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S706 CONCEPTUAL STRUCTURES

Indepth combinatorial analysis of admissible sets for abstract argumentation

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Abstract

In this paper we investigate, from a graph theoretical point of view, the notion of acceptability in Dung semantics for abstract argumentation frameworks. We advance the state of the art by introducing and analyzing combinatorial structures exploited for taming, in particular cases, the exponential blowout of acceptance algorithms. We conclude the paper by a series of observations allowing to deepen the intuition with respect to the practical use of Dung acceptance based semantics.

Keywords Acceptability · Graph theory · Dung semantics · Argumentation frameworks

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1 Introduction

Dung's *Theory of Formal Argumentation* [13] is founded on *admissible sets* in *argumentation frameworks*. An argumentation framework is a digraph with vertices interpreted as (abstract) *arguments* and directed edges as *attacks* between arguments. A set S of arguments in an argumentation framework is an *admissible set* if it is *conflict-free* – that is, there is no attack between members of S– and, for any attack from an argument a outside S on an argument b in S, there is an argument in S attacking a. In order to make a distinction among *acceptable* and *unacceptable* arguments in a given argumentation framework, Dung defined the *admissibility-based semantics*, using different families of admissible sets (referred as *extensions*) and representing sets of collectively accepted arguments. Most decision problems concerning the acceptability of a specified argument in a given argumentation framework (also referred as *reasoning tasks*) was later proved to be **NP**-complete, Σ_2^P-complete, or Π_2^P-complete: [9, 12, 16], see also the survey [18]. In order to speed-up the exponential combinatorial search to solve them, we make in this paper a graph-theoretical in-depth study of the family of admissible sets in Dung's Argumentation Frameworks.

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The two main contributions of this paper are as follows:

1. In Section 3 we prove that if a set of arguments A in an argumentation framework D induces an argumentation framework having exactly one preferred extension (maximal admissible set), then there is also a unique maximal admissible set in D contained in A (Theorem 1). The above hypothesis holds trivially for $A = S$, where S is a conflict-free set in the argumentation framework D. The unique maximal admissible set in D contained in S, denoted S^{adm}, is called the *admissibility basis* of the conflict-free set S. Ideal extension, [14], or eager extension, [6], are admissibility basis of the intersection of all preferred extensions, respectively of all semi-stables extensions (Theorem 2, Remark 1). We show (in Theorem 3) that the admissibility basis of a given conflict-free set can be found in polynomial-time (with respect to the number of arguments in the argumentation framework containing this conflict-free set). Using this, we develop an efficient algorithm for skeptical acceptance of a given argument in a bipartite argumentation framework (Lemma 1, Proposition 1, Theorem 4).
2. In Section 4 we consider a specific family of conflict-free sets, *out-and-out conflict-free sets*. A conflict-free set S is *out-and-out* if any argument outside S is either attacked by an argument in S or is attacked by an argument attacked by S. We give a linear algorithm to construct such a conflict-free set in a given argumentation framework (Theorem 5 and its proof) and show that if the argumentation framework is *transitive* (that is, for any distinct arguments a, b, c, if a attacks b and b attacks c, then a attacks c) then any out-and-out conflict-free set is a stable extension (Theorem 6) and, as a consequence, any preferred extension is a stable extension. The same result can be extended to what we call *near-transitive* argumentation framework (Theorem 7), that is argumentation framework in which for any *intransitive triple* a, b, c (three distinct arguments a, b, c with a attacking b, b attacking c, and a not attacking c), if c attacks an argument d, then the set $\{a\} \cup \{d\}$ is not conflict-free. We show that in general argumentation frameworks any preferred extension is contained in an out-and-out conflict-free set (Theorem 8). We call such an out-and-out conflict-free set *sound* and deduce that the ideal extension is the admissibility basis of the intersection of all sound out-and-out conflict free sets (Proposition 3).

In Section 5 we further our analysis by a set of observations meant to deepen practical insights when employing admissibility based semantics in applications. We discuss the asymmetry in the strength of attacks when they are used to defeat or to attack a conflict-free set in an argumentation framework. Also, we introduce a decomposition of the set of arguments outside of a conflict free set S into different sets of arguments determined by the number of attacked and attacking arguments from S. This fine-grained decomposition can be used as a dynamic labeling of the arguments in the search algorithms for deciding the acceptability of a given argument, and also to define a score to distinguish between different extensions of the same type. We conclude the paper with Section 6.

2 Background

In this section we present the basic concepts used for defining classical semantics in abstract argumentation frameworks introduced by Dung in 1995, [13].

Definition 1 [13] An **argumentation framework** is a digraph $D = (\text{ARG}(D), \text{ATT}(D))$, where the vertices in $\text{ARG}(D)$ are called **arguments** and, if $(a, b) \in \text{ATT}(D)$

is a directed edge, then (a, b) is called an **attack** and referred as: **argument** a **attacks argument** b, **argument** b **is attacked by argument** a, and **a is an attacker of b**. The set of attackers of an argument $a \in \mathrm{ARG}(D)$ is denoted by a^-, and the set of arguments attacked by a is denoted by a^+. If $S \subseteq \mathrm{ARG}(D)$ then $S^- = \cup_{a \in S} a^-$ and $S^+ = \cup_{a \in S} a^+$. The set of arguments S **defends** an argument a if $a^- \subseteq S^+$ (i.e., any a's attacker is attacked by an argument in S). The set of arguments defended by a set of arguments $S \subseteq \mathrm{ARG}(D)$ is denoted by $F(S)$. The map F, assigning to each set of arguments, S, the set $F(S)$ of arguments defended by S, is called the **characteristic function**.

A **conflict-free set** in D is a set $S \subseteq \mathrm{ARG}(D)$ such that $S^+ \cap S = \emptyset$. An **admissible set** is a conflict-free set such that $S^- \subseteq S^+$.

An admissible set S is called a **complete extension** if $F(S) = S$. A **preferred extension** is a maximal (w.r.t. set inclusion) complete extension. A **grounded extension** is a minimal (w.r.t. set inclusion) complete extension. A **stable extension** is a conflict-free set S such that $\mathrm{ARG}(D) - S = S^+$. An argument $a \in \mathrm{ARG}(D)$ is **credulously admissible acceptable** in D if there is an admissible set S such that $a \in S$.

For $\sigma \in \{$*complete, grounded, preferred, stable*$\}$, an argument $a \in \mathrm{ARG}(D)$ is **credulously σ-acceptable** in D if there is a σ extension S such that $a \in S$. An argument $a \in \mathrm{ARG}(D)$ is **skeptically acceptable w.r.t. preferred semantics** in D if it belongs to each preferred extension.

Definition 2 *(Other admissibility based semantics)* Let D be an argumentation framework.

- [14] The **ideal extension** of D is the unique maximal (w.r.t. set-inclusion) admissible set contained in every preferred extension in D.
- [5] A **semi-stable extension** in D is a complete extension S with the property that $S \cup S^+$ is maximal w.r.t. set-inclusion (if T is a complete extension in D such that $S \cup S^+ \subseteq T \cup T^+$, then $S \cup S^+ = T \cup T^+$).
- [6] The **eager extension** of D is the unique maximal (w.r.t. set-inclusion) admissible set contained in every semi-stable extension in D.

Throughout this paper we consider only finite argumentation frameworks (that is, argumentation frameworks D with $\mathrm{ARG}(D)$ finite) without self-attacking arguments (that is, there is no argument $a \in \mathrm{ARG}(D)$ with $(a, a) \in \mathrm{ATT}(D)$). Also if D is an argumentation framework and $A \subseteq \mathrm{ARG}(D)$, then we denote by $D[A]$ the argumentation framework (induced in D by A) with $\mathrm{ARG}(D[A]) = A$ and $\mathrm{ATT}(D[A]) = A \times A \cap \mathrm{ATT}(D)$. We prefer this notation although most authors use $D \downarrow_A$.

3 Admissibility basis

Let $S \subset \mathrm{ARG}(D)$ be an admissible set in the argumentation framework D. It is not difficult to see that S remains an admissible set in the argumentation framework $D[A]$, for any set of arguments A such that $S \subseteq A \subseteq \mathrm{ARG}(D)$. However, if S is a maximal admissible set in D contained in A (i.e., if $T \subseteq A$ is an admissible set in D and $S \subseteq T$, then $S = T$) it is possible that S is not a preferred extension in $D[A]$. Nevertheless, if $D[A]$ has exactly one preferred extension, then there is also a unique maximal admissible set in D contained in A, as the following theorem shows.

