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ERRATUM TO "AFFINE SURFACES IN $\mathbf{R}^4$ WITH PLANAR GEODESICS WITH RESPECT TO THE AFFINE METRIC"*

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Abstract. In this work a mistake in the paper is corrected. There is also a new proof of the main theorem which classifies the non-degenerate affine surfaces in $\mathbf{R}^4$ having planar geodesics with respect to the affine metric.

1. Introduction

In the paper the following theorem is stated.

THEOREM 1. *The only nondegenerate surfaces in* $\mathbf{R}^4$ *whose geodesics with respect to the Levi-Civita connection of the affine metric are planar are the complex parabola* $x(u,v)=(u,v,uv,u^2-v^2)$ *and the product of two parabolas* $x(u,v)=(u,v,u^2,v^2)$.

The theorem remains true but there is an essential mistake in the proof. It turns out that the statement just after equation (3.6) is not true in general. We said that we can replace X_1, h^i, τ^i_j, S_1 by $\bar{X}_1$, $\bar{h}^i$, $\bar{\tau}^i_j$ and $\bar{S}_1$, respectively, in conditions (3.4), (3.5) and (3.6). In fact we could apply such replacements

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if the terms $\tau^i_j(X_k)$ have depended on ξ_i at the point x_0 but they depend on the behaviour of ξ_i in a neighbourhood of x_0 along the geodesic γ.

Now we give a proper proof. We again use Burstin–Mayer transversal bundle whose induced connection ∇ is defined by $\nabla_X Y = \hat{\nabla}_X Y + \frac{1}{2}g(X,Y)\Delta_g x$, where $\hat{\nabla}$ is the Levi-Civita connection of the affine metric g. By $\{X_1, X_2\}$ we denote a local orthonormal frame, that is $g(X_1,X_1)=\varepsilon$, $g(X_1,X_2)=0$ and $g(X_2,X_2)=1$,where $\varepsilon=1$ for definite surfaces and $\varepsilon=-1$ for indefinite ones. Let x_0 be a fixed point of the surface, and v an arbitrary unit vector in $T_{x_0}M$ with respect to the affine metric (if the metric is indefinite, $g(v,v)=-1$). Let $\gamma=\gamma(t)$ be a geodesic passing through x_0 for $t=0$ with the velocity v. Let X_1 be a unit vector field along γ such that $X_1(x_0)$ $=v$. Let X_2 be a vector field along γ such that X_1, X_2 are orthonormal. If $\{\xi_1,\xi_2\}$ denotes the transversal frame associated to $\{X_1,X_2\}$, we have

$$\gamma'' = D_{\gamma'}\gamma' = \hat{\nabla}_{\gamma'}\gamma' + \frac{1}{2}g(\gamma',\gamma')\Delta_g x + h(\gamma',\gamma'). \tag{1}$$

Consequently,

$$\gamma''' = D_{\gamma'}\gamma'' = \frac{1}{2}\varepsilon\nabla_{\gamma'}\Delta_g x + \frac{1}{2}\varepsilon h(X_1,\Delta_g x) - S_{h(\gamma',\gamma')}\gamma' + \nabla^{\perp}_{\gamma'}h(\gamma',\gamma'). \tag{2}$$

The image of a geodesic γ is included in a plane if and only if at every point there exist $\alpha,\beta\in\mathbf{R}$ such that $\gamma'''=\alpha\gamma'+\beta\gamma''$. This condition yields that the following equalities holds:

$$\alpha\gamma' + \frac{1}{2}\varepsilon\beta\Delta_g x = \frac{1}{2}\varepsilon\nabla_{\gamma'}\Delta_g x - S_{h(\gamma',\gamma')}\gamma', \tag{3}$$

$$\beta h(\gamma',\gamma') = \frac{1}{2}\varepsilon h(\gamma',\Delta_g x) + \nabla^{\perp}_{\gamma'}h(\gamma',\gamma'). \tag{4}$$

