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EGERVÁRY'S THEOREMS FOR HARMONIC TRINOMIALS

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Abstract. We study the arrangements of the roots in the complex plane for the lacunary harmonic polynomials called harmonic trinomials. We provide necessary and sufficient conditions so that two general harmonic trinomials have the same set of roots up to a rotation around the origin in the complex plane, a reflection over the real axis, or a composition of the previous both transformations. This extends the results of Jenő Egerváry given in [19] for the setting of trinomials to the setting of harmonic trinomials.

1. Introduction, main results and their consequences

1.1. Introduction. The computation and the quantitative location of the roots for polynomials are important in many research areas, and therefore a vast literature in both pure and applied mathematics has been produced; we refer to [6,40,44–47] and the references therein.

Given two positive integers m and n, a *trinomial* of degree $n+m$ is a lacunary polynomial with three terms of the form

$$T(z) := Az^{n+m} + Bz^m + C \quad \text{for all } z \in \mathbb{C}, \tag{1.1}$$

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where A, B and C are non-zero complex numbers. Despite the apparent simplicity of (1.1), the well-known works of P. Ruffini, N. H. Abel and É. Galois imply that for $n+m \geq 5$ and generic trinomials of the form (1.1) there is no formula for their roots in terms of the so-called radicals. For the literature reporting geometric, topological, quantitative and qualitative behavior of the roots for trinomials of the form (1.1) we refer to [2–5,7–10, 14–20,27,32,33,41–43,48,49] and the references therein.

In [19] J. Egerváry analyzes the roots of general trinomials. More precisely, he studied

(I) the arrangements of the roots of (1.1) in the complex plane, that is, provides necessary and sufficient conditions so that two general *trinomials* possess the same set of roots up to a rotation around the origin in the complex plane, a reflection over the real axis, or a composition of the previous both transformations. The latter is an equivalence relation, which in the sequel we refer to as its Egerváry equivalent, see Definition 1.1 below;

(II) the description of geometric sectors for the localization of the roots of (1.1).

Since [19] is written in Hungarian, many of the results given there have been rediscovered afterwards. We refer to [48] for an English review of [19].

In this paper, we extend (I) to the setting of harmonic trinomials; more precisely, to the setting of lacunary harmonic polynomials of the form

$$H(z) := Az^{n+m} + B\overline{z}^{m} + C \quad \text{for all } z \in \mathbb{C}, \tag{1.2}$$

where A, B and C are non-zero complex numbers, and $\overline{\zeta}$ denotes the complex conjugate of the given complex number ζ, see Theorem 1.2 below. As a consequence of Theorem 1.2 we obtain the following results.

(a) A characterization of the class of harmonic trinomials of the form (1.2) which are Egerváry equivalent with a harmonic trinomial with real coefficients, see Corollary 1.4 below.

(b) A harmonic trinomial of the form (1.2) with different roots having the same complex modulus is Egerváry equivalent to a harmonic trinomial with real coefficients, see Corollary 1.5 below.

(c) A harmonic trinomial of the form (1.2) with a root of multiplicity at least two is Egerváry equivalent to a harmonic trinomial with real coefficients, see Corollary 1.7 below.

In addition, Theorem 1.2 with the help of the following results in [1], Lemma 2.6, Lemma 2.11, Lemma A.3 and Proposition 2.3, yields the following statements.

(d) A geometric degenerate triangle condition on the modulus of the coefficients of (1.2) so that (1.2) is Egerváry equivalent to a harmonic trinomial with real coefficients, see Theorem 1.8 below.

(e) Two harmonic trinomials of the form (1.2) with roots having the same complex modulus (such roots may be different) and satisfying that the ratio between the complex modulus of their respective coefficients with the same degree is constant, are Egerváry equivalent, see Theorem 1.9 below.

By Bézout's Theorem ([52, Theorems 1, 5]) it follows that (1.2) has at most $(n+m)^2$ roots. Recently, in [1, Corollary 1.4], it is shown that (1.2) has at most $n+3m$ roots. Moreover, such bound is sharp in the sense that there exist harmonic trinomials with exactly $n+3m$ roots. In general, there exist harmonic polynomials with exactly $(n+m)^2$ roots, see for instance [52, Section 2] or [12, p. 2080].

Recently, the corresponding geometric sectors as in (II) for harmonic trinomials of the form (1.2) with $A=1$, $B \in \mathbb{C} \setminus \{0\}$, $C=-1$ has been derived in [21]. For references about location, counting, geometry, and lower/uppers bounds for the moduli of roots for harmonic polynomials including probabilistic approaches and numerical experiments, we refer to [1,11–13,21–26, 28–31,34–39,50–52] and the references therein.

1.2. Preliminaries and main results. In this subsection, we present the preliminaries and state the results of this manuscript. Given $m, n \in \mathbb{N} := \{1, 2, \ldots\}$, we consider two harmonic trinomials

$$h_1(z) := A_1 z^{n+m} + B_1 \overline{z}^m + C_1 \quad \text{for all } z \in \mathbb{C}, \tag{1.3}$$

and

$$h_2(z) := A_2 z^{n+m} + B_2 \overline{z}^m + C_2 \quad \text{for all } z \in \mathbb{C}, \tag{1.4}$$

where A_1, A_2, B_1, B_2, C_1, C_2 are non-zero complex numbers.

We start with the following definition, which rigorously encodes the arrangements of roots that are equivalent.

Definition 1.1 (Egerváry equivalent). Let h_1 and h_2 be the harmonic polynomials given in (1.3) and (1.4), respectively. We say h_1 and h_2 are Egerváry equivalent if and only if the set of roots of h_1 differs of the set of roots of h_2 by

(a) a rotation around the origin in the complex plane,
(b) a reflection over the real axis,
(c) a composition of both transformations given in (a) and (b).

