A DIFFUSIVE SVEIR EPIDEMIC MODEL WITH TIME DELAY AND GENERAL INCIDENCE*

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Abstract In this paper, we consider a delayed diffusive SVEIR model with general incidence. We first establish the threshold dynamics of this model. Using a Nonstandard Finite Difference (NSFD) scheme, we then give the discretization of the continuous model. Applying Lyapunov functions, global stability of the equilibria are established. Numerical simulations are presented to validate the obtained results. The prolonged time delay can lead to the elimination of the infectiousness.

Key words SVEIR model; vaccination; Lyapunov function; nonstandard finite difference method

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1 Introduction

Vaccination is an effective way of controlling the transmission of infectious diseases such as tuberculosis and tetanus etc.. Thus, many countries provide routine vaccination against all of these diseases. However, vaccine-induced immunity may wane as time goes on. To better understand this phenomenon, mathematical models have been developed. Kribs-Zaleta and Velasco-Hernández [1] considered an SIS disease model with vaccination. Arino et al. [2] investigated an SIRS model with vaccination. Li et al. [3] indicated that vaccine effectiveness plays a key role in disease prevention and control. To describe vaccination strategy, Liu et al. [4] considered SVIR epidemic models. Let S, V, I and R be the susceptible, vaccinated,

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infectious and recovered individuals, respectively. Furthermore, Li and Yang [5] proposed the following model for $t > 0$:

$$\begin{cases} \frac{dS}{dt} = \mu pA - \mu S - \beta SI - \alpha S + \eta V, \\ \frac{dV}{dt} = \mu qA + \alpha S - \sigma \beta VI - (\mu + \eta)V, \\ \frac{dI}{dt} = \beta I(S + \sigma V) - (\gamma + \mu + \delta)I, \quad \frac{dR}{dt} = \gamma I - \mu R. \end{cases} \quad (1.1)$$

Here μ and A represent the death rate and the birth rate, respectively. $q < 1$ denotes the fraction of the vaccinated newborns, p is the unvaccinated newborns, $0 < \sigma < 1$ represents that the vaccine is not completely effective, β is the transmission coefficient of the susceptible, γ is the recovery rate, δ is the per capita disease-induced death rate. The susceptible population is vaccinated at a constant rate α and the vaccine-induced immunity wanes at rate η . Li and Yang discussed the global dynamics of system (1.1) by applying Lyapunov functions.

As seen from the existing models, incidence rates play a very important role in determining model dynamics; for example, the bilinear incidence rate is applicable to Hand-Foot-and-Mouth disease [6], H5N1 [7] and SARS [8], but not to sexually transmitted diseases [9]. To model the effect of behavioural changes, Liu et al. [10] proposed an incidence rate $\frac{\beta SI^p}{1+\alpha I^q}$. To model the cholera epidemics in Bari, Capasso and Serio [11] considered the incidence rate $p = q = 1$. Due to a diseases latency, or factors of immunity, infection processes are not instantaneous. Hence, time delay is important in studying infectious disease dynamics. Hattaf et al. [12] studied a delayed SIR model with general incidence. Wang et al. [13] proposed a delayed SVEIR model with nonlinear incidence. Recently, Hattaf [14] proposed a generalized viral infection model with multi-delays and humoral immunity. For more works on delayed epidemic models with vaccination, we refer readers to [15–19].

All of the above mentioned works are location independent, but location-dependent phenomenon are not uncommon in mathematical biology (see [20–22]). Webb [23] pointed out that infectious cases can first be found at one location and can then spread to other areas. Therefore, it is interesting to study epidemic models with spatial diffusion. Xu and Ai [24] considered an influenza disease model with spatial diffusion and vaccination. Abdelmalek and Bendoukha [25] proposed a diffusive SVIR epidemic model allowing continuous immigration of all classes of individuals. Xu et al. [26] discussed a vaccination model with spatial diffusion and nonlinear incidence.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $D_i (i = 1, 2, 3, 4, 5)$ be the diffusion rate and Δ be the Laplace operator. Then, motivated by the aforementioned works, particularly [13, 23], we study the delayed SVEIR model with spatial diffusion as follows:

$$\begin{cases} \frac{\partial S(x,t)}{\partial t} = D_1 \Delta S + \mu pA - Sf(I) - (\mu + \alpha)S, & x \in \Omega, t > 0, \\ \frac{\partial V(x,t)}{\partial t} = D_2 \Delta V + \mu qA + \alpha S - g(I)V - (\beta + \mu)V, & x \in \Omega, t > 0, \\ \frac{\partial E(x,t)}{\partial t} = D_3 \Delta E + f(I)S + g(I)V - e^{-\mu\tau}[f(I(t-\tau))S(t-\tau) \\ \quad + g(I(t-\tau))V(t-\tau)] - \mu E, & x \in \Omega, t > 0, \\ \frac{\partial I(x,t)}{\partial t} = D_4 \Delta I + e^{-\mu\tau}[f(I(t-\tau))S(t-\tau) + g(I(t-\tau))V(t-\tau)] \\ \quad - (\delta + \gamma + \mu)I, & x \in \Omega, t > 0, \\ \frac{\partial R(x,t)}{\partial t} = D_5 \Delta R + \beta V + \delta I - \mu R, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial n} = \frac{\partial V}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \quad (1.2)$$

Here, τ represents the latent period of the disease. The other parameters are as described for system (1.1). Denote by $\vec{n}$ the outward unit normal vector of $\partial\Omega$ as in [20, 21]. We further consider model (1.2) with initial condition

$$\begin{aligned} S(x, \theta) &= \varphi_1(x, \theta), \quad V(x, \theta) = \varphi_2(x, \theta), \quad E(x, \theta) = \varphi_3(x, \theta), \\ I(x, \theta) &= \varphi_4(x, \theta), \quad R(x, \theta) = \varphi_5(x, \theta), \quad x \in \overline{\Omega}, \quad \theta \in [-\tau, 0], \end{aligned}$$

where $\varphi_i (i = 1, 2, 3, 4, 5)$ are uniformly continuous and bounded. Functions g and f satisfy $g(0) = f(0) = 0$ and

(H1) for $I > 0$, $g(I) > 0$ and $f(I) > 0$;

(H2) for $I \geq 0$, $g'(I) > 0$ and $f'(I) > 0$, $g''(I) \leq 0$ and $f''(I) \leq 0$.

Epidemiologically, (H1) means that individuals are positive. (H2) implies that the incidences of $Sf(I)$ and $Vg(I)$ become faster with an increase in the number of the infectious individuals. However, the per capita infection rate will slow down because of a certain inhibition effect, since $g''(I), f''(I) \leq 0$ imply that $(\frac{f(I)}{I})', (\frac{g(I)}{I})' < 0$. For example, the commonly used nonlinear incidence function $\frac{x}{1+\alpha x} (\alpha > 0)$ satisfies both (H1) and (H2).

In this study, in addition to model (1.2), we will also investigate the discrete analogue, due to the fact that epidemiological data is usually collected daily, monthly, or even yearly, but not continuously. Hence, it is more reasonable to use a discrete model to study the transmission mechanism of infectious disease. Furthermore, it is an interesting problem as to whether or not a selected difference scheme can preserve the positivity, boundedness and global stability for the corresponding continuous model. In this regard, some researchers have applied the NSFD scheme proposed by Mickens [27] to discuss the dynamical behaviors of different epidemic models ([28–37]).

The rest of the paper is organized as follows: in Section 2, we establish the global dynamics of the continuous model (1.2). In Section 3, we derive the discretization of (1.2) by the NSFD scheme and establish the positivity and boundedness of the solution. By using discrete Lyapunov functionals, we discuss the global stability of the equilibria of the discretised model in Section 4. This is then followed by numerical simulations in Section 5 to illustrate the obtained results.

2 The Continuous Model

From system (1.2), we only discuss the following system:

$$\begin{cases} \frac{\partial S(x, t)}{\partial t} = D_1 \Delta S + \mu p A - Sf(I) - (\mu + \alpha)S, & x \in \Omega, \quad t > 0, \\ \frac{\partial V(x, t)}{\partial t} = D_2 \Delta V + \mu q A + \alpha S - g(I)V - (\beta + \mu)V, & x \in \Omega, \quad t > 0, \\ \frac{\partial I(x, t)}{\partial t} = D_4 \Delta I + e^{-\mu\tau} [f(I(t-\tau))S(t-\tau) + g(I(t-\tau))V(t-\tau)] \\ \quad - (\delta + \gamma + \mu)I, & x \in \Omega, \quad t > 0, \\ \frac{\partial S}{\partial \vec{n}} = \frac{\partial V}{\partial \vec{n}} = \frac{\partial I}{\partial \vec{n}} = 0, & x \in \partial\Omega, \quad t > 0, \end{cases} \quad (2.1)$$

with

$$S(x, \theta) = \varphi_1(x, \theta), \quad V(x, \theta) = \varphi_2(x, \theta), \quad I(x, \theta) = \varphi_4(x, \theta), \quad x \in \overline{\Omega}, \quad \theta \in [-\tau, 0]. \quad (2.2)$$

