Z}\}$ and by $\mathbb{Z}^{(\mathbb{N})} = \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{Z} \text{ and } x_i = 0 \text{ for almost all } i\}$ the direct sum of countably many copies of $\mathbb{Z}$. Denote by $\dim(A)$ the dimension of a compact abelian group A, that is, the dimension of the $\mathbb{Q}$-module $\mathbb{Q} \otimes \widehat{A}$ as per [13, Definitions 8.23]. Note also from [13, Corollary 7.58] that a connected compact abelian group A of finite dimension is characterized to be the direct sum of finitely many copies of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Definition 1.2 [9, p. 489]. A (discrete) torsion-free abelian group G is *slender*, if for every homomorphism $\alpha\colon (e_i)_{i\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} \mapsto \alpha((e_i)_{i\in\mathbb{N}}) \in G$ we have $\alpha((e_i)_{i\in\mathbb{N}}) = 0$ for almost all i, where $(e_i)_{i\in\mathbb{N}}$ is the sequence with the i-th component equal to 1 and 0 elsewhere.

Our first main result can be now formulated:

Theorem 1.3 (First Main Theorem). *Let G be a compactly generated locally compact abelian group. With the notations of Theorem* 1.1, *the following statements are satisfied*:

(a) *If G is slender, then $G \in \mathfrak{E}_0$. Viceversa, if $G \in \mathfrak{E}_0$ and $K = 0$, then G is slender.*

(b) *Assume that K is connected. Then $G \in \mathfrak{E}_{<\infty}$ if and only if $G \simeq \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$ for some nonnegative integers d, m, s.*

Note that computations of the topological entropy of continuous automorphisms (not endomorphisms) of $\mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$ are available in [18, pp. 475–476]. Also [4,17] contain computations of the topological entropy of continuous endomorphims, but mostly of Lie groups. We go ahead and describe the finiteness of the topological entropy for some nonabelian locally compact groups, looking at the behaviour of the Sylow p-subgroups. This is our second main result.

Theorem 1.4 (Second Main Theorem). *The continuous automorphisms of a nilpotent periodic locally compact p-group G have finite topological entropy whenever* $\mathrm{rank}_p(G)$ *is finite.*

We can always find periodic locally compact p-groups G of $\mathrm{rank}_p(G) = r$ in $\mathfrak{E}_{<\infty}$, looking at the direct sum $G = \mathbb{Z}_p^r$ of r copies of the additive group of p-adic integers $\mathbb{Z}_p$. On the other hand, it is possible to find periodic locally compact p-groups of nilpotency class two and of finite p-rank, looking at Heisenberg p-groups $\mathbb{H}_n(\mathbb{Q}_p)$ constructed with upper triangular $(n+2) \times (n+2)$ matrices with coefficients in the field of p-adic rationals $\mathbb{Q}_p$. These are neither abelian nor compact groups, and have finite topological entropy and finite p-rank large enough.

Theorem 1.5 (Third Main Theorem). *The Heisenberg group* $\mathbb{H}_n(\mathbb{Q}_p)$ *is a periodic locally compact nonabelian p-group of nilpotency class* 2 *of* $\mathrm{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)) = 2n$, *where* n *is an arbitrary positive integer. Moreover* $\mathbb{H}_n(\mathbb{Q}_p)$ *belongs to* $\mathfrak{E}_{<\infty}$, *but not to* $\mathfrak{E}_0$.

Terminology and notations are standard and follow [9,12,13,15,19,20]. After the statement of the main results in Section 1, the theory of slender groups is summarized in Section 2 from [9,20] and some recent results on the finiteness of the topological entropy for periodic locally compact groups are summarized in Section 3 from [1,3,7,8,10,22]. Section 4 is devoted to construct $\mathbb{H}_n(\mathbb{Q}_p)$ and to prove some results on the p-rank of these groups. Then we end with the proofs of Theorems 1.3, 1.4 and 1.5 in Section 5.

2. Previous results on slender groups

We recall properties of slender groups, originally noted by Nunke, Los and Sasiada, see [9].

Lemma 2.1 [9, Ch. 13, §2]. (i) *Subgroups of slender groups are slender*;

(ii) *Slender groups are torsion-free*;

(iii) $\mathbb{Q}$, $\mathbb{Z}_p$ *and* $\mathbb{Z}^{\mathbb{N}}$ *are not slender*;

(iv) *A group which is slender cannot contain a subgroup isomorphic to* $\mathbb{Q}$, $\mathbb{Z}_p$, *or* $\mathbb{Z}^{\mathbb{N}}$;

(v) *Direct products of slender groups are slender. In particular*, $\mathbb{Z}^{(\mathbb{N})}$ *is slender*;

(vi) *A torsion-free abelian group* G *is slender if for every homomorpshim* $f\colon \mathbb{Z}^{\mathbb{N}} \to G$ *the image* $f(\mathbb{Z}^{\mathbb{N}})$ *is a discrete finitely generated abelian group.*

From [12,13,15,19], we may consider a periodic locally compact p-group G (not necessarily abelian) with k positive integer and introduce the subgroups

$$\Omega_k(G) = \overline{\langle g \mid g^{p^k} = 1\rangle} \quad \text{and} \quad \Omega^k(G) = \langle g^{p^k} \mid g \in G\rangle, \tag{2.1}$$

which are fully invariant in G and satisfy $G/\Omega_k(G) = \Omega^k(G)$. This allows us to introduce also

$$\mathrm{Div}(G) = \bigcap_{k\in\mathbb{N}} \Omega^k(G), \tag{2.2}$$

which turns out to be useful for various reasons. For instance, if G is an (discrete) abelian group (not necessarily a periodic locally compact p-group), $\mathrm{Div}(G)$ as above is still well defined and we say that G is *divisible*, if $\mathrm{Div}(G) \supseteq G$, or that G is *reduced*, if the trivial subgroup of G is the only divisible subgroup of G (see [13, Appendix 1, Definition A1.29]).

Lemma 2.2 [13, Corollary 8.5]. *For a compact abelian group* G, *the following conditions are equivalent*:

(i) G *is totally disconnected*;

(ii) $\mathrm{Div}(G) = 0$;

(iii) $\widehat{G}$ *is a torsion group.*

Nunke and Sasiada [9, Chapter 13, §2] showed that slender groups cannot be divisible. The reader can refer to [20, Exercise 4.4.9]; their result is summarized below.

