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A semiotic perspective on polysemy

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Abstract

This paper extends Semiotic-Conceptual Analysis (SCA) to provide a means for comparing and evaluating semiotic relations (i.e. sets of signs) with respect to their ability of and efficiency of expressing conceptual structures. One contributing factor for efficiency is polysemy which refers to reusing representamens of signs in different contexts. Two different types of polysemy are identified: ambiguous polysemy which encodes a view from part signs to compound signs and simultaneous polysemy for analysing how compound signs simultaneously denote more than one meaning. Two detailed examples are included.

Keywords Formal concept analysis · Semiotic-conceptual analysis · Polysemy · diagrams · Observational advantage

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1 Introduction

Semiotic-Conceptual Analysis (SCA) is a mathematical formalisation of basic semiotic notions. SCA defines a *sign* as a triple of three components which are called *representamen*, *denotation* and *interpretation* (Priss [6]). A set of signs using shared sets of representamens, denotations and interpretations is called a *semiotic relation*. The definition of a sign was inspired by the semiotics of the American philosopher Charles S. Peirce but SCA does not intend to describe an exact formalisation of his ideas. Furthermore SCA reduces each notion to an abstract mathematical role. SCA can be applied to linguistic data in which case representamens might correspond to words (or lexemes), denotations to word meanings and interpretations to a mapping from words to their meanings. But SCA can also be applied to non-linguistic data, for example graphical representations, in which case "representamen", "denotation" and so on are structural labels. The only condition for a semiotic relation is that interpretations are partial functions (as explained below) that map representamens onto denotations. These notions are thus abstractions of their usual philosophical or linguistic meanings. In the same manner, the notion of *polysemy* presented in this paper focuses on

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structural properties and is an abstraction of its linguistic meaning. More traditional linguistic applications of SCA have been discussed elsewhere (Priss [7]) and are not the main topic of this paper.

Being a partial function for an interpretation means that it does not need to be applicable to all representamens but if it is applicable to a representamen then it must yield a unique denotation. A semiotic relation usually contains not just one but a set of interpretations. Two different interpretations applied to the same representamen can yield different denotations. But each pair of an interpretation and a representamen uniquely identifies a denotation. Altogether this implies that many kinds of triadic relations might be considered semiotic relations. Therefore, in order for SCA to be "semiotic", a further condition can be added stating that the data should be collected in some kind of communication context. But such a condition cannot be mathematically formalised. SCA is meant to be used as a data analysis method. Users can decide which sets to use as representamens and denotations and how to model interpretations. The manner of defining a sign in SCA is similar to how Formal Concept Analysis (FCA) as described by Ganter & Wille [3] defines concepts and conceptual hierarchies. These notions are more abstract and maybe shallower than their philosophical or linguistic counterparts, but they result in mathematical precision.

SCA is intended to provide a means for evaluating and comparing semiotic relations. Sets of representamens must be sufficiently small so that sign users can memorise them (which might depend on the users' age: natural languages learnt by children have larger sets of representamens than traffic signs learnt by adults). A set of n representamens together with m interpretations can be mapped onto or express a maximum of $m \times n$ denotations. If simple representamens are combined to form compound representamens according to some syntax or grammar, then even more denotations can be expressed. For example, if compounds of exactly one noun and verb are formed using k nouns, j verbs and m interpretations, then a maximum of $m \times k \times j$ denotations can be expressed. Such numbers can be calculated using basic combinatorics. Formal languages and information theory address similar questions, but are more aimed at calculating the expressiveness of types of languages whereas SCA concentrates on calculating values for actual semiotic data. Furthermore SCA starts with a distinction between representamens and interpretations.

The focus of this paper is on reusing representamens with different interpretations which is called *polysemy*. Two different kinds of polysemy are identified. The notion of *ambiguous polysemy* is coined for reusing representamens or their parts in different contexts whereas the notion of *simultaneous polysemy* refers to compound signs that simultaneously have different meanings within a single context similar to the proverbial "picture being worth a 1000 words". Ambiguous polysemy considers how parts can be combined to form different compounds whereas simultaneous polysemy describes what partial meanings can be observed from a compound sign.

Pictures and diagrams are examples of compound signs which often engage simultaneous polysemy. In the field of diagrams research the notion of *observational advantage* has been studied (e.g. Stapleton et al. [9]), for example with respect to how in mathematics, a single diagram sometimes displays a whole proof. It is not easy to define the exact mechanisms of such proofs but Barwise & Etchemendy [1] present a systematic approach for some types of visual proofs. The focus of SCA is not on proof theory but on the relationships between representamens and denotations which are usually modelled as concepts belonging to a conceptual structure, for example in the sense of FCA. SCA then facilitates to investigate what conceptual structure belongs to a semiotic relation, how signs are mapped into a conceptual structure and how much of the conceptual structure is covered by a semiotic relation.

Measures of polysemy provide additional information for determining the efficiency of a representation of concepts with respect to a semiotic relation.

