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RESEARCH

Decompositions of Infinite-Dimensional $A_{\infty,\infty}$ Quiver Representations

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Abstract

Gabriel's Theorem states that the quivers which have finitely many isomorphism classes of indecomposable representations are exactly those with underlying graph one of the ADE Dynkin diagrams and that the indecomposables are in bijection with the positive roots of this graph. When the underlying graph is A_n, these indecomposable representations are thin (either 0 or 1 dimensional at every vertex) and in bijection with the connected subquivers. Using linear algebraic methods we show that every (possibly infinite-dimensional) representation of a quiver with underlying graph $A_{\infty,\infty}$ is infinite Krull-Schmidt, i.e. a direct sum of indecomposables, as long as the arrows in the quiver eventually point outward. We furthermore prove that these indecomposable are again thin and in bijection with both the connected subquivers and the limits of the positive roots of $A_{\infty,\infty}$ with respect to a certain uniform topology on the root space. Finally we give an example of an $A_{\infty,\infty}$ quiver which is not infinite Krull-Schmidt and hence necessarily is not eventually-outward.

Keywords Quiver representations · Krull-Schmidt · Dynkin diagram · Infinite-dimensional representations

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1 Introduction

Most early techniques in the study of representations of quivers (e.g. Auslander-Reiten theory, the Euler form, etc.) assume that the quiver has finitely many vertices and that the vector

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spaces of these representations are finite-dimensional. In particular, Gabriel's Theorem [1] characterizes the quivers of finite-type as exactly those with underlying graph the ADE Dynkin diagrams. A nice survey of Gabriel's Theorem can be found in [2], and we adopt most of Brion's notation in this paper. More recently there have been several results which weaken these finiteness hypotheses.

In [3], Bautista, Liu, and Paquette observed that any **locally finite-dimensional** representation (meaning each vector space is finite-dimensional) of various infinite quivers including an eventually outward $A_{\infty,\infty}$ quiver is infinite Krull-Schmidt (a unique direct sum of indecomposables) and identified the indecomposables as being in bijection with the connected full subquivers.

In [4], Ringel was able to remove the finite-dimensional hypothesis from one direction of Gabriel's Theorem. There he proved that infinite-dimensional representations of (**finite**) ADE Dynkin quivers are infinite Krull-Schmidt, and showed that these indecomposables are the same as those given in Gabriel's Theorem which correspond to the positive roots of the corresponding root system.

In [5], Enochs, Estrada, and Rozas studied **injective** quiver representations over a general (not necessarily commutative) ring R. They proved that the indecomposable injective representations of certain infinite quivers (which include eventually outward $A_{\infty,\infty}$ quivers) called *transfinite tree quivers* are in bijection with a subset of the connected full subquivers. Note that as they are working over a general ring, there is no locally finite-dimensional hypothesis. Here the term "injective" refers to injective objects in the category of representations and in fact (see Prop. 2.1 in [5]) the linear maps in these representations are necessarily surjective.

More recently, Igusa, Rock, and Todorov proved in [6] that pointwise finite-dimensional representations of certain **continuous** generalizations $A_{\mathbb{R}}$ of $A_{\infty,\infty}$ quivers are infinite Krull-Schmidt and classified the indecomposable representations for these quivers, which, similarly to all aforementioned generalizations, are thin and in bijection with the connected subquivers of $A_{\mathbb{R}}$.

Even more recently, Botnan and Crawley-Boevey [7, 8] prove that locally finite-dimensional representations of *any* quiver are infinite Krull-Schmidt, and further that the indecomposables in the case of $A_{\infty,\infty}$ with any orientation are "interval representations", which are exactly those corresponding to a connected subquiver.

In this paper, we simultaneously weaken the two finiteness conditions (those requiring finite quivers and locally finite-dimensional representations) as well as the injective representation requirement in the type-A case over a field. More specifically we show that every (possibly infinite-dimensional) representation of an eventually outward quiver with underlying graph $A_{\infty,\infty}$ is infinite Krull-Schmidt. This eventually outward condition is in fact necessary: there exist representations of a not eventually outward $A_{\infty,\infty}$ quiver which are not infinite Krull-Schmidt, an example of which is given at the end of the paper in Section 8. We furthermore give a description of the indecomposables in this case, which are in bijection with the connected full subquivers. These can be thought of as limits of the classical positive roots of A_n. These match those from [8] and [3], and also those from [5] if we restrict to injective representations. Our proof furthermore gives an algorithm for decomposing a given quiver representation into indecomposables.

In Sections 2 we review the necessary quiver representation theory background. The remainder of the paper can be divided into two parts.