More precisely, there exist a non-zero complex number c and a real number δ satisfying

$$h_1(z) = c h_2(e^{\mathrm{i}\delta} z) \quad \text{for all } z \in \mathbb{C} \tag{1.5}$$

or

$$h_1(z) = c \overline{h}_2(e^{\mathrm{i}\delta} z) \quad \text{for all } z \in \mathbb{C}, \tag{1.6}$$

where i denotes the imaginary unit and

$$\overline{h}_2(z) := \overline{A_2} z^{n+m} + \overline{B_2}\overline{z}^m + \overline{C_2} \quad \text{for all } z \in \mathbb{C}.$$

We note that Definition 1.1 defines an equivalence relation. In addition, we observe that (1.5) and (1.6) are mutually exclusive whenever some of the coefficients A_2, B_2 or C_2 is not a real number. Indeed, if (1.5) and (1.6) both hold true, we have

$$h_1(z) = ch_2(e^{\mathsf{i}\delta} z) = c\overline{h}_2(e^{\mathsf{i}\delta} z) \quad \text{for all } z \in \mathbb{C},$$

which yields

$$(A_2 - \overline{A_2}) z^{n+m} + (B_2 - \overline{B_2})\overline{z}^m + (C_2 - \overline{C_2}) = 0 \quad \text{for all } z \in \mathbb{C}.$$

The latter implies A_2, B_2 and C_2 are real numbers.

Along this manuscript, $|\zeta|$ denotes the complex modulus of the given complex number ζ. We recall that the polar representation of ζ is given by $\zeta = |\zeta| e^{\mathsf{i}\varphi}$, where $\varphi \in [0, 2\pi)$ is the argument of ζ. Moreover, for any real numbers x and y, we write that

$$x \equiv y \pmod{2\pi} \quad \text{if and only if} \quad x - y = 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

The first main result of this manuscript is the following extension of the results given in Equations (2) and (3) [19, p. 37] or the survey [48, Theorem 1], to the setting of harmonic trinomials. It reads as follows.

THEOREM 1.2 (Egerváry's theorem for harmonic trinomials). *Let h_1 and h_2 be the harmonic polynomials given in* (1.3) *and* (1.4), *respectively. Then the following holds true*: *h_1 and h_2 are Egerváry equivalent if and only if*

$$\left|\frac{A_1}{A_2}\right| = \left|\frac{B_1}{B_2}\right| = \left|\frac{C_1}{C_2}\right| \tag{1.7}$$

and

$$m(\alpha_1 \pm \alpha_2) + (n+m)(\beta_1 \pm \beta_2) - (n+2m)(\gamma_1 \pm \gamma_2) \equiv 0 \pmod{2\pi}, \tag{1.8}$$

where α_1, α_2, β_1, β_2, γ_1 and γ_2 are the arguments in the polar representation of A_1, A_2, B_1, B_2, C_1 and C_2, respectively.

REMARK 1.3 (about the choice of $\pm$). We point out that (1.8) reads

$$m(\alpha_1 + \alpha_2) + (n+m)(\beta_1 + \beta_2) - (n+2m)(\gamma_1 + \gamma_2) \equiv 0 \pmod{2\pi}$$

or

$$m(\alpha_1 - \alpha_2) + (n+m)(\beta_1 - \beta_2) - (n+2m)(\gamma_1 - \gamma_2) \equiv 0 \pmod{2\pi}.$$

The following corollary is an analog of the results given in Equations (1), (5) and (6) in [19, p. 38] or [48, p. 100] to the setting of harmonic trinomials. It reads as follows.

Corollary 1.4 (the class of harmonic trinomials with real coefficients). *Let* $h(z) = Az^{n+m} + B\overline{z}^m + C$ *for all* $z \in \mathbb{C}$ *be a harmonic trinomial whose coefficients* A, B *and* C *are non-zero complex numbers. We consider the polar representation of* A, B *and* C, *that is,* $A = |A|e^{i\alpha}$, $B = |B|e^{i\beta}$, $C = |C|e^{i\gamma}$. *Then the following statements are equivalent.*

(i) *The harmonic trinomial* h *is Egerváry equivalent to a harmonic trinomial with real coefficients.*

(ii) *The angular relation*

$$m\alpha + (n+m)\beta - (n+2m)\gamma \equiv 0 \pmod{\pi} \tag{1.9}$$

holds true.

(iii) *The coefficients* A, B *and* C *satisfy that*

$$\frac{A^m B^{n+m}}{C^{n+2m}} \quad \text{is a real number.}$$

The following corollary is an analog of Statement III in [19, p. 40].

Corollary 1.5 (different roots with the equal modulus). *Let* $h(z) = Az^{n+m} + B\overline{z}^m + C$ *for all* $z \in \mathbb{C}$ *be a harmonic trinomial whose coefficients* A, B *and* C *are non-zero complex numbers. We consider the polar representation of* A, B *and* C, *that is,* $A = |A|e^{i\alpha}$, $B = |B|e^{i\beta}$, $C = |C|e^{i\gamma}$. *Assume that* h *has at least two different roots with the same modulus. Then* h *is Egerváry equivalent to a harmonic trinomial with real coefficients.*

In the sequel, we remark that the converse of the Statement III in [19, p. 40] does not hold true in general.

Remark 1.6 (converse of Corollary 1.5 for degree three or more). Let $h(z) = z^2 + \frac{\sqrt{3}}{6}\overline{z} - \frac{1}{4}$ for all $z \in \mathbb{C}$. By [1, Corollary 1.4] we have that h has at most four different roots in $\mathbb{C}$. In fact, a straightforward computation yields that h has only two roots, which are given by

$$z_1 := \frac{-1-\sqrt{13}}{2\sqrt{12}} \approx -0.664 \quad \text{and} \quad z_2 := \frac{-1+\sqrt{13}}{2\sqrt{12}} \approx 0.376.$$

Since $|z_1| \neq |z_2|$, the converse of Corollary 1.5 is not valid when $n + m = 2$, i.e., $n = m = 1$.

For $n + m \in \mathbb{N} \setminus \{1, 2\}$ we claim that all the roots of $h(z) = Az^{n+m} + B\overline{z}^m + C$ for all $z \in \mathbb{C}$, where $A, B, C \in \mathbb{R} \setminus \{0\}$ cannot be real numbers. Indeed, by Descartes' rule of signs h has at most two real roots and hence h has at least one complex root ζ. It is not hard to see that $\overline{\zeta}$ is also a root of h and hence the converse of Corollary 1.5 holds true.

The following corollary is an analog of Statement IV in [19, p. 40].