Lemma 2.3 ([9, Lemma 2.3], Sasiada's Theorem). *An abelian group which is slender must be reduced. In addition, if the group is countable, then the condition of being reduced is necessary and sufficient to conclude that the group is slender.*

Following the discussion in [9, Chapter 1, §7] and [6, §1], we may consider an abelian group G and a filter $\mathcal{F}$ in the subgroups lattice $\mathrm{L}(G)$ of G. Automatically $\mathcal{F}$ defines a topology on G, if we declare $\mathcal{B} = \{U \mid U \in \mathcal{F}\}$ to be a basis of open neighborhoods at the identity of G and if for every $g \in G$ the cosets $g\mathcal{B} = \{gU \mid U \in \mathcal{B}\}$ form a basis of open neighborhoods at $g \in G$. This topology is said to be a *linear topology* on G (or more precisely a *linear* $\mathcal{F}$*-topology* on G). *Linear groups* (in the sense of Orsatti and De Marco) are abelian groups with linear topologies. A linear group G is *complete*, if it is Hausdorff and every Cauchy net in G has a limit in G. De Marco and Orsatti [6] studied Hausdorff linear groups:

Definition 2.4 [6]. An abelian group G belongs to the class $L\Omega$ if it admits a *linear complete and nondiscrete, Hausdorff topology.* We say that G belongs to the class $L\Omega_1$, if it belongs to $L\Omega$ and in addition its topology is metrizable.

In fact the conditions of Definition 2.4 are not verified simultaneously, that is, there are abelian linear groups which are not Hausdorff, or abelian linear groups which are not complete and so on. Of course, abelian groups in $L\Omega_1$ are also in $L\Omega$, but examples can show that the viceversa is false.

Theorem 2.5 [6, Theorem 2.3]. *A torsion-free abelian group possesses a metrizable linear complete nondiscrete topology if and only if it contains a copy of* $\mathbb{Z}_p$, *or of* $\mathbb{Z}^{\mathbb{N}}$ *as subgroup.*

Note that all groups of $L\Omega_1$ are classified by Theorem 2.5. Moreover Lemma 2.1 shows that both $\mathbb{Z}_p$ and $\mathbb{Z}^{\mathbb{N}}$ are not slender, hence Theorem 2.5 implies that G cannot be slender, if it is possible to endow G of a metrizable linear complete nondiscrete topology. This is reported below:

Theorem 2.6 ([6], De Marco and Orsatti). *Let* G *be a reduced torsion-free abelian group. Then* G *is slender if and only if* G *does not belong to* $L\Omega_1$.

Thanks to what we have seen until now:

LEMMA 2.7. *There are no nontrivial compact abelian slender groups.*

PROOF. Assume that G is a compact abelian slender group. Lemma 2.3 along with Lemma 2.2 (a) and (b) imply that G is totally disconnected. Then G should be profinite by [13, Theorem 1.34], hence projective limit of finite groups. Profinite abelian groups are not slender; $\mathbb{Z}_p$ is a counterexample. From the contradiction, there are no nontrivial compact slender groups. $\square$

3. Previous results on locally compact groups

When we have a totally disconnected locally compact group G, van Dantzig [5] proved that

$$\mathcal{U}(G) = \big\{ V \leq G \mid V \text{ compact and open} \big\} \tag{3.1}$$

is contained in $\mathcal{CT}(G)$ and is local basis. From [8, Proposition 3.4], we have that

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \sup \Big\{ \lim_{n\to\infty} \Big(\frac{\log |V : C_n(\varphi, V)|}{n} \Big) \Bigm| V \in \mathcal{U}(G) \Big\}, \tag{3.2}$$

where $C_n(\varphi, V) \in \mathcal{U}(G)$ and the index $|V : C_n(\varphi, V)|$ is finite. In fact, the set $\mathsf{E}_{\mathsf{top}}(G)$ turns out to be a countable subset of the real half-line in this situation.

Some relevant facts are reported below. The first regards discrete groups.

REMARK 3.1 [8, Remark 2.4]. Discrete groups belong to $\mathfrak{E}_0$.

The second regards the additive group of p-adic integers.

COROLLARY 3.2 [8, Corollary 2.2]. *Let G be a locally compact group and $\varphi \in \mathrm{End}(G)$. If $\mathcal{S} \subseteq \mathcal{CT}(G)$ is a local basis of G and $\mathcal{S}$ is realized by φ-invariant subgroups, then $\mathsf{h}_{\mathsf{top}}(\varphi) = 0$. In particular, this applies to $\mathbb{Z}_p^n$, hence $\mathbb{Z}_p^n \in \mathfrak{E}_0$.*

The computation of the topological entropy of continuous endomorphisms is somehow harder than that of continuous automorphisms, but we have results for totally disconnected groups.

COROLLARY 3.3 ([8, Lemma 2.3, Theorem 3.11], [10, Corollary 1.3]). *Let G be a locally compact group and $\varphi \in \mathrm{End}(G)$.*

(a) *If H is a φ-invariant closed subgroup of G, then $\mathsf{h}_{\mathsf{top}}(\varphi_{|_H}) \leq \mathsf{h}_{\mathsf{top}}(\varphi)$, and, if in addition H is normal, then $\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/H}) \leq \mathsf{h}_{\mathsf{top}}(\varphi)$, where $\bar{\varphi}_{G/H} : G/H \to G/H$ is induced by φ.*

(b) *If* $\mathcal{S} \subseteq \mathcal{U}(G)$ *is a local basis of* G *such that* $\varphi^{-n}(V)$ *is normal in* G *for all* n *and* $V \in \mathcal{S}$, *then* $\mathsf{h}_{\mathsf{top}}(\varphi) = \mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/\ker\varphi})$.

(c) *If* G *is totally disconnected and* $\varphi \in \mathrm{Aut}(G)$, *then* $\mathsf{h}_{\mathsf{top}}(\varphi) = \mathsf{h}_{\mathsf{top}}(\varphi_{|N}) + \mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/N})$, *where* N *is a closed normal subgroup of* G.