2.1 Threshold dynamics

In this section we assume that $D_1 = D_2 = D_4 = D$. Let $\mathbb{X} := C(\overline{\Omega}, \mathbb{R}^3)$ be a Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$ and $\tau > 0$. Let $\mathbf{C}_\tau := C([- \tau, 0], \mathbb{X})$ with the norm $\|\tilde{\phi}\| := \max_{\theta \in [-\tau, 0]} \|\tilde{\phi}(\theta)\|_{\mathbb{X}}, \forall \tilde{\phi} \in \mathbf{C}_\tau$. Define $\mathbb{X}^+ := C(\overline{\Omega}, \mathbb{R}_+^3)$ and $\mathbf{C}_\tau^+ := C([- \tau, 0], \mathbb{X}^+)$. Then $(\mathbb{X}, \mathbb{X}^+)$ and $(\mathbf{C}_\tau, \mathbf{C}_\tau^+)$ are strongly ordered spaces. For any given function $\tilde{\phi}(t) : [-\tau, \varsigma) \rightarrow \mathbb{X} (\varsigma > 0)$, we denote $\tilde{\phi}_t \in \mathbf{C}_\tau$ by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \forall \theta \in [-\tau, 0].$$

Let $\mu_1 = \alpha + \mu$, $\mu_2 = \beta + \mu$ and $\mu_3 = \delta + \gamma + \mu$. Define $\tilde{T}_j(t) : C(\overline{\Omega}, \mathbb{R}) \rightarrow C(\overline{\Omega}, \mathbb{R})$ by the C_0 semigroups with $D\Delta - \mu_j$ ($j = 1, 2, 3$). With $\forall t \geq 0$ and $\tilde{\phi} \in C(\overline{\Omega}, \mathbb{R})$, we have

$$\tilde{T}_j(t)\tilde{\phi}(x) = e^{-\mu_j t} \int_{\Omega} \tilde{\Gamma}(x, t, s)\tilde{\phi}(s)ds, \quad j = 1, 2, 3,$$

where $\tilde{\Gamma}$ is the Green function associated with $D\Delta$. According to [35, Corollary 4], $\tilde{T}_j(t)$ are strongly positive and compact ($j = 1, 2, 3, \forall t > 0$). Taking $\forall x \in \overline{\Omega}$ and $\tilde{\phi} = (\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+$, let $\tilde{F} = (F_1, F_2, F_3) : \mathbb{X}^+ \rightarrow \mathbb{X}$ be

$$\begin{aligned} F_1(\tilde{\phi})(x) &= \mu p A - f(\phi_3(x, 0))\phi_1(x, 0) - (\alpha + \mu)\phi_1(x, 0), \\ F_2(\tilde{\phi})(x) &= \mu q A + \alpha\phi_1(x, 0) - g(\phi_3(x, 0))\phi_2(x, 0) - (\beta + \mu)\phi_2(x, 0), \\ F_3(\tilde{\phi})(x) &= e^{-\mu\tau}(f(\phi_3(x, -\tau))\phi_1(x, -\tau) + g(\phi_3(x, -\tau))\phi_2(x, -\tau)) - (\delta + \gamma + \mu)\phi_3(x, 0). \end{aligned}$$

Then, system (2.1) can be rewritten as

$$\tilde{Z}(x, t) = \tilde{T}(t)\tilde{\phi}(x) + \int_0^t \tilde{T}(t-s)\tilde{F}(\tilde{Z}(x, s))ds,$$

where $\tilde{Z}(x, t) = ((S(x, t), V(x, t), I(x, t)))$ and

$$\tilde{T} = \begin{pmatrix} \tilde{T}_1 & 0 & 0 \\ 0 & \tilde{T}_2 & 0 \\ 0 & 0 & \tilde{T}_3 \end{pmatrix}.$$

Theorem 2.1 System (2.1)–(2.2) admits a unique solution $\tilde{T}(\cdot, t, \tilde{\phi})$ on $[0, \infty)$ satisfying $\tilde{Z}(\cdot, 0, \tilde{\phi}) = \tilde{\phi}$ for all $\tilde{\phi} \in \mathbb{X}^+$. Given $\forall x \in \overline{\Omega}$ and $t \geq 0$, the semiflow

$$\Theta(t)\tilde{\phi} = (S(\cdot, t, \tilde{\phi}), V(\cdot, t, \tilde{\phi}), I(\cdot, t, \tilde{\phi}))$$

is point dissipative.

Proof Taking $\forall \tilde{\phi} \in \mathbb{X}^+$ and $k > 0$ (sufficiently small), we have

$$\begin{aligned} \tilde{\phi}(x, 0) + kF(\tilde{\phi})(x) &= \begin{pmatrix} \phi_1(x, 0) + k(\mu p A - \phi_1(x, 0)f(\phi_3(x, 0)) - (\mu + \alpha)\phi_1(x, 0)) \\ \phi_2(x, 0) + k(\mu q A + \alpha\phi_1(x, 0) - \phi_2(x, 0)g(\phi_3(x, 0)) - (\mu + \beta)\phi_2(x, 0)) \\ \phi_3(x, 0) + k(e^{-\mu\tau}(f(\phi_3(x, -\tau))\phi_1(x, -\tau) + g(\phi_3(x, -\tau))\phi_2(x, -\tau)) - (\mu + \delta + \gamma)\phi_3(x, 0)) \end{pmatrix} \\ &\geq \begin{pmatrix} \phi_1(x, 0)(1 - k((\alpha + \mu) + f(\phi_3(x, 0)))) \\ \phi_2(x, 0)(1 - k((\mu + \beta) + g(\phi_3(x, 0)))) \\ \phi_3(x, 0)(1 - k(\mu + \delta + \gamma)) \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The above inequality implies that

$$\lim_{k \rightarrow 0^+} \text{dist}(\tilde{\phi} + kF(\tilde{\phi}), \mathbf{C}_\tau^+) = 0.$$

According to [38, Corollary 4], one can derive that system (2.1) has a unique mild solution $\tilde{Z}(\cdot, t, \tilde{\phi}) \in \mathbb{X}^+$ for $t \in [0, \tau_{\tilde{\phi}})$.

Define $W(x, t) = S(x, t) + V(x, t)$. Then, we can get

$$\frac{\partial W(x, t)}{\partial t} = D\Delta W + \mu A - d_1 W,$$

where $d_1 = \min\{\mu + \alpha, \mu + \beta\}$. Then, $S(x, t)$ and $V(x, t)$ are bounded on $[0, \tau_{\tilde{\phi}})$ by using a comparison principle. Define $\tilde{G}(x, t) = S(x, t - \tau) + V(x, t - \tau) + e^{\mu\tau}I(x, t)$. Then, we have

$$\frac{\partial \tilde{G}(x, t)}{\partial t} \leq D\Delta \tilde{G} + \mu A - d_2 \tilde{G}(x, t),$$

where $d_2 = \min\{\mu, \mu + \beta, \mu + \delta + \gamma\}$. Thus, $\tilde{G}(x, t)$ are bounded on $[0, \tau_{\tilde{\phi}})$, by the comparison principle. This implies that $I(x, t)$ are also bounded on $[0, \tau_{\tilde{\phi}})$. The remaining proofs are similar to Theorem 2.1 of Zhou et al. [34], which we omit here. $\square$

2.2 Existence of equilibria

Define

$$S_0 = \frac{\mu p A}{\mu + \alpha}, \quad V_0 = \frac{\mu A(q(\mu + \alpha) + \alpha p)}{(\mu + \alpha)(\mu + \beta)}.$$

Then, $E_0(S_0, V_0, 0)$ is the disease-free equilibrium of system (2.1). The basic reproduction number is

$$\mathcal{R}_0 = \frac{S_0 f'(0)}{e^{\mu\tau}(\mu + \delta + \gamma)} + \frac{V_0 g'(0)}{e^{\mu\tau}(\mu + \delta + \gamma)}. \quad (2.3)$$

The endemic equilibrium should satisfy

$$\begin{cases} \mu p A - f(I)S - (\alpha + \mu)S = 0, \\ \mu q A + \alpha S - g(I)V - (\beta + \mu)V = 0, \\ g(I)V + f(I)S - e^{\mu\tau}I(\gamma + \mu + \delta) = 0. \end{cases} \quad (2.4)$$

Theorem 2.2 If $\mathcal{R}_0 < 1$, then system (2.1) has a unique disease-free equilibrium $E_0(S_0, V_0, 0)$; if $\mathcal{R}_0 > 1$, then system (2.1) has a unique endemic equilibrium $E^*(S^*, V^*, I^*)$ with $S^* = \frac{\mu p A}{\mu + \alpha + f(I^*)}$ and $V^* = \frac{\mu q A(\mu + \alpha + f(I^*)) + \alpha \mu p A}{(\mu + \beta + g(I^*))(\mu + \alpha + f(I^*))}$, except for E_0 .

Proof When $\mathcal{R}_0 < 1$, the result is obvious.