Some further research that is relevant for SCA should be mentioned: Barwise & Seligman's [2] information flow theory and Goguen's [4] algebraic semiotics have somewhat related goals to SCA. But information flow and algebraic semiotics do not start with a distinction between representamens and interpretations as discussed in this paper. Moody [5] proposes a "scientific basis for constructing visual notations". He provides a detailed list of characteristics of visualisations using perceptual, cognitive and structural features. Some of his notions can be mapped onto SCA, but at least in this paper, SCA is mainly applied to quantitative features whereas many of Moody's features are of a qualitative nature. A more detailed comparison of Moody's and similar work and SCA will be left for a separate paper.

The next section summarises basic SCA notions. Section 3 defines compound signs and some measures for evaluating semiotic relations. Section 4 discusses observational advantages. Section 5 provides an example. While Section 2 repeats notions from Priss [7], most of the material of Sections 3–5 is new and has not been previously published.

2 Basic notions

All relevant SCA definitions are repeated in this paper. Further details, motivating examples and explanations can be found in Priss ([6]). As mentioned in the introduction, SCA can be applied to a variety of linguistic or non-linguistic data. The following examples of four signs are used in this section:

$s_1 = (i_1$, "pickup", "small van")
$s_2 = (i_2$, "PICKUP","a casual encounter with a stranger")
$s_3 = (i_3$, "pickup", "an act of collecting goods in a small van")
$s_4 = (i_3$, "truck", "heavy road vehicle")

The sign s_1 consists of the string "pickup" as representamen, the string "small van" as denotation and an interpretation i_1 with i_1("pickup")= "small van" corresponding to some context and sign user. The sign s_3 has the same representamen as s_1 but a different interpretation and denotation. Data collection usually involves some abstraction which requires equivalence relations on the data. For example "pickup" and "PICKUP" might be considered equivalent and not be further distinguished. The interpretations of s_1, s_2 and s_3 have to be different because they have equivalent representamens but different denotations. But s_4 can have the same interpretation (in this case i_3) as one of the other signs. In general, anything that is stated about a sign and its components requires some judgement and modelling. Therefore from a philosophical viewpoint, signs start an infinite chain of interpretation where interpretations themselves can be further interpreted and so on. From an SCA viewpoint, such an infinite chain has been stopped at the point at which the data is collected and modelled. SCA focuses on analysing provided data and not on how the data has been collected.

Definition 1 For a set R (called *representamens*), a set D (called *denotations*) and a set I of partial functions $i : R \rightarrow D$ (called *interpretations*), a *semiotic relation* S is a relation $S \subseteq I \times R \times D$. A relation instance $(i, r, d) \in S$ with $i(r) = d$ is called a *sign*. For a semiotic relation, an equivalence relation $\approx_R$ on R, an equivalence relation $\approx_I$ on I, and a tolerance relation $\sim_D$ on D are defined.

Thus a semiotic relation involves a structure $(S, I, R, D, \approx_I, \approx_R, \sim_D)$. Additionally, a non-mathematical condition is assumed stating that the data for SCA has been collected in some kind of communication context. For each interpretation some further parameters about context, time, place, sign user and so on might be available. In the examples above, i_k might correspond to sign user u_k at time t_k in location l_k. If i_1 and i_2 refer to the same time and location, then s_1 and s_2 might indicate a miscommunication where a sign user u_1 has a totally different understanding of "pickup" from sign user u_2. For denotations an equivalence relation or even equality could be defined but quite often a weaker tolerance relation is sufficient. A tolerance relation expresses similarity (in this case synonymy) because it is reflexive and symmetric, but need not be transitive. For example "small van" might be similar to "an act of collecting goods in a small van" which might be similar to "a casual encounter with a stranger" but the last one is unlikely to be similar to "small van". The originally linguistic definitions have been abstracted in SCA as follows:

Definition 2 For a semiotic relation, two signs (i_1, r_1, d_1) and (i_2, r_2, d_2) are:

- *synonyms* $\Leftrightarrow d_1 \sim_D d_2$,
- *polysemous* $\Leftrightarrow r_1 \approx_R r_2$ and $d_1 \sim_D d_2$,
- *homographs* $\Leftrightarrow r_1 \approx_R r_2$ and $d_1 \not\sim_D d_2$.

Thus continuing with the example, s_1 and s_4 might be synonyms. The signs s_1 and s_3 might be polysemous, but s_1 and s_2 are most likely homographs instead of polysemous. It should be noted that contrary to standard linguistic terminology, in SCA it is not required for synonyms to have different representamens. Thus a sign is both polysemous and synonymous to itself and, in SCA, polysemy is a special kind of synonymy.

The question remains as to how one decides whether "small van" $\sim_D$ "heavy road vehicle" and so on. In the SCA, the denotations are not usually strings as in the example above, but concepts that are modelled according to some kind of formalisation (formal ontology, FCA or other logical conceptual formalism). SCA does not prescribe which formalisation should be used but only defines *conceptual classes* as a means for describing some aspects that a modelling of conceptual structures must provide. Figure 1 presents an overview of how the sign components are mapped into conceptual classes. Representamens are mapped onto formal objects of a conceptual class. If human sign users are involved, it is not possible to obtain exact information about the denotations because they only exist in the minds of the sign users. It is therefore preferable not to map from denotations to concepts but the other way around in order to approximate the hypothetical denotations.