Theorem 1 *(Unique maximal admissible set contained in a set inducing an argumentation framework with exactly one preferred extension)*

Let $A \subset \textsc{Arg}(D)$ be a set of arguments in the argumentation framework D such that $D[A]$ has exactly one preferred extension. Then, there is $A^{adm} \subseteq A$, an admissible set in D that contains any admissible set in D contained in A.

Proof Let $\mathcal{A}$ be the set of all admissible sets S in D satisfying $S \subseteq A$. $\mathcal{A}$ is non-empty since $\emptyset \in \mathcal{A}$. Let $A^{adm} = \cup_{S \in \mathcal{A}} S$. We show that A^{adm} is an admissible set in D.

Clearly, $A^{adm} \subseteq A$. Since each $S \in \mathcal{A}$ is an admissible set in D, we have $S^- \subseteq S^+$. Hence $(A^{adm})^- = \cup_{S \in \mathcal{A}} S^- \subseteq \cup_{S \in \mathcal{A}} S^+ = (A^{adm})^+$. It remains to prove that A^{adm} is a conflict-free set.
Let $P \subseteq A$ the unique preferred extension of $D[A]$. For each $S \in \mathcal{A}$ we have $S^- \subseteq S^+$, hence $S^- \cap A \subseteq S^+ \cap A$. Also, S remains conflict-free in $D[A]$. It follows that each $S \in \mathcal{A}$ is a subset of P. Therefore $A^{adm} \subseteq P$, and therefore it is a conflict-free set. □

Prominent examples of sets A satisfying the hypothesis of the above theorem are those with the property that there is no even circuit in $D[A]$ (by a result of [16]) and conflict-free sets. The case of conflict-free sets deserves a particular investigation as the rest of this section shows.

Theorem 2 *(Admissible basis of a conflict-free set)*

Let $S \subset \textsc{Arg}(D)$ be an arbitrary conflict-free set in the argumentation framework D. There is an admissible set $S^{adm} \subseteq S$ that contains any admissible set contained in S. Moreover, if S satisfies $F(S) \subseteq S$, then S^{adm} is a complete extension.

Proof Since S is a conflict-free set in D, we have $\textsc{Att}(D[S]) = \emptyset$, S is the unique preferred extension in $D[S]$, and Theorem 1 applies. For the second assertion, we prove that if $F(S) \subseteq S$ then $F(S^{adm}) = S^{adm}$. Indeed, since S^{adm} is an admissible set, we have $S^{adm} \subseteq F(S^{adm})$. Let $a \in F(S^{adm})$. From $S^{adm} \subseteq S$, we have $F(S^{adm}) \subseteq F(S)$. By hypothesis $F(S) \subseteq S$, hence $a \in S$. It follows that $S^{adm} \cup \{a\} \subseteq S$ is a conflict-free set and because $a \in F(S^{adm})$ and S^{adm} is an admissible set we obtain that $S^{adm} \cup \{a\}$ is an admissible set contained in S. Therefore $S^{adm} \cup \{a\} \subseteq S^{adm}$, that is $a \in S^{adm}$. Hence $F(S^{adm}) \subseteq S^{adm}$. □

By the proof of Theorem 2, S^{adm} is unique and will be referred to as the *admissible basis* of the conflict-free set S. Its significance is revealed by the following remark (see also [17]).

Remark 1 (Ideal and Eager Semantics) Let D be an argumentation framework.

- [14] If S is the conflict free set obtained by intersecting all preferred extensions of D, then its admissible basis, S^{adm}, is the **ideal extension** of D.
- [6] If S is the conflict free set obtained by intersecting all complete extensions T of D with the property that $T \cup T^+$ are maximal with respect to set-inclusion (semi-stable extensions), then its admissible basis, S^{adm}, is the **eager extension** of D.

By Theorem 2, we can define a function $S \rightarrow S^{adm}$, which associates to each conflict free set S its admissible basis S^{adm}. Interesting enough, this function can be evaluated in polynomial time, as the following theorem shows.

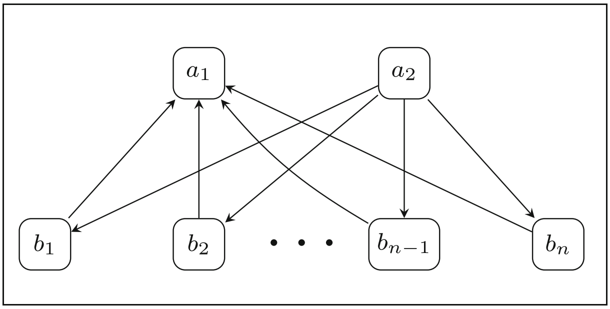

Fig. 1 Bipartite argumentation framework

Theorem 3 *(Computing the admissible basis) For every conflict-free set S in the argumentation framework D, its admissible basis, S^{adm}, is given by the following algorithm:*

$S^{adm} \leftarrow S$
while S^{adm} *is not admissible* ***do***
  $S^{adm} \leftarrow S^{adm} - ((S^{adm})^- - (S^{adm})^+)^+$
return S^{adm}.

Proof If the current conflict-free set, S^{adm}, is an admissible set, then the algorithm returns it. Otherwise, the set $(S^{adm})^- - (S^{adm})^+$, of attackers of S^{adm} that are not counterattacked by S^{adm}, is not empty. The arguments of S^{adm} attacked by this set can not be members of any admissible set contained in S^{adm}. Hence they can be deleted from S^{adm}. At each *while* iteration, at least one argument is deleted from the current conflict-free set, hence the algorithm terminates. □

We note that, when the uniqueness of the maximal (w.r.t. inclusion) admissible set contained in S is assured, an equivalent (fixed point) form of the above algorithm is given in [17].

Theorems 2 and 3 can be combined to decide in polynomial time the acceptability of an argument in a bipartite argumentation framework. An argumentation framework D is bipartite if $\text{ARG}(D)$ is the disjoint union of two conflict-free sets L and R, $\text{ARG}(D) = L\dot{\cup}R$. An example is given in Fig. 1, where $L = \{a_1, a_2\}$ and $R = \{b_1, b_2, \ldots, b_n\}$. It is not difficult to see that, in this case, we have $L^{adm} = L$ and $R^{adm} = \emptyset$.

Parts of the following lemma are well-known, but some of them are new. We prove all of them for the sake of completeness.

Lemma 1 *(Admissible sets in bipartite AFs)*

i) *(Graph theory folklore,* [13]*) In every bipartite argumentation framework there is a stable extension.*
ii) *Let D be an argumentation framework, $S \subseteq \text{ARG}(D)$ be an admissible set, and $D' = D[\text{ARG}(D) - (S \cup S^+)]$. If D' is bipartite, then S is contained in a stable extension of D.*
iii) [16] *In a bipartite argumentation framework any preferred extension is a stable extension.*

iv) *If S is an admissible set in the bipartite argumentation framework D, $\textsc{Arg}(D) = L\dot{\cup}R$ with L and R conflict-free sets, then $S_L = S \cap L$ and $S_R = S \cap R$ are both admissible sets in D. Conversely, if $S_L \subseteq L$ and $S_R \subseteq R$ are admissible sets in D and $S = S_L \cup S_R$ is a conflict-free set, then S is an admissible set in D.*

v) *Let D be a bipartite argumentation framework with bipartition $\textsc{Arg}(D) = L\dot{\cup}R$. Then $R - R^{adm} \subseteq (L^{adm})^+$ and $L - L^{adm} \subseteq (R^{adm})^+$.*

Proof i) We use induction on $|\textsc{Arg}(D)|$. In the inductive step, if every $a \in \textsc{Arg}(D) = L\dot{\cup}R$ satisfies $a^- \neq \emptyset$, then both L and R are stable extensions. Otherwise, let $a \in \textsc{Arg}(D)$ with $a^- = \emptyset$. The argumentation framework $D' = D[\textsc{Arg}(D) - (\{a\} \cup a^+)]$ has a stable extension S (by the induction hypothesis). Then, $S \cup \{a\}$ is a stable extension in D.