According to the first two equations of (2.4), one can get

$$S = \frac{A\mu p}{f(I) + \mu + \alpha}, \quad V = \frac{\mu q A(\alpha + f(I) + \mu) + \alpha \mu p A}{(\beta + g(I) + \mu)(f(I) + \mu + \alpha)}.$$

Then, we have

$$h(I) = \frac{\mu p A}{f(I) + \mu + \alpha} f(I) + \frac{\mu q A(\alpha + f(I) + \mu) + \alpha \mu p A}{(g(I) + \mu + \beta)(f(I) + \mu + \alpha)} g(I) - e^{\mu\tau}(\delta + \gamma + \mu)I.$$

Obviously, $h(+\infty) = -\infty$ and $h(0) = 0$. It follows from $h'(0) > 0$ that $h(I) = 0$ has at least one positive solution denoted by I^* , where

$$h'(0) = -e^{\mu\tau}(\delta + \gamma + \mu) + \frac{\mu p A f'(0)}{\alpha + \mu} + \frac{\mu q A(\alpha + \mu) + \alpha \mu p A}{(\alpha + \mu)(\beta + \mu)} g'(0)$$

$$= e^{\mu\tau}(\gamma + \mu + \delta)(\mathcal{R}_0 - 1) > 0.$$

This is equivalent to $\mathcal{R}_0 > 1$. Thus, (2.4) has at least one positive solution with

$$S^* = \frac{\mu p A}{\mu + \alpha + f(I^*)}, \quad V^* = \frac{\mu q A(\mu + \alpha + f(I^*)) + \alpha \mu p A}{(\mu + \beta + g(I^*))(\mu + \alpha + f(I^*))}.$$

We now prove that the endemic equilibrium is unique. Note that

$$\begin{aligned} h'(I) &= \frac{\mu p A(\alpha + \mu)}{(f(I) + \mu + \alpha)^2} f'(I) - \frac{\alpha \mu p A}{(f(I) + \mu + \alpha)^2 (\beta + g(I) + \mu)} g(I) f'(I) \\ &\quad + \left(\mu q A + \frac{A \alpha \mu p}{\alpha + f(I) + \mu} \right) \frac{(\mu + \beta)}{(\beta + g(I) + \mu)^2} g'(I) - e^{\mu\tau}(\delta + \gamma + \mu), \end{aligned}$$

and

$$\begin{aligned} h''(I) &= \left(\mu p A(\mu + \alpha) - \frac{A p \alpha \mu g(I)}{g(I) + \mu + \beta} \right) \frac{(\alpha + \mu + f(I)) f''(I) - 2(f'(I))^2}{(\alpha + \mu + f(I))^3} \\ &\quad + (\mu + \beta) \left(\mu q A + \frac{\alpha \mu p A}{\mu + \alpha + f(I)} \right) \frac{g''(I)(\beta + \mu + g(I)) - 2(g'(I))^2}{(\beta + \mu + g(I))^3} \\ &\quad - \frac{2 \alpha \mu p A(\beta + \mu) g'(I) f'(I)}{(\alpha + \mu + f(I))^2 (\beta + \mu + g(I))^2} \\ &= \mu p A \left(\frac{(\beta + \mu) \alpha}{\beta + \mu + g(I)} + \mu \right) \frac{(\alpha + \mu + f(I)) f''(I) - 2(f'(I))^2}{(\alpha + \mu + f(I))^3} \\ &\quad + (\mu + \beta) \left(\mu q A + \frac{\alpha \mu p A}{\mu + \alpha + f(I)} \right) \frac{g''(I)(\beta + \mu + g(I)) - 2(g'(I))^2}{(\beta + \mu + g(I))^3} \\ &\quad - \frac{2 \alpha \mu p A(\beta + \mu)}{(\alpha + \mu + f(I))^2 (\beta + \mu + g(I))^2} g'(I) f'(I). \end{aligned}$$

By (H2), we know that $h''(I) < 0$ for $I > 0$. If there exists more than one positive equilibrium, then there must exist a point $E_*(S_*, V_*, I_*)$ such that $h''(I_*) = 0$. We obtain a contradiction. $\square$

2.3 Local stability

Let $0 = \mu_0 < \mu_i < \mu_{i+1}$ be the eigenvalues of $-\Delta$ on Ω , and $E(\mu_i)$ be the space of eigenfunctions with μ_i ($i = 1, 2, \dots$). Then, we define the orthonormal basis of $E(\mu_i)$ ($i = 1, 2, \dots$) by $\{\phi_{ij} : j = 1, 2, \dots, \dim E(\mu_i)\}$ as follows:

$$\mathbb{X} = \bigoplus_{i=1}^{\infty} \mathbb{X}_i, \quad \mathbb{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbb{X}_{ij}.$$

Here, $\mathbb{X}_{ij} = \{\mathbf{c} \phi_{ij} : \mathbf{c} \in \mathbb{R}^3\}$. In a fashion similar to [20, Theorem 3.1], one gets the following result:

Theorem 2.3 If $\mathcal{R}_0 < 1$, then E_0 of system (2.1) is locally asymptotically stable.

Proof Linearizing system (2.1) at E_0 , we get

$$\frac{\partial \tilde{Z}(x, t)}{\partial t} = \mathcal{S} \Delta \tilde{Z}(x, t) + \mathcal{A} \tilde{Z}(x, t) + \mathcal{B} \tilde{Z}_\tau(x, t),$$

where

$$\mathcal{S} = \begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} -(\alpha + \mu) & 0 & -S_0 f'(0) \\ \alpha & -(\mu + \beta) & -V_0 g'(0) \\ 0 & 0 & -(\mu + \delta + \gamma) \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{pmatrix},$$

and $\rho = e^{-\mu\tau}(S_0 f'(0) + V_0 g'(0))$.

Thus, we get

$$(\lambda + \mu_i D + \mu + \alpha)(\lambda + \mu_i D + \mu + \beta)(\lambda + \mu_i D + \mu + \delta + \gamma - \rho e^{-\lambda\tau}) = 0. \quad (2.5)$$

Clearly, (2.5) has eigenvalues $\lambda_1 = -(\mu_i D + \mu + \alpha) < 0$ and $\lambda_2 = -(\mu_i D + \mu + \beta) < 0$. The other eigenvalue λ_3 satisfies

$$\lambda + \mu_i D + \mu + \delta + \gamma - \rho e^{-\lambda\tau} = 0. \quad (2.6)$$

Define

$$\lambda_3(\lambda, i) = -\mu_i D - (\mu + \delta + \gamma)(1 - \mathcal{R}_0 e^{-\lambda\tau}) = 0.$$

If $\mathcal{R}_0 < 1$, then we have

$$\lambda_3(0, i) = -\mu_i D - (\mu + \delta + \gamma)(1 - \mathcal{R}_0) < 0,$$

and

$$\frac{\partial \lambda_3(\lambda, i)}{\partial \lambda} = -(\mu + \delta + \gamma)\mathcal{R}_0 \tau e^{-\lambda\tau} < 0, \quad \forall \lambda \geq 0.$$

Thus, (2.6) has no positive real root.

Assume that (2.6) has a complex root $\lambda = \omega_1 + i\omega_2$ with $\omega_1 \geq 0$; substituting it into (2.6), one has

$$\begin{aligned} (\delta + \gamma + \mu)\mathcal{R}_0 e^{-\omega_1\tau} \sin \omega_2\tau &= -\omega_2, \\ (\delta + \gamma + \mu)\mathcal{R}_0 e^{-\omega_1\tau} \cos \omega_2\tau &= \omega_1 + \mu_i D + \mu + \delta + \gamma. \end{aligned}$$

Squaring and adding these equations together, we obtain

$$(\mu + \delta + \gamma)^2 \mathcal{R}_0^2 e^{-2\omega_1\tau} = \omega_2^2 + (\omega_1 + \mu_i D + \mu + \delta + \gamma)^2.$$

Using $\omega_1 \geq 0$ and $\mu_i \geq 0$, we have

$$(\mu + \delta + \gamma)^2 \mathcal{R}_0^2 e^{-2\omega_1\tau} < \omega_2^2 + (\omega_1 + \mu_i D + \mu + \delta + \gamma)^2,$$

when $\mathcal{R}_0 < 1$. This is a contradiction. Therefore, (2.6) has no complex root with a non-negative real part. Considering $i = 0$ and the space $\mathbb{X}_0$ corresponding to $\mu_0 = 0$, we get

$$\lambda_3(0, 0) = -(\mu + \delta + \gamma)(1 - \mathcal{R}_0) > 0,$$

and

$$\lim_{\lambda \rightarrow +\infty} \lambda_3(\lambda, 0) = -(\mu + \delta + \gamma) < 0,$$

when $\mathcal{R}_0 > 1$. Therefore, there exists a constant $\lambda_0 > 0$ such that $\lambda_3(\lambda_0, 0) = 0$, yielding that (2.6) has at least one positive root. $\square$

2.4 Global stability

Define $\Phi(x) = x - 1 - \ln x$. It is clear that $\Phi(x) \geq 0$ for all $x > 0$. It follows from (H2) that $g'(I)$ is nonincreasing, so one can obtain $g(I) = g(I) - g(0) = g'(\eta)(I - 0) \leq g'(0)I$, where η is between 0 and I . Similarly, one has $f(I) \leq f'(0)I$.

Theorem 2.4 If $\mathcal{R}_0 \leq 1$, then E_0 of system (2.1) is globally asymptotically stable.