Now we use the cubic form, which is defined by $C(X,Y,Z)=\nabla^{\perp}_X h(Y,Z)$ $-\,h(\nabla_X Y,Z)-h(Y,\nabla_X Z)$. It is well known that it is a $(0,3)$ tensor and is totally symmetric. Thus

$$\nabla^{\perp}_{\gamma'}h(\gamma',\gamma') = C(\gamma',\gamma',\gamma') + 2h(\nabla_{\gamma'}\gamma',\gamma')$$

$$= C(\gamma',\gamma',\gamma') + 2h\left(\hat{\nabla}_{\gamma'}\gamma' + \frac{1}{2}\varepsilon\Delta_g x,\gamma'\right) = C(\gamma',\gamma',\gamma') + 2h\left(\frac{1}{2}\varepsilon\Delta_g x,\gamma'\right).$$

Hence at $t=0$, equality (4) is equivalent to $\beta h(v,v)=3h\big(v,\frac{1}{2}\varepsilon\Delta_g x\big)+$ $C(v,v,v)$ and

$$h^1(v,v)\left(\frac{3}{2}\varepsilon h^2(v,\Delta_g x) + C^2(v,v,v)\right) \tag{5}$$

$$= h^2(v,v)\left(\frac{3}{2}\varepsilon h^1(v,\Delta_g x) + C^1(v,v,v)\right)$$

for $C = C^1\xi_1 + C^2\xi_2$. From this point we can do the proof of the main theorem as in the original paper because equality (3) and the last equality contain only tensors, but here we give a simpler reasoning using the result [1] of Vrancken. We will always assume for the contradiction that the Laplacian is different from zero at x_0. The fact that the Laplacian vanishes completes the proof because the connections ∇ and $\hat{\nabla}$ coincide.

2. Definite case

Let c be a function such that $\frac{1}{2}\Delta_g x = cX_1$, and let t, s denote $\cos(u)$ and $\sin(u)$ for $u \in \mathbf{R}$, resp. Then $v = tX_1 + sX_2$ is a unit vector and equation (5) is equivalent to

$$(t^2 - s^2)(3cs(t^2+s^2) + t^3C^2_{111} + 3t^2sC^2_{112} + 3ts^2C^2_{122} + s^3C^2_{222})$$

$$= 2ts(3ct(t^2+s^2) + t^3C^1_{111} + 3t^2sC^1_{112} + 3ts^2C^1_{122} + s^3C^1_{222})$$

where $C^i_{jkl} = C^i(X_j, X_k, X_l)$. Since the above equation is homogeneous with respect to t and s, we consider the coefficients at t^4s, t^2s^3 and s^5 and get

$$3C^2_{112} + 3c = 6c + 2C^1_{111}, \quad -3C^2_{112} + C^2_{222} = 6c + 6C^1_{122}, \quad -3c - C^2_{222} = 0.$$

These equations together with one of the conditions determining the Burstin–Mayer transversal bundle, $2C^1_{122} + C^1_{111} - C^2_{211} = 0$ give $C^1_{111} = -15c$, $C^1_{122} = 3c$. Since there exists a local function a that $\hat{\nabla}_{X_1}X_1 = aX_2$ and $\hat{\nabla}_{X_1}X_2 = -aX_1$, we have $\nabla_{X_1}X_1 = aX_2 + cX_1$ and $\nabla_{X_1}X_2 = -aX_1$. Using the expression $C^1(X,Y,Z) = (\nabla_X h^1)(Y,Z) + \tau^1_1(X)h^1(Y,Z) + \tau^1_2(X)h^2(Y,Z)$, we obtain

$$-15c = C^1_{111} = -2h^1(aX_2 + cX_1, X_1) + \tau^1_1(X_1) = -2c + \tau^1_1(X_1)$$

$$3c = C^1_{122} = -2h^1(-aX_1, X_2) - \tau^1_1(X_1) = -\tau^1_1(X_1),$$

whence $c = 0$ which contradicts the assumption $\Delta_g x \neq 0$ at x_0.

3. Indefinite case

Let t, s denote $\cosh(u)$ and $\sinh(u)$ for $u \in \mathbf{R}$, resp. Then $v = tX_1 + sX_2$ is a unit vector with $g(v, v) = -1$. In this case we will use one of the equations determining the Burstin–Mayer transversal bundle:

$$-2C^1_{122} + C^1_{111} - C^2_{112} = 0. \tag{6}$$

We distinguish three cases.

