Annals of Mathematics and Artificial Intelligence (2019) 87:227–232
https://doi.org/10.1007/s10472-019-09635-9

Impossibility results for belief contraction

Sven Ove Hansson[1]

Published online: 14 June 2019

Abstract
Three seemingly weak and plausible conditions on an operation of contraction on belief sets are shown to be logically incompatible: (1) there is at least one sentence that can be successfully removed by the operation, (2) both the original belief set and the outcome of the operation have finite representations, and (3) a non-tautologous sentence can be removed without loss of all its non-tautologous logical consequences.

Keywords Belief change · Contraction · Eradication · Success postulates · Finite-based outcome

Mathematics Subject Classification (2010) 03B42

1 Introduction

The purpose of this contribution is to show that three seemingly weak and plausible conditions on an operation of contraction on belief sets are logically incompatible. Following the tradition in studies of belief change [1, 4, 7], by a contraction is meant an operation $\div$ on a logically closed set (belief set) K, such that for any sentence p in the object language, the contraction outcome, denoted $K \div p$, is another belief set. The logic is assumed to be classical and compact. It is represented by a Tarskian consequence operation Cn (alternatively denoted relationally as $\vdash$). The following are considered to be fundamental postulates for characterizing the properties of operations of contraction:

$$\mathrm{Cn}(K \div p) = K \div p \; (closure)$$

$$K \div p \subseteq K \; (inclusion)$$

$$p \notin (K \div p) \setminus \mathrm{Cn}(\varnothing) \; (success)$$

These are three of the six basic AGM postulates, which characterize the most thoroughly studied operation of contraction, partial meet contraction [1]. The other three basic postulates will not be referred to here. Furthermore, *closure* is the only AGM postulate that will be assumed to hold. Only much weakened versions of *success* and *inclusion* will be used.

✉ Sven Ove Hansson
soh@kth.se

1 Kungliga Tekniska Högskolan, Stockholm, Sweden

On the other hand, two additional types of postulates will be introduced in order to capture important intuitions that are not covered in the AGM postulates.

The following three sections will introduce the conditions to be used in the impossibility results: two weakened success conditions (Section 2), finite representability (Section 3), and the possibility of contracting a sentence without losing all its non-tautologous logical consequences (Section 4). The impossibility results are presented and discussed in Section 5.

2 First condition: some minimal degree of success

The postulate of *success* has been questioned by authors who maintain that some non-tautologous sentences may be so strongly held that attempts to remove them cannot succeed. [16], p. 54, [3, 14] Operations that have been constructed to capture this intuition do not satisfy *success*, but they tend to satisfy weakened versions of it, such as the following:

There is some $p \in K$ such that $K \div p \nvdash p$ (*minimal success*),

According to *minimal success*, there is at least one sentence that the operation can remove. We will also use the following condition:

$\mathrm{Cn}(K \div p) \subset K$ for some sentence p (*reducibility*),

It says that there is at least one sentence whose contraction results in a proper subset of the original belief set.

3 Second condition: finite representability

The AGM model [1] is the dominating model of belief change. It does not stipulate that belief sets should have finite representations. However, all three AGM authors have, on different occasions, endorsed the standpoint that "it is only by working on some finite generator or representative of the theory that the outcome of a process such as contraction can ever in practice be determined". ([2], pp. 21-22; cf. [5], p. 90, and [13], p. 384) For our present purposes, this amounts to two conditions. First, we will assume that the original belief set K is finite-based, i.e. there is some finite set B of sentences such that $K = \mathrm{Cn}(B)$. Secondly, we will assume that the following postulate ([6], p. 604) holds:

If K is finite-based, then so is $K \div p$. (*finite-based outcome*)

This is a highly plausible postulate. Its violation would mean that the removal of a single sentence, an operation that is normally performed in order to reduce the amount of information contained in the belief set, instead adds an infinite amount of information to the belief set. However, *finite-based outcome* is not satisfied by AGM contraction. [9]

4 Third condition: no eradication

One important property of human belief systems is that they can harbour beliefs that are essentially unrelated with each other. We can think of our beliefs as belonging to different compartments of our minds. It should be possible to perform a change in one compartment while leaving another compartment unchanged. Parikh [11, 12, 15] This means that some of our beliefs are *independently removable*. For a simple example, let m represent your belief that there is milk in your fridge and p your belief that Paris is the capital of France. Presumably, these two elements of your belief set belong to different compartments. They are fully independent of each other, and we expect that one of them can be removed while the other is retained.