Proof Define

$$L(t) = \int_{\Omega} (L_1(x, t) + L_2(x, t)) dx,$$

where

$$\begin{aligned} L_1(x, t) &= e^{\mu\tau} I(x, t) + S_0 \Phi\left(\frac{S(x, t)}{S_0}\right) + V_0 \Phi\left(\frac{V(x, t)}{V_0}\right), \\ L_2(x, t) &= \int_0^\tau (f(I(x, t - \theta))S(x, t - \theta) + g(I(x, t - \theta))V(x, t - \theta)) d\theta. \end{aligned}$$

According to $\ln x \leq x - 1$ and

$$\mu pA = (\mu + \alpha)S_0, \quad \mu qA + \alpha S_0 = (\mu + \beta)V_0,$$

we have

$$\begin{aligned} \frac{\partial L_1(x, t)}{\partial t} &= \left(1 - \frac{S_0}{S}\right) (D\Delta S + (\alpha + \mu)(S_0 - S) - Sf(I)) \\ &\quad + \left(1 - \frac{V_0}{V}\right) (D\Delta V + \alpha(S - S_0) + (\mu + \beta)(V_0 - V) - Vg(I)) \\ &\quad + e^{\mu\tau} D\Delta I + S_\tau f(I_\tau) + V_\tau g(I_\tau) - e^{\mu\tau} (\delta + \gamma + \mu)I, \\ \frac{\partial L_2(x, t)}{\partial t} &= Sf(I) + Vg(I) - S_\tau f(I_\tau) - V_\tau g(I_\tau), \end{aligned}$$

where $S_\tau = S(x, t - \tau)$, $V_\tau = V(x, t - \tau)$ and $I_\tau = I(x, t - \tau)$.

Thus, we get

$$\begin{aligned} \frac{\partial(L_1 + L_2)(x, t)}{\partial t} &= \left[(\mu + \alpha)S_0 \left(2 - \frac{S_0}{S} - \frac{S}{S_0}\right) + (\mu + \beta)V_0 \left(2 - \frac{V_0}{V} - \frac{V}{V_0}\right) \right. \\ &\quad \left. + \alpha S_0 \left(\frac{S}{S_0} + \frac{V_0}{V} - \frac{V_0 S}{V S_0} - 1\right) + S_0 f(I) + V_0 g(I) - e^{\mu\tau} (\mu + \delta + \gamma)I \right] \\ &\quad + D \left(1 - \frac{S_0}{S}\right) \Delta S + D \left(1 - \frac{V_0}{V}\right) \Delta V + e^{\mu\tau} D\Delta I \\ &\leq - \left[(\mu + \alpha)S_0 \left(\Phi\left(\frac{S_0}{S}\right) + \Phi\left(\frac{S}{S_0}\right)\right) + (\mu + \beta)V_0 \left(\Phi\left(\frac{V}{V_0}\right) + \Phi\left(\frac{V_0}{V}\right)\right) \right. \\ &\quad \left. - \alpha S_0 \left(\Phi\left(\frac{S}{S_0}\right) + \Phi\left(\frac{V_0}{V}\right) - \Phi\left(\frac{V_0 S}{V S_0}\right)\right) - (S_0 f'(0) + V_0 g'(0)) \right. \\ &\quad \left. - e^{\mu\tau} (\mu + \delta + \gamma)I \right] + D \left(1 - \frac{S_0}{S}\right) \Delta S + D \left(1 - \frac{V_0}{V}\right) \Delta V + e^{\mu\tau} D\Delta I. \end{aligned}$$

Since

$$\int_{\Omega} \Delta S dx = \int_{\Omega} \Delta V dx = 0, \quad \int_{\Omega} \frac{\Delta S}{S} dx = \int_{\Omega} \frac{|\nabla S|^2}{S^2} dx \geq 0, \quad \int_{\Omega} \frac{\Delta V}{V} dx = \int_{\Omega} \frac{|\nabla V|^2}{V^2} dx \geq 0,$$

we have

$$\begin{aligned} \frac{dL(t)}{dt} &= \int_{\Omega} \frac{\partial(L_1 + L_2)(x, t)}{\partial t} dx \\ &\leq - \int_{\Omega} \left[(\mu + \alpha)S_0 \Phi\left(\frac{S_0}{S}\right) + \mu S_0 \Phi\left(\frac{S}{S_0}\right) + (\mu + \beta)V_0 \Phi\left(\frac{V}{V_0}\right) + \mu qA \Phi\left(\frac{V_0}{V}\right) \right. \\ &\quad \left. + \alpha S_0 \Phi\left(\frac{V_0 S}{V S_0}\right) + e^{\mu\tau} (\mu + \delta + \gamma)(1 - \mathcal{R}_0)I \right] dx \leq 0. \end{aligned}$$

Clearly, the largest invariant subset of $\left\{\frac{dL(t)}{dt} = 0\right\}$ is $\{E_0\}$. The conclusion is correct. $\square$

Theorem 2.5 If $\mathcal{R}_0 > 1$, then E^* of system (2.1) is globally asymptotically stable.

Proof Define

$$H(t) = \int_{\Omega} (H_1(x, t) + H_2(x, t)) dx,$$

where

$$\begin{aligned} H_1(x, t) &= S^* \Phi\left(\frac{S}{S^*}\right) + V^* \Phi\left(\frac{V}{V^*}\right) + e^{\mu\tau} I^* \Phi\left(\frac{I}{I^*}\right), \\ H_2(x, t) &= S^* f(I^*) \int_0^\tau \Phi\left(\frac{S_\theta f(I_\theta)}{S^* f(I^*)}\right) d\theta + V^* g(I^*) \int_0^\tau \Phi\left(\frac{V_\theta g(I_\theta)}{V^* g(I^*)}\right) d\theta. \end{aligned}$$

Clearly, $H \geq 0$ with the equality holds if and only if $S = S^*, V = V^*$ and $I = I^*$.

It follows from (2.4) that

$$\begin{aligned} \frac{\partial H_1(x, t)}{\partial t} &= \left(1 - \frac{S^*}{S}\right) (D\Delta S + (\alpha + \mu)S^* + S^* f(I^*) - Sf(I) - (\alpha + \mu)S) \\ &\quad + \left(1 - \frac{V^*}{V}\right) (D\Delta V + \alpha(S - S^*) + (\mu + \beta)(V^* - V) + g(I^*)V^* - g(I)V) \\ &\quad + \left(1 - \frac{I^*}{I}\right) (D\Delta I + S_\tau f(I_\tau) + V_\tau g(I_\tau) - e^{\mu\tau}(\mu + \delta + \gamma)I), \\ \frac{\partial H_2(x, t)}{\partial t} &= Sf(I) - S_\tau f(I_\tau) + S^* f(I^*) \ln\left(\frac{S_\tau f(I_\tau)}{S f(I)}\right) \\ &\quad + g(I)V - g(I_\tau)V_\tau + g(I^*)V^* \ln\left(\frac{g(I_\tau)V_\tau}{g(I)V}\right). \end{aligned}$$

Notice that

$$\int_{\Omega} \Delta S dx = \int_{\Omega} \Delta V dx = \int_{\Omega} \Delta I dx = 0,$$

and

$$\int_{\Omega} \frac{\Delta S}{S} dx = \int_{\Omega} \frac{|\nabla S|^2}{S^2} dx \geq 0, \quad \int_{\Omega} \frac{\Delta V}{V} dx = \int_{\Omega} \frac{|\nabla V|^2}{V^2} dx \geq 0, \quad \int_{\Omega} \frac{\Delta I}{I} dx = \int_{\Omega} \frac{|\nabla I|^2}{I^2} dx \geq 0,$$

so

$$\begin{aligned} \frac{dH(t)}{dt} &= \int_{\Omega} \frac{\partial(H_1(x, t) + H_2(x, t))}{\partial t} dx \\ &\leq \int_{\Omega} \left[(\mu + \alpha)S^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*}\right) + (\mu + \beta)V^* \left(2 - \frac{V^*}{V} - \frac{V}{V^*}\right) \right. \\ &\quad \left. + \alpha S^* \left(\frac{S}{S^*} + \frac{V^*}{V} - \frac{SV^*}{S^*V} - 1\right) \right. \\ &\quad \left. + S^* f(I^*) \left(2 - \frac{S^*}{S} + \frac{f(I)}{f(I^*)} - \frac{I^* S_\tau f(I_\tau)}{I S^* f(I^*)} - \frac{I}{I^*} + \ln\left(\frac{S_\tau f(I_\tau)}{S f(I)}\right)\right) \right. \\ &\quad \left. + V^* g(I^*) \left(2 - \frac{V^*}{V} + \frac{g(I)}{g(I^*)} - \frac{I^* V_\tau g(I_\tau)}{I V^* g(I^*)} - \frac{I}{I^*} + \ln\left(\frac{V_\tau g(I_\tau)}{V g(I)}\right)\right) \right] dx \\ &= - \int_{\Omega} \left[\mu S^* \left(\Phi\left(\frac{S^*}{S}\right) + \Phi\left(\frac{S}{S^*}\right)\right) + \alpha S^* \left(\Phi\left(\frac{S^*}{S}\right) + \Phi\left(\frac{SV^*}{S^*V}\right)\right) \right. \\ &\quad \left. + \mu q A \Phi\left(\frac{V^*}{V}\right) + (\mu + \beta) V^* \Phi\left(\frac{V}{V^*}\right) \right. \\ &\quad \left. + S^* f(I^*) \left(\Phi\left(\frac{I^* S_\tau f(I_\tau)}{I S^* f(I^*)}\right) + \Phi\left(\frac{S^*}{S}\right) - \frac{f(I)}{f(I^*)} + \frac{I}{I^*} - \ln\left(\frac{I f(I^*)}{I^* f(I)}\right)\right) \right] dx \end{aligned}$$

$$+ V^* g(I^*) \left(\Phi \left(\frac{I^* V_\tau g(I_\tau)}{I V^* g(I^*)} \right) - \frac{g(I)}{g(I^*)} + \frac{I}{I^*} - \ln \left(\frac{I g(I^*)}{I^* g(I^*)} \right) \right) dx.$$