Corollary 1.7 (root with multiplicity at least two). *Let $h(z) = Az^{n+m} + B\overline{z}^m + C$ for all $z \in \mathbb{C}$ be a harmonic trinomial whose coefficients A, B and C are non-zero complex numbers. We consider the polar representation of A, B and C, that is, $A = |A|e^{\mathrm{i}\alpha}$, $B = |B|e^{\mathrm{i}\beta}$, $C = |C|e^{\mathrm{i}\gamma}$. Assume that h has a root of multiplicity at least two with modulus r. Then h is Egerváry equivalent to a harmonic trinomial with real coefficients. Moreover,*

$$\frac{A^m B^{n+m}}{C^{n+2m}} = \frac{(-1)^{n+m}}{r^{2m(n+m)}} \frac{m^m (n+m)^{n+m}}{n^{n+2m}}.$$

Analogues of the following theorems are not given in [19]. We state them here since they are interesting on their own. They are deduced using Theorem 1.2 together with the following results in [1, Lemmas 2.6, 2.11, A.3 and Proposition 2.3].

Theorem 1.8 (geometric degenerate condition). *Let $h(z) = Az^{n+m} + B\overline{z}^m + C$ for all $z \in \mathbb{C}$ be a harmonic trinomial whose coefficients A, B and C are non-zero complex numbers. We consider the polar representation of A, B and C, that is, $A = |A|e^{\mathrm{i}\alpha}$, $B = |B|e^{\mathrm{i}\beta}$, $C = |C|e^{\mathrm{i}\gamma}$. Assume that there exists a root of h with modulus r such that $|A|r^{n+m}$, $|B|r^m$ and $|C|$ are the side lengths of some degenerate triangle. Then h is Egerváry equivalent to a harmonic trinomial with real coefficients of the form*

$$g_{u,v}(z) := u|A|z^{n+m} + v|B|\overline{z}^m + |C|, \quad z \in \mathbb{C}, \tag{1.10}$$

for some $u, v \in \{-1, 1\}$. Moreover, if n and m are co-prime numbers, then

(a) *for $|C| = |A|r^{n+m} + |B|r^m$ it follows that h is Egerváry equivalent to*

$$g(z) := |A|z^{n+m} + |B|\overline{z}^m - |C|, \quad z \in \mathbb{C},$$

(b) *for $|A|r^{n+m} = |B|r^m + |C|$, it follows that h is Egerváry equivalent to*

$$g(z) := |A|z^{n+m} - |B|\overline{z}^m - |C|, \quad z \in \mathbb{C},$$

(c) *for $|B|r^m = |A|r^{m+n} + |C|$, it follows that h is Egerváry equivalent to*

$$g(z) := |A|z^{n+m} - |B|\overline{z}^m + |C|, \quad z \in \mathbb{C}.$$

Theorem 1.9 (common root with the same modulus). *Let h_1 and h_2 be the harmonic polynomials given in* (1.3) *and* (1.4), *respectively. Assume that* (1.7) *holds true. In addition, assume that there exist ζ_1 and ζ_2 roots of h_1 and h_2, respectively, and satisfying $|\zeta_1| = |\zeta_2|$. Then*

$$m(\alpha_1 \pm \alpha_2) + (n+m)(\beta_1 \pm \beta_2) - (n+2m)(\gamma_1 \pm \gamma_2) \equiv 0 \pmod{2\pi}. \tag{1.11}$$

In particular, h_1 and h_2 are Egerváry equivalent.

2. Proofs of the results

PROOF OF THEOREM 1.2. Assume that h_1 and h_2 are Egerváry equivalent. By Definition 1.1 it is not hard to see that there exist a non-zero complex number c and a real number δ satisfying

$$h_1(e^{-\mathrm{i}\delta}z) = ch_2(z) \quad \text{for all } z \in \mathbb{C} \tag{2.1}$$

or

$$\overline{h}_1(e^{-\mathrm{i}\delta}z) = \overline{c}h_2(z) \quad \text{for all } z \in \mathbb{C}. \tag{2.2}$$

By (2.1) we have

$$\frac{A_1}{A_2} = \frac{B_1}{B_2}e^{\mathrm{i}(n+2m)\delta} = \frac{C_1}{C_2}e^{\mathrm{i}(n+m)\delta}, \tag{2.3}$$

which easily implies (1.7). By (2.3) we have

$$\alpha_1 - \alpha_2 \equiv \beta_1 - \beta_2 + (2m+n)\delta \equiv \gamma_1 - \gamma_2 + (n+m)\delta \pmod{2\pi}.$$

In particular, we obtain

$$\begin{cases} \gamma_1 - \gamma_2 \equiv \beta_1 - \beta_2 + m\delta \pmod{2\pi} \\ \alpha_1 - \alpha_2 \equiv \beta_1 - \beta_2 + (2m+n)\delta \pmod{2\pi}. \end{cases} \tag{2.4}$$

By (2.4) we have

$$\begin{aligned} &m(\alpha_1 - \alpha_2) + (n+m)(\beta_1 - \beta_2) - (n+2m)(\gamma_1 - \gamma_2) \qquad (2.5)\\ &\equiv m(\beta_1 - \beta_2 + (2m+n)\delta) + (n+m)(\beta_1 - \beta_2) - (n+2m)(\beta_1 - \beta_2 + m\delta) \\ &\equiv 0 \pmod{2\pi}. \end{aligned}$$

Analogously, (2.2) implies

$$m(\alpha_1 + \alpha_2) + (n+m)(\beta_1 + \beta_2) - (n+2m)(\gamma_1 + \gamma_2) \equiv 0 \pmod{2\pi}. \tag{2.6}$$

By (2.5) and (2.6) we deduce (1.8).