In the first part we discuss representations of a general quiver Ω which is an eventually outward finitely branched tree, i.e. a union of finitely many journeys, the arrows of which eventually point away from its start. More specifically, in Section 3 we define a certain partial order on the set $\mathcal{C}$ of connected full subquivers of Ω by taking the poset closure of two "moves" which we call *reduction* and *enhancement*. Using the eventually outward condition we prove that this is in fact a **well-founded** poset.

In Section 4 we then define a poset filtration of any representation V of Ω, i.e. an order-preserving function F from $\mathcal{C}$ to the poset of subrepresentations of V. We will show that the successive quotients $F^\alpha / \sum_{\beta<\alpha} F^\beta$ of this poset filtration are isotypic, meaning a direct sum of indecomposables all of which are isomorphic.

In Section 5 we prove that for any connected full subquiver $\alpha \in \mathcal{C}$, the successive quotient $F^\alpha / \sum_{\beta<\alpha} F^\beta$ is supported on α and under mild conditions the morphism $F^\alpha \to F^\alpha / \sum_{\beta<\alpha} F^\beta$ lifts, i.e. that the subrepresentation $\sum_{\beta<\alpha} F^\beta$ is complemented, say by W^α, inside of F^α. Thus we obtain an *almost gradation* W of F, i.e. a function $W : \mathcal{C} \to \mathrm{Sub}(V)$ such that for all $\alpha \in \mathcal{C}$ we have $F^\alpha = W^\alpha \oplus \sum_{\beta<\alpha} F^\beta$. We then prove that each W_α is an isotypic subrepresentation. Finally, using the fact that $\mathcal{C}$ is a well-founded poset, we use induction to prove that our poset filtration *spans* V. This means that $V = \sum_{\alpha\in\mathcal{C}} W^\alpha$.

In the second part of the paper, we specialize to the case where Ω is an eventually outward quiver of type $A_{\infty,\infty}$, and prove that any representation of such a quiver is infinite Krull-Schmidt. We furthermore classify the indecomposables. Specifically, in Section 6 we show that if Ω is eventually outward and of type $A_{\infty,\infty}$, then the partial order on the set $\mathcal{C}$ of connected subquivers of Ω is actually the product of two total orders on the set Ω_1 of arrows of Ω. It follows that, if V is any representation of Ω, then the poset filtration $F : \mathcal{C} \to \mathrm{Sub}(V)$ is the intersection of two linear filtrations $L, R : \Omega_1 \to \mathrm{Sub}(V)$. We then prove the general theorem that **any** almost gradation of a poset filtration which is the intersection of two linear filtrations is independent. All together this proves our main result that $V = \bigoplus_{\alpha\in\mathcal{C}} W_\alpha$ and thus every (possibly infinite dimensional) representation of an eventually outward type $A_{\infty,\infty}$ quiver Ω is infinite Krull-Schmidt.

In Section 7 we describe the isomorphism classes of the indecomposable subrepresentations of an eventually outward type $A_{\infty,\infty}$ quiver. As for any finite quiver of type A_n, they are in bijection with the connected subquivers, and are *thin*, meaning one-dimensional at every vertex in their support. Note that this means as long as the quiver is eventually outward, the indecomposables are independent of the directions of the arrows. This agrees with the indecomposables found in [3] for a locally finite-dimensional representation of an eventually outward type $A_{\infty,\infty}$ quiver, and with those found in [5] for a (possibly infinite-dimensional) injective representation of an eventually outward type $A_{\infty,\infty}$ quiver. Note the dimensions of these are 1 at each vertex contained in the subquiver and 0 otherwise. The finite subquivers correspond to what are normally called positive roots, which are finite roots of Tits length 1, or equivalently the image of simple roots under finitely many Weyl reflections. The infinite quivers are naturally limits of these in the appropriate topology.

Finally in Section 8 we give an example of a representation of an $A_{\infty,\infty}$ quiver which is not infinite Krull-Schmidt. This quiver is not eventually outward, as is required by the results of Section 6. The example is closely related to a construction in [6].

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2 Background

In this section we review the standard definitions in the theory of quivers and their representations and fix the notation that will be used throughout the paper.

2.1 Quivers

A *quiver*, denoted by Ω, consists of a set Ω_0 of *vertices*, a set Ω_1 of *arrows*, a *source function* $s : \Omega_1 \to \Omega_0$, and a *target function* $t : \Omega_1 \to \Omega_0$. We assume that all quivers have no multiple arrows and no loops, and then we say that the *underlying graph* of Ω is the graph with a vertex for each vertex of Ω and an undirected edge for each arrow of Ω. The quiver is a tree if the underlying graph is a tree.

A vertex $x \in \Omega_0$ is a *source* if it is not the target of any arrow, and a *sink* if it is not the source of any arrow.