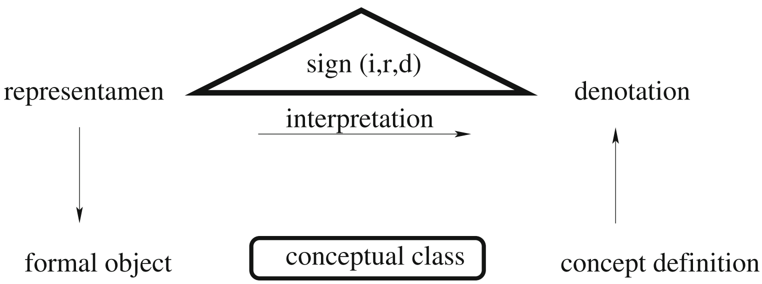

Fig. 1 Sign components and how they can be investigated

A conceptual class is a formalisation of conceptual knowledge containing extensional information (some elements, sets, relations, functions, ...) and intensional information (some logical statements, axioms, implications, clauses, algorithms, ...). Because a purpose of SCA is to analyse data, all sets are assumed to be finite. Only some aspects of a conceptual class are required:

Definition 3 A *conceptual class* is a structure containing some sets $\{O_1, O_2, ..., O_n\}$ with a set $\mathcal{O} := \bigcup\{O_1, O_2, ..., O_n\}$, a set $\mathcal{A}$ of statements according to the rules of some logical language L and a relation $J \subseteq \mathcal{O} \times A$ with $oJa :\Longleftrightarrow a(o)$ is true. The elements of O are called *formal objects*. The set $\mathcal{A}$ contains a subset $\mathcal{C}$ whose elements are called *concept definitions*. $\mathcal{O}$ is called *extensional description* and $\mathcal{A}$ *intensional description* of the conceptual class.

Because n-ary relations and functions are also sets and operations can be modelled as n-ary functions, formal objects can be of a variety of complex types. An example of a conceptual class is a formal context in the sense of FCA where attributes are considered as predicates. Other examples of conceptual classes are using description logic for the statements in $\mathcal{A}$ and set-theoretical models for $\mathcal{O}$ or a relational database as a conceptual class where each O_i is a table and $\mathcal{A}$ might be empty.

3 Compound signs

Ideally, sign users should be able to produce a maximal number of signs with a minimal amount of information that they need to memorise about a semiotic relation. This section discusses the use of compound signs as a means for reducing the cognitive load of a semiotic relation. Measures similar to the ones provided in this section most likely already exist in other fields. But, as far as we know, they have not yet been discussed in a similar setting of triadic signs.

Definition 4 Let $\mathcal{L}(S)$ denote a measure for the *cognitive load* involved in using the signs of a semiotic relation S, such as memorising, parsing, selecting, understanding and so on.

There are many possibilities for specifying $\mathcal{L}(S)$. Because R and I together determine denotations, they need to be memorised. In analogy to (but not identical to) Peirce's triad of symbol, index and icon, in SCA interpretations can be divided into arbitrary[1], algorithmic and observational interpretations. Observations use perceptual features of representamens. For example, the meaning of onomatopoeic words (such as "cuckoo") can sometimes be directly observed without having to learn the word. Arbitrary interpretations need to be fully memorised whereas for algorithmic interpretations only some rules need to be memorised from which other interpretations can be derived. For example, if "a" is interpreted as 1 and "b" as 2, then it is possible to guess what "c" might be interpreted as. The distinctions between arbitrary, algorithmic and observational are not mathematically precise and depend on the socio-cultural background of sign users. Even for arbitrary interpretations contextual clues are relevant, for example in order to express metaphors. Thus the actual cognitive load can differ for different types of interpretations.

[1] Following Saussure's notion of "arbitraire".

The complexity of parsing and of applying interpretations is also relevant. From a perceptual viewpoint, it may be impossible to parse representamens that are too small or too quietly uttered. In this paper a constant $c(S)$ is assumed to summarise perceptual and other qualitative aspects of a semiotic relation. If the set of representamens is large, then $c(S)$ will be insignificant compared to $|R|$. Further details of $c(S)$ are left for future research.

Theorem 1 *The maximal number of possible signs in any semiotic relation is* $|S| \leq |I| \times |R|$. *Furthermore,* $\mathcal{L}(S) \leq |I| + |R| + c(S)$.

Proof: according to Def. 1, $i(r) = d$ which means that if a pair (i, r) exists, it uniquely identifies d. The maximal number of possible combinations of r and i is $|R| \times |I|$. But not all signs that are theoretically possible need to actually exist in a semiotic relation. The cognitive load mostly depends on memorisation of representamens and interpretations.

The importance of Theorem 1 is that the cognitive load is calculated using addition whereas the number of possible signs is calculated using multiplication which tends to be multitudes larger than addition at least for large numbers.