In the sequel, we assume that (1.7) and (1.8) are valid. In particular, we have

$$\frac{|A_1|}{|A_2|} = \frac{|B_1|}{|B_2|} = \frac{|C_1|}{|C_2|} =: r \tag{2.7}$$

and

$$m(\alpha_1 - \alpha_2) + (n+m)(\beta_1 - \beta_2) - (n+2m)(\gamma_1 - \gamma_2) \equiv 0 \pmod{2\pi}. \tag{2.8}$$

Without loss of generality, we can assume that $r=1$. Since $r=1$, (2.7) implies the existence of real numbers θ_1, θ_2 and θ_3 such that

$$A_1 = e^{\mathrm{i}\theta_1} A_2, \quad B_1 = e^{\mathrm{i}\theta_2} B_2 \quad \text{and} \quad C_1 = e^{\mathrm{i}\theta_3} C_2. \tag{2.9}$$

Then we have

$$\alpha_1 - \alpha_2 \equiv \theta_1 \pmod{2\pi}, \quad \beta_1 - \beta_2 \equiv \theta_2 \pmod{2\pi} \\ \gamma_1 - \gamma_2 \equiv \theta_3 \pmod{2\pi}. \tag{2.10}$$

By (2.8) and (2.10) we obtain

$$m(\theta_1 - \theta_3) + (n+m)(\theta_2 - \theta_3) \equiv 0 \pmod{2\pi}. \tag{2.11}$$

By Lemma A.1 in Appendix A we have that the solutions of (2.11) are parametrized as follows:

$$\theta_1 \equiv \theta_3 + (n+m)\delta \pmod{2\pi} \quad \text{and} \quad \theta_2 \equiv \theta_3 - m\delta \pmod{2\pi} \tag{2.12}$$

for some $\delta \in \mathbb{R}$. By (2.9) and (2.12) we obtain

$$\begin{aligned} h_1(z) &= A_1 z^{n+m} + B_1 \overline{z}^m + C_1 = A_2 e^{\mathrm{i}\theta_1} z^{n+m} + B_2 e^{\mathrm{i}\theta_2} \overline{z}^m + C_2 e^{\mathrm{i}\theta_3} \\ &= e^{\mathrm{i}\theta_3}(A_2 e^{\mathrm{i}(n+m)\delta} z^{n+m} + B_2 e^{-\mathrm{i}m\delta} \overline{z}^m + C_2) = e^{\mathrm{i}\theta_3} f_2(e^{\mathrm{i}\delta} z) \end{aligned}$$

for all $z \in \mathbb{C}$. In the case of

$$m(\alpha_1 + \alpha_2) + (n+m)(\beta_1 + \beta_2) - (n+2m)(\gamma_1 + \gamma_2) \equiv 0 \pmod{2\pi}$$

the proof is analogous and we omit it. □

Proof of Corollary 1.4. We start proving that (i) implies (ii). Since (i) holds true, h is Egerváry equivalent to a harmonic trinomial with real coefficients g. We write $g(z) = A_1 z^{n+m} + B_1 \overline{z}^m + C_1$, $z \in \mathbb{C}$, where A_1, B_1 and C_1 are real numbers. In particular,

$$\alpha_1 \equiv 0 \pmod{\pi}, \quad \beta_1 \equiv 0 \pmod{\pi} \quad \text{and} \quad \gamma_1 \equiv 0 \pmod{\pi}, \tag{2.13}$$

where α_1, β_1 and γ_1 are the arguments A_1, B_1 and C_1, respectively. By Theorem 1.2 (applied to h and g) we have

$$m(\alpha \pm \alpha_1) + (n+m)(\beta \pm \beta_1) - (n+2m)(\gamma \pm \gamma_1) \equiv 0 \pmod{2\pi},$$

which implies

$$m\alpha + (n+m)\beta - (n+2m)\gamma \equiv \mp m\alpha_1 \mp (n+m)\beta_1 \pm (n+2m)\gamma_1 \pmod{2\pi}. \tag{2.14}$$

By (2.13) for any $\ell_1, \ell_2, \ell_3 \in \{-1, 1\}$ we obtain

$$\ell_1 m\alpha_1 + \ell_2(n+m)\beta_1 + \ell_3(n+2m)\gamma_1 \equiv 0 \pmod{\pi}. \tag{2.15}$$

Hence (2.14) with the help of (2.15) yields (1.9). The proof of (i) $\Rightarrow$ (ii) is finished.

Now, we show that (ii) implies (i). Since (1.9) holds true, we have

$$m(\alpha-\gamma) + (n+m)(\beta-\gamma) \equiv 0 \pmod{\pi}.$$

By Lemma A.2 in Appendix A we obtain

$$\alpha \equiv \gamma + (n+m)\delta \pmod{\pi} \quad \text{and} \quad \beta \equiv \gamma - m\delta \pmod{\pi}$$

for some $\delta \in \mathbb{R}$. Then for any $z \in \mathbb{C}$ we obtain

$$\begin{aligned} h(z) &= |A|e^{\mathrm{i}\alpha} z^{n+m} + |B|e^{\mathrm{i}\beta}\overline{z}^m + |C|e^{\mathrm{i}\gamma} \\ &= e^{\mathrm{i}\gamma}\big(|A|e^{\mathrm{i}(\alpha-\gamma)} z^{n+m} + |B|e^{\mathrm{i}(\beta-\gamma)}\overline{z}^m + |C|\big) \\ &= e^{\mathrm{i}\gamma}\big(u|A|e^{\mathrm{i}(n+m)\delta} z^{n+m} + v|B|e^{-\mathrm{i}m\delta}\overline{z}^m + |C|\big) \\ &= e^{\mathrm{i}\gamma}\big(u|A|(e^{\mathrm{i}\delta}z)^{n+m} + v|B|(\overline{e^{\mathrm{i}\delta}z})^m + |C|\big), \end{aligned} \tag{2.16}$$

where $u, v \in \{-1, 1\}$. By (2.16) we deduce that h is Egerváry equivalent to the harmonic trinomial

$$g(z) := u|A|z^{n+m} + v|B|\overline{z}^m + |C| \quad \text{for all } z \in \mathbb{C}.$$

The proof of (ii) $\Rightarrow$ (i) is complete.