By Assumption (H2) and $\ln x \leq x - 1$, we can get

$$\ln \left(\frac{I G(I^*)}{I^* G(I)} \right) + \frac{G(I)}{G(I^*)} - \frac{I}{I^*} \leq \frac{I G(I^*)}{I^* G(I)} + \frac{G(I)}{G(I^*)} - \frac{I}{I^*} - 1 \leq 0,$$

where $G = \{f, g\}$. Hence,

$$\begin{aligned} \frac{dH(t)}{dt} &= \int_{\Omega} \frac{\partial(H_1(x, t) + H_2(x, t))}{\partial t} dx \\ &\leq - \int_{\Omega} \left[\mu S^* \left(\Phi \left(\frac{S^*}{S} \right) + \Phi \left(\frac{S}{S^*} \right) \right) + \alpha S^* \left(\Phi \left(\frac{S^*}{S} \right) + \Phi \left(\frac{S V^*}{S^* V} \right) \right) \right. \\ &\quad \left. + \mu q A \Phi \left(\frac{V^*}{V} \right) + (\mu + \beta) V^* \Phi \left(\frac{V}{V^*} \right) \right. \\ &\quad \left. + S^* f(I^*) \left(\Phi \left(\frac{S^*}{S} \right) + \Phi \left(\frac{I^* S_\tau f(I_\tau)}{I S^* f(I^*)} \right) \right) + V^* g(I^*) \Phi \left(\frac{I^* V_\tau g(I_\tau)}{I V^* g(I^*)} \right) \right] dx \\ &\leq 0. \end{aligned}$$

In a manner similar to the proof of Theorem 2.4, the conclusion is proved. $\square$

3 A Discretized Model

Define $\Omega = [a, b]$, $\Delta x = (b - a)/M$ and $m = \lceil \tau/\Delta t \rceil$. Δt is the time stepsize. The mesh points are (x_n, t_k) , where $x_n = a + n\Delta x$ and $t_k = k\Delta t$ with $n \in \{0, 1, \dots, M\}$ and $k \in \mathbb{N}$. Denote $S(x_n, t_k)$, $V(x_n, t_k)$ and $I(x_n, t_k)$ by S_n^k , V_n^k and I_n^k , respectively. We use a $(M+1)$ -dimensional vector

$$U^k = (U_0^k, U_1^k, \dots, U_M^k)^T$$

to denote the approximation S, V and I at time t_k . $()^T$ is the transposition of a vector. According to the NSFD scheme, the discretization of system (2.1) is

$$\begin{cases} \frac{S_n^{k+1} - S_n^k}{\Delta t} = D_1 \frac{S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}}{(\Delta x)^2} + \mu p A - S_n^{k+1} f(I_n^k) - (\mu + \alpha) S_n^{k+1}, \\ \frac{V_n^{k+1} - V_n^k}{\Delta t} = D_2 \frac{V_{n+1}^{k+1} - 2V_n^{k+1} + V_{n-1}^{k+1}}{(\Delta x)^2} + \mu q A + \alpha S_n^{k+1} - V_n^{k+1} g(I_n^k) - (\mu + \beta) V_n^{k+1}, \\ \frac{I_n^{k+1} - I_n^k}{\Delta t} = D_4 \frac{I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}}{(\Delta x)^2} \\ \quad + e^{-\mu\tau} [S_n^{k+1-m} f(I_n^{k-m}) + V_n^{k+1-m} g(I_n^{k-m})] - (\mu + \delta + \gamma) I_n^{k+1}, \end{cases} \quad (3.1)$$

with initial condition

$$S_n^k = \phi_n^k \geq 0, \quad V_n^k = \psi_n^k \geq 0 \text{ and } I_n^0 = \varphi_n^k \geq 0, \quad (3.2)$$

where $k \in \{-m, -m+1, \dots, 0\}$ and $n \in \{0, 1, \dots, M\}$ and the boundary condition is

$$S_{-1}^k = S_0^k, \quad S_M^k = S_{M+1}^k, \quad V_{-1}^k = V_0^k, \quad V_M^k = V_{M+1}^k, \quad I_{-1}^k = I_0^k, \quad I_M^k = I_{M+1}^k, \quad k \in \mathbb{N}. \quad (3.3)$$

The equilibria of system (3.1) is the same as for (2.1). Applying M-matrix theory [39], we have the following result:

Theorem 3.1 For any $\Delta x > 0$ and $\Delta t > 0$, the solution of system (3.1) with (3.2) and (3.3) is nonnegative and bounded.

Proof According to (3.1), we get

$$\mathcal{A}^k S^{k+1} = S^k + \mu p A \Delta t e,$$

where

$$\mathcal{A}^k = \begin{pmatrix} a_0^k & a & 0 & \dots & 0 & 0 & 0 \\ a & a_1^k & a & \dots & 0 & 0 & 0 \\ 0 & a & a_2^k & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{M-2}^k & a & 0 \\ 0 & 0 & 0 & \dots & a & a_{M-1}^k & a \\ 0 & 0 & 0 & \dots & 0 & a & a_M^k \end{pmatrix}$$

and $\mathbf{e} = (1, 1, \dots, 1)^T$. We have the coefficients $a = -D_1 \Delta t / (\Delta x)^2$, $a_0^k = 1 + D_1 \Delta t / (\Delta x)^2 + \Delta t(f(I_0^k) + \mu + \alpha)$, $a_M^k = 1 + D_1 \Delta t / (\Delta x)^2 + \Delta t(f(I_M^k) + \mu + \alpha)$ and $a_i^k = 1 + 2D_1 \Delta t / (\Delta x)^2 + \Delta t(f(I_i^k) + \mu + \alpha)$ ($i = 1, 2, \dots, M-1$). Since $\mathcal{A}^k$ is a strictly diagonally dominant matrix, we can obtain

$$S^{k+1} = (\mathcal{A}^k)^{-1}(S^k + \mu p A \Delta t e) > 0.$$

From (3.1) again, one has

$$\mathcal{B}^k V^{k+1} = V^k + (\mu q A \mathbf{e} + \alpha S^{k+1}) \Delta t,$$

where

$$\mathcal{B}^k = \begin{pmatrix} b_0^k & b & 0 & \dots & 0 & 0 & 0 \\ b & b_1^k & b & \dots & 0 & 0 & 0 \\ 0 & b & b_2^k & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{M-2}^k & b & 0 \\ 0 & 0 & 0 & \dots & b & b_{M-1}^k & b \\ 0 & 0 & 0 & \dots & 0 & b & b_M^k \end{pmatrix},$$

with $b = -D_2 \Delta t / (\Delta x)^2$, $b_0^k = 1 + D_2 \Delta t / (\Delta x)^2 + \Delta t(g(I_0^k) + \mu + \beta)$, $b_M^k = 1 + D_2 \Delta t / (\Delta x)^2 + \Delta t(g(I_M^k) + \mu + \beta)$, and $b_i^k = 1 + 2D_2 \Delta t / (\Delta x)^2 + \Delta t(g(I_i^k) + \mu + \beta)$ ($i = 1, 2, \dots, M-1$). Since $\mathcal{B}^k$ is a M-matrix, we get

$$V^{k+1} = (\mathcal{B}^k)^{-1}(V^k + (\mu q A \mathbf{e} + \alpha S^{k+1}) \Delta t).$$

Similarly, we also have

$$\mathcal{C} I^{k+1} = e^{-\mu \tau} \Delta t T^{k+1} + I^k.$$

Here $T^{k+1} = (S_0^{k+1-m} f(I_0^{k-m}) + V_0^{k+1-m} g(I_0^{k-m}), \dots, S_M^{k+1-m} f(I_M^{k-m}) + V_M^{k+1-m} g(I_M^{k-m}))^T$

and

$$\mathcal{C} = \begin{pmatrix} c_1 & c_2 & 0 & \dots & 0 & 0 & 0 \\ c_2 & c_3 & c_2 & \dots & 0 & 0 & 0 \\ 0 & c_2 & c_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_3 & c_2 & 0 \\ 0 & 0 & 0 & \dots & c_2 & c_3 & c_2 \\ 0 & 0 & 0 & \dots & 0 & c_2 & c_1 \end{pmatrix},$$

with $c_1 = 1 + D_4\Delta t/(\Delta x)^2 + \Delta t(\mu + \delta + \gamma)$, $c_2 = -D_4\Delta t/(\Delta x)^2$ and $c_3 = 1 + 2D_4\Delta t/(\Delta x)^2 + \Delta t(\mu + \delta + \gamma)$. Since $\mathcal{C}$ is a M-matrix, one has

$$I^{k+1} = \mathcal{C}^{-1}(e^{-\mu\tau}\Delta t T^{k+1} + I^k).$$

Thus, the solution of system (3.1) remains nonnegative.

Define

$$G^k = \sum_{n=0}^M (S_n^k + V_n^k + e^{\mu\tau} I_n^{k+m}).$$

By (3.1), we get

$$\begin{aligned} \frac{G^{k+1} - G^k}{\Delta t} &= \mu A(M+1) - \sum_{n=0}^M (\mu S_n^{k+1} + (\mu + \beta)V_n^{k+1} + e^{\mu\tau}(\mu + \delta + \gamma)I_n^{k+m+1}) \\ &< \mu A(M+1) - \mu G^{k+1}. \end{aligned}$$

Thus,

$$G^{k+1} \leq \frac{\mu A(M+1)\Delta t + G^k}{1 + \mu\Delta t}.$$

It can be concluded by induction that $\limsup_{k \rightarrow +\infty} G^k \leq (M+1)A$. Therefore, for all $k \in \mathbb{N}$, $\{S^k\}, \{V^k\}, \{I^k\}$ are bounded. $\square$

4 Global Stability of the Discretized System (3.1)

In this section, we discuss the global stability of equilibria for system (3.1).