Important consequences of these assumptions can be expressed with the notion of eradication. An operation $\div$ that takes us from a belief set K to a new belief set K' *eradicates* an element q of K if and only if q and all its non-tautological logical consequences are absent from K', i.e. if and only if $K' \cap \text{Cn}(\{q\}) = \text{Cn}(\varnothing)$. An *eradication* is an operation of contraction such that when applied to a sentence p, it removes not only p but also all its non-tautologous logical consequences, or in other words, a contraction $\div$ that satisfies the postulate $(K \div p) \cap \text{Cn}(\{p\}) = \text{Cn}(\varnothing)$. Hansson [8, 10][1] Alternatively, eradication by p can be described as a multiple contraction by the set $\text{Cn}(\{p\}) \setminus \text{Cn}(\varnothing)$.

Under the assumption that our belief sets contain independently removable beliefs, we have reasons to expect reasonable operations of contraction not to be eradications. To see why this is so, let us again consider the two beliefs m (there is milk in your fridge) and p (Paris is the capital of France). These are independently removable, so when you contract by m, p should remain in the belief set. But if the operation of contraction is an eradication, then that is not possible. Since $m \vdash m \vee p$, an operation that eradicates m will remove $m \vee p$ from your belief set, and consequently p (that implies $m \vee p$) also has to go.[2] Thus, after eradicating m you cannot retain your seemingly independent belief in p. This absurd result is reason enough to impose the following requirement on an operation $\div$ of contraction:

If $p \in K \setminus \text{Cn}(\varnothing)$, then p is not eradicated in $K \div p$. (*no self-eradication*)

No self-eradication does not appear to be sufficiently strong. Extending the previous example, let e denote that there are eggs in the fridge. A visitor who lives temporarily in your flat sends a text message, saying: "I cannot make pancakes. You said there would be both milk and eggs in the fridge, but there isn't." You then have to contract by $m\&e$. Suppose that when you do so, m is eradicated but e is retained. Although such a contraction does not violate *no self-eradication*, it has equally mutilating effects. Again, since m is eradicated, $m \vee p$ and therefore also p will be lost, and you are bereft of your belief that Paris is the capital of France. This example can be used to justify the following strengthening of *no self-eradication*:

If $p \in K \setminus \text{Cn}(\varnothing)$, then there is no non-tautologous sentence q such that $p \vdash q$ and q is eradicated in $K \div p$. (*no implied eradication*)

The argument can be extended to eradication by sentences that are not logically implied by the contracted sentence. Suppose that when you find reason to give up $m\&e$, you also give up some sentence s that is not a logical consequence of $m\&e$. (s can, for instance, denote that your partner, who often takes food from the fridge, has not been at home since last time you filled the fridge.) Furthermore, suppose that s is not only removed, but also eradicated. Then $s \vee p$ has to go, and so has p. In this case as well, you are deprived of your belief

[1] Note that this does not imply that $K \div p = \text{Cn}(\varnothing)$. For any sentence $r \in K \setminus \text{Cn}(\{p\})$, we have $p \rightarrow r \notin \text{Cn}(\{p\})$, and the postulate is therefore compatible with $p \rightarrow r \in K \div p$.

[2] As usual in the theory of belief change, when we say that a sentence p is removed from the belief set K by an operation $\div$, this means that p is not an element of the new belief set that results from the operation.

that Paris is the capital of France. This gives us reason to further strengthen the condition, as follows:

There is no non-tautologous sentence q that is eradicated in $K \div p$. (*no eradication*)

5 The impossibility results

The following two observations show that both *no implied eradication* and *no eradication* are incompatible with other, highly plausible postulates of belief contraction, which we have presented above.