2.2 Subquivers

A *subquiver* of a quiver Ω is a quiver Ω' such that $\Omega'_0 \subseteq \Omega_0$, $\Omega'_1 \subseteq \Omega_1$, $s' = s\ |_{\Omega'_1}$, and $t' = t\ |_{\Omega'_1}$. A *full* subquiver is one that contains every arrow whose source and target are contained in it.

From now on, we will assume that the underlying graphs of all quivers in this paper are trees. Notice that means the removal of any arrow e of a quiver divides the quiver into two connected subquivers, the one including $t(e)$ which we call the subquiver *in front* of e and the one containing $s(e)$ which we call the subquiver *behind* e. In general, any vertex, arrow, or subquiver contained in the former will be said to be *in front of* e and e will be said to *point towards it*, while any in the latter will be said to be *behind* e, and e will be said to *point away from it*.

2.3 Walks and Paths

A (finite or infinite) *walk* w in a quiver with an underlying graph is a map from a connected (finite or infinite) interval of the integers to Ω_0 so that if i and $i+1$ are in the interval, there is an arrow whose source and target are w_i and w_{i+1}. We then say w *contains* that arrow e. We say that e is *oriented consistently* if $s(e) = w_i$ and $t(e) = w_{i+1}$, and *oriented inconsistently* if $s(e) = w_{i+1}$ and $t(e) = w_i$. Adding an integer to every element of the interval results in a new map we will consider the same walk. If the interval has a minimal i we define $s(w) = w_i$ and call the walk a *journey*, if the interval has a maximal j we define $t(e) = w_j$. If w has a target and w' has a source and these two vertices are the same we can define the *concatenation* walk $w'w$ in the obvious way. An *injective walk* is one where the map is injective.

2.4 Types of Quivers and Walks

Since Ω is a tree then every two vertices are connected by a unique minimal walk. We say that Ω is a *finitely-branched* tree if it is a union of finitely many injective paths, or equivalently for any vertex x it is the union of finitely many injective journeys starting at x. Finally, if every injective journey contains at most finitely many arrows oriented inconsistently, we say that Ω is *eventually outward*.

Example 1 By *the mountain* we refer to the quiver Θ (Fig. 1) which has underlying graph $A_{\infty,\infty}$ and in which all arrows point away from the vertex x_0. It is clear that Θ is an eventually outward finitely branching tree.

$$\cdots \xleftarrow{a_{-3}} x_{-2} \xleftarrow{a_{-2}} x_{-1} \xleftarrow{a_{-1}} x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} x_2 \xrightarrow{a_3} \cdots$$

Fig. 1 The Quiver Θ

These two types of quivers can be helpfully redefined in terms of a strict partial order $\prec$ on the set Ω_1 of arrows of Ω defined by $f \prec e$ if and only if f is behind e and e is in front of f.

Lemma 2.1 *Ω is eventually outward if and only if its $\prec$ is well-founded (no infinite descending chains). If Ω is a tree, then it is finitely branched if and only if its $\prec$ has no infinite antichains (subsets in in which no two elements are related).*

Proof If Ω contains an injective journey with infinitely many arrows inconsistently oriented, the set of inconsistently oriented arrows in that journey is an infinite descending chain. If $\cdots \prec e_3 \prec e_2 \prec e_1$ is an infinite downward chain, there is a unique injective journey p_i from the target of each e_i to the source of each e_{i+1}, and these are all distinct because if $j > i$ then p_j is behind e_{i+1} and p_i is in front. Concatenating $\cdots p_2 e_2^{-1} p_1 e_1^{-1}$ gives a journey with each e_i oriented inconsistently.

If Ω is finitely branched, let $p_1, \ldots, p_n$ be a set of injective paths covering it. If E is an antichain, there can be at most two elements of E contained in any given path, or else one would be in front of the other. Thus E is finite. On the other hand if Ω is not finitely branched, then for some x there is an infinite sequence $p_1, p_2, \ldots$ of injective journeys from x such that each journey p_i contains a first arrow e_i not in any of the previous journeys, and after that all the arrows are not in any other journeys. Of those infinitely many arrows, either an infinite set of them point away from x or an infinite set point towards x. Each of these sets separately forms an antichain, because the path connecting e_i to e_j will follow the reverse of p_i to the final point of shared intersection with p_j, then out p_j. Thus there is an infinite antichain. □

Example 2 The partial order $\succ$ on Θ_1 is given by $a_1 \prec a_2 \prec \ldots$ and $a_{-1} \prec a_{-2} \prec \ldots$ but a_n and a_m are unrelated to each other if $n > 0$ and $m < 0$. Note that this is in fact a well-founded order in which every antichain has at most two elements, as required by Lemma 2.1.