ii) By i), D' has a stable extension S'. Since S is an admissible set, it follows that $S \cup S'$ is a stable extension in D.

iii) Let S be a preferred extension in the bipartite argumentation framework D. In the above proof of ii) S is a subset of the stable extension $S \cup S'$. The only possibility is that $S' = \emptyset$, that is $\textsc{Arg}(D')$ is empty, and, therefore, S is a stable extension.

iv) Since S is a conflict-free set and S_L and S_R are subsets of S, it follows that these are also conflict-free sets. Since S is an admissible set, we have $S^- \subseteq S^+$, that is $S^- \cap L \cup S^- \cap R \subseteq S^+ \cap L \cup S^+ \cap R$. Since $S^- \cap L = S_R^-$, $S^- \cap R = S_L^-$, $S^+ \cap L = S_R^+$, and $S^+ \cap R = S_L^+$, we obtain $S_R^- \cup S_L^- \subseteq S_R^+ \cup S_L^+$. Since L and R are disjoint, the last inclusion holds if and only if $S_R^- \subseteq S_R^+$ and $S_L^- \subseteq S_L^+$. It follows that S_R and S_L are admissible sets.

Conversely, since S_L and S_R are admissible sets, we have $S_L^- \subseteq S_L^+$ and $S_R^- \subseteq S_R^+$. Hence $S^- = S_L^- \cup S_R^- \subseteq S_L^+ \cup S_R^+ = S^+$. By hypothesis S is a conflict-free set, therefore it is an admissible set in D.

v) Let S be a stable extension in D (by i), S exists). By iv), $S_L = S \cap L$ and $S_R = S \cap R$ are admissible sets contained in L, respectively R, satisfying $R - S_R \subseteq S_L^+$. By Theorem 2, $S_L \subseteq L^{adm}$ and $S_R \subseteq R^{adm}$. Hence, $R - R^{adm} \subseteq R - S_R \subseteq S_L^+ \subseteq (L^{adm})^+$. Similarly, we can prove that $L - L^{adm} \subseteq (R^{adm})^+$ holds. □

Proposition 1 *(Admissible basis and bipartite credulous evaluation) The following algorithm is correct:*

Bipartite_ Credulous_ Acceptance

Input: $a \in \textsc{Arg}(D) = L\dot{\cup}R$; *$L$ and R conflict-free sets in D*

Output: YES, if a is credulously accepted, NO, otherwise

if $a \in L$ ***then*** $S \leftarrow L$ ***else*** $S \leftarrow R$;

Compute S^{adm};

if $a \in S^{adm}$ ***then*** *return YES* ***else*** *return NO.*

Proof Suppose that $a \in L$, that is, $S = L$ (the case $a \in R$ is similarly). Then, by Lemma 1 iv), there is an admissible set T such that $a \in T$ if and only if $a \in T_L$. By Theorem 2, this happens if and only if $a \in L^{adm}$. □

We note that, essentially, the above algorithm has been previously also given in [16] (see also [15]).

Theorem 4 *(Admissible bases and bipartite skeptical evaluation) The following algorithm is correct:*

Bipartite_ Skeptical_ Acceptance
Input: $a \in \text{ARG}(D) = L\dot{\cup}R$; *L and R conflict-free sets in D*
Output: YES, if a *is skeptically accepted, NO, otherwise*
if $a \in L$ **then** $S \leftarrow L, T \leftarrow R$ **else** $S \leftarrow R, T \leftarrow L$;
Compute S^{adm};
if $a \notin S^{adm}$ **then** *return NO* ;
Compute T^{adm};
if $a^- \cap T^{adm} \neq \emptyset$ **then** *return NO* ;
return YES.

Proof Suppose that $a \in L$, that is, $S = L$ and $T = R$ (the case $a \in R$ is similarly). By the above theorem, if $a \notin L^{adm}$ then a is not credulously accepted, and the answer is NO. Suppose $a \in L^{adm}$ and there is a preferred extension U such that $a \notin U$. Then, by Lemma 1 iii), U is a stable extension and, therefore $a^- \cap U \neq \emptyset$. By Lemma 1 iv), we have $a^- \cap U_R \neq \emptyset$, and the algorithm returns NO, since $a^- \cap U_R \subseteq a^- \cap R^{adm}$, by Theorem 2. Hence if $a \in L^{adm}$ and $a^- \cap R^{adm} = \emptyset$, then every preferred extension contains a and, therefore, the algorithm returns YES. □

Please note that the above algorithm improves that given in [7] (see also, [15]), since it avoids testing if there is an argument in a^- that is credulously accepted, by the use of the admissible bases. For example, testing if a_1 is skeptical accepted in the bipartite argumentation framework in Fig. 1, needs only to compute $L^{adm} = \{a_1, a_2\}$ and, since $R^{adm} = \emptyset$, the answer is positive. On the other hand, the algorithm in [7] requires $O(n)$ credulously acceptance testing of the arguments $b_i \in a_1^-$. It is well-known that the acceptability problems on bipartite argumentation problems can be solved in polynomial time (see, e.g., [18]) but the speed-up of a factor of $|\text{ARG}(D)|$ of the worst time complexity is important.

4 Out-and-out conflict free sets

In this section we consider a specific family of conflict-free sets and use it to obtain results on the existence of stable extensions in special argumentation frameworks. A conflict-free set S in an argumentation framework D is called an *out-and-out conflict-free set* if any argument in $\text{ARG}(D) - (S \cup S^+)$ is attacked by an argument in S^+. This corresponds to *quasi-kernels* in Graph Theory terminology (introduced by [10], see [11]) and appears also in the Theory of Voting in the study of *uncovered sets* (see, e.g., [3]).

Definition 3 *(Out-and-out conflict-free set)* Let D be an argumentation framework.
A conflict-free set S in D is called an **out-and-out conflict-free set** if

$$S \cup S^+ \cup (S^+)^+ = \text{ARG}(D).$$

Example. In the argumentation framework D in Fig. 2, $S_1 = \{a, c, e\}$ is an out-and-out conflict-free set, since $\{a, b, c, d, e, f\} = \text{ARG}(D) = S_1 \cup S_1^+$. Also, $S_2 = \{a, d, f\}$ is an out-and-out conflict-free set, since $\text{ARG}(D) - (S_2 \cup S_2^+) = \{c\}$ and $c \in (S_2^+)^+$.

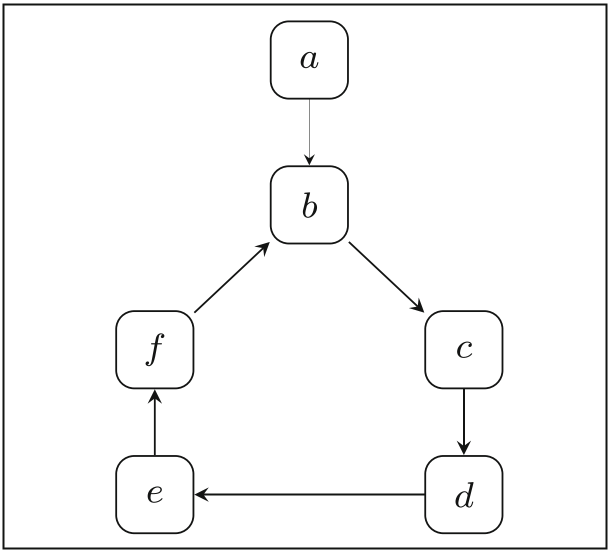

Fig. 2 $\{a, d, f\}$ and $\{a, c, e\}$ are out-and-out conflict free sets

Theorem 5 *(Out-and-out conflict-free sets in AFs)*
In every argumentation framework, D, there is an out-and-out conflict-free set.

Proof Let $\{a_1, a_2, \ldots, a_n\}$ be an arbitrary ordering of the arguments in $\text{ARG}(D)$, and let us consider the following algorithm

Construction of an out-and-out conflict-free set

$S \leftarrow \emptyset$; $Discard \leftarrow \emptyset$;
for $i = 1$ **to** n **do**
  if $a_i \notin Discard$ **then**
    $S \leftarrow S \cup \{a_i\}$
    $Discard \leftarrow Discard \cup (a_i^+ \cap \{a_{i+1}, \ldots, a_n\})$;
for $i = n$ **downto** 2 **do**
  if $a_i \notin Discard$ **then**
    $S' \leftarrow S \cap (a_i^+ \cap \{a_1, \ldots, a_{i-1}\})$
    $S \leftarrow S - S'$
    $Discard \leftarrow Discard \cup S'$;
return S.