Finally, the equivalence of (ii) and (iii) is straightforward due to the relation

$$\frac{A^m B^{n+m}}{C^{n+2m}} = \frac{|A|^m |B|^{n+m}}{|C|^{n+2m}} e^{\mathrm{i}(m\alpha + (n+m)\beta - (n+2m)\gamma)}. \quad \square$$

Proof of Corollary 1.5. Assume that h has two different roots with modulus $r > 0$. After a rotation, one can see that h is Egerváry equivalent to a harmonic trinomial $\widetilde{h}$ with roots $\zeta_1 = r$ and $\zeta_2 = re^{\mathrm{i}\theta}$, where $\theta \in (0, 2\pi)$. Without loss of generality, we assume that $h = \widetilde{h}$. Since $h(\zeta_1) = h(\zeta_2) = 0$, we have

$$A = -\frac{C}{r^{n+m}} \frac{e^{\mathrm{i}m\theta} - 1}{e^{\mathrm{i}(n+2m)\theta} - 1} \quad \text{and} \quad B = -\frac{C}{r^m} \frac{e^{\mathrm{i}m\theta}(e^{\mathrm{i}(n+m)\theta} - 1)}{e^{\mathrm{i}(n+2m)\theta} - 1}.$$

Recall the identity

$$e^{\mathrm{i}t} - 1 = 2\mathrm{i} \cdot \sin\Big(\frac{t}{2}\Big) e^{\mathrm{i}\frac{t}{2}} \quad \text{for any } t \in \mathbb{R}.$$

Then we have

$$\frac{A^m B^{n+m}}{C^{n+2m}} = \frac{1}{r^{2m(n+m)}}(-1)^n \frac{\sin^m(\frac{m\theta}{2})\sin^{n+m}(\frac{(n+m)\theta}{2})}{\sin^{n+2m}(\frac{(n+2m)\theta}{2})},$$

which with the help of (iii) of Corollary 1.4 yields the statement. □

Proof of Corollary 1.7. After a rotation, without loss of generality, we can assume that h has a real root $r > 0$ with multiplicity at least two. The function $\gamma(x) := Ax^{n+m} + Bx^m + C$, $x \in \mathbb{R}$ represents a curve in the complex plane $\mathbb{C}$. Since r is a root of h with multiplicity at least two and $h(x) = \gamma(x)$ for all $x \in \mathbb{R}$, we have $\gamma(r) = \gamma'(r) = 0$, where γ' denotes the derivative of γ. The latter reads as follows

$$Ar^{n+m} + Br^m + C = 0 \quad \text{and} \quad (n+m)Ar^{n+m-1} + mBr^{m-1} = 0,$$

so $A = \frac{mC}{r^{n+m}n}$, $B = -\frac{(n+m)C}{r^m n}$. A straightforward computation yields

$$\frac{A^m B^{n+m}}{C^{n+2m}} = \frac{(-1)^{n+m}}{r^{2m(n+m)}} \frac{m^m (n+m)^{n+m}}{n^{n+2m}},$$

which with the help of (iii) of Corollary 1.4 yields the statement. □

Proof of Theorem 1.8. By (ii) of Corollary 1.4, it is enough to show (1.9). By hypothesis we have that $h(r) = 0$. We assume that n and m are co-prime numbers. Since $|A|r^{n+m}$, $|B|r^m$ and $|C|$ are the side lengths of some degenerate triangle, the contrapositive of [1, Lemma 2.11] applied to $\widetilde{h}(z) := e^{-\mathrm{i}\gamma}h(z)$, $z \in \mathbb{C}$ yields

$$(n+m)(\beta - \gamma) + m(\alpha - \gamma) \equiv 0 \pmod{\pi}.$$

The latter implies (1.9). Moreover, by (2.16) we have that h is Egerváry equivalent to $g_{u,v}$ for some $u, v \in \{-1, 1\}$, where $g_{u,v}$ is defined in (1.10).

We continue with the proof when $d := \gcd(n+m, m) \in \{2, \ldots, m\}$. Observe that $\gcd(n, m) = d$. Let $n' := n/d$ and $m' := m/d$ and note that $\gcd(n', m') = 1$. Since h has a root of modulus r, the harmonic trinomial

$$H(z) := A_1 z^{n'+m'} + B_1 \overline{z}^{m'} + C_1 \quad \text{for all } z \in \mathbb{C}$$

has a root of modulus r^d. Then the previous discussion for the co-prime case implies

$$(n'+m')(\beta - \gamma) + m'(\alpha - \gamma) \equiv 0 \pmod{\pi}.$$

Multiplying by d in both sides the preceding inequality yields

$$(n+m)(\beta - \gamma) + m(\alpha - \gamma) \equiv 0 \pmod{\pi}.$$

In addition, H is Egerváry equivalent to

$$\widetilde{g}_{u,v}(z) := u|A|z^{n'+m'} + v|B|\overline{z}^{m'} + |C|, \quad z \in \mathbb{C},$$

for some $u, v \in \{-1, 1\}$. The change of variable $z \mapsto z^d$ yields that h is Egerváry equivalent to $g_{u,v}$ for some $u, v \in \{-1, 1\}$.

In the sequel, we show (a). We start with the following observation. The relation $|C| = |A|r^{n+m} + |B|r^m$ holds true if and only if $g_{-1,-1}(r) = 0$. By Descartes' rule of signs we have that r is the unique positive real number satisfying $g_{-1,-1}(r) = 0$. By Theorem 1.2 one can verify that $g_{-1,-1}$ is Egerváry equivalent to

$$\begin{cases} g_{-1,1} & \text{if and only if } n+m \text{ is an even number,} \\ g_{1,1} & \text{if and only if } n \text{ is an even number,} \\ g_{1,-1} & \text{if and only if } m \text{ is an even number,} \end{cases} \tag{2.17}$$

where $g_{u,v}$ is defined in (1.10).

Now, we assume that n and m are co-prime numbers. Then we claim that $g_{-1,1}$, $g_{1,1}$ and $g_{1,-1}$ are never Egerváry equivalent between them. Indeed, we start assuming that $n + m$ is an even number. By (2.17) we have $g_{-1,-1}$ is Egerváry equivalent to $g_{-1,1}$. Since n and m are co-prime numbers, the assumption that $n + m$ is an even number imply that n and m are odd numbers. Recall that being Egerváry equivalent is an equivalence relation. Hence (2.17) yields that $g_{-1,1}$ cannot be Egerváry equivalent to $g_{1,1}$ neither $g_{1,-1}$ when $n + m$ is an even number.