The third regards p-adic rationals. Denoting the p-adic norm with $|-|_p$, Yuzvinski's Formula [16,24] helps with the following computations:

THEOREM 3.4 [16]. *For* $n \in \mathbb{N}$ *and* $\varphi \in \mathrm{End}(\mathbb{Q}_p^n)$, *we have*

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \sum_{|\lambda_i|_p > 1} \log |\lambda_i|_p, \tag{3.3}$$

where λ_i (*with* $1 \leq i \leq n$) *is eigenvalue of* φ *in a finite extension of* $\mathbb{Q}_p$. *In particular, we have that* $\mathbb{Q}_p^n \in \mathfrak{E}_{<\infty}$.

Further criteria of finiteness are related to the notion of of finite p-rank.

THEOREM 3.5 [12, Theorem 3.97]. *A locally compact abelian* p*-group* G *has finite* p*-rank if and only*

$$G \simeq \mathbb{Z}_p^{\alpha} \times \mathbb{Q}_p^{\beta} \times \mathbb{Z}(p^{\infty})^{\gamma} \times E_p \tag{3.4}$$

for some nonnegative integers α, β, γ, δ *and a finite* p*-group* E_p *of* $\mathrm{rank}_p(E_p) = \delta$. *In particular,* G *belongs to* $\mathfrak{E}_{<\infty}$ *and*

$$\mathrm{rank}_p(G) = \alpha + \beta + \gamma + \delta. \tag{3.5}$$

The case of G *in* $\mathfrak{E}_0$ *is characterized by the condition* $\beta = 0$.

The above result shows that the p-rank is preserved under Pontryagin duality. In fact we have

$$\widehat{G} = (\mathbb{Z}_p^{\alpha} \times \mathbb{Q}_p^{\beta} \times \mathbb{Z}(p^{\infty})^{\gamma} \times F_p)^{\wedge} \cong \mathbb{Z}_p^{\gamma} \times \mathbb{Q}_p^{\beta} \times \mathbb{Z}(p^{\infty})^{\alpha} \times F_p, \tag{3.6}$$

and so $\mathrm{rank}_p(\widehat{G}) = \mathrm{rank}_p(G)$. In particular, it can be seen that $\mathbb{Q}_p^{\beta} \in \mathfrak{E}_{<\infty} \setminus \mathfrak{E}_0$, $\mathbb{R}^d \in \mathfrak{E}_{<\infty} \setminus \mathfrak{E}_0$, $\mathbb{Z}_p^{\gamma} \in \mathfrak{E}_0$, $F_p \in \mathfrak{E}_0$ and $\mathbb{Z}(p^{\infty})^{\alpha} \in \mathfrak{E}_0$. Further results are reported below in the abelian case.

THEOREM 3.6 [8, Theorems 1.1, 1.2]. *Let* G *be a locally compact abelian group.*

(i) *If* G *belongs to* $\mathfrak{E}_{<\infty}$, *then its dimension should be finite*;

(ii) *The viceversa of* (i) *above is true when* G *is compact and* G/G_0 *belongs to* $\mathfrak{E}_{<\infty}$;

(iii) *If* G *belongs to* $\mathfrak{E}_0$, *then* G *is totally disconnected*; *moreover a profinite group belongs to* $\mathfrak{E}_0$ *if and only if it belongs to* $\mathfrak{E}_{<\infty}$;

(iv) *If* G *is periodic, then* $G \in \mathfrak{E}_0$ *iff all its* p*-Sylow subgroups* G_p *do the same.*

In the arguments which are used to prove Theorem 3.6, the main logic is to find decompositions of the endomorphisms in portions where we can control the finiteness of the topological entropy. In fact we say that *the Addition Theorem holds* for (G, φ, H) of a locally compact group G with $\varphi \in \mathrm{End}(G)$ and a φ-invariant closed normal subgroup N of G, if

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \mathsf{h}_{\mathsf{top}}(\varphi_{|_N}) + \mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{G/N}), \tag{3.7}$$

or briefly, we write that *$AT(G, \varphi, N)$ holds.* Of course, (3.7) is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & G/N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{|_N}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \bar{\varphi}_{G/N}} & & \\ 0 & \longrightarrow & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & G/N & \longrightarrow & 0 \end{array} \tag{3.8}$$

Similarly, *$AT(G)$ holds* if $AT(G, \varphi, N)$, which is depending on φ and N in general, is satisfied by all φ and N. From [8, Proposition 3.6], if N is a fully invariant open subgroup of G and $AT(N)$ holds, then also $AT(G)$ holds.

At this point it is important that we pause and look closely at the structure of compactly generated locally compact abelian groups of Theorem 1.1. First, we note that the groups that appear in the decomposition are either compact or totally disconnected, or isomorphic to $\mathbb{R}^d$ for some nonnegative integer d. Because of this observation, we record the following result:

LEMMA 3.7 [8, Lemma 3.1]. *Let A, B be two locally compact groups that either are compact, or totally disconnected or isomorphic to $\mathbb{R}^d$ for some nonnegative integer d, and $f \in \mathrm{End}(A)$, $g \in \mathrm{End}(B)$. Consider $A \times B$ with the product topology and $f \times g \in \mathrm{End}(A \times B)$. Then*

$$\mathsf{h}_{\mathsf{top}}(f \times g) = \mathsf{h}_{\mathsf{top}}(f) + \mathsf{h}_{\mathsf{top}}(g). \tag{3.9}$$

Again the situation is computationally clear for locally compact abelian groups.

THEOREM 3.8 [8, Theorems 1.8, 1.9]. *Let G be a totally disconnected locally compact abelian group. Then, for every $\varphi \in \mathrm{End}(G)$, we have*

$$\mathsf{h}_{\mathsf{top}}(\varphi) = \sum_{p \in \mathbb{P}} \mathsf{h}_{\mathsf{top}}(\varphi_{|_{G_p}}). \tag{3.10}$$

If G is also periodic, then $AT(G)$ holds if and only if $AT(G_p)$ holds for all p-Sylow subgroups G_p.