Case 1: $g(\Delta_g x, \Delta_g x) < 0$ at x_0. Let c be such a function that $\frac{1}{2}\Delta_g x = cX_1$, and equation (5) is equivalent to

$$(t^2+s^2)\big(-3cs(t^2-s^2)+t^3C^2_{111}+3t^2sC^2_{112}+3ts^2C^2_{122}+s^3C^2_{222}\big)$$
$$=2ts\big(-3ct(t^2-s^2)+t^3C^1_{111}+3t^2sC^1_{112}+3ts^2C^1_{122}+s^3C^1_{222}\big).$$

Like in the definite case we obtain

$$3C^2_{112}-3c=-6c+2C^1_{111},\quad 3C^2_{112}+C^2_{222}=6c+6C^1_{122},\quad 3c+C^2_{222}=0.$$

These together with (6) give $C^1_{111} = 15c$ and $C^1_{122} = 3c$. We also have $\nabla_{X_1}X_1 = \hat{\nabla}_{X_1}X_1 - \frac{1}{2}\Delta_g x = aX_2 - cX_1$ and $\nabla_{X_1}X_2 = \hat{\nabla}_{X_1}X_2 = aX_1$ for a function a. Hence

$$15c = C^1_{111} = -2h^1(aX_2 - cX_1, X_1) + \tau^1_1(X_1) = 2c + \tau^1_1(X_1)$$
$$3c = C^1_{122} = -2h^1(aX_1, X_2) - \tau^1_1(X_1) = -\tau^1_1(X_1),$$

which again gives $c = 0$.

Case 2: $g(\Delta_g x, \Delta_g x) > 0$ at x_0. Then we take c such that $\frac{1}{2}\Delta_g x = cX_2$, and equation (5) is equivalent to

$$(t^2+s^2)\big(-3ct(t^2-s^2)+t^3C^2_{111}+3t^2sC^2_{112}+3ts^2C^2_{122}+s^3C^2_{222}\big)$$
$$=2ts\big(-3cs(t^2-s^2)+t^3C^1_{111}+3t^2sC^1_{112}+3ts^2C^1_{122}+s^3C^1_{222}\big).$$

As in Case 1, we obtain $C^1_{222} = -3c$ and $C^1_{112} = C^1_{211} = 0$. Using the fact that $\nabla_{X_2}X_2 = \hat{\nabla}_{X_2}X_2 + \frac{1}{2}\Delta_g x = bX_1 + cX_2$ and $\nabla_{X_2}X_1 = \hat{\nabla}_{X_2}X_1 = bX_2$ for a function b, we use cubic forms and get

$$-3c = C^1_{222} = -2h^1(bX_1 + cX_2, X_2) + \tau^1_1(X_2) = -2c + \tau^1_1(X_2)$$
$$0 = C^1_{211} = -2h^1(bX_2, X_1) + \tau^1_1(X_2) = -\tau^1_1(X_1).$$

This gives $c=0$.

Case 3: $g(\Delta_g x, \Delta_g x) = 0$ at x_0. Then we can take c such that $\frac{1}{2}\Delta_g x = c(X_1 + \varepsilon X_2)$ ($\varepsilon = 1$ or -1), and equation (5) is equivalent to

$$(t^2+s^2)\big(-3(\varepsilon ct+cs)(t^2-s^2)+t^3C^2_{111}+3t^2sC^2_{112}+3ts^2C^2_{122}+s^3C^2_{222}\big)$$
$$=2ts\big(-3(ct+\varepsilon)cs(t^2-s^2)+t^3C^1_{111}+3t^2sC^1_{112}+3ts^2C^1_{122}+s^3C^1_{222}\big).$$

As in the previous cases, we obtain $C^1_{111} = 15c$ and $C^1_{122} = 3c$. Using the cubic form gives $c = 0$ again.

Reference

[1] L. Vrancken, Affine surfaces whose geodesics are planar curves, *Proc. Amer. Math. Soc.,* **123** (1995), 3851–3854.