Theorem 2 *For a semiotic relation with a maximal number of possible signs (i.e.* $|S| = |I| \times |R|$*), the value* $|I| + |R|$ *is minimal if* $|I| = |R| = \sqrt{|S|}$. *Thus if* $\mathcal{L}(S) \geq 2\sqrt{|S|} + c(S)$, *a semiotic relation* S' *exists which can express the same number of signs as* S *but with a smaller cognitive load.*

Proof: For a maximal number of possible signs $|S| = |R| \times |I|$ the term $|I| + |R|$ is supposed to be minimal. After substituting $|I|$ by $|S|/|R|$ the minimum of $|I| + |R| = |S|/|R| + |R|$ with respect to $|R|$ can be calculated as a point where its derivative is zero. The derivative is $1 - |S|/|R|^2$ which leads to $1 = |S|/|R|^2 \iff |R|^2 = |S| \iff |R| = \sqrt{|S|}$. Thus $|I| = |R| = \sqrt{|S|}$.

A single representamen being interpreted onto different denotations is a case of polysemy as defined above. Theorem 2 indicates that the cognitive load would be minimal if the polysemy of signs were to correspond to the square root of the number of signs of a semiotic relation. The English language has 200.000 - 500.000 entries[2] or word-sense pairs. The most polysemous word "run" has about 650 different senses[3] which is approximately $\sqrt{|S|}$. But most natural language words only have a few different senses. According to WordNet[4] the average polysemy of words is only between 1 and 3 senses. Polysemy of natural language words also depends on how it is counted. A word such as "today" would be infinitely polysemous if each day was a different denotation. Considering denotations as concepts lowers the count because then the denotation of "today" is not an actual day but a concept for "current day" which has actual days in its extension.

Two interpretations can be considered equivalent ($\approx_I$) if they encode the same usage context, for example, the same time, place, sign user and physical sign context. Again deciding which interpretations are equivalent is a modelling decision made by a user of SCA with respect to an application.

Definition 5 For a semiotic relation, two signs (i_1, r_1, d_1) and (i_2, r_2, d_2) are:

- *ambiguously polysemous* ⇔ polysemous and $i_1 \not\approx_I i_2$,

[2] https://en.wikipedia.org/wiki/List_of_dictionaries_by_number_of_words

[3] https://public.oed.com/blog/word-stories-go/

[4] https://wordnet.princeton.edu/documentation/21-wnstats7wn

- *simultaneously polysemous* ⇔ polysemous and $i_1 \approx_I i_2$.

This section develops a measure based on ambiguous polysemy. Simultaneous polysemy is discussed in the next section. Both types of polysemy often involve compound signs: parts are ambiguously polysemous with respect to their use in compounds, for example a letter used in different words or a word in different sentences. The different meanings of a compound sign, however, can be explained as simultaneous polysemy.

Definition 6 For a set R of representamens with a partial order $(R, \leq)$, a representamen r is called a *compound representamen* if $\exists r_p \in R$ with $r_p < r$. The set of minimal elements of $(R, \leq)$ is denoted by R_π (set of *part representamens*) and $R_\omega := R \setminus R_\pi$. For each $r \in R_\omega$, a set $\phi(r)$ is defined with $(r_p < r \iff \exists n \in \mathbb{N}, i \in I_\pi : (n, i, r_p) \in \phi(r))$ where n is a unique index number within $\phi(r)$ and I_π is a set of interpretations. The set R_ω is *constructable* if each $r \in R_\omega$ is uniquely identified by its set $\phi(r)$.

It follows that $R = R_\pi \cup R_\omega$ and $R_\pi \cap R_\omega = \{\}$. In Def. 6 the interpretations indicate roles that the parts have in the compound, for example, parts of speech of a sentence. The set $\phi(r)$ represents a result of parsing r into a list of parts with their interpretations. Each pair (i, r_p) has a distinct denotation but can occur more than once for a compound. Therefore, the purpose of the index numbers is to distinguish duplicate pairs. Otherwise index numbers are not intended to provide denotational information. Table 1 shows some examples: 48 as a decimal number or Roman numeral and 9:16 as a time on an analogue clock where the minute hand is meant to have 60 different interpretations and the hour hand 12 interpretations.

For a constructable R_ω, the compounds can be parsed into their parts and the parts then be put together again to form the compound. Graphical representamens such as paintings may not be constructable because it may be difficult to define how the parts constitute the compound. In the remainder of this paper it is assumed that R_ω is constructable. It is also assumed that I_π has been designed by a user of SCA to contain as few interpretations as needed.

Definition 7 For a semiotic relation S with a constructable R_ω, the following measures are defined:

- $\mu_\phi(r) := |\{i \mid \exists r_p, n : (n, i, r_p) \in \phi(r)\}|$
- $\mu_\phi(S) := max_r(\mu_\phi(r))$
- $rpt(r_p) := max_{(r,i)}|\{(n \mid (n, i, r_p) \in \phi(r)\}| - 1$
- $rpt(S) := max_{r_p}(rpt(r_p))$

The measure μ_ϕ counts the number of different interpretations per compound. In natural languages $\mu_\phi(r)$ might correspond to counting the different parts of speech that a piece

Table 1 Examples of parsing compound representamens

type of compound	meaning	$\phi(r)$
decimal number	48	{ (1,10^1,4), (2,10^0,8) }
Roman numeral	48	{ (1,-,X), (2,+,L), (3,+,V), (4,+,I), (5,+,I), (6,+,I) }
analogue clock	9:16	{ (1,9, hour hand), (2,16, minute hand) }

of text contains. If $R = R_\pi$ then $\mu_\phi(S) = 0$ and $I_\pi := \{\}$ can be assumed. The measure $rpt(r_p)$ (for "repetition") detects if a part r_p is used more than once with the same interpretation in a compound. For example, I is used 3 times with the interpretation + in the Roman numeral for 48. Presumably reusing a part many times with the same interpretation can render a compound difficult to perceptually parse. Therefore ideally $rpt(r_p) = 0$ and $rpt(S) = 0$.