Theorem 3.8 (ii) shows that Addition Theorems may be reduced to Addition Theorems on p-Sylow subgroups. This means that the presence of a decomposition helps to determine groups in $\mathfrak{E}_0$ or $\mathfrak{E}_{<\infty}$, just looking at Sylow p-subgroups in $\mathfrak{E}_0$ or $\mathfrak{E}_{<\infty}$.

REMARK 3.9. For compactly generated locally compact abelian groups, Theorem 1.1 shows that Lemma 3.7 can be applied and so we have an Addition Theorem. This helps to reduce the computation of the topological entropy of continuous endomorphisms to the topological entropy of continuous endomorphisms arising from factors.

4. Heisenberg groups on p-adic rationals

As application of Corollary 3.2, we have that a compact p-group G with local basis $\{\Omega^n(G) \mid n \in \mathbb{N}\} \subseteq \mathcal{U}(G)$ should belong to $\mathfrak{E}_0$. Note that this applies to $\mathbb{Z}_p^n \times F_p$, where F_p is finite p-group.

REMARK 4.1. Groups of the form $\mathbb{Z}_p \times F_p$ for F_p finite nonabelian p-group are among the easiest examples of infinite nilpotent compact p-groups which can be produced in $\mathfrak{E}_0$. Looking at [15, Section 3.1], a finite p-group F_p is of maximal class if $|p^n|$ with $n > 3$ and its nilpotency class is $c = n - 1$. Their costruction can be found in [15, Examples 3.1.5]. Now $\mathbb{Z}_p \times F_p$ has nilpotency class exactly n by Fitting's Lemma [15, Lemma 1.1.21]. This means that we have already an example of an infinite nonabelian compact p-group of nilpotency class arbitrarily large in $\mathfrak{E}_0$.

Given a commutative unitary topological ring R, the *Heisenberg group* on R is the group of all $(n+2) \times (n+2)$-matrices of the following form

$$M(A,B;c) = \left(\begin{array}{c|ccccc|c} 1 & a_1 & a_2 & \dots & a_n & & c \\ \hline 0 & 1 & 0 & \dots & 0 & & b_1 \\ 0 & 0 & 1 & \dots & 0 & & b_2 \\ \dots & \dots & \dots & \dots & \dots & & \dots \\ 0 & 0 & 0 & \dots & 1 & & b_n \\ \hline 0 & 0 & 0 & 0 & 0 & & 1 \end{array}\right) = \begin{pmatrix} 1 & A & c \\ O & I_n & B \\ 0 & O & 1 \end{pmatrix}, \tag{4.1}$$

where the block O is of all zeros, I_n denotes a identity matrix $n \times n$, A the n-tuple row $(a_1, \dots, a_n)$, B the n-tuple column $(b_1, \dots, b_n)$. Of course, for $n = 1$ we get the usual representation of the Heisenberg group as group of matrices 3×3.

In particular, the matrices (4.1) have coefficients m_{ij} such that

$$m_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i > j, \text{ or } 1 < i < j < n-1. \end{cases} \tag{4.2}$$

Note that $\mathrm{GL}(R^{n+2})$ is the general linear group of dimension $n+2$ of all invertible matrices with coefficients in R, and the set of all matrices (4.1) is denoted by $\mathbb{H}_n(R)$ and equipped with the product topology induced by the product topology in $R^{(n+2)^2}$. In particular, one can check that $\mathbb{H}_n(R)$ is nilpotent of class 2, since the center

$$Z(\mathbb{H}_n(R)) = \overline{[\mathbb{H}_n(R), \mathbb{H}_n(R)]} = \left\{ \begin{pmatrix} 1 & O & c \\ O & I_n & O \\ 0 & O & 1 \end{pmatrix} \,\middle|\, c \in R \right\} \tag{4.3}$$

is topologically isomorphic to $(R, +)$ and the central quotient

$$\mathbb{H}_n(R)/Z(\mathbb{H}_n(R)) \cong \underbrace{(R, +) \times (R, +) \times \ldots \times (R, +)}_{2n\text{-times}}, \tag{4.4}$$

is topologically isomorphic to $2n$ copies of $(R, +)$. Note that for $R = \mathbb{Z}$, or $\mathbb{Z}_p$, or $\mathbb{Z}(p)$, (4.4) is topologically generated by the matrices of the following form

$$\left(\begin{array}{c|ccccc|c} 1 & 1 & 0 & \ldots & 0 & & 1 \\ \hline 0 & 1 & 0 & \ldots & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & & 1 \end{array}\right), \quad \left(\begin{array}{c|ccccc|c} 1 & 0 & 1 & \ldots & 0 & & 1 \\ \hline 0 & 1 & 0 & \ldots & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & & 1 \end{array}\right), \tag{4.5}$$

$$\ldots, \quad \left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \ldots & 1 & & 1 \\ \hline 0 & 1 & 0 & \ldots & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & & 1 \end{array}\right),$$

along with the corresponding ones where the role of A is played by B in the last column

$$\left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \ldots & 0 & & 1 \\ \hline 0 & 1 & 0 & \ldots & 0 & & 1 \\ 0 & 0 & 1 & \ldots & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & & 1 \end{array}\right), \quad \left(\begin{array}{c|ccccc|c} 1 & 0 & 0 & \ldots & 0 & & 1 \\ \hline 0 & 1 & 0 & \ldots & 0 & & 0 \\ 0 & 0 & 1 & \ldots & 0 & & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & & 1 \end{array}\right), \tag{4.6}$$

$$\dots,\quad \left(\begin{array}{c|cccc|c} 1 & 0 & 0 & \dots & 0 & 1 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

In particular, we describe the nonabelian compact p-group $\mathbb{H}(\mathbb{Z}_p)$ below for $n = 1$.

Note from [19, Chapter 2] that the *Frattini subgroup* $\mathrm{Frat}(G)$ of a profinite group G is defined as the intersection of all its maximal open subgroups. Moreover it is a characteristic subgroup of G. An element g of a profinite group G is a *nongenerator* if it can be omitted from every generating set of G, that is, whenever $G = \overline{\langle X, g\rangle}$, then $G = \overline{\langle X\rangle}$. In particular,

Remark 4.2. We have that the set of all nongenerators of a profinite group G coincides with $\mathrm{Frat}(G)$, see [19, Lemma 2.8.1]. Moreover [19, Lemma 2.8.6] shows that the minimal number of generators of a topologically finitely generated profinite group G agrees with the minimal number of generators of $G/\mathrm{Frat}(G)$. For compact p-group G, this means that $\mathrm{rank}_p(G) = \mathrm{rank}_p(G/\mathrm{Frat}(G))$.