Clearly, the set S constructed by the algorithm is non-empty: in the first scan at least a_1 is added to S, and in the second scan at least the last argument added to S in the first scan, remains in S.

Also, S is a conflict-free set: in the first scan, the attacks $(a_i, a_j) \in \text{ATT}(D)$ with $a_i, a_j \in S$ and $i < j$ are cleared, and in the second scan the attacks $(a_i, a_j) \in \text{ATT}(D)$ with $a_i, a_j \in S$ and $i > j$ are avoided.

Clearly, at the end of the algorithm, we have $\text{ARG}(D) = S \dot{\cup} Discard$. The arguments in $Discard$ are those attacked by the returned set S, that is S^+, and those attacked by the arguments added to S in first scan and deleted from S in the second scan. Therefore, these arguments belongs to $(S^+)^+$, and the theorem holds. □

For the argumentation framework D in Fig. 2, taking $\{a_1, \ldots, a_6\} = \{b, c, d, e, f, a\}$, the set S constructed by the above algorithm in the first scan is $\{b, d, f, a\}$, and after the second scan, the algorithm returns $\{a, f, d\}$. Also, taking $\{a_1, \ldots, a_6\} = \{a, b, c, d, e, f\}$, we obtain after the first scan $\{a, c, e\}$, which is not modified by the second scan. We already verified that these are out-and-out conflict-free sets.

Out-and-out conflict-free sets can be used to obtain admissibility results in particular argumentation frameworks.

An argumentation framework D is *transitive* if for any distinct arguments $a, b, c \in \text{ARG}(D)$,

$$\text{if } (a, b), (b, c) \in \text{ATT}(D) \text{ then } (a, c) \in \text{ATT}(D).$$

Theorem 6 *(Admissible sets in transitive argumentation frameworks)*

i) *In every transitive argumentation framework there is a stable extension.*
ii) *Let $S \subseteq \text{ARG}(D)$ be an admissible set in the argumentation framework D and let $D' = D[\text{ARG}(D) - (S \cup S^+)]$. If D' is transitive, then S is contained in a stable extension of D.*
iii) *In a transitive argumentation framework any preferred extension is a stable extension.*

Proof i) By Theorem 5, there is in D an out-and-out conflict-free set S such that $S \cup S^+ \cup (S^+)^+ = \text{ARG}(D)$. If there is an argument $c \in (S^+)^+ - (S \cup S^+)$, this means that there is $a \in S$ and $b \in S^+$ such that $(a, b), (b, c) \in \text{ATT}(D)$ and $(a, c) \notin \text{ATT}(D)$. But, this contradicts the hypothesis that D is transitive. Hence $(S^+)^+ - (S \cup S^+) = \emptyset$. It follows that $S \cup S^+ = \text{ARG}(D)$, that is, S is a stable extension.

ii) Since D' is a transitive argumentation framework, by i), D' has a stable extension S'. Since S is an admissible set, no argument in $\text{ARG}(D')$ attacks S, therefore we obtain that $S \cup S'$ is a conflict-free set in D attacking all arguments outside it, that is, it is stable extension in D.

iii) Let S be a preferred extension in the transitive argumentation framework D. Note that D', as defined in ii), is also a transitive argumentation framework. In the above proof of ii) S is a subset of the stable extension $S \cup S'$. Since S is a maximal admissible set, it follows that $S' = \emptyset$, that is, $\text{ARG}(D')$ is empty. Hence, S is a stable extension in D. □

The above proof can be adapted to obtain similar results for a class of argumentation frameworks containing the transitive ones.

An *intransitive triple* in the argumentation framework D is a set of three distinct arguments $\{a, b, c\} \subseteq \text{ARG}(D)$, such that $(a, b), (b, c) \in \text{ATT}(D)$ and $(a, c) \notin \text{ATT}(D)$.

The argumentation framework D is called *near-transitive* if, for any intransitive triple $\{a, b, c\}$, if $(c, d) \in \text{ATT}(D)$, then the set $\{a\} \cup \{d\}$ is not conflict-free. This means that $d \neq a$ (since there are no self-attacking arguments) and that $(a, d) \in \text{ATT}(D)$ or $(d, a) \in \text{ATT}(D)$. Since in a transitive argumentation framework there are no intransitive triples, it follows that any transitive argumentation framework is a near-transitive argumentation framework. On the other hand, there are near-transitive argumentation frameworks which are not transitive. A simple example is 4-cycle $D = (\{x, y, z, t\}, \{(x, y), (y, z), (z, t), (t, x)\})$.

Theorem 7 *(Admissible sets in near-transitive AFs)*

i) *In every near-transitive argumentation framework there is a stable extension.*

ii) *Let $S \subseteq \text{ARG}(D)$ be an admissible set in the argumentation framework D and let $D' = D[\text{ARG}(D) - (S \cup S^+)]$. If D' is near-transitive, then S is contained in a stable extension of D.*

iii) *In a near-transitive argumentation framework any preferred extension is a stable extension.*

Proof i) By Theorem 5, there is in D an out-and-out conflict-free set S such that $S \cup S^+ \cup (S^+)^+ = \text{ARG}(D)$. The set $U = (S^+)^+ - (S \cup S^+)$ can be decomposed $U = U_1 \dot{\cup} U_2$, where $U_1 = U \cap S^-$ and $U_2 = U - U_1$. If there is $c \in U_1$, this means that there is $a \in S$ and $b \in S^+$ such that $(a, b), (b, c) \in \text{ATT}(D)$ and $(a, c) \notin \text{ATT}(D)$, that is, $\{a, b, c\}$ is an intransitive triple. Since $c \in S^-$ it follows that there is $d \in S$ such that $(c, d) \in \text{ATT}(D)$. Because D is a near-transitive argumentation framework, it follows that the set $\{a\} \cup \{d\} \subseteq S$ is not conflict-free, a contradiction. It follows that $U_1 = \emptyset$, that is S is an admissible set. Moreover, it follows that: (*) there is no attack between an argument in S and an argument in U.

Let $c \in U$, and as above, the intransitive triple $\{a, b, c\}$ implied by the definition of U. There is no attack from c against an argument in U, that is U is a conflict-free set. Otherwise, if there is $d \in U$ such that $(c, d) \in \text{ATT}(D)$, then (since D is a near-transitive argumentation framework) the set $\{a\} \cup \{d\}$ is not conflict-free, contradicting the above remark (*) . It follows that $S \cup U$ is a conflict-free set in D, hence a stable extension, since $\text{ARG}(D) - (S \cup U) = S^+ = (S \cup U)^+$.

ii) Similar to the proof of Theorem 6 ii).

iii) Similar to the proof of Theorem 6 iii), with the observation that if D is a near-transitive argumentation framework, then D' is also a near-transitive argumentation framework. □

Le us note here the algorithmic importance of Theorem 6 ii) and Theorem 7 ii). Clearly, verifying if the argumentation framework D' is transitive or near-transitive can be done in polynomial time (e.g., testing the conditions in their definitions for each triple of arguments). Also, by the proofs of Theorem 6 i) and Theorem 7 i), the stable extension containing the admissible set S in the (arbitrary) argumentation framework D can be constructed in polynomial time with the algorithm in the proof of Theorem 5. Hence, the stable extension containing the admissible set S can be constructed in polynomial time whenever the (polynomial) test whether D' is (near)transitive is positive.

Theorem 8 *(Out-and-out conflict-free set extending a preferred extension) Let P be a preferred extension in the argumentation framework D. There is an out-and-out conflict-free set $S \subseteq \text{ARG}(D)$ such that $P \subseteq S$.*

Proof Let $U = \text{ARG}(D) - (P \cup P^+)$. If $U = \emptyset$, then P is a stable extension. Since any stable extension is an out-and-out conflict-free set, we can take $S = P$.