Theorem 4.1 For any $\Delta x > 0$ and $\Delta t > 0$, if $\mathcal{R}_0 \leq 1$, then E_0 of system (3.1) is globally asymptotically stable.

Proof Define $L^k = L_1^k + L_2^k$, where

$$L_1^k = \frac{1}{\Delta t} \sum_{n=0}^M \left[S_0 \Phi\left(\frac{S_n^k}{S_0}\right) + V_0 \Phi\left(\frac{V_n^k}{V_0}\right) + e^{\mu\tau}(1 + (\mu + \delta + \gamma)\Delta t)I_n^k \right]$$

and

$$L_2^k = \sum_{n=0}^M \sum_{j=k-m}^{k-1} (S_n^{j+1} f(I_n^j) + V_n^{j+1} g(I_n^j)).$$

Clearly, $L^k \geq 0$ with equality holds if and only if $S_n^k = 0, V_n^k = 0$ and $I_n^k = 0$ for all $k \in \mathbb{N}$ and $n \in \{1, 2, \dots, M\}$.

Applying $\mu pA = (\mu + \alpha)S_0$, $\mu qA + \alpha S_0 = (\mu + \beta)V_0$, we can get

$$\begin{aligned} L_1^{k+1} - L_1^k &\leq \frac{1}{\Delta t} \sum_{n=0}^M \left[\left(1 - \frac{S_0}{S_n^{k+1}} \right) (S_n^{k+1} - S_n^k) + \left(1 - \frac{V_0}{V_n^{k+1}} \right) (V_n^{k+1} - V_n^k) \right. \\ &\quad \left. + e^{\mu\tau} (1 + (\mu + \delta + \gamma)\Delta t) (I_n^{k+1} - I_n^k) \right] \\ &= \sum_{n=0}^M \left[\left(1 - \frac{S_0}{S_n^{k+1}} \right) ((\mu + \alpha)(S_0 - S_n^{k+1}) - S_n^{k+1} f(I_n^k)) \right. \\ &\quad + \left(1 - \frac{V_0}{V_n^{k+1}} \right) (\alpha(S_n^{k+1} - S_0) + (\mu + \beta)(V_0 - V_n^{k+1}) - V_n^{k+1} g(I_n^k)) \\ &\quad + S_n^{k+1-m} f(I_n^{k-m}) + V_n^{k+1-m} g(I_n^{k-m}) - e^{\mu\tau} (\mu + \delta + \gamma) I_n^{k+1} \\ &\quad \left. + e^{\mu\tau} (\mu + \delta + \gamma) (I_n^{k+1} - I_n^k) \right] + \Delta \bar{L}^k, \end{aligned}$$

where

$$\begin{aligned} \Delta \bar{L}^k &= \sum_{n=0}^M \frac{1}{(\Delta x)^2} \left[D_1 \left(1 - \frac{S_0}{S_n^{k+1}} \right) (S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}) \right. \\ &\quad + D_2 \left(1 - \frac{V_0}{V_n^{k+1}} \right) (V_{n+1}^{k+1} - 2V_n^{k+1} + V_{n-1}^{k+1}) \\ &\quad \left. + e^{\mu\tau} D_4 (I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}) \right], \end{aligned}$$

and

$$L_2^{k+1} - L_2^k = \sum_{n=0}^M (S_n^{k+1} f(I_n^k) + V_n^{k+1} g(I_n^k) - S_n^{k+1-m} f(I_n^{k-m}) - V_n^{k+1-m} g(I_n^{k-m})).$$

For $n \in \{0, 1, \dots, M-1\}$, we have

$$\frac{S_{n+1}^{k+1}}{S_n^{k+1}} - 2 + \frac{S_n^{k+1}}{S_{n+1}^{k+1}} \geq 0, \quad \frac{V_{n+1}^{k+1}}{V_n^{k+1}} - 2 + \frac{V_n^{k+1}}{V_{n+1}^{k+1}} \geq 0.$$

Then,

$$\begin{aligned} \Delta \bar{L}^k &= \sum_{n=0}^M \frac{1}{(\Delta x)^2} \left[D_1 (S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}) - S_0 D_1 \left(\frac{S_{n+1}^{k+1}}{S_n^{k+1}} - 2 + \frac{S_{n-1}^{k+1}}{S_n^{k+1}} \right) \right. \\ &\quad + D_2 (V_{n+1}^{k+1} - 2V_n^{k+1} + V_{n-1}^{k+1}) - V_0 D_2 \left(\frac{V_{n+1}^{k+1}}{V_n^{k+1}} - 2 + \frac{V_{n-1}^{k+1}}{V_n^{k+1}} \right) \\ &\quad \left. + e^{\mu\tau} D_4 (I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}) \right] \\ &\leq \frac{1}{(\Delta x)^2} \left[D_1 (S_{M+1}^{k+1} - S_M^{k+1} + S_{-1}^{k+1} - S_0^{k+1}) - S_0 D_1 \left(\frac{S_{-1}^{k+1}}{S_0^{k+1}} - 2 + \frac{S_{M+1}^{k+1}}{S_M^{k+1}} \right) \right. \\ &\quad + D_2 (V_{M+1}^{k+1} - V_M^{k+1} + V_{-1}^{k+1} - V_0^{k+1}) - V_0 D_2 \left(\frac{V_{-1}^{k+1}}{V_0^{k+1}} - 2 + \frac{V_{M+1}^{k+1}}{V_M^{k+1}} \right) \\ &\quad \left. + e^{\mu\tau} D_4 (I_{M+1}^{k+1} - I_M^{k+1} + I_{-1}^{k+1} - I_0^{k+1}) \right] = 0. \end{aligned}$$

Thus,

$$\begin{aligned}
L^{k+1} - L^k &\leq \sum_{n=0}^M \left[(\mu + \alpha) S_0 \left(2 - \frac{S_0}{S_n^{k+1}} - \frac{S_n^{k+1}}{S_0} \right) + (\mu + \beta) V_0 \left(2 - \frac{V_0}{V_n^{k+1}} - \frac{V_n^{k+1}}{V_0} \right) \right. \\
&\quad + \alpha S_0 \left(\frac{S_n^{k+1}}{S_0} + \frac{V_0}{V_n^{k+1}} - \frac{V_0 S_n^{k+1}}{V_n^{k+1} S_0} - 1 \right) + S_0 f(I_n^k) + V_0 g(I_n^k) \\
&\quad \left. - e^{\mu\tau} (\mu + \delta + \gamma) I_n^k \right] \\
&\leq - \sum_{n=0}^M \left[(\mu + \alpha) S_0 \left(\Phi \left(\frac{S_0}{S_n^{k+1}} \right) + \Phi \left(\frac{S_n^{k+1}}{S_0} \right) \right) + (\mu + \beta) V_0 \left(\Phi \left(\frac{V_0}{V_n^{k+1}} \right) \right. \right. \\
&\quad \left. \left. + \Phi \left(\frac{V_n^{k+1}}{V_0} \right) \right) - \alpha S_0 \left(\Phi \left(\frac{S_n^{k+1}}{S_0} \right) + \Phi \left(\frac{V_0}{V_n^{k+1}} \right) - \Phi \left(\frac{V_0 S_n^{k+1}}{V_n^{k+1} S_0} \right) \right) \right. \\
&\quad \left. - (S_0 f'(0) + V_0 g'(0) - e^{\mu\tau} (\mu + \delta + \gamma)) I_n^k \right] \\
&= - \sum_{n=0}^M \left[(\mu + \alpha) S_0 \Phi \left(\frac{S_0}{S_n^{k+1}} \right) + \mu S_0 \Phi \left(\frac{S_n^{k+1}}{S_0} \right) + (\mu + \beta) V_0 \Phi \left(\frac{V_n^{k+1}}{V_0} \right) \right. \\
&\quad \left. + \mu q A \Phi \left(\frac{V_0}{V_n^{k+1}} \right) + \alpha S_0 \Phi \left(\frac{V_0 S_n^{k+1}}{V_n^{k+1} S_0} \right) \right] + e^{\mu\tau} (\mu + \delta + \gamma) (1 - \mathcal{R}_0) I_n^k.
\end{aligned}$$

It is clear that $L^{k+1} - L^k \leq 0$, when $\mathcal{R}_0 \leq 1$. Then $\{L^k\}$ is a non-increasing sequence. There must exist $\tilde{L} > 0$ such that $\lim_{k \rightarrow +\infty} L^k = \tilde{L}$, meaning that $\lim_{k \rightarrow +\infty} (L^{k+1} - L^k) = 0$. Thus, we have

$$\lim_{k \rightarrow +\infty} S_n^k = S_0 \quad \text{and} \quad \lim_{k \rightarrow +\infty} V_n^k = V_0.$$

Furthermore, we can get $\lim_{k \rightarrow +\infty} I_n^k = 0$. □

Theorem 4.2 For any $\Delta x > 0$ and $\Delta t > 0$, if $\mathcal{R}_0 > 1$, then E^* of system (3.1) is globally asymptotically stable.