Example 4.3. For any prime p, consider a separated bilinear map

$$\omega : (x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p \to \omega(x, y) \in \mathbb{Z}_p \tag{4.7}$$

and the set $\mathbb{Z}_p^3$ endowed with the binary operation

$$\begin{aligned} \square\colon ((x_1, y_1, z_2), (x_2, y_2, z_2)) \in \mathbb{Z}_p^3 \times \mathbb{Z}_p^3 &\mapsto (x_1, y_1, z_1) \,\square\, (x_2, y_2, z_2) \\ &= \big(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2 + \omega(x_1, y_2)\big) \in \mathbb{Z}_p^3. \end{aligned} \tag{4.8}$$

See terminology in [2, Definitions 2.1, 2.2]. In particular, $\square$ is not a commutative operation and $(\mathbb{Z}_p^3, \square)$ satisfies the algebraic axioms of group. Of course, the construction depends on ω and $(\mathbb{Z}_p^3, \square)$ is topologically isomorphic to $\mathbb{H}(\mathbb{Z}_p)$ with the matrix product. The center, the Frattini subgroup and the derived subgroup of $\mathbb{H}(\mathbb{Z}_p)$ satisfy

$$\begin{aligned} \mathrm{Frat}(\mathbb{H}(\mathbb{Z}_p)) \supseteq Z(\mathbb{H}(\mathbb{Z}_p)) &= \{M(0, 0; c) \mid c \in \mathbb{Z}_p\} \\ &= \overline{[\mathbb{H}(\mathbb{Z}_p), \mathbb{H}(\mathbb{Z}_p)]} \simeq \mathbb{Z}_p. \end{aligned} \tag{4.9}$$

We can now look at topological generators and relations for $\mathbb{H}(\mathbb{Z}_p)$, finding that

$$\begin{aligned}(4.10)\qquad & \mathbb{H}(\mathbb{Z}_p)\\ &= \overline{\langle M(1,0;0), M(0,1;0), M(0,0;1) \mid [M(1,0;0), M(0,1;0)] = M(0,0;1),}\\ &\overline{[M(1,0;0), M(0,0;1)] = [M(0,1;0), M(0,0;1)] = I_3\rangle}.\end{aligned}$$

From Remark 4.2, the p-rank of $\mathbb{H}(\mathbb{Z}_p)$ can be reduced to the Frattini quotient, i.e.

$$\operatorname{rank}_p(\mathbb{H}(\mathbb{Z}_p)) = \operatorname{rank}_p(\mathbb{H}(\mathbb{Z}_p)/\operatorname{Frat}(\mathbb{H}(\mathbb{Z}_p))) = 2. \tag{4.11}$$

Indeed, $\operatorname{Frat}(G) = \overline{\Omega^1(G)[G,G]}$ for any compact p-group G by [19, Lemma 2.8.7 (c)]. Since this is true of course when $G = \mathbb{H}(\mathbb{Z}_p)$, in (4.9) one can compute $\operatorname{Frat}(\mathbb{H}(\mathbb{Z}_p))$ as follows. Now $[\mathbb{H}(\mathbb{Z}_p), \mathbb{H}(\mathbb{Z}_p)] = Z(\mathbb{H}(\mathbb{Z}_p))$ and

$$\Omega^1(\mathbb{H}(\mathbb{Z}_p)) = \begin{pmatrix} 1 & p\mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 1 & p\mathbb{Z}_p \\ 0 & 0 & 1 \end{pmatrix} \tag{4.12}$$

are compact, so $\operatorname{Frat}(\mathbb{H}(\mathbb{Z}_p)) = \Omega^1(\mathbb{H}(\mathbb{Z}_p))Z(\mathbb{H}(\mathbb{Z}_p))$, since the subgroup $\Omega^1(\mathbb{H}(\mathbb{Z}_p))Z(\mathbb{H}(\mathbb{Z}_p))$ is compact. Therefore, $\mathbb{H}(\mathbb{Z}_p)/\operatorname{Frat}(\mathbb{H}(\mathbb{Z}_p)) \cong \mathbb{Z}(p) \times \mathbb{Z}(p)$. This proves the second equality on (4.11). Example 4.3 holds more generally than $R = \mathbb{Z}_p$, see [11, Theorem 2.5, Lemma 5.5] and [2, §4]. Now we look at $\mathbb{H}_n(\mathbb{Z}_p)$ and $\mathbb{H}_n(\mathbb{Q}_p)$ for n large enough.

LEMMA 4.4. *The Heisenberg group $\mathbb{H}_n(\mathbb{Q}_p)$ is a locally compact nonabelian p-group of nilpotency class two and*

$$\operatorname{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)) = \operatorname{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))) = 2n.$$

PROOF. Looking at (4.1), (4.2), (4.3) and (4.4) with $R = \mathbb{Q}_p$, it is clear that $\mathbb{H}_n(\mathbb{Q}_p)$ is a locally compact nonabelian p-group of nilpotency class two. Now consider (4.1) and observe that

$$H_1 = Z(\mathbb{H}_n(\mathbb{Q}_p)) \times K_2 \simeq \mathbb{Q}_p \times \mathbb{Q}_p^n \ \text{ and } \ K_1 \simeq \mathbb{Q}_p^n. \tag{4.13}$$

Moreover H_1 is a closed normal subgroup such that

$$\begin{aligned}(4.14)\qquad \mathbb{H}_n(\mathbb{Q}_p) &= H_1 \rtimes K_1 = \left\{ h_1k_1 \mid h_1 \in H_1 \text{ and } k_1 \in K_1 \right\}\\ &= \left\{ zk_2k_1 \mid z \in Z(\mathbb{H}_n(\mathbb{Q}_p)),\ k_2 \in K_2,\ k_1 \in K_1 \right\}\\ &= \left\{ [u_2, u_1]k_2k_1 \mid k_2, u_2 \in K_2 \text{ and } k_1, u_1 \in K_1 \right\},\end{aligned}$$

because we have

$$Z(\mathbb{H}_n(\mathbb{Q}_p)) = \overline{[\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]} = \overline{[K_2, K_1]}. \tag{4.15}$$