If $U \neq \emptyset$, let $D' = D[U]$ be the argumentation framework induced by U in D. By Theorem 5, there is T, an out-and-out conflict-free set in D'. Since there are no attacks between the arguments in P and U, it follows that $S = P \cup T$ is a conflict-free set. It is easy to see that $S \cup S^+ \cup S^{++} = \text{ARG}(D)$. □

Note that, for the out-and-out conflict-free set S in the above theorem, we have $S^{adm} = P$.

Proposition 2 *(Maximal out-and-out conflict-free sets)*
Let S be a maximal (w.r.t set-inclusion) out-and-out conflict-free set $S \subseteq \mathrm{ARG}(D)$. *Then, for any argument* $a \in \mathrm{ARG}(D) - S$, *either* $a \in S^+$ *or* $a \in S^- - S^+$.

Proof Let $U = \mathrm{ARG}(D) - (S \cup S^+)$ and $U = U_1 \cup U_2$, where $U_1 = \{a \in U \mid \exists \text{ no } b \in S \text{ s.t. } (a, b) \in \mathrm{ATT}(D)\}$, and $U_2 = U - U_1$. If we prove that $U_1 = \emptyset$, the theorem holds.

Suppose that $U_1 \neq \emptyset$, and let $D' = D[U_1]$ be the argumentation framework induced by U_1 in D. By Theorem 5, there is T, an out-and-out conflict-free set in D'. Since there are no attacks between the arguments in S and U_1, it follows that $S' = S \cup T$ is a conflict-free set. By the construction of T, S' is an out-and-out conflict set, a contradiction to the hypothesis that S is a maximal out-and-out conflict-free set. □

Definition 4 *(Sound out-and-out conflict-free set)* Let D be an argumentation framework. A maximal (w.r.t set-inclusion) out-and-out conflict-free set $S \subseteq \mathrm{ARG}(D)$ is called a **sound conflict-free set** if S^{adm} is a preferred extension of D. The set of all sound conflict-free sets of D is denoted by $\mathcal{S}_D$.

Let $\mathcal{P}_D$ the set of all preferred extensions in the argumentation framework D. By Theorem 8, for every $P \in \mathcal{P}$ there is $S_P \in \mathcal{S}_D$ such that $P \subseteq S_P$. It follows that $\cap_{P \in \mathcal{P}_D} P \subseteq \cap_{P \in \mathcal{P}_D} S_P$. By the Definition 4, the set $\{S_P \mid P \in \mathcal{P}_D\}$ is exactly $\mathcal{S}_D$. Hence, the following inclusion holds

$$\bigcap_{P \in \mathcal{P}_D} P \subseteq \bigcap_{S \in \mathcal{S}_D} S.$$

In fact, we have equality here.[1] Indeed, suppose that there is $x \in \cap_{S \in \mathcal{S}_D} S - \cap_{P \in \mathcal{P}_D} P$. It follows that there is $P_0 \in \mathcal{P}$ such that $x \notin P_0$ and P_0 is not a stable extension (since a stable extension is a sound conflict-free set). Since P_0 is a preferred extension, there is $y \in \mathrm{ARG}(D) - (P_0 \cup P_0^+)$ such that $(y, x) \in \mathrm{ATT}(D)$. In the argumentation framework $D[\mathrm{ARG}(D) - (P_0 \cup P_0^+)]$ we construct an out-and-out conflict free S_0 by applying the algorithm in the proof of Theorem 5 with any ordering of its arguments starting with y. Clearly, in the first scan of this algorithm x is discarded from S_0. It follows that $P_0 \cup S_0$ is a sound out-and-out conflict free set not containing x, contradicting its choice ($x \in \cap_{S \in \mathcal{S}_D} S$).

Hence $\bigcap_{P \in \mathcal{P}_D} P = \bigcap_{S \in \mathcal{S}_D} S$, and using Remark 1, we obtain the following result.

Proposition 3 *(Ideal extension and the family of sound out-and-out conflict-free sets)*
The admissible basis of the intersection of all sound out-and-out conflict free sets in an argumentation framework D is exactly the ideal extension of D.

5 Further observations

In this section we conclude our analysis by two types of observations aimed at deepening the intuitive understanding of how such admissibility notions can play out in practice.

[1] Thanks to an anonymous reviewer for the suggestion.

To this end we will:

1. Discuss our assumption that the argumentation frameworks considered have no self-attacks and, also, show that we can not change the admissibility acceptance of an argument a by adding (missing) attacks issued from the attackers of a. These two simple remarks (new to our knowledge) could be very useful from an algorithmic point of view.
2. Decompose the set of arguments outside of a conflict free set S into different sets of arguments determined by the number of attacked and attacking arguments from S. This fine-grained decomposition can be used in two ways in practice. First, as a dynamic labeling of the arguments (and, consequently, as a topological searching space by using appropriate heuristics) and, second, in order to define a score to distinguish between different extensions of the same type.

5.1 Self-Attacks and Reinforced Attackers

By the Definition 1 of an argumentation framework, it is possible to have self-attacking arguments. An argument $a \in \text{ARG}(D)$ is a self-attacking argument if $(a, a) \in \text{ATT}(D)$. Clearly, no self-attacking argument can be member of a conflict-free set, therefore no self-attacking argument is acceptable in an argumentation framework. However, a self-attacking argument can have a decisive influence in the non-acceptance of some other arguments. The following proposition shows that we can restrict to argumentation frameworks without self-attacking arguments without any loss of the generality.

Proposition 4 *(No need for self-attacks) Let $D = (\text{ARG}(D), \text{ATT}(D))$ be an argumentation framework and $a \in \text{ARG}(D)$ such that $(a, a) \in \text{ATT}(D)$. Let D' be the argumentation framework obtained from D by adding two new arguments a_1, a_2, and replacing the attack (a, a) with the attacks (a, a_1), (a_1, a_2) and (a_2, a). Then, $S \subseteq \text{ARG}(D)$ is an admissible set in D if and only if it is an admissible set in D'.*

Proof Let $S \subseteq \text{ARG}(D)$ be an admissible set in D. Since $(a, a) \in \text{ATT}(D)$ and S is a conflict-free set in D, it follows that $S \subseteq \text{ARG}(D) - \{a\}$. By the construction of D', S^+ and S^- are the same in D and D'. It follows that S is an admissible set in D'.

Conversely, let $S \subseteq \text{ARG}(D)$ be an admissible set in D'. If $a \in S$, then $a_2 \in S^-$ in D'. Since a_1 is the only attacker of a_2 in D', and $a_1 \notin S$, it follows that S does not defend a against the attack (a_1, a), a contradiction. Therefore, $S \subseteq \text{ARG}(D) - \{a\}$. As above, we obtain that S is an admissible set in D. [2] □

The strength of Dung's collective view of arguments acceptance is illustrated by the following proposition which surprisingly shows that the admissibility based acceptability of a given argument is not influenced by adding attacks issued from the attackers of the specified argument, against other arguments.

Proposition 5 *(Attackers reinforcement is worthless) Let D be an argumentation framework, $a \in \text{ARG}(D)$, and D' the argumentation framework obtained from D by adding some*

[2] As one anonymous reviewer observed, if we add to D' all attacks from the arguments in a^-(in D) to a_1 and a_2, and all attacks from a_1 and a_2 to the arguments in a^+(in D), then the admissible sets in D' are exactly the admissible sets in D.

missing attacks from the arguments in a^- *to other arguments in* $\textsc{Arg}(D)$*. i.e.* $\textsc{Arg}(D) = \textsc{Arg}(D')$, $\textsc{Att}(D) \subseteq \textsc{Att}(D')$ *and, if* $(b, c) \in \textsc{Att}(D') - \textsc{Att}(D)$*, then* $(b, a) \in \textsc{Att}(D)$*. Then, there is an admissible set containing* a *in* D *if and only if there is an admissible set containing* a *in* D'*.*

Proof We specify the argumentation framework in the notations x^+ (x^-, S^+, S^-) by writing x^{+_D} $(x^{-_D}, S^{+_D}, S^{-_D})$.