The next definition extends the partial order amongst representamens to a partial order amongst signs.

Definition 8 For a semiotic relation S with $(R, \leq)$, $\sim_D$ and $\approx_I$, a sign $s := (i, r, d)$ is called a *compound sign* if $\phi(r) \neq \{\}$ and $\forall (n_p, i_p, r_p) \in \phi(r)$:
a) $s_p := (i_p, r_p, i_p(r_p)) \in S$; *b)* $\exists i_1 (i_1, r, i_p(r_p)) \in S$; *c)* $i \approx_I i_p$ and $d \sim_D i_p(r_p)$.
In that case the notation is $s_p <_S s$ and, as usual, $s_p \leq_S s :\Longleftrightarrow s_p <_S s$ or $s_p = s$.

The conditions mean that a) each part representamen yields a sign itself; b) the denotations of part signs can also be extracted from the compound sign; and c) compound and part signs are simultaneously polysemous. It follows that a compound sign has at least as many interpretations as it has parts. For example a compound sign for the Roman numeral in Table 1 is (i, XLVIII, 48). It has a part (-,X,-10) and the -10 can also be read from the compound sign: (i_1, XLVIII, -10). The denotational similarity required for simultaneous polysemy can be quite weak. For Roman numerals all numbers might be considered similar.

Theorem 3 *$(S, \leq_S)$ is a partial order if $\sim_D$ is an equivalence relation.*

Proof: $(S, \leq_S)$ has to be shown to be reflexive, antisymmetric and transitive. For $s_1 = (i_1, r_1, d_1)$ and $s_2 = (i_2, r_2, d_2)$: Reflexive because $s_1 = s_2 \Rightarrow s_1 \leq_S s_2$. Antisymmetric: if $s_1 \leq_S s_2$, then either $s_1 <_S s_2 \Rightarrow r_1 < r_2 \Rightarrow r_2 \not< r_2$ or $s_1 = s_2$. Transitive because conditions a) and b) follow from the transitivity of $(R, \leq)$ and c) is fulfilled if $\sim_D$ is an equivalence relation.

Theorem 4 *For a semiotic relation S with a constructable R_ω, an upper threshold for the cognitive load is $\mathcal{L}(S) \leq |R_\pi| + |I_\pi| + |I| + c(S)$*

Proof: The threshold holds because according to Theorem 1: $\mathcal{L}(S) \leq |R| + |I| + c(S)$ and every representamen in R_ω can be constructed from representamens in R_π and interpretations in I_π.

For decimal numbers $|I_\pi| = \mu_\phi(S)$ according to Table 1 because the largest number uses every interpretation. For analogue clocks $|I_\pi| = 12 + 60$ but $\mu_\phi(S) = 2$. Analogue clocks involve algorithmic interpretations. It is not necessary to memorise 60 interpretations for the minute hand, but only the principle of how to read one of these interpretation. Thus $\mu_\phi(S)$ may be more relevant for the cognitive load than $|I_\pi|$. On the other hand, a high $rpt(S)$-value is not an indication of ambiguous polysemy but of repetition. The following measure intends to improve the threshold of Theorem 4 by an estimation which is demonstrated in the examples below. A small $|R_\pi|$ with a large $|I_\pi|$ but small $\mu_\phi(S)$ results in a particularly low value of the measure which is desirable.

Definition 9 For a semiotic relation S with a constructable R_ω, a *measure of ambiguous polysemy* is defined as $\mu_{ap}(S) := |R_\pi| + \mu_\phi(S) + |I| + rpt(S)$.

Table 2 Semiotic relations for numbers between 1 and 100

semiotic relation	R_π	$\|R_\pi\|$	$\mu_\phi(S)$	$rpt(S)$	$\|I\|$	$\mu_{ap}(S)$
S_1: decimal num.	0, 1, 2, ..., 9	10	$log\|S\|$+1=3	0	1	14
S_2: tally marks	I, 𝍸	2	1	$\|S\|/5 - 1 = 19$	1	23
S_3: Roman num.	I, V, X, L, C	$2log(\|S\|)$+1=5	2	3-1=2	1	10
S_4	1, 2, ..., 100	$\|S\|$ =100	0	0	1	101
S_5	a, b, c, ..., j	$\sqrt{\|S\|}$ =10	0	0	$\sqrt{\|S\|}$	20

The example of the first rows of Table 1 is discussed further in Table 2 for the integers from 1 to 100. S_1 is a semiotic relation for decimal numbers (1, 2, ..., 100). S_2 uses tally marks: (I, II, III, IIII, 𝍸, ...). S_3 utilises Roman numerals (I, II, III, IV, V, ..., C). The semiotic relations S_1-S_3 contain compound signs; the semiotic relations S_4 and S_5 do not. In S_4 all numbers are considered non-compound representamens which all share a single interpretation. S_5 contains 10 representamens with 10 interpretations.