Therefore the p-rank of $\mathbb{H}_n(\mathbb{Q}_p)$ is reduced to that of K_1 plus that of K_2, i.e., $2n$. □

Lemma 4.4 can be proved with the idea of Example 4.3, that is, noting that

$$\mathrm{Frat}(\mathbb{H}_n(\mathbb{Q}_p)) \supseteq Z(\mathbb{H}_n(\mathbb{Q}_p)), \tag{4.16}$$

and that the quotient $\mathbb{H}_n(\mathbb{Q}_p)/\mathrm{Frat}(\mathbb{H}_n(\mathbb{Q}_p))$ has p-rank $2n$, but we gave an argument based on the structure of semidirect product for Heisenberg groups. Moreover also here one could argue that $\mathrm{Frat}(\mathbb{H}_n(\mathbb{Q}_p)) = \overline{\Omega^1(\mathbb{H}_n(\mathbb{Q}_p))[\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]}$, even if in this sitation we don't have a compact p-group but a periodic locally compact p-group.

Remark 4.5. Looking at [2, Lemma 2.4, Theorem 2.5], one can show that $\mathbb{H}(\mathbb{Z}_p)$ possesses abelian maximal subgroups of the following form

$$H_1 = Z(\mathbb{H}(\mathbb{Z}_p)) \oplus \overline{\langle M(1,0;0)\rangle} \simeq {\mathbb{Z}_p}^2, \ H_2 = Z(\mathbb{H}(\mathbb{Z}_p)) \oplus \overline{\langle M(0,1;0)\rangle} \simeq {\mathbb{Z}_p}^2 \tag{4.17}$$

satisfying the following conditions:

$$H_1 \cap H_2 = Z(\mathbb{H}(\mathbb{Z}_p)), \ H_1 \cap \overline{\langle M(0,1;0)\rangle} = 1, \ H_2 \cap \overline{\langle M(1,0;0)\rangle} = 1, \tag{4.18}$$

$$\mathbb{H}(\mathbb{Z}_p) = H_1 \rtimes \overline{\langle M(0,1;0)\rangle} = H_2 \rtimes \overline{\langle M(1,0;0)\rangle} \simeq {\mathbb{Z}_p}^2 \rtimes \mathbb{Z}_p. \tag{4.19}$$

In particular (4.19) shows that any element of $\mathbb{H}(\mathbb{Z}_p)$ can be written uniquely as product of an element of H_1 and of one of $\overline{\langle M(0,1;0)\rangle} = K_1$, but any element of H_1 can be also written uniquely as product of an element of $Z(\mathbb{H}(\mathbb{Z}_p))$ and of one of $\overline{\langle M(1,0;0)\rangle} = K_2$ by (4.17). In Fig. 1 we identify the aforementioned subgroups in the lattice of closed subgroups $\mathcal{SUB}(\mathbb{H}(\mathbb{Z}_p))$ of $\mathbb{H}(\mathbb{Z}_p)$. At the first level (beginning from the bottom of Fig. 1) we find the trivial subgroup. At the second level there are three subgroups isomorphic to $\mathbb{Z}_p$. At the third level there are two subgroups isomorphic to the additive group ${\mathbb{Z}_p}^2$. At the fourth level we find the entire group. Note that Fig. 1 shows only the subgroups that can be directly deduced from (4.19) and not all the subgroups of $\mathbb{H}(\mathbb{Z}_p)$.

In fact one can see that, given the cardinality of the continuum $\mathfrak{c}$ and fixed $\xi \in \mathbb{Z}_p$, the subset M_ξ of all matrices $M(a, \xi a; t) \in \mathbb{H}(\mathbb{Z}_p)$ is a maximal abelian subgroup of $\mathbb{H}(\mathbb{Z}_p)$, and of course there are $\mathfrak{c}$ of this type.

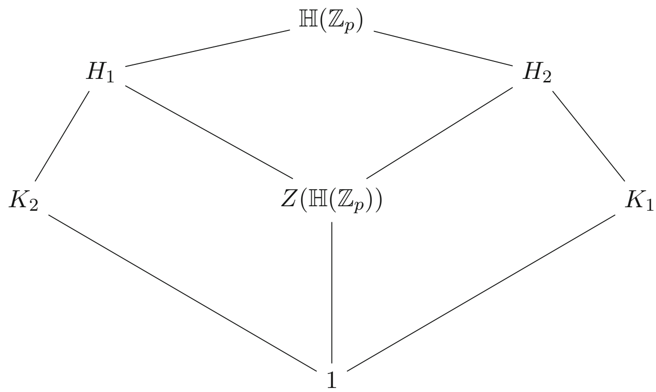

Fig. 1: Some relevant subgroups in $\mathcal{SUB}(\mathbb{H}(\mathbb{Z}_p))$

5. Proofs of main theorems

Proof of Theorem 1.3. (a) If G is a compactly generated locally compact abelian group, then Theorem 1.1 implies $G \cong \mathbb{R}^d \oplus \mathbb{Z}^m \oplus K$ for a compact abelian group K and nonnegative integers m, d. Assume in addition that G is slender. Lemma 2.1 shows that subgroups of slender groups are slender. Then Lemma 2.3 implies $n = 0$, that is, $G \simeq \mathbb{Z}^m \oplus K$. Lemma 2.7 implies $K = 0$. Hence $G \simeq \mathbb{Z}^m$, and since $\mathbb{Z}^m \in \mathfrak{E}_0$, the first part of the result follows. Assume now that $G \in \mathfrak{E}_0$ and that $G \cong \mathbb{R}^d \oplus \mathbb{Z}^m$. Since $\mathbb{R}^d \in \mathfrak{E}_\infty \setminus \mathfrak{E}_0$, G should be totally disconnected by Theorem 3.6 (iii) and so $G \simeq \mathbb{Z}^m$ which is slender. The result follows completely.