Suppose that there is an admissible set $S \subseteq \textsc{Arg}(D)$ with $a \in S$ (S is conflict free in D and $S^{-_D} \subseteq S^{+_D}$). Since $\textsc{Arg}(D) = \textsc{Arg}(D')$, we have $S \subseteq \textsc{Arg}(D')$. Since $a \in S$ and the attacks added to D to obtain D' are from the arguments in $a^{-_D} \subseteq \textsc{Arg}(D) - S$ to other arguments, it follows that S is a conflict-free set in D'. Also, $S^{+_{D'}} = S^{+_D}$. Since $a^{-_D} \subseteq S^-_D$ it follows that $S^{-_{D'}} = S^{-_D}$. Therefore, from $S^{-_D} \subseteq S^{+_D}$ we obtain $S^{-_{D'}} \subseteq S^{+_{D'}}$, that is, S is an admissible set in D' containing a.

Conversely, suppose that there is an admissible set $S \subseteq \textsc{Arg}(D')$ with $a \in S$ (S is conflict free in D' and $S^{-_{D'}} \subseteq S^{+_{D'}}$). Since $\textsc{Arg}(D') = \textsc{Arg}(D)$, we have $S \subseteq \textsc{Arg}(D)$. Since $\textsc{Att}(D) \subseteq \textsc{Att}(D')$ and S is a conflict-free set in D', it follows that S is a conflict-free set in D. Since $a \in S$ and in D' only attacks from a^{-_D} are added, it follows that $S^{+_D} = S^{+_{D'}}$. Hence $S^{-_{D'}} \subseteq S^{+_D}$, and because $S^{-_D} \subseteq S^{-_{D'}}$, it follows that $S^{-_D} \subseteq S^{+_D}$, that is, S is an admissible set in D' containing a. □

5.2 Dynamic labelling

An equivalent way to express Dung's extension-based semantics is using argument labeling as proposed by [4] (originally introduced in [24]). The idea underlying the labeling-based approach is to assign to each argument a label from the set $\{I, O, U\}$. The label I (i.e., In) means the argument is accepted, the label O (i.e., Out) means the argument is rejected, and the label U (i.e., Undecided) means one abstains from an opinion on whether the argument is accepted or rejected.

We consider here a related approach, more appropriate for the combinatorial search of σ-extensions containing a specified argument in a given argumentation framework.

Definition 5 *(Hashing* $\textsc{Arg}(D)$ *by a conflict-free set)*
Let $D = (\textsc{Arg}(D), \textsc{Att}(D))$ be an argumentation framework and $S \subseteq \textsc{Arg}(D)$ be a conflict-free set in D. S decomposes $\textsc{Arg}(D)$ into disjoint (possible empty) sets

$$\textsc{Arg}(D) = S \,\dot\cup\, \dot\bigcup_{i,j=0}^{|S|} \overline{S}_{i,j}\ ,$$

where,

$$\overline{S}_{i,j} = \{a \in \textsc{Arg}(D) - S \mid |a^- \cap S| = i \text{ and } |a^+ \cap S| = j\}$$

is the set of arguments of D outside S attacked by i arguments from S and attacking j arguments in S.

In order to capture Dung's extension-based semantics, we need a coarser decomposition of the argument set of an argumentation framework, as follows.

Definition 6 *(A coarser split of* $\textsc{Arg}(D)$ *by a conflict-free set)*
Let $D = (\textsc{Arg}(D), \textsc{Att}(D))$ be an argumentation framework and $S \subseteq \textsc{Arg}(D)$ be a conflict-free set in D. S decomposes $\textsc{Arg}(D)$ into four disjoint (possible empty) sets

$$\textsc{Arg}(D) = S \,\dot\cup\, \mathit{Discard}(S) \,\dot\cup\, \mathit{Attacker}(S) \,\dot\cup\, \mathit{Neutral}(S), \text{ where,}$$

- $Discard(S) := \dot{\cup}_{i=1}^{|S|} \dot{\cup}_{j=0}^{|S|} \overline{S}_{i,j}$, is the set of all arguments collectively attacked by S, i.e., $Discard(S) := S^+$,
- $Attacker(S) := \dot{\cup}_{j=1}^{|S|} \overline{S}_{0,j}$, is the set of all arguments attacking at least one argument in S and not counterattacked by an argument in S, i.e., $Attacker(S) := S^- - S^+$,
- $Neutral(S) := \overline{S}_{0,0}$ is the set of all arguments outside S, not attacking the arguments in S and not attacked by an argument in S.

It is not difficult to verify (using the Definition 6) that the following important property holds.

Remark 2 Let S and S' be conflict-free sets such that $S \subseteq S'$. Then,

$$S' - S \subseteq Neutral(S) \text{ and } Neutral(S') \subseteq Neutral(S).$$

The argumentation semantics in Definition 1 can be equivalently characterized using our decomposition. Some parts of the the following proposition are obvious, some parts are already established (e.g. Proposition 3.2 in [2]) but they are interesting from a combinatorial search point of view.

Proposition 6 *(Rereading argumentation semantics) Let $S \subseteq \text{ARG}(D)$ be a conflict-free set in the argumentation framework $D = (\text{ARG}(D), \text{ATT}(D))$. Then S is*

i) an admissible set if and only if $Attacker(S) = \emptyset$;
ii) complete extension if and only if it is an admissible set and for all $a \in Neutral(S)$ we have $a^- \cap Neutral(S) \neq \emptyset$;
iii) preferred extension if and only if it is a complete extension and, in the argumentation framework induced in D by the arguments in $Neutral(S)$, the empty set is only admissible set.
iv) grounded extension if and only if it is a complete extension and there is a linear order $<$ on S such that $\forall a \in S,$ if $b \in a^-$ then there is $a' \in S \cap b^-$ with $a' < a$.
v) stable extension if and only if $Discard(S) = \text{ARG}(D) - S$.

Proof i) and v) are obvious. ii) is Caminada's characterization of a complete labeling [4]. iv) follows from Dung's [13] characterization of grounded extension as the unique least fixed point of the characteristic function. We prove only iii).

iii) Let $D' = D[Neutral(S)]$ the argumentation framework induced by $Neutral(S)$ in D.

If S is a preferred extension in D then it is a complete extension and there is no complete extension S' such that $S \subset S'$ ($S \neq S'$). Suppose that $T \neq \emptyset$ is an admissible set in D'. Then $S \cup T$ is an admissible set in D and (by Remark 2) $Neutral(S \cup T) \subseteq Neutral(S)$. Since S is a complete extension, by (ii), we have $a^- \cap Neutral(S \cup T) = \emptyset$, for every $a \in Neutral(S \cup T)$. Hence, again by (ii), $S \cup T$ is a complete extension, contradicting the maximality of S.

Conversely, suppose that the empty set is the only admissible set in D'. If there is a complete extension S' such that $S' - S \neq \emptyset$, then (by Remark 2) $S' - S \subseteq Neutral(S)$ and $Attacker(S' - S) \cap Neutral(S) = \emptyset$, that is $S' - S$ is a non-empty admissible set in D', a contradiction. □

We note here that, inherently, similar decompositions appear in all "labeling-approach" algorithms or heuristics-guided backtracking algorithms for solving reasoning problems in abstract argumentation, e.g., [8, 22], or [25].

Our intention is to use the decomposition in Definition 6 as a dynamic labeling of the arguments (having as parameter the conflict-free set): each argument of the argumentation framework has one of the following labels: **S**, **A**($ttacker$), **N**($eutral$) and **D**($iscard$). Since the conflict-free sets containing S can be obtained only by adding arguments from *Neutral*(S), deciding the acceptability of a given argument can be done as follows.

Let us denote by **S** the set of all conflict-free sets in D containing a given argument a_0. We can decide the (credulously) acceptability of a_0 by considering search algorithms that operate on the *state space* $\mathcal{S} = (S_I, \mathbf{S}^*, succ)$, where $S_I = \{a_0\} \in \mathbf{S}$ is the initial state, $\mathbf{S}^* \subseteq \mathbf{S}$ is the set of *goal states* (that are conflict-free sets corresponding the various admissibility based semantics, as characterized in Theorem 6) and $succ$ is a *successor function* that maps each state $S \in \mathbf{S}$ (i.e., a conflict-free set in D containing a_0) to a finite (possible empty) set of successor states: $succ(S) = \{S \cup \{a\} | a \in \mathit{Neutral}(S)\}$. Note that for preferred semantics, a goal state can be recognized only by solving a set of search problems of the same type.