Observation 1 Let K be a finite-based belief set. Then there is no sentential operation $\div$ on K that satisfies

There is some $p \in K$ such that $K \div p \nvdash p$ (*minimal success*),
If K is finite-based, then so is $K \div p$ (*finite-based outcome*), and
If $p \in K \setminus \text{Cn}(\varnothing)$, then there is no non-tautologous sentence q such that $p \vdash q$ and q is eradicated in $K \div p$ (*no implied eradication*)

Proof of Observation 1: (For any finite-based set A, $\&A$ is a sentence such that $\text{Cn}(\{\&A\}) = \text{Cn}(A)$.) Due to *minimal success* we can choose p such that $K \div p \nvdash p$, and then use *finite-based outcome* to obtain $\nvdash \&(K \div p) \rightarrow p$. In order to show that $\&(K \div p) \rightarrow p$ is eradicated, let $\&(K \div p) \rightarrow p \vdash x$ and $K \div p \vdash x$. *Finite-based outcome* yields $\&(K \div p) \vdash x$. It follows truth-functionally from $\&(K \div p) \rightarrow p \vdash x$ and $\&(K \div p) \vdash x$ that $\vdash x$. □

Observation 2 Let K be a finite-based belief set. Then there is no sentential operation $\div$ on K that satisfies

$\text{Cn}(K \div p) \subset K$ for some sentence p (*reducibility*),
If K is finite-based, then so is $K \div p$ (*finite-based outcome*), and
There is no non-tautologous sentence $q \in K$ that is eradicated in $K \div p$ (*no eradication*)

Proof of Observation 2: Due to *reducibility* we can choose p such that $\text{Cn}(K \div p) \subset K$. *Finite-based outcome* and the logical closure of K yield $\nvdash \&(K \div p) \rightarrow \&K$. We are going to show that $\&(K \div p) \rightarrow \&K$ is eradicated. Let $\&(K \div p) \rightarrow \&K \vdash x$ and $K \div p \vdash x$. *Finite-based outcome* yields $\&(K \div p) \vdash x$. It follows truth-functionally from $\&(K \div p) \rightarrow \&K \vdash x$ and $\&(K \div p) \vdash x$ that $\vdash x$. □

The above two observations are disturbing from the viewpoint of knowledge representation. As shown in Section 4, we should expect a belief set to contain beliefs that are unrelated and can therefore be removed independently of each other. For such operations the postulates listed in the two observations seem reasonable enough. Four major approaches are available for resolving these impossibility results. We can reject (1) the success-related postulates *minimal success* and *reducibility*, (2) the postulate *finite-based outcome*, (3) the eradication-avoiding postulates *no eradication* and *no implied eradication*, or (4) the general framework in which these derivations are performed. The first two of these options do not appear to be very promising since they involve the repudiation of weak, plausible, and well understood postulates. The third option has better prospects since the notion of contracting without eradicating is more intricate than what first impressions might suggest. The following observation identifies the eradicated sentences whose existence follows from Observations 1 and 2 for a wide range of contraction operations.

Observation 3 Let K and $K \div p$ be finite-based, logically closed sets such that $K \div p \subset K$. Then:

(1) A sentence $x \in K$ is eradicated in $K \div p$ if and only if $\&(K \div p) \rightarrow \&K \vdash x$, and
(2) If $p \in K \setminus (K \div p)$ and $p \vdash x$, then x is eradicated in $K \div p$ if and only if $\&(K \div p) \rightarrow p \vdash x$.