(b) From Theorem 1.1 and the assumption that K is a connected compact abelian group, we have that $G \simeq \mathbb{R}^d \oplus \mathbb{Z}^m \oplus K$ with K of $\dim(K)$ eventually infinite. Then

$$\dim(G) = \dim(\mathbb{R}^d) + \dim(\mathbb{Z}^m) + \dim(K) = d + 0 + \dim(K) \tag{5.1}$$

and this shows that $\dim(G) < \infty$ if and only if $\dim(K) < \infty$ if and only if $K = \mathbb{T}^s$ for some nonnegative integer s, see [13, Corollary 8.22(5)]. From Theorem 3.6(i), this means that if $G \in \mathfrak{E}_{<\infty}$, then $\dim(G) < \infty$ hence $\dim(K) < \infty$, and so $G \simeq \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$. Conversely, assume that $G \simeq \mathbb{R}^d \oplus \mathbb{Z}^m \oplus \mathbb{T}^s$. We may apply Lemma 3.7 with summands $\mathbb{R}^d \in \mathfrak{E}_{<\infty}$, $\mathbb{Z}^m \in \mathfrak{E}_0$ and $\mathbb{T}^s \in \mathfrak{E}_{<\infty}$, concluding $G \in \mathfrak{E}_{<\infty}$. Note that the computations of topological entropy, which allows us to have $\mathbb{R}^d \in \mathfrak{E}_{<\infty}$, $\mathbb{Z}^m \in \mathfrak{E}_0$ and $\mathbb{T}^s \in \mathfrak{E}_{<\infty}$, are well known, see [3,16,22]. The result follows. □

Proof of Theorem 1.4. First assume that G has $\mathrm{rank}_p(G) < \infty$. We note that closed subgroups and quotients of G are again periodic locally compact p-groups. The topological lower central series of G of length c has closed characteristic p-subgroups $\overline{\gamma_i(G)}$ (with $i = 1, 2, \dots, c$) such that

$$G = \overline{\gamma_1(G)} \geq \overline{\gamma_2(G)} = \overline{[G,G]} \geq \overline{\gamma_3(G)}$$
$$= \overline{[[G,G],G]} \geq \cdots \geq \overline{\gamma_c(G)} \geq \overline{\gamma_{c+1}(G)} = 1$$

and $\overline{\gamma_i(G)}/\overline{\gamma_{i+1}(G)}$ are locally compact abelian p-groups for all i. Note also that closed subgroups and quotients of a periodic locally compact p-group of finite p-rank have finite p-rank. This means that if G has finite p-rank, then $\overline{\gamma_i(G)}/\overline{\gamma_{i+1}(G)}$ are of the form of those in Theorem 3.5, and in particular continuous automorphisms of $\overline{\gamma_i(G)}/\overline{\gamma_{i+1}(G)}$ have finite topological entropy. Now we do induction on c. Assume $c = 1$. Then G is a locally compact abelian group of finite p-rank and the result is true by Theorem 3.5, because in this situation the continuous automorphisms of G should have finite topological entropy. Assume $c > 1$ and that the result is true for all periodic nilpotent locally compact p-groups of derived length at most $c - 1$. Then the continuous automorphisms of $N = \overline{\gamma_c(G)}$ have finite topological entropy, since N is abelian, but also those of G/N have finite topological entropy, since G/N is a locally compact abelian p-group of finite p-rank. From Addition Theorem for continuous automorphisms of totally disconnected locally compact abelian groups (see [18, Addition Theorem 10], or Corollary 3.3 (b)) we conclude that $AT(G, \varphi, N)$ holds for every continuous automorphism φ of G. The result follows. $\square$

Proof of Theorem 1.5. From Lemma 4.4, the Heisenberg group $\mathbb{H}_n(\mathbb{Q}_p)$ is a periodic locally compact nonabelian p-group of nilpotency class two and $\mathrm{rank}_p(\mathbb{H}_n(\mathbb{Q}_p)) = 2n$. Then we shall only prove that $\mathbb{H}_n(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$, but not to $\mathfrak{E}_0$.

Assume that $n = 1$. From [8, Theorem 6.8] we know that $\mathbb{H}(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$, but not to $\mathfrak{E}_0$. Then there exists a subgroup S of $\mathbb{H}_n(\mathbb{Q}_p)$ which is isomorphic to $\mathbb{H}(\mathbb{Q}_p)$ as topological group, for instance S can be realized putting in (4.1) the condition $a_i = b_i = 0$ for all $i = 2, 3, \dots, n$. This is sufficient to show that $\mathbb{H}_n(\mathbb{Q}_p)$ cannot be in $\mathfrak{E}_0$, since it contains a subgroup S which is not in $\mathfrak{E}_0$. It remains to check that $\mathbb{H}_n(\mathbb{Q}_p)$ belongs to $\mathfrak{E}_{<\infty}$ and we adapt the argument of [8, Proof of Theorem 6.8] for this scope.

Consider $\varphi \in \mathrm{End}(\mathbb{H}_n(\mathbb{Q}_p))$ and $N = \ker \varphi$; we claim that $\mathsf{h}_{\mathsf{top}}(\varphi) < \infty$.

Assume that $N = 1$. We claim that $\varphi \in \mathrm{Aut}(\mathbb{H}_n(\mathbb{Q}_p))$. Since $Z(\mathbb{H}_n(\mathbb{Q}_p))$ is fully invariant, $\varphi_{|Z(\mathbb{H}_n(\mathbb{Q}_p))}$ is injective, hence $\varphi_{|Z(\mathbb{H}_n(\mathbb{Q}_p))}$ is a continuous automorphism of $Z(\mathbb{H}_n(\mathbb{Q}_p))$. In particular, $\varphi^{-1}(Z(\mathbb{H}_n(\mathbb{Q}_p))) = Z(\mathbb{H}_n(\mathbb{Q}_p))$ and so $\bar{\varphi}_{|\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))}$ on $\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))$ is injective. In fact it is a continuous automorphism of $\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p)) \simeq \mathbb{Q}_p^{2n}$. Now $\mathbb{H}_n(\mathbb{Q}_p)$ is a totally disconnected locally compact group, which can be also realized as union of countably many compact sets, and so φ is a continuous automorphism by the Open Mapping Theorem [13, Appendix 1, Exercise EA1.21]. We may apply Addition Theorems on closed normal subgroups for continuous automorphisms of locally compact groups as per Corollary 3.3(c), concluding $\mathsf{h}_{\mathsf{top}}(\varphi) < \infty$ from the fact that both $\mathsf{h}_{\mathsf{top}}(\varphi_{|Z(\mathbb{H}_n(\mathbb{Q}_p))}) < \infty$ and $\mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{\mathbb{H}_n(\mathbb{Q}_p)/Z(\mathbb{H}_n(\mathbb{Q}_p))}) < \infty$ by Theorem 3.5.