We now claim that $g_{1,1}$ and $g_{1,-1}$ do not have a root of modulus r. We start showing that $g_{1,-1}$ does not have a root of modulus r. Indeed, by contradiction assume that there exists $\zeta = re^{\mathrm{i}\theta}$ with $\theta \in [0, 2\pi)$ such that $g_{1,-1}(\zeta) = 0$, that is,

$$|A|r^{n+m}e^{\mathrm{i}(\theta(n+m))} + |B|r^m e^{-\mathrm{i}(\theta m+\pi)} + |C| = 0.$$

Since $|C| = |A|r^{n+m} + |B|r^m$, Lemma A.3 in [1, Appendix A] implies

$$\theta(n + m) \equiv \pi \pmod{2\pi} \quad \text{and} \quad -\theta m - \pi \equiv \pi \pmod{2\pi}.$$

Then we have

$$\theta = \frac{(2k+1)\pi}{n+m} = \frac{2\pi k'}{m}$$

for some $k, k' \in \mathbb{Z}$. Hence, $(2k + 1)m = 2k'(n + m)$, which is a contradiction since m and $2k + 1$ are odd numbers.

Now, we prove that $g_{1,1}$ has no root of modulus r. By contradiction, assume that there exists a root of $g_{1,1}$ of the form $\zeta = re^{\mathrm{i}\theta}$ with $\theta \in [0, 2\pi)$. Then it follows that

$$|A|r^{n+m}e^{\mathrm{i}(\theta(n+m))} + |B|r^m e^{-\mathrm{i}(\theta m)} + |C| = 0.$$

Similarly to the previous case, we obtain

$$\theta(n+m) \equiv \pi \pmod{2\pi} \quad \text{and} \quad -\theta m \equiv \pi \pmod{2\pi},$$

which implies $(2k+1)m = (2k'+1)(n+m)$ for some $k, k' \in \mathbb{Z}$. This yields a contradiction since m and $2k+1$ are odd numbers and $n+m$ is an even number.

Observe that (1.10) yields that h is Egerváry equivalent to $g_{u,v}$ for some $u, v \in \{-1, 1\}$. The preceding analysis implies that h is Egerváry equivalent to $g_{-1,-1}$, which is also Egerváry equivalent to $g_{-1,1}$.

The proof when n is an even number and the proof when m is an even number follow similarly and we omit them. In summary, the proof of (a) is complete. Moreover, the proofs of (b) and (c) are analogous. □

Proof of Theorem 1.9. Since (1.7) is valid, without loss of generality we assume that

$$\left|\frac{A_1}{A_2}\right| = \left|\frac{B_1}{B_2}\right| = \left|\frac{C_1}{C_2}\right| = 1,$$

that is, $|A_1| = |A_2|$, $|B_1| = |B_2|$ and $|C_1| = |C_2|$. Let $r > 0$ be fixed. Then the following straightforward remark is true: $|A_1|r^{n+m}$, $|B_1|r^m$ and $|C_1|$ are the side lengths of a triangle Δ_1 (it may be degenerate), if and only if, $|A_2|r^{n+m}$, $|B_2|r^m$ and $|C_2|$ are the side lengths of a triangle Δ_2. In fact, Δ_1 and Δ_2 are congruent.

By hypothesis, h_1 and h_2 have roots (such roots may be different) of modulus r for some $r > 0$. The proof is divided in three cases accordingly to $|A_1|r^{n+m}$, $|B_1|r^m$ and $|C_1|$ are the side lengths of some triangle.

We now assume that n and m are co-prime numbers.

Case (1). Assume that $|A_1|r^{n+m}$, $|B_1|r^m$ and $|C_1|$ are not the side lengths of any triangle. By [1, Lemma 2.6] we have that there is no root of modulus r for the harmonic trinomial h_1 and h_2, which yields a contradiction.

Case (2). Assume that $|A_1|r^{n+m}$, $|B_1|r^m$ and $|C_1|$ are the side lengths of some triangle. For each $j \in \{1, 2\}$, we set the corresponding pivotals

$$\text{(2.18)} \qquad \begin{cases} P_{*,j} = \dfrac{(n+m)(\beta_j - \gamma_j - \pi) + m(\alpha_j - \gamma_j - \pi)}{2\pi}, \\[2ex] \omega_{*,j} = \dfrac{(n+m)w_1 - mw_2}{2\pi}, \end{cases}$$

where w_1 and w_2 are the angles opposite to the side lengths $|A_1|r^{n+m}$ and $|A_2|r^m$, respectively. We note that $\omega_{*,1}(r) = \omega_{*,2}(r)$. By [1, Proposition 2.3] for each $j = 1, 2$ we have that $P_{*,j} + \omega_{*,j}$ or $P_{*,j} - \omega_{*,j}$ are integers numbers. If $P_{*,1} + \omega_{*,1}(r)$ and $P_{*,2} + \omega_{*,2}(r)$ are integers, then $P_{*,1} - P_{*,2}$ is an integer and by (2.18) we deduce

$$(n+m)(\beta_1 - \beta_2) + m(\alpha_1 - \alpha_2) - (n+2m)(\gamma_1 - \gamma_2) \equiv 0 \pmod{2\pi}.$$

The remaining cases are similar and hence we omit their proofs.

Case (3). Assume that $|A_1|r^{n+m}$, $|B_1|r^m$ and $|C_1|$ are the side lengths of some degenerate triangle. By (a), (b) and (c) of Theorem 1.8, h_1 and h_2 are Egerváry equivalent. Hence (1.8) in Theorem 1.2 yields (1.11).

By Case (1), Case (2) and Case (3) we finish the proof for the co-prime setting.

We continue with the proof of (1.11) for $d := \gcd(n+m, m) \in \{2, \ldots, m\}$. Let $n' := n/d$ and $m' := m/d$ and note that $\gcd(n', m') = 1$. Since h_1 and h_2 have roots (such roots may be different) of modulus r for some $r > 0$, the harmonic trinomials

$$H_1(z) := A_1 z^{n'+m'} + B_1 \overline{z}^{m'} + C_1 \quad \text{for all } z \in \mathbb{C}$$

and

$$H_2(z) := A_2 z^{n'+m'} + B_2 \overline{z}^{m'} + C_2 \quad \text{for all } z \in \mathbb{C}$$

have roots (such roots may be different) of modulus r^d. Then the previous discussion for the co-prime case implies

$$m'(\alpha_1 \pm \alpha_2) + (n'+m')(\beta_1 \pm \beta_2) - (n'+2m')(\gamma_1 \pm \gamma_2) \equiv 0 \pmod{2\pi}. \tag{2.19}$$

Multiplying by d in both sides of (2.19) gives (1.11). Finally, Theorem 1.2 yields that h_1 and h_2 are Egerváry equivalent. □

Appendix A. Tools

This section contains auxiliary results that help us to make this paper more fluid.