Proof of Observation 3: Part 1: It was shown in the proof of Observation 2 that $\&(K \div p) \rightarrow \&K$ is eradicated, and then so are all its non-tautologous logical consequences. For the other direction, let x be eradicated. Since $x \vee \&(K \div p)$ is implied by x and is also an element of $K \div p$, it must be a tautology. It follows truth-functionally from $\vdash x \vee \&(K \div p)$ and $\&K \vdash x$ that $\&(K \div p) \rightarrow \&K \vdash x$.
Part 2: Let $p \in K \setminus (K \div p)$. It was shown in the proof of Observation 1 that $\&(K \div p) \rightarrow p$ is eradicated, and then so are all its non-tautologous logical consequences. For the other direction, we assume that $p \vdash x$ and that x is eradicated in $K \div p$. Due to Part 1, $\&(K \div p) \rightarrow \&K \vdash x$. It follows truth-functionally from $\&(K \div p) \rightarrow \&K \vdash x$ and $p \vdash x$ that $\&(K \div p) \rightarrow p \vdash x$.

□

Using this observation, let us reconsider the milk-and-eggs example in Section 4. For simplicity, we can assume that the original belief set was $K = \text{Cn}(\{m, e, p\})$. Furthermore, suppose that contraction by $m\&e$ removes m but not e, and that $K \div (m\&e) = \text{Cn}(\{e, p\})$. It follows from Observations 1 and 2 that some sentences were eradicated in that operation. We can now conclude from Observation 3 that $e\&p \rightarrow m$ was eradicated. This example illustrates that in order for eradication to leave a part of the belief set unchanged, the eradicated sentence has to be logically related to that part. Eradicating K by m removes both e and p, whereas eradicating it by $e\&p \rightarrow m$ does not affect them. This may be surprising since it could be expected that mentioning a sentence in the contractee should be a means to have it removed, rather than a means to retain it. Although eradication is a form of contraction, the effects of eradicating a specific sentence are quite different from those of contracting by it. Possibly, a solution to the above impossibility results should be sought in that direction.

References

1. Alchourrón, C., Makinson, D., Gärdenfors, P.: On the logic of theory change: partial meet contraction and revision functions. J. Symb. Log. **50**, 510–530 (1985)
2. Alchourrón, C., Makinson, D.: On the logic of theory change: Contraction functions and their associated revision functions. Theoria **48**(1), 14–37 (1982)
3. Fermé, E., Hansson, S.O.: Shielded contraction. In: Rott, H., Williams, M.-A. (eds.) Frontiers of Belief Revision, pp. 85–107. Kluwer Academic Publishers, Dordrecht (2001)
4. Fermé, E., Hansson, S.O.: Belief Change. Introduction and Overview. Springer, Berlin (2018)

5. Gärdenfors, P., Makinson, D.: Revisions of knowledge systems using epistemic entrenchment. In: Proceedings of the 2nd Conference on Theoretical Aspects of Reasoning about Knowledge, pp. 83–95. Morgan Kaufmann Publishers, Los Angeles (1988)
6. Hansson, S.O.: Theory contraction and base contraction unified. J. Symb. Log. **58**, 602–625 (1993)
7. Hansson, S.O.: A Textbook of Belief Dynamics. Theory Change and Database Updating. Kluwer, Dordrecht (1999)
8. Hansson, S.O.: In praise of full meet contraction. Análisis Filosófico **26**, 134–146 (2006)
9. Hansson, S.O.: Specified meet contraction. Erkenntnis **69**, 31–54 (2008)
10. Hansson, S.O.: Eradication. J. Appl. Log. **10**, 75–84 (2012)
11. Hansson, S.O.: Finite contractions on infinite belief sets. Stud. Logica. **100**, 907–920 (2012)
12. Hansson, S.O., Wassermann, R.: Local change. Stud. Logica. **70**, 49–76 (2002)
13. Makinson, D.: On the status of the postulate of recovery in the logic of theory change. J. Philos. Log. **16**(4), 383–394 (1987)
14. Makinson, D.: Screened revision. Theoria **63**, 14–23 (1997)
15. Parikh, R.: Beliefs, belief revision, and splitting languages, pp. 266-278 in Lawrence S. Moss, Jonathan Ginzburg, and Maarten de Rijke (eds) Logic, language, and computation, vol 2, CSLI lecture notes 96. Stanford, CA: CSLI (1999)
16. Rott, H.: Preferential belief change using generalized epistemic entrenchment. J. Log. Lang. Inf. **1**(1), 45–78 (1992)