Now assume that $N = \ker \varphi \neq 1$. First we show that $N \cap Z(\mathbb{H}_n(\mathbb{Q}_p))$ is nontrivial and then that $Z(\mathbb{H}_n(\mathbb{Q}_p)) \subseteq N$. If there exists some $y \in N \setminus Z(\mathbb{H}_n(\mathbb{Q}_p))$, then there exists $x \in \mathbb{H}_n(\mathbb{Q}_p)$ such that $[x, y]$ is nontrivial. This implies that $N \cap [\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]$ is nontrivial, because $[x, y] \in N$. The claim follows and $N \cap Z(\mathbb{H}_n(\mathbb{Q}_p))$ is a nontrivial closed subgroup of $Z(\mathbb{H}_n(\mathbb{Q}_p))$, hence $Z(\mathbb{H}_n(\mathbb{Q}_p))/(N \cap Z(\mathbb{H}_n(\mathbb{Q}_p)))$ is torsion because nontrivial quotient of $\mathbb{Q}_p$. On the other hand, $Z(\mathbb{H}_n(\mathbb{Q}_p))/(N \cap Z(\mathbb{H}_n(\mathbb{Q}_p))) \cong \varphi(Z(\mathbb{H}_n(\mathbb{Q}_p)))$ is a subgroup of $\mathbb{H}_n(\mathbb{Q}_p)$ (up to continuous isomorphisms), hence torsion-free. Consequently $Z(\mathbb{H}_n(\mathbb{Q}_p))/(N \cap Z(\mathbb{H}_n(\mathbb{Q}_p)))$ is trivial, and the other claim $Z(\mathbb{H}_n(\mathbb{Q}_p)) \subseteq N$ follows. Since N contains $Z(\mathbb{H}_n(\mathbb{Q}_p)) = [\mathbb{H}_n(\mathbb{Q}_p), \mathbb{H}_n(\mathbb{Q}_p)]$, we may apply Addition Theorems as per Corollary 3.3(b), hence $\mathsf{h}_{\mathsf{top}}(\varphi) = \mathsf{h}_{\mathsf{top}}(\bar{\varphi}_{\mathbb{H}_n(\mathbb{Q}_p)/N})$ is finite by Theorem 3.5. Therefore the result follows. $\square$

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, Topological entropy, *Trans. Amer. Math. Soc.*, **114** (1965), 309–319.

[2] M. Bonatto and D. Dikranjan, Generalized Heisenberg groups and self-duality, *Quest. Answ. Gen. Topol.*, **37** (2019), 89–108.

[3] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.*, **153** (1971), 401–414.

[4] A. Caldas and M. Patrão, Entropy and its variational principle for locally compact metrizable systems, *Ergodic Theory Dynam. Systems*, **38** (2018), 540–565.

[5] D. van Dantzig, *Studien over Topologische Algebra*, Dissertation (Amsterdam, 1931).

[6] G. De Marco and A. Orsatti, Complete linear topologies on abelian groups, *Sympos. Math.*, **13** (1974),153–161.

[7] D. Dikranjan and M. Sanchis, Dimension and entropy in compact topological groups, *J. Math. Anal. Appl.*, **476** (2019), 337–366.

[8] D. Dikranjan, A. Giordano Bruno and F. G. Russo, Finiteness of topological entropy for locally compact abelian groups, *Glasgow Math. J.*, **361** (2020), 403–442.

[9] L. Fuchs, *Abelian Groups*, Springer (Cham, 2015).

[10] A. Giordano Bruno and S. Virili, Topological entropy in totally disconnected locally compact groups, *Ergod. Theory Dynam. Systems*, **37** (2017), 2163–2186.

[11] T. Grundhöfer and M. Stroppel, Automorphisms of Verardi groups: small upper triangular matrices over rings, *Beitr. Algebra Geom.*, **49** (2008), 1–31.

[12] W. Herfort, K. H. Hofmann and F. G. Russo, *Periodic Locally Compact Groups*, de Gruyter (Berlin, 2019).

[13] K. H. Hofmann and S. A. Morris, *The Structure of Compact Groups*, de Gruyter (Berlin, 2020).

[14] B. M. Hood, Topological entropy and uniform spaces, *J. London Math. Soc.*, **8** (1974), 633–641.

[15] C. R. Leedham-Green and S. McKay, *The Structure of Groups of Prime Power Order*, Oxford University Press (Oxford, 2002).
[16] D. A. Lind and T. Ward, Automorphisms of solenoids and p-adic entropy, *Ergod. Theory Dynam. Systems*, **8** (1988), 411–419.
[17] M. Patrão, The topological entropy of endomorphisms of Lie groups, *Israel J. Math.*, **234** (2019), 55–80.
[18] J. Peters, Entropy of automorphisms on locally compact abelian groups, *Pacific J. Math.*, **96** (1981), 475–488.
[19] L. Ribes and P. Zalesskii, *Profinite Groups*, Springer (Berlin, 2000).
[20] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer (New York, 1996).
[21] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser (Basel, 1995).
[22] S. Virili, Entropy for endomorphisms of locally compact abelian groups, *Topology Appl.*, **159** (2012), 2546–2556.
[23] P. Walters, *An Introduction to Ergodic Theory*, Springer (Berlin, 1969).
[24] S. Yuzvinski, Metric properties of endomorphisms of compact groups, *Izv. Acad. Nauk SSSR, Ser. Mat.*, **29** (1965), 1295–1328 (in Russian); translation in *Amer. Math. Soc. Transl.*, **66** (1968), 63–98.