Table 2 calculates μ_{ap} for all five semiotic relations. In S_1, the digit 1 can occur in 3 positions which are counted as 3 interpretations (or $log(|S|) + 1$ positions/interpretations). For S_2, the tally mark I can occur at most 4 times. The mark 𝍸 can occur up to $|S|/5 = 20$ times. Each mark has only 1 interpretation, but $rpt(\text{I}) = 3$ and $rpt(𝍸) = 19$. Tally marks can become difficult to visually parse as indicated by the high $rpt(S)$ value. The next row of the table shows Roman numerals. The numerals I and X have 2 interpretations (subtractively before or additively after) which can be applied up to 3 times. Table 2 shows that for numbers below 100, Roman numerals have the best measure of ambiguous polysemy. For larger numbers, decimal numbers outperform Roman numerals even if one were to invent more representamens because one would need $2log(|S|) + 1$ representamens.

4 Observational advantages

The polysemy measure $\mu_{ap}(S)$ is just one type of measure for semiotic relations. While $\mu_{ap}(S)$ focuses on a view from parts to compounds, the opposite view is to consider how much can be represented with a single compound sign. In both views it is an advantage for a semiotic relation to express many denotations with few representamens. In this section, the number of denotations for a fixed set of signs is counted. As mentioned before, observational interpretations need not be learned but are derived from contextual clues and prior knowledge. The term "observation" in SCA was chosen in analogy to Stapleton et al.'s, ([9]) "observation" as a generalisation of "free rides" in the context of diagrams. While parts of a compound sign are ambiguously polysemous with respect to the different manners in which they can be employed, a compound sign is simultaneously polysemous with respect to the information that can be observed from it.

First, equivalence classes of signs are defined. Denotational similarity and thus synonymy and polysemy are only tolerance relations, not equivalence relations. But it is always possible to generate an equivalence relation from a tolerance relation by assigning all elements that can be reached from an element (through any chain of pairs of the tolerance relation) to the same equivalence class. For synonymy this might not be useful because in the worst case, all signs of a semiotic relation might belong to a single equivalence class.

But for polysemy, the equivalence classes are much smaller because they are constrained by the fact that their representamens are equivalent.

Definition 10 For a semiotic relation with a sign $s := (i, r, d)$:
$S_p(s) := \{s\} \cup \{s_2 \mid \exists s_1 \in S_p(s) : s_1$ polysemous to $s_2\}$.
$S_{sp}(s) := \{s\} \cup \{s_2 \mid \exists s_1 \in S_{sp}(s) : s_1$ polysemous to s_2 and $i \approx_I i_1\}$.

Theorem 5 *For each $s \in S$, the sets $S_p(s)$ and $S_{sp}(s)$ are equivalence classes on S of equivalence relations $\approx_{S_p}$ and $\approx_{S_{sp}}$. The equivalence relation $\approx_{S_{sp}}$ is a finer grained partition of $\approx_{S_p}$ which is finer grained than $\approx_R$. If homographs do not exist in S, then each equivalence class of $\approx_{S_p}$ corresponds to an equivalence class of $\approx_R$.*

Proof: $\sim_D$ is not an equivalence relation, but the construction of $S_p(s)$ generates the transitive closure of $\sim_D$ with respect to a selected s and the transitive closure of a tolerance relation is an equivalence relation. S_{sp} is an intersection of equivalence relations, $S_p(s)$ and $\approx_I$, and thus an equivalence relation.

Next, for each class of equivalent signs, the set of denotations belonging to that class is determined.

Definition 11 For a semiotic relation S, the sets $D_p(s) := \{d_1 \in D \mid \exists (i_1, r_1, d_1) \in S_p(s)\}$ and $D_{sp}(s) := \{d_1 \in D \mid \exists (i_1, r_1, d_1) \in S_{sp}(s)\}$ are defined. For $S_1 \subseteq S$, the sets $S_{sp}(S_1) := \bigcup_{s \in S_1} S_{sp}(s)$ and $D_{sp}(S_1) := \bigcup_{s \in S_1} D_{sp}(s)$ are defined.

A further condition might be to require all interpretations in $S_{sp}(S_1)$ to belong to a single equivalence class of $\approx_I$. But that depends on applications. Last but not least observational advantages and efficiency are defined as a translation of Stapleton et al.'s, [9] notions into SCA:

Definition 12 For a semiotic relation S with signs $s := (i, r, d)$, $s_1 := (i_1, r_1, d_1)$, $i \approx_I i_1$ and $d = d_1$, the sign s has an *observational advantage* over s_1 if $|D_{sp}(s)| > |D_{sp}(s_1)|$. For two semiotic relations S_1 and S_2 whose interpretations all belong to the same equivalence class, S_1 has a *higher observational efficiency* over S_2 if $D_{sp}(S_1) = D_{sp}(S_2)$ and $|S_{sp}(S_1)| < |S_{sp}(S_2)|$. A semiotic relation S has *maximal observational advantage* over a conceptual class if $D_{sp}(S)$ contains the set of true statements of the intensional description of the conceptual class.