$\mathcal{S}$ can be organized as a *state space topology* [21] by defining heuristic functions, $h : \mathbf{S} \to \mathbf{R}_0^+ \cup \{\infty\}$, in order to speed up the search for a goal state or to add fast pruning criteria that allow some states to be discarded.

For example, an obvious heuristic, h^1, estimates the "degree of inadmissibility" of any conflict-free set $S \in \mathbf{S}$ by the number of arguments attacking S and not counterattacked, i.e., $h^1(S) = |\mathit{Attacker}(S)|$. Clearly, $h^1(S) = 0$ if and only if S is an admissible set in D. Assuming that states with lower h^1 values are on a path to the closest goal state, the successor state of an arbitrary (non goal) state S will be $S \cup \{x_S\}$, selected in a *greedy best first search* (GBFS) manner. Obviously, if $h^1(S) > 0$ but $\mathit{Neutral}(S) = \emptyset$, then S is a *dead-end*: there is no way to extend S to goal state (no admissible set extends S).

The decomposition in the Definition 5 can be also used to define the score of a conflict-free set S with respect to an argument $a \in \text{ARG}(D) - S$. If $a \in \overline{S}_{i,j}$ then $Score(S, a) = i - j$. Equivalently, we have the following definition.

Definition 7 *(Score of a conflict-free set against an outside argument)*
Let $D = (\text{ARG}(D), \text{ATT}(D))$ be an argumentation framework and $S \subseteq \text{ARG}(D)$ be a conflict-free set in D. The *score* of S with respect to an argument $a \in \text{ARG}(D) - S$ is

$$Score(S, a) = |a^- \cap S| - |a^+ \cap S|.$$

Note that a non-negative score, $Score(S, a) \geq 0$, shows that the conflict-free set S counter-attacks every attack from a. This is not the same as graded defense (or neutrality) introduced in [20], but shares the motivation and significance, excellently discussed in their paper.

Examples.

1. Let $D_n = (\text{ARG}(D_n), \text{ATT}(D_n))$ be the argumentation framework with $\text{ARG}(D_n) = \{a_0, \ldots, a_{2n}\}$ and $\text{ATT}(D_n)) = \{(a_0, a_i) | i \in \{1, \ldots, n\} \cup \{(a_i, a_{n+i}), (a_{n+i}, a_i) | i \in \{1, \ldots, n\}$ (adapted from [19]; D_4 is depicted in Fig. 3). D_n has 2^n stable extensions: $S_0 = \{a_0, a_{n+1}, \ldots, a_{2n}\}$ and $S_A = A \cup \{a_{n+1}, \ldots, a_{2n}\} - \{a_{n+i} | a_i \in A\}$, for each $A \subseteq \{a_1, \ldots, a_n\}$, $A \neq \emptyset$. Note that each argument is credulously preferred (stable) accepted in D_n. However, despite of the exponential number of preferred extension, no argument is skeptically accepted, since for each argument a there is a preferred extension not containing a. The same conclusion holds for the ideal or eager extension (which are empty). Let us analyze the score of each such stable extensions with respect to the outside arguments.

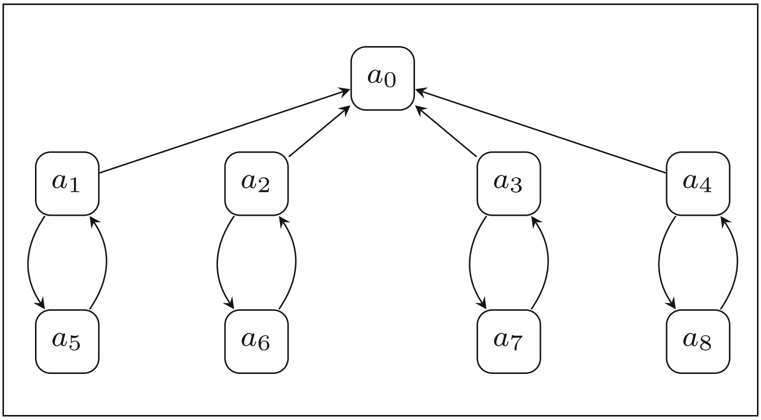

Fig. 3 D_4 with $2 \cdot 4 + 1$ arguments and 2^4 stable extensions (not all with the same strength)

- $\mathbf{S_0}$: $\text{ARG}(D_n) - S_0 = \{a_1, \ldots, a_n\}$; for each $i \in \{1, \ldots, n\}$, $a_i^- \cap S_0 = \{a_{n+i}\}$ and $a_i^+ \cap S_0 = \{a_0, a_{n+i}\}$, hence $Score(S_0, a_i) = 1 - 2 = -1$.
- $\mathbf{S_A}$, for $A = \{a_1, \ldots, a_k\}$:
  $\text{ARG}(D_n) - S_A = \{a_0\} \cup \{a_{k+1}, \ldots, a_n\} \cup \{a_{n+1}, \ldots, a_{n+k}\}$. $a_0^- \cap S_A = A$ and $a_0^+ \cap S_A = \emptyset$, hence $Score(S_A, a_0) = |A|$. For $a \in \{a_{k+1}, \ldots, a_n\} \cup \{a_{n+1}, \ldots, a_{n+k}\}$, $a^- \cap S_A = a^+ \cap S_A$ and, therefore $Score(S_A, a) = 0$. Similar results are obtained for other $A \subseteq \{a_1, \ldots, a_n\}$, $A \neq \emptyset$.

It follows that $S_{\{a_1,\ldots,a_n\}} = \{a_1, \ldots, a_n\}$ is the only stable extension having non-negative scores with respect to the arguments outside it and maximizing the sum of these scores.

2. Let D_n (n integer, $n \geq 2$) be the argumentation framework with $\text{ARG}(D_n) = \{a_0, \ldots, a_n\} \cup \{b\}$ and $\text{ATT}(D_n)) = \{(a_0, b)\} \cup \{(b, a_i) | i \in \{1, \ldots, n\}\}$ (D_4 is depicted in Fig. 4). Clearly, $S = \{a_0, \ldots, a_n\}$ is the grounded extension in D_n. Since $\text{ARG}(D_n) - S = \{b\}$, the score of S against b is $Score(S, b) = |b^- \cap S| - |b^+ \cap S| = 1 - n < 0$. On the other hand, $S_0 = \{a_0\}$ and $S_i = \{a_0, a_i\}$ are admissible sets with non-negative scores with respect to arguments outside them: $Score(S_0, b) = 1$ and $Score(S_0, a_i) = 0$, for all $i \geq 1$; $Score(S_i, b) = 0$ and $Score(S_i, a_j) = 0$, for all $j \geq 1$ and $j \neq i$. It follows that S_0 is an admissible set containing and defending the argument a_0 and its score against any other argument is non-negative (which is not the case for the grounded extension). This can be more convincing for the acceptance of the argument a_0 in some applications (of course, for non logical argumentation frameworks). Similarly, S_i can be used as a better justification for the acceptability of a_i.

We can also use the scores to pass beyond of the notion of admissibility, as follows.

Definition 8 *(Strict admissibility)* Let $D = (\text{ARG}(D), \text{ATT}(D))$ be an argumentation framework with $n = |\text{ARG}(D)|$ arguments and let k be an integer, $0 \leq k \leq n - 1$. A conflict-free set $S \subseteq \text{ARG}(D)$ is **k-strict admissible set** in D if $Score(S, x) \geq k$, for every argument $x \in \text{ARG}(D) - S$. An argument $a \in \text{ARG}(D)$ is **k-strict admissible acceptable** if there is a k-strict admissible set containing a in D. For $k = 0$, 0-strict admissibility will be referred, simply, as **strict admissibility**.