Proof Define $H^k = H_1^k + H_2^k$, where

$$H_1^k = \frac{1}{\Delta t} \sum_{n=0}^M \left[S^* \Phi \left(\frac{S_n^k}{S^*} \right) + V^* \Phi \left(\frac{V_n^k}{V^*} \right) + e^{\mu\tau} I^* \Phi \left(\frac{I_n^k}{I^*} \right) \right]$$

and

$$\begin{aligned}
H_2^k &= \sum_{n=0}^M \sum_{j=k-m}^{k-1} \left[S^* f(I^*) \Phi \left(\frac{S_n^{j+1} f(I_n^j)}{S^* f(I^*)} \right) + V^* g(I^*) \Phi \left(\frac{V_n^{j+1} g(I_n^j)}{V^* g(I^*)} \right) \right. \\
&\quad \left. + (f(I^*) S^* + g(I^*) V^*) \Phi \left(\frac{I_n^k}{I^*} \right) \right].
\end{aligned}$$

We conclude that $H^k \geq 0$ if and if only $S_n^k = S^*, V_n^k = V^*$ and $I_n^k = I^*$ for all $k \in \mathbb{N}$ and $n \in \{0, 1, \dots, M\}$.

By (2.4), we obtain

$$\begin{aligned}
H_1^{k+1} - H_1^k &\leq \sum_{n=0}^M \frac{1}{\Delta t} \left[\left(1 - \frac{S^*}{S_n^{k+1}} \right) (S_n^{k+1} - S_n^k) + \left(1 - \frac{V^*}{V_n^{k+1}} \right) (V_n^{k+1} - V_n^k) \right. \\
&\quad \left. + e^{\mu\tau} \left(1 - \frac{I^*}{I_n^{k+1}} \right) (I_n^{k+1} - I_n^k) \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^M \left[\left(1 - \frac{S^*}{S_n^{k+1}} \right) ((\mu + \alpha)S^* + S^*f(I^*) - S_n^{k+1}f(I_n^k) - (\mu + \alpha)S_n^{k+1}) \right. \\
&\quad + \left(1 - \frac{V^*}{V_n^{k+1}} \right) (\alpha S_n^{k+1} - \alpha S^* + (\mu + \beta)(V^* - V_n^{k+1}) + V^*g(I^*) - V_n^{k+1}g(I_n^k)) \\
&\quad \left. \left(1 - \frac{I^*}{I_n^{k+1}} \right) (f(I_n^{k-m})S_n^{k+1-m} + g(I_n^{k-m})V_n^{k+1-m} - e^{\mu\tau}(\mu + \delta + \gamma)I_n^{k+1}) \right] + \Delta \bar{H}^k \\
&= \sum_{n=0}^M \left[(\mu + \alpha)S^* \left(2 - \frac{S^*}{S_n^{k+1}} - \frac{S_n^{k+1}}{S^*} \right) + (\mu + \beta)V^* \left(2 - \frac{V^*}{V_n^{k+1}} - \frac{V_n^{k+1}}{V^*} \right) \right. \\
&\quad + \alpha S^* \left(\frac{S_n^{k+1}}{S^*} + \frac{V^*}{V_n^{k+1}} - \frac{S_n^{k+1}V^*}{S^*V_n^{k+1}} - 1 \right) + S_n^{k+1-m}f(I_n^{k-m}) + V_n^{k+1-m}g(I_n^{k-m}) \\
&\quad - S_n^{k+1}f(I_n^k) - V_n^{k+1}g(I_n^k) \\
&\quad + S^*f(I^*) \left(2 - \frac{S^*}{S_n^{k+1}} + \frac{f(I_n^k)}{f(I^*)} - \frac{I^*S_n^{k+1-m}f(I_n^{k-m})}{I_n^{k+1}S^*f(I^*)} - \frac{I_n^{k+1}}{I^*} \right) \\
&\quad \left. + V^*g(I^*) \left(2 - \frac{V^*}{V_n^{k+1}} + \frac{g(I_n^k)}{g(I^*)} - \frac{I^*V_n^{k+1-m}g(I_n^{k-m})}{I_n^{k+1}V^*g(I^*)} - \frac{I_n^{k+1}}{I^*} \right) \right] + \Delta \bar{H}^k,
\end{aligned}$$

where

$$\begin{aligned}
\Delta \bar{H}^k &= \frac{1}{(\Delta x)^2} \sum_{n=0}^M \left[D_1 \left(1 - \frac{S^*}{S_n^{k+1}} \right) (S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}) \right. \\
&\quad + D_2 \left(1 - \frac{V^*}{V_n^{k+1}} \right) (V_{n+1}^{k+1} - 2V_n^{k+1} + V_{n-1}^{k+1}) \\
&\quad + e^{\mu\tau} D_4 \left(1 - \frac{I^*}{I_n^{k+1}} \right) (I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}) \\
&= \frac{1}{(\Delta x)^2} \sum_{n=0}^M \left[D_1 (S_{n+1}^{k+1} - 2S_n^{k+1} + S_{n-1}^{k+1}) - S^* D_1 \left(\frac{S_{n+1}^{k+1}}{S_n^{k+1}} - 2 + \frac{S_{n-1}^{k+1}}{S_n^{k+1}} \right) \right. \\
&\quad + D_2 (V_{n+1}^{k+1} - 2V_n^{k+1} + V_{n-1}^{k+1}) - V^* D_2 \left(\frac{V_{n+1}^{k+1}}{V_n^{k+1}} - 2 + \frac{V_{n-1}^{k+1}}{V_n^{k+1}} \right) \\
&\quad \left. + e^{\mu\tau} D_4 (I_{n+1}^{k+1} - 2I_n^{k+1} + I_{n-1}^{k+1}) - I^* D_4 \left(\frac{I_{n+1}^{k+1}}{I_n^{k+1}} - 2 + \frac{I_{n-1}^{k+1}}{I_n^{k+1}} \right) \right],
\end{aligned}$$

and

$$\begin{aligned}
H_2^{k+1} - H_2^k &= \sum_{n=0}^M \left[S_{n+1}^{k+1}f(I_n^k) - S_n^{k+1-m}f(I_n^{k-m}) + S^*f(I^*) \ln \frac{S_n^{k+1-m}f(I_n^{k-m})}{S_n^{k+1}f(I_n^k)} \right. \\
&\quad + V_{n+1}^{k+1}g(I_n^k) - V_n^{k+1-m}g(I_n^{k-m}) + V^*g(I^*) \ln \frac{V_n^{k+1-m}g(I_n^{k-m})}{V_n^{k+1}g(I_n^k)} \\
&\quad \left. + (S^*f(I^*) + V^*g(I^*)) \left(\frac{I_n^{k+1}}{I^*} - \frac{I_n^k}{I^*} + \ln \frac{I_n^k}{I^*} \right) \right].
\end{aligned}$$

When $n \in \{0, 1, \dots, M-1\}$, we have

$$\frac{S_{n+1}^{k+1}}{S_n^{k+1}} - 2 + \frac{S_n^{k+1}}{S_{n+1}^{k+1}} \geq 0, \quad \frac{V_{n+1}^{k+1}}{V_n^{k+1}} - 2 + \frac{V_n^{k+1}}{V_{n+1}^{k+1}} \geq 0 \quad \text{and} \quad \frac{I_{n+1}^{k+1}}{I_n^{k+1}} - 2 + \frac{I_n^{k+1}}{I_{n+1}^{k+1}} \geq 0.$$