A non-mathematical condition for Def. 12 is that only signs with *observational* interpretations should be considered. The formulas of Def. 12 are still reasonable for other types of interpretations, but the word *observational* might then be misleading. Single words or mathematical formulas tend to not produce any observations at all (other than perceiving the sign itself). In exceptional cases, they produce a small number of observations. A formula

Table 3 Conceptual classes for numbers

	formal objects	iterator	constructor	operations	properties
Int	1, 2, ...	nextNum()		+, -, ×, <, =,...	...
Int_{10}	1, 2, ...	nextNum()	$n_0 + 10n_1 + 100n_3 + ...$	+, -, ×, <, =, ...	...
$\text{Int}_5^\star$	1, 2, ...	nextNum()	$n_0 + 5n_1$	+, -, ×, <, =, ...	...

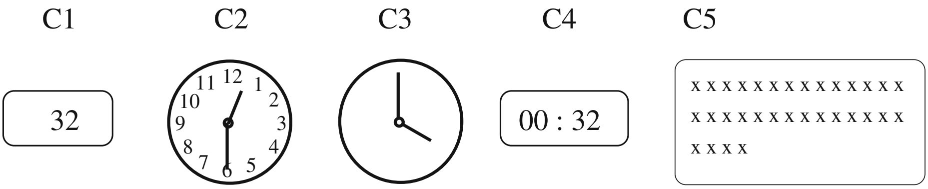

Fig. 2 Types of clocks

$A \subseteq B \subseteq C$ can be considered having a denotation of "$A \subseteq B$, $B \subseteq C$" and an additional observational denotation of $A \subseteq C$. For the representamen "$A \subseteq B$, $B \subseteq C$", however, $A \subseteq C$ is only an implication but not an observation.

Observability also depends on what conceptual class corresponds to a semiotic relation and how well the semiotic relation covers the content of a conceptual class. Table 3 shows three different conceptual classes for the semiotic relations of Table 2. They all contain numbers and an iterator function which retrieves the next number for any number (by adding 1) as formal objects. The usual operations (addition, subtraction), comparison operations (<, =) and properties of integers can be assumed for these conceptual classes. The only difference is a constructor method: in Int_{10} numbers are represented as base-ten digits and in $\text{Int}_5^{\star}$ numbers are represented as a sum of a number ≤ 4 plus the remainder as a multiple of 5. Each conceptual class is assumed to provide functions to convert numbers between its specific representation and a plain integer representation. The first column of Table 4 shows that each of the conceptual classes belongs to one of the semiotic relations S_1-S_4.

With respect to observability, someone who has never seen tally marks, may be able to deduce their denotations directly from the representamens. This is also the case for the first three Roman numerals. For tally marks the functions for progressing from n to $n + 1$, addition, subtraction and size comparisons are also easy to observe. The construction mechanisms of decimal numbers and tally marks correspond to their representamen structures. With some background knowledge, it is possible to observe whether a decimal number is divisible by 2, 5 or 10 based on the last digit. For all four semiotic relations one can observe whether two numbers (of the same semiotic relation) are equal or not. Table 4 shows that tally marks support more observations than signs from the other semiotic relations. But larger numbers expressed by tally marks are difficult to perceptually distinguish (resulting in a high value for $\mu_{ap}(S)$). The results from Tables 2–4 taken together, each semiotic relation has some advantages and disadvantages. Further investigations can be conducted with respect to the complexity of functions for operations and properties. For example, decimal numbers are better suited for multiplication than the other semiotic relations.

Table 4 Semiotic relations for numbers with their conceptual classes and observability

	class	observability				
		objects	iterator	constructor	operations	properties
S_1 decimal	Int_{10}			yes	=	$10\|n$, $5\|n$, $2\|n$
S_2 tally marks	$\text{Int}_5^{\star}$	all	yes	yes	+,-,<,=	$5\|n$
S_3 Roman	Int_{10}	1-3	?		=	
S_4	Int				=	

Table 5 Conceptual classes for time

	time point	iterator	constructor	concepts
T1	1,2,...	nextMin()		minute
T2	1:00-12:59	nextMin()	n_0+60n_1	minute, (full/half/quarter) hour
T3	0:00-23:59	nextMin()	n_0+10n_1+60n_2	1/10 minutes/seconds, (full/half/quarter) hour, noon, midnight

5 An example of types of clocks

This section discusses a further example of how to apply the SCA measures that are defined in this paper. Analogue clocks provide a visualisation of minutes, hours and days that is clearly closely connected to the vocabulary about and conceptualisation of time. For example, words exist for the angles of an analogue clock that are easy to visually detect, such as 90 degrees for a quarter hour. In a digital clock, the passing of a quarter hour can only be calculated but not visualised. Clocks are therefore an interesting example for a semiotic analysis. The example in Fig. 2. is similar to Goguen's [4] example. A naive digital clock (C1) displays just the number of minutes that have passed since a starting point. A military time clock (C4) shows the time as a string of hours, a colon and minutes. A unary clock (C5) prints one character "x" per minute on a computer terminal. We are also considering analogue clocks (C3 and C4) which were not part of Goguen's original analysis. Goguen discusses the differences between the clocks mainly by constructing morphisms which corresponds to an analysis of conceptual classes in SCA. We would argue that other semiotic aspects are missing from Goguen's modelling.