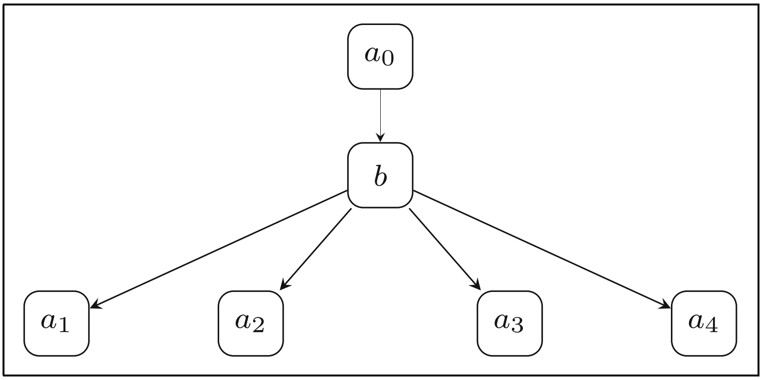

Fig. 4 The grounded extension has not the best scores

Clearly, every strict admissible set S is an admissible set: if $a \in \text{ARG}(D) - S$ is an attacker of S, then there is at least one argument in S attacking a, since $Score(S, a) \geq 0$. Converse is not true, not every admissible set is strict admissible. A simple example is given in Fig. 5. *Strict admissibility*, imposing an external condition is different from *strong admissibility* introduced in [1], which imposes a supplementary internal condition on a Dung's admissible set.

There are particular argumentation frameworks D and $a \in \text{ARG}(D)$ such that every admissible set containing a is also a strict admissible set. A prominent example is the argumentation framework in the standard proof of the polynomial time reduction of the **SAT** problem to the problem of deciding if a given argument in a given argumentation framework is credulously accepted [12, 16].

Let $F = C_1 \wedge \ldots \wedge C_m$ be an arbitrary instance of **SAT**, where for each $i \in \{1, \ldots, m\}$, the clause C_i is a a disjunction of literals, $C_i = l_{i_1} \vee \ldots \vee l_{i_{k_i}}$, and a literal l is either a variable x_j or its negation $\overline{x}_j$, where $\{x_1, \ldots, x_n\}$ is the set of Boolean variables occurring in F. An argumentation framework D_F and $a_F \in \text{ARG}(D_F)$ can be constructed in polynomial time such that F is satisfiable if and only if a_F is (credulously) admissible acceptable in D_F. Firstly, the special argument a_F is added to $\text{ARG}(D_F)$. Second, for each clause C_i an argument a_{C_i} is added to $\text{ARG}(D_F)$ and an attack (a_{C_i}, a_F) is added to $\text{ATT}(D_F)$). Finally, for each Boolean variable x_j occurring in F, two arguments a_{x_j} and $a_{\overline{x}_j}$ are added to $\text{ARG}(D_F)$, attacking each other in D_F and adding all the attacks (a_{x_j}, a_{C_i}), if x_j is a

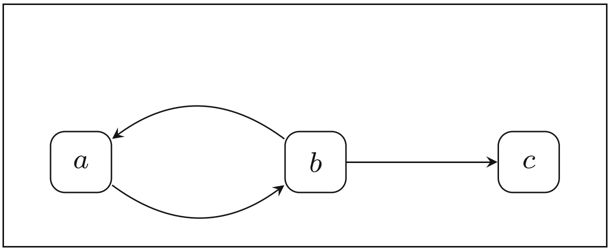

Fig. 5 The argument c is not strict admissible acceptable, since the conflict-free sets $S_1 = \{c\}$ and $S_2 = \{a, c\}$, containing it, are not strict admissible sets (S_1 is not admissible, S_2 is admissible, but $Score(S_2, b) = 1 - 2 = -1 < 0$)

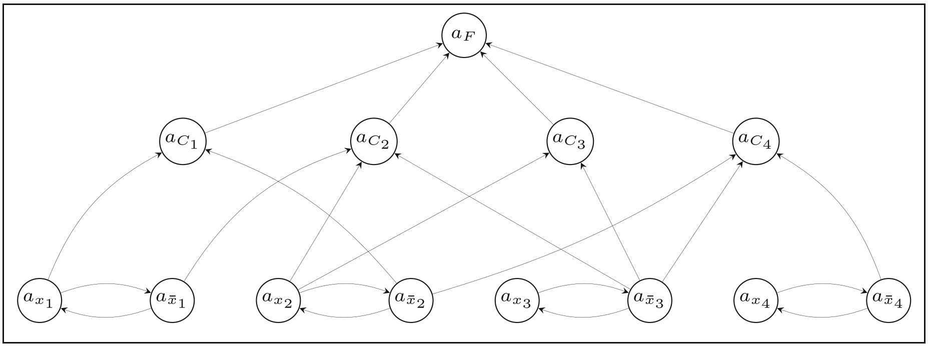

Fig. 6 The argumentation framework D_F associated to instance $F = C_1 \wedge C_2 \wedge C_3 \wedge C_4$, where $C_1 = x_1 \vee \bar{x}_2$, $C_2 = \bar{x}_1 \vee x_2 \vee \bar{x}_3$, $C_3 = x_2 \vee \bar{x}_3$, $C_4 = \bar{x}_2 \vee \bar{x}_3 \vee \bar{x}_4$

literal in C_i, and $(a_{\overline{x}_j}, a_{C_i})$, if $\overline{x}_j$ is a literal in C_i, for all clauses C_i. Clearly, D_F can be constructed in polynomial time in the size of the instance of **SAT**. An example of this construction is illustrated in Fig. 6.

It is not difficult to see that any admissible set in D_F containing a_F is of the form $S = \{a_F\} \cup T$, where T selects for each variable x_j at most one of the arguments a_{x_j} or $a_{\overline{x}_j}$ such that for each clause C_i, the argument a_{C_i} has at least one attacker in S. But this is equivalent to the fact that F is satisfiable (by assigning the truth value to the literals corresponding to the arguments in T). The arguments a outside S are either of the type a_{x_j} and $a_{\overline{x}_j}$ not selected by the set T (for which $Score(S, a) = 0$), or of the form a_{C_i} (for each clause C_i), for which $Score(S, a) \geq 0$, since a attacks a_F, but it is attacked by at least one argument in T.

It follows that we proved the hardness of the problem of deciding if a specified argument in a given argumentation framework is strictly accepted, by exhibiting a polynomial time reduction from the CNF satisfiability problem **SAT**. Since the membership of this problem to **NP** is obvious, the following theorem holds.

Theorem 9 *(Strict admissibility acceptance is NP-complete) Deciding if a specified argument in a given argumentation framework is strictly accepted is an* ***NP****-complete problem.*

6 Discussion

In this paper we presented a combinatorial analysis of the notion of admissibility in Dung semantics for abstract argumentation frameworks. The two main contributions were the study of admissibility basis of conflict free sets and the study of a specific family of conflict free sets - the out-and-out conflict free sets. The first contribution allowed for the development of an efficient skeptical acceptance algorithm in bipartite argumentation frameworks. The second allowed for the development of linear algorithms to compute stable extensions in transitive argumentation frameworks and the further study of admissibility in near transitive argumentation frameworks. Despite of the fact that the results concern very specific sub-classes of argumentation frameworks, since their algorithmic recognition can be made in polynomial time, they are applicable as sub-tasks in the algorithms dedicated to reasoning with general argumentation frameworks (see, e.g., Lemma 1 i), Theorem 6 ii), Theorem 7 ii)). In the same line of idea, [23] reported remarkable experimental results on checking skeptical admissibility in general argumentation frameworks using bipartite argumentation framework recognition. It is therefore not necessary to show that they correspond to interesting applications of argumentation (in order to sustain the practical applicability of these results).

We closed the paper by providing two types of observations relevant for the practical use of admissibility notions. In fact, the paper is conceived as a theoretical basis for developing an efficient argumentation solver using a dedicated combinatorial search space (as suggested in Section 5.2). Several software systems for deciding acceptability in abstract argumentation frameworks have been developed and, also, an international competition[3] for empirical evaluation and comparison of solvers appeared in 2015. In the survey [8], the existing implementation methods are classified into two categories: *reduction approaches* and *direct approaches*. The reduction approaches transform (in polynomial time, sometimes iteratively) the graph model into inputs for established solvers (SAT, CSP, or ASP) assessed for well-studied problem domains. The direct methods access the framework directly (without having the overhead of transformation and, as a result, a potential loss of structural information). In a future work (carrying on this paper) we intend to introduce a new direct approach based on a systematic and informative search method on the space of conflict-free sets of arguments associated to the input of the problem of deciding acceptability.

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Declarations

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