Hence,

$$\begin{aligned} \Delta \overline{H}^k &\leq \frac{1}{(\Delta x)^2} \left[D_1(S_{M+1}^{k+1} - S_M^{k+1} + S_{-1}^{k+1} - S_0^{k+1}) - S^* D_1 \left(\frac{S_{-1}^{k+1}}{S_0^{k+1}} - 2 + \frac{S_{M+1}^{k+1}}{S_M^{k+1}} \right) \right. \\ &\quad + D_2(V_{M+1}^{k+1} - V_M^{k+1} + V_{-1}^{k+1} - V_0^{k+1}) - V^* D_2 \left(\frac{V_{-1}^{k+1}}{V_0^{k+1}} - 2 + \frac{V_{M+1}^{k+1}}{V_M^{k+1}} \right) \\ &\quad \left. + D_4(I_{M+1}^{k+1} - I_M^{k+1} + I_{-1}^{k+1} - I_0^{k+1}) - I^* D_4 \left(\frac{I_{-1}^{k+1}}{I_0^{k+1}} - 2 + \frac{I_{M+1}^{k+1}}{I_M^{k+1}} \right) \right] \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} &H^{k+1} - H^k \\ &\leq \sum_{n=0}^M \left[(\alpha + \mu) S^* \left(2 - \frac{S^*}{S_n^{k+1}} - \frac{S_n^{k+1}}{S^*} \right) + (\mu + \beta) V^* \left(2 - \frac{V^*}{V_n^{k+1}} - \frac{V_n^{k+1}}{V^*} \right) \right. \\ &\quad + \alpha S^* \left(\frac{S_n^{k+1}}{S^*} + \frac{V^*}{V_n^{k+1}} - \frac{S_n^{k+1} V^*}{S^* V_n^{k+1}} - 1 \right) \\ &\quad + \left(2 - \frac{S^*}{S_n^{k+1}} + \frac{f(I_n^k)}{f(I^*)} - \frac{I^* S_n^{k+1-m} f(I_n^{k-m})}{I_n^{k+1} S^* f(I^*)} - \frac{I_n^k}{I^*} + \ln \frac{I^k S_n^{k+1-m} f(I_n^{k-m})}{I_n^{k+1} S_n^{k+1} f(I_n^k)} \right) S^* f(I^*) \\ &\quad + \left(2 - \frac{V^*}{V_n^{k+1}} + \frac{g(I_n^k)}{g(I^*)} - \frac{I^* V_n^{k+1-m} g(I_n^{k-m})}{I_n^{k+1} V^* g(I^*)} - \frac{I_n^k}{I^*} + \ln \frac{I^k V_n^{k+1-m} g(I_n^{k-m})}{I_n^{k+1} V_n^{k+1} g(I_n^k)} \right) V^* g(I^*) \left. \right] \\ &= - \sum_{n=0}^M \left[(\mu + \alpha) S^* \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{S_n^{k+1}}{S^*} \right) \right) + (\mu + \beta) V^* \left(\Phi \left(\frac{V^*}{V_n^{k+1}} \right) + \Phi \left(\frac{V_n^{k+1}}{V^*} \right) \right) \right. \\ &\quad - \alpha S^* \left(\Phi \left(\frac{S_n^{k+1}}{S^*} \right) + \Phi \left(\frac{V^*}{V_n^{k+1}} \right) - \Phi \left(\frac{S_n^{k+1} V^*}{S^* V_n^{k+1}} \right) \right) \\ &\quad + S^* f(I^*) \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{I^* S_n^{k+1-m} f(I_n^{k-m})}{I_n^{k+1} S^* f(I^*)} \right) - \frac{f(I_n^k)}{f(I^*)} + \frac{I_n^k}{I^*} - \ln \frac{I_n^k f(I^*)}{I^* f(I_n^k)} \right) \\ &\quad + V^* g(I^*) \left(\Phi \left(\frac{V^*}{V_n^{k+1}} \right) + \Phi \left(\frac{I^* V_n^{k+1-m} g(I_n^{k-m})}{I_n^{k+1} V^* g(I^*)} \right) - \frac{g(I_n^k)}{g(I^*)} + \frac{I_n^k}{I^*} - \ln \frac{I_n^k g(I^*)}{I^* g(I_n^k)} \right) \left. \right] \\ &= - \sum_{n=0}^M \left[\mu S^* \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{S_n^{k+1}}{S^*} \right) \right) + \alpha S^* \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{S_n^{k+1} V^*}{S^* V_n^{k+1}} \right) \right) \right. \\ &\quad + \mu q A \Phi \left(\frac{V^*}{V_n^{k+1}} \right) + (\mu + \beta) V^* \Phi \left(\frac{V_n^{k+1}}{V^*} \right) \\ &\quad + S^* f(I^*) \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{I^* S_n^{k+1-m} f(I_n^{k-m})}{I_n^{k+1} S^* f(I^*)} \right) - \frac{f(I_n^k)}{f(I^*)} + \frac{I_n^k}{I^*} - \ln \frac{I_n^k f(I^*)}{I^* f(I_n^k)} \right) \\ &\quad + V^* g(I^*) \left(\Phi \left(\frac{I^* V_n^{k+1-m} g(I_n^{k-m})}{I_n^{k+1} V^* g(I^*)} \right) - \frac{g(I_n^k)}{g(I^*)} + \frac{I_n^k}{I^*} - \ln \frac{I_n^k g(I^*)}{I^* g(I_n^k)} \right) \left. \right]. \end{aligned}$$

Applying Assumption (H2) and that $\ln x \leq x - 1$, we can get

$$\frac{\mathcal{G}(I_n^k)}{\mathcal{G}(I^*)} - \frac{I_n^k}{I^*} + \ln \left(\frac{I_n^k \mathcal{G}(I^*)}{I^* \mathcal{G}(I_n^k)} \right) \leq \frac{\mathcal{G}(I_n^k)}{\mathcal{G}(I^*)} - \frac{I_n^k}{I^*} + \frac{I_n^k \mathcal{G}(I^*)}{I^* \mathcal{G}(I_n^k)} - 1 \leq 0,$$

where $\mathcal{G} = \{f, g\}$. Therefore,

$$\begin{aligned} H^{k+1} - H^k \leq & - \sum_{n=0}^M \left[\mu S^* \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{S_n^{k+1}}{S^*} \right) \right) + \alpha S^* \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{S_n^{k+1} V^*}{S^* V_n^{k+1}} \right) \right) \right. \\ & + \mu q A \Phi \left(\frac{V^*}{V_n^{k+1}} \right) + (\mu + \beta) V^* \Phi \left(\frac{V_n^{k+1}}{V^*} \right) \\ & + S^* f(I^*) \left(\Phi \left(\frac{S^*}{S_n^{k+1}} \right) + \Phi \left(\frac{I^* S_n^{k+1-m} f(I_n^{k-m})}{I_n^{k+1} S^* f(I^*)} \right) \right) \\ & \left. + V^* g(I^*) \Phi \left(\frac{I^* V_n^{k+1-m} g(I_n^{k-m})}{I_n^{k+1} V^* g(I^*)} \right) \right] \leq 0. \end{aligned}$$

Clearly, H^k is a non-increasing sequence. There exists $\mathcal{H} > 0$ such that $\lim_{k \rightarrow \infty} H^k = \mathcal{H}$, yielding that $\lim_{k \rightarrow \infty} (H^{k+1} - H^k) = 0$. This means that $\lim_{k \rightarrow \infty} S_n^k = S^*$, $\lim_{k \rightarrow \infty} V_n^k = V^*$, $\lim_{k \rightarrow \infty} I_n^k = I^*$. $\square$

5 Numerical Simulations

From (3.1), set $f(I) = \frac{\beta_1 I}{1+I}$, $g(I) = \frac{\beta_2 I}{1+I}$, $\Delta x = 0.2$ and $\Delta t = 0.1$. Referring to [13, 26], we take $\delta = 0$, $p = 1$, $q = 0$, $D_1 = D_2 = D_4 = D = 1$ and the other parameters as follows:

$$\beta_1 = 0.0004, \beta_2 = 0.000012, A = 175000, \mu = 0.04, \beta = 0.05, \gamma = 0.02.$$

Case 1 Choose $\alpha = 9, \tau = 20$ and initial condition

$$S(n, k) = 100 \sin n + 100, V(n, k) = 7000(1 + \cos n),$$

$$I(n, k) = \sin(0.5n) + 1, n \in \{0, 1, \dots, 200\}, k \in \{-m, -m+1, \dots, 0\}.$$

By (2.3) and simple calculations, we have that $\mathcal{R}_0 = 0.9278 < 1$ and that $E_0 = (77.4336, 7743.3628, 0)$. Using Theorem 4.1, E_0 is globally asymptotically stable. One gets that the disease is extinct (see Figure 1).

Figure 1 The disease-free equilibrium $E_0 = (77.4336, 7743.3628, 0)$ of system (3.1) is globally asymptotically stable when $\mathcal{R}_0 = 0.9278 < 1$

Case 2 Choose $\alpha = 0.9, \tau = 20$ and initial condition

$$S(n, k) = 700 \sin n + 700, \quad V(n, k) = 8000(1 + \cos n),$$

$$I(n, k) = \sin(0.5n) + 2, \quad n \in \{0, 1, \dots, 200\}, \quad k \in \{-m, -m+1, \dots, 0\}.$$

We obtain that $\mathcal{R}_0 = 2.8999 > 1$ and that $E^* = (744.4733, 7444.0831, 1.8991)$, respectively. Thus, E^* is globally asymptotically stable, by Theorem 4.2. Hence, the disease will eventually become endemic (see Figure 2).

Figure 2 The disease-free equilibrium $E^* = (744.4733, 7444.0831, 1.8991)$ of system (3.1) is globally asymptotically stable when $\mathcal{R}_0 = 2.8999 > 1$

Case 3 Effect of time delay.

Choose $\tau = 5, 10, 15, 20$ with $\alpha = 0.9$ and an initial condition as in Case (2). We obtain that $\mathcal{R}_0 = 5.2840, 4.3262, 3.5420, 2.8999$ and that $I^* = 4.2821, 3.3247, 2.5408, 1.8991$, respectively. Here, we give the simulations of solutions of the infectious I at $x = 10$ with different values of τ . We observe that the number of those who are infectious decreases with an increase of τ (see Figure 3). Biologically, this delay can play an important role in eliminating the number of people who are infectious. By increasing the delay, we can decrease the number of people who are infectious.

Figure 3 The solutions of the infectious I at $x = 10$ with different τ in Case (3)

6 Conclusions

In this paper, we proposed a diffusive SVEIR epidemic model with time delay and general incidence. For this model, we first considered the global dynamics of the continuous case. Then, by using the NSFD scheme, we derived the discretization of the model. It has been shown that the global stability of the equilibria is completely determined by the basic reproduction number $\mathcal{R}_0$: if $\mathcal{R}_0 \leq 1$, then the disease-free equilibrium E_0 is globally asymptotically stable; if $\mathcal{R}_0 > 1$, then the endemic equilibrium E^* is globally asymptotically stable. One sees that the NSFD scheme can preserve the global properties of solutions for an original continuous model, such as the positivity and ultimate boundedness of solutions, and global stability of the equilibria. It is our intention to use this method to study other delayed diffusive epidemic models.

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