Table 5 shows three different conceptual classes that are relevant for this example. How to chose conceptual classes is a matter of judgement in SCA. Whether or not these classes are appropriately described is a question that is external to SCA. SCA focuses on how well a semiotic relation fits to a given conceptual class. The conceptual class T1 only counts minutes, T2 and T3 also have concepts for hours and maybe for parts of hours. T2 repeats after 12 hours, T3 after 24 hours.

Table 6 calculates the measure of ambiguous polysemy for the five different types of clocks. C5 is clearly poorly designed compared to the other 4 and not commonly used as a clock at all. C1, C2 and C4 are all similar but C3 is best. Table 7 shows that the clocks C2, C3 and C4 outperform the other clocks with respect to how many concepts can be observed from them. The only advantage of C1 and C5 is that they allow an easy comparison of length of time ($<$) because it is not necessary to convert between different units. For C5 it is possible to observe whether one clock shows a larger amount of time, but it is

Table 6 Ambiguous polysemy of different types of clocks for minutes over a 1 day period

	$\|R\|$	$\|I\|$	$\|R_\pi\|$	$\mu_\phi(S)$	$rpt(S)$	$\mu_{ap}(S)$
C1	1440	1	10	4	0	15
C2	720	2	14	2	0	18
C3	720	2	2	2	0	6
C4	1440	1	11	4	0	16
C5	1440	1	1	1	1399	1442

Table 7 Simultaneous polysemy of different types of clocks

	class	extension	iterator	constructor	concepts	properties
C1	T1	integer	change	n/a	minutes	=, (<)
C2	T2	nr + angle	move	yes	minutes, full/half/quarter hour	=
C3	T2	angle	move	yes	minutes, full/half/quarter hour	=
C4	T3	integer tuple	change	yes	minutes, 10 minutes, hour	=
C5	T1	string	add string	n/a	minutes	=,<

difficult to visually determine the exact amount of time. Whether two clocks show exactly the same time (with respect to the timeframe that can be expressed in each conceptual class), is observable for all types of clocks. For digital clocks, concepts of quarter/half/three quarter hour can only be calculated and not observed. But a progress of 10 minutes or 1 hour is easily observable. In total, C3 may be more advantageous than C2 because both clocks share a conceptual class but C3 uses fewer part representamens. But C3 requires users to memorise where the hour digits exist on an analogue clock whereas they can be read from C2. It is probably not surprising that clocks such as C1 and C5 do not actually exist for use by humans. C1 is used internally by computers.

6 Conclusion

SCA as described in this paper attempts to provide quantitative measures for comparing representations of signs and qualitative means for comparing efficiency of semiotic relations with respect to underlying conceptual classes. The main application domain for SCA are graphical representations because they contain more observational interpretations than textual representations. Graphical mathematical representations are particularly suitable because their concepts are defined with mathematical precision. It is intended to apply the results from this paper to a further development of methods for analysing and structuring teaching materials in mathematics education as suggested by Priss [8]. Other applications, for example, in the domain of usability and user experience would also be possible applications of SCA in the future.

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References

1. Barwise, J., Etchemendy, J.: Hyperproof. CSLI. Stanford, CA (1994)
2. Barwise, J., Seligman, J.: Information Flow. The Logic of Distributed Systems. Cambridge University Press, Cambridge (1997)
3. Ganter, B., Wille, R.: Formal Concept Analysis. Mathematical Foundations. Springer, Berlin (1999)
4. Goguen, J.: An introduction to algebraic semiotics, with application to user interface design. Computation for metaphors, analogy, and agents, pp 242–291. Springer, Berlin (1999)
5. Moody, D.: The “physics” of notations: toward a scientific basis for constructing visual notations in software engineering. IEEE Transactions on software engineering **35**(6), 756–779 (2009)
6. Priss, U.: Semiotic-Conceptual Analysis: a proposal. Int. J. Gen. Syst. **46**(5), 569–585 (2017)
7. Priss, U., Old, J.: Semiotic-Conceptual Analysis of a lexical field. In: Alam, Sotropa (eds.) Graph-Based Representation and Reasoning, Proceedings of ICCS’19, Springer Verlag, LNCS, vol. 11530, pp. 239–247 (2019)
8. Priss, U.: Conceptual Schemata as a Means for Structuring Teaching Materials. In: Third International Workshop ‘Concepts in Action: Representation, Learning, and Application’ (CARLA 2021) (2021)
9. Stapleton, G., Jamnik, M., Shimojima, A.: What makes an effective representation of information:, a formal account of observational advantages. Journal of Logic, Language and Information **26**(2), 143–177 (2017)