LEMMA A.1 (linear Diophantine solutions I). *Let $n, m \in \mathbb{N}$ be fixed. Then the solutions x_1, x_2, x_3 of the linear Diophantine equation*

$$m(x_1 - x_3) + (n+m)(x_2 - x_3) \equiv 0 \pmod{2\pi} \tag{A.1}$$

can be parametrized as

$$x_1 \equiv x_3 + (n+m)\delta \pmod{2\pi} \quad \text{and} \quad x_2 \equiv x_3 - m\delta \pmod{2\pi} \tag{A.2}$$

for some $\delta \in \mathbb{R}$.

Proof. Note that (A.1) reads as follows

$$m(x_1 - x_3) + (n+m)(x_2 - x_3) = 2\pi k \quad \text{for some } k \in \mathbb{Z}. \tag{A.3}$$

Let $k \in \mathbb{Z}$ be fixed. We observe that the solutions of the homogeneous equation

$$m(x_1 - x_3) + (n+m)(x_2 - x_3) = 0$$

can be parametrized by $x_1 - x_3 = (n+m)\delta$ and $x_2 - x_3 = -m\delta$ for $\delta \in \mathbb{R}$. If $k = 0$ we immediately obtain (A.2). Then we assume that $k \in \mathbb{Z} \setminus \{0\}$. One can see that the solutions of (A.3) can be parametrized by

$$x_1 - x_3 = (n+m)\delta + 2\pi z_1 \quad \text{and} \quad x_2 - x_3 = -m\delta + 2\pi z_2 \quad \text{for } \delta \in \mathbb{R}, \tag{A.4}$$

where $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$ is a particular solution of the linear Diophantine equation $mz_1 + (n+m)z_2 = k$. Now, we assume that $\gcd(n+m, m) = 1$. Then Bézout's Identity implies that there exist $z_1^* \in \mathbb{Z}$ and $z_2^* \in \mathbb{Z}$ satisfying $mz_1^* + (n+m)z_2^* = k$. Choosing $z_1 = z_1^*$ and $z_2 = z_2^*$ in (A.4), we obtain (A.2).

We continue with the proof of (A.2) for $d := \gcd(n+m, m) \in \{2, \ldots, m\}$. Let $n' := n/d$ and $m' := m/d$ and note that $\gcd(n', m') = 1$. We rewrite (A.3) as $m'(x_1' - x_3') + (n' + m')(x_2' - x_3') = 2\pi k$ for some $k \in \mathbb{Z}$, where $x_1' = x_1 d$, $x_2' := x_2 d$ and $x_3' := x_3 d$. Since $\gcd(n' + m', m') = 1$, the previous reasoning yields

$$x_1' - x_3' = (n' + m')\delta + 2\pi z_1' \quad \text{and} \quad x_2' - x_3' = -m'\delta + 2\pi z_2' \quad \text{for } \delta \in \mathbb{R}, \tag{A.5}$$

and some integers z_1' and z_2'. Multiplying by d in both sides of the equalities given in (A.5) we obtain (A.2). □

Lemma A.2 (linear Diophantine solutions II). *Let $n, m \in \mathbb{N}$ be fixed. Then the solutions x_1, x_2, x_3 of the linear Diophantine equation*

$$m(x_1 - x_3) + (n+m)(x_2 - x_3) \equiv 0 \pmod{\pi}$$

can be parametrized as

$$x_1 \equiv x_3 + (n+m)\delta \pmod{\pi} \quad \text{and} \quad x_2 \equiv x_3 - m\delta \pmod{\pi}$$

for some $\delta \in \mathbb{R}$.

Proof. The proof follows step by step that of Lemma A.1 replacing 2π by π. □

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References

[1] G. Barrera, W. Barrera and J. P. Navarrete, On the number of roots of harmonic trinomials, *J. Math. Anal. Appl.*, **514** (2022), Paper No. 126313, 14 pp.

[2] G. Barrera, W. Barrera and J. P. Navarrete, The stability region for Schur stable trinomials with general complex coefficients, arXiv:2304.09147 (2023).

[3] W. Barrera, J. C. Magaña and J. P. Navarrete, Trinomials, torus knots and chains, *Trans. Amer. Math. Soc.*, **376** (2023), 2963–3004.

[4] D. Belkić, All the trinomial roots, their powers and logarithms from the Lambert series, Bell polynomials and Fox–Wright function: illustration for genome multiplicity in survival of irradiated cells, *J. Math. Chem.*, **57** (2019), 59–106.

[5] P. Bohl, Zur Theorie der trinomischen Gleichungen, *Math. Ann.*, **65** (1908), 556–566.

[6] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Graduate Texts in Math. vol. 161, Springer-Verlag (New York, 1995).

[7] V. Botta and J. da Silva, On the behavior of roots of trinomial equations, *Acta Math. Hungar.*, **157** (2019), 54–62.

[8] M. Brilleslyper, J. Brooks, M. Dorff, R. Howell and L. Schaubroeck, Zeros of a one-parameter family of harmonic trinomials, *Proc. Amer. Math. Soc. Ser. B*, **7** (2020), 82–90.

[9] M. Brilleslyper and L. Schaubroeck, Counting interior roots of trinomials, *Math. Mag.*, **91** (2018), 142–150.

[10] M. Brilleslyper and L. Schaubroeck, Locating unimodular roots, *College Math. J.*, **45** (2014), 162–168.

[11] J. Brooks, M. Dorff, A. Hudson, E. Pitts, C. Whiffen and A. Woodall, Zeros of a family of complex-valued harmonic trinomials, *Bull. Malays. Math. Sci. Soc.*, **45** (2022), 1079–1091.

[12] D. Bshouty, W. Hengartner and T. Suez, The exact bound on the number of zeros of harmonic polynomials, *J. Anal. Math.*, **67** (1995), 207–218.

[13] D. Bshouty and A. Lyzzaik, On Crofoot–Sarason's conjecture for harmonic polynomials, *Comput. Methods Funct. Theory*, **4** (2004), 35–41.

[14] J. Čermák, Stability conditions for linear delay difference equations: a survey and perspective, *Tatra Mt. Math. Publ.*, **63** (2015), 1–29.

[15] J. Čermák, L. Fedorková and M. Kureš, Complete classification scheme for the distribution of trinomial zeros with respect to their moduli, *Publ. Math. Debrecen*, **101** (2022), 119–146.

[16] J. Čermák and L. Fedorková, On a nearly forgotten polynomial result by P. Bohl, *Amer. Math. Monthly*, **130** (2022), 176–181.

[17] J. Čermák and J. Jánský, Explicit stability conditions for a linear trinomial delay difference equation, *Appl. Math. Lett.*, **43**, (2015), 56–60.

[18] J. Čermák, J. Jánský and L. Nechvátal, Exact versus discretized stability regions for a linear delay differential equation, *Appl. Math. Comput.*, **347** (2019), 712–722.

[19] J. Egerváry, A trinom egyenletről, *Matematikai és Fizikai Lapok*, **37** (1930), 36–57 (in Hungarian).

[20] H. Fell, The geometry of zeros of trinomial equations, *Rend. Circ. Mat. Palermo*, **29** (1980), 303–336.

[21] L. Gao, J. Gao and G. Liu, Location of the zeros of harmonic trinomials, *Bull. Malays. Math. Sci. Soc.*, **46** (2023), 549–555.

[22] H. L. Geleta and O. A. Alemu, Location of the zeros of certain complex-valued harmonic polynomials, *J. Math.*, 2022, Paper No. 4886522, 5 pp.

[23] H. L. Geleta and O. A. Alemu, Zeros of a two-parameter family of harmonic quadrinomials, *SINET: Ethiop. J. Sci.*, **45** (2022), 105–114.

[24] L. Geyer, Sharp bounds for the valence of certain harmonic polynomials, *Proc. Amer. Math. Soc.*, **136** (2008), 549–555.

[25] C. Harrat, Zeros of harmonic polynomials with complex coefficients, *Colloq. Math.*, **168** (2022), 119–126.

[26] J. Hauenstein, A. Lerario, E. Lundberg and D. Mehta, Experiments on the zeros of harmonic polynomials using certified counting, *Exp. Math.*, **24** (2015), 133–141.

[27] R. Howell and D. Kyle, Locating trinomial zeros, *Involve*, **11** (2018), 711–720.

[28] D. Khavinson, S. Lee and A. Saez, Zeros of harmonic polynomials, critical lemniscates, and caustics, *Complex Anal. Synerg.*, **4** (2018), 1–20.

[29] D. Khavinson and G. Neumann, From the fundamental theorem of algebra to astrophysics: a "harmonious" path, *Notices Amer. Math. Soc.*, **55** (2008), 666–675.

[30] D. Khavinson and G. Neumann, On the number of zeros of certain rational harmonic functions, *Proc. Amer. Math. Soc.*, **134** (2006), 1077–1085.

[31] D. Khavinson and G. Świaątek, On the number of zeros of certain harmonic polynomials, *Proc. Amer. Math. Soc.*, **131** (2003), 409–414.

[32] M. Kipnis and R. Nigmatullin, Stability of the trinomial linear difference equations with two delays, *Avtomat. i Telemekh.*, no. 11 (2004), 25–39 (in Russian); translation in *Autom. Remote Control*, **65** (2004), 1710–1723.

[33] S. Kuruklis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, *J. Math. Anal. Appl.*, **188** (1994), 719–731.

[34] S.-Y. Lee, A. Lerario and E. Lundberg, Remarks on Wilmshurst's theorem, *Indiana Univ. Math. J.*, **64** (2015), 1153–1167.

[35] S.-Y. Lee and A. Saez, A new lower bound for the maximal valence of harmonic polynomials, *Methods Funct. Theory*, **17** (2017), 139–149.

[36] A. Lerario and E. Lundberg, On the zeros of random harmonic polynomials: the truncated model, *J. Math. Anal. Appl.*, **438** (2016), 1041–1054.

[37] W. Li and A. Wei, On the expected number of zeros of a random harmonic polynomial, *Proc. Amer. Math. Soc.*, **137** (2009), 195–204.

[38] E. Lundberg, The valence of harmonic polynomials viewed through the probabilistic lens, *Proc. Amer. Math. Soc.*, **151** (2023), 2963–2973.

[39] E. Lundberg and A. Thomack, On the average number of zeros of random harmonic polynomials with i.i.d. coefficients: precise asymptotics, arXiv:2308.10333 (2023).

[40] M. Marden, *Geometry of Polynomials*, Mathematical Surveys and Monographs, vol. 3, Amer. Math. Soc. (Providence, RI, 1949).

[41] H. Matsunaga and C. Hajiri, Exact stability sets for a linear difference system with diagonal delay, *J. Math. Anal. Appl.*, **369** (2010), 616–622.

[42] A. Melman, Geometry of Polynomials, *Pacific J. Math.*, **259** (2012), 141–159.
[43] P. Nekrassoff, Über trinomische Gleichungen, *Math. Ann.*, **29** (1887), 413–430.
[44] V. Pan, Solving a polynomial equation: some history and recent progress, *SIAM Rev.*, **39** (1997), 187–220.
[45] V. Prasolov, Roots of polynomials, in: *Polynomials*, Translated from the 2001 Russian second edition by Dimitry Leites, Algorithms Comput. Math., vol. 11, Springer-Verlag (Berlin, 2004).
[46] Q. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Mathematical Society Monographs, New Series, vol. 26, Clarendon Press, Oxford University Press (Oxford, 2002).
[47] T. Sheil-Small, *Complex Polynomials*, Cambridge Studies in Advanced Math., vol. 75, Cambridge University Press (2002).
[48] P. Szabó, On the roots of the trinomial equation, *Cent. Eur. J. Oper. Res.*, **18** (2010), 97–104.
[49] T. Theobald and T. de Wolff, Norms of roots of trinomials, *Math. Ann.*, **366** (2016), 219–247.
[50] A. Thomack, On the zeros of random harmonic polynomials: the naive model, arXiv:1610.02611 (2016).
[51] A. Thomack and Z. Tyree, On the zeros of random harmonic polynomials: the Weyl model, *Anal. Math. Phys.*, **8** (2018), 237–253.
[52] A. Wilmshurst, The valence of harmonic polynomials, *Proc. Amer. Math. Soc.*, **126** (1998), 2077–2081.