Lemma 2.2 *If α is a connected full nonempty subquiver of an eventually outward finitely branching tree, then there are only finitely many arrows that point towards α.*

Proof Being finitely branching, the quiver is a union of finitely many journeys from an $x \in \alpha$. Since α is connected all arrows that point towards α point towards x, therefore the result follows from the property of being eventually outward. □

2.5 Quiver Representations

A *representation* (V, f) of a quiver Ω over a field $\mathbb{F}$ consists of an $\mathbb{F}$-vector space V_i at every vertex $i \in \Omega_0$ and a linear map $f_e : V_{s(e)} \to V_{t(e)}$ for every arrow $e \in \Omega_1$. If (V, f) and (W, g) are representations of Ω, a *morphism of representations* $T : V \to W$ consists of a linear map $T_i : V_i \to W_i$ for all vertices $i \in \Omega_0$ such that for all arrows $e \in \Omega_1$, we have $T_{t(e)} \circ f_e = g_e \circ T_{s(e)}$. This is equivalent to requiring that the appropriate diagrams commute.

Example 3 We give two examples of representations of the mountain quiver Θ which we will carry throughout the paper.

1. Define a representation M of Θ by setting $M(x_n) = \mathbb{F}^{(\mathbb{N})} = \bigoplus_{i\in\mathbb{N}} \mathbb{F}$ for all $n \in \mathbb{Z}$ and letting $M(a_n)$ be the projection map which sends $(c_1, c_2, \ldots) \mapsto (c_2, c_3, \ldots)$ also for all $n \in \mathbb{N}$. Note that M is not locally finite-dimensional.
2. Now define a representation N of Θ by setting $N(x_n) = \mathbb{F}^{|n|+2}$ for all $n \in \mathbb{Z}$ and for all $n \neq 0$ letting $N(a_n)$ be the inclusion map which sends $(c_1, c_2, \ldots c_{|n|+1}) \mapsto (c_1, c_2, \ldots c_{|n|+1}, 0)$.

Direct sums of representations of Ω are defined in the obvious way. A quiver representation is called *indecomposable* if it is nonzero and is not a direct sum of two proper, non-zero subrepresentations, and is called *infinite Krull-Schmidt* if it is a direct sum of indecomposable subrepresentations in a unique way, up to ordering and isomorphism.

2.6 The Transport of a Subspace

Let Ω be a quiver, V a representation of Ω, w a finite injective walk in Ω, and $W \subseteq V_{s(w)}$ a subspace. We define the *transport* of W via w to $t(w)$ recursively as follows:

- If w contains a single arrow e oriented consistently then the transport of W is $f_e[W] \subseteq V_{t(w)}$.
- If w contains a single arrow e oriented inconsistently then the transport of W is $f_e^{-1}[W] \subseteq V_{t(w)}$.
- If w is the concatenation of path w_1 and w_2 then the transport of w is the transport via w_2 of the transport via w_1 of W.

Notice that if $W' \subseteq W \subseteq V_{s(w)}$ then the transport of W' is contained in the transport of W.

Example 4

1. Consider the representation M of the mountain quiver Θ from Example 3 (1). The transport of the zero subspace in $M(x_n)$ to x_m for $|m| \geq |n|$ is simply zero, and the transport to x_m for $|m| < |n|$ is $\mathrm{Span}(e_1, \ldots, e_{|n|-|m|})$ where e_i denotes the ith standard basis vector in $\mathbb{F}^{(\mathbb{N})}$.
2. Now consider the representation N of the mountain quiver Θ from Example 3 (2). Given $n > 0$, the transport of $N(x_n)$ to x_m for $m > n$ is the subspace $\mathrm{Span}(e_1, \ldots, e_{n+2})$, for $0 < m < n$ is $N(x_m)$, and for $m \leq 0$ is $\mathrm{Span}(e_1, e_2)$. An analogous description holds for negative n.

Lemma 2.3 *Let Ω be a quiver which is a tree, V a representation of Ω, $i \in \Omega_0$ a vertex, and $W_i \subseteq V_i$ a subspace. For each vertex $j \in \Omega_0$, define W_j to be the transport of W_i to j along the unique injective walk starting at i and ending at j. Then W is the maximal subrepresentation of V whose value at i is W_i and such that for each arrow e pointing away from i, f_e restricted to W is onto. It is the minimal subrepresentation W of V whose value at i is W_i and such that for each arrow e pointing towards i, the map from $(V/W)_{s(e)} \to (V/W)_{t(e)}$ induced by f_e is injective.*

Proof Elementary. □

3 The Partial Order of Connected Components

Let Ω be an eventually outward finitely branched tree quiver. Let $\mathcal{C}$ be the set of non-empty connected full subquivers of Ω. If $\alpha \in \mathcal{C}$, the *complement* of α in Ω, denoted by α', is the full subquiver of Ω containing the vertices which are not in α. Then α' is a disjoint union of finitely many components, with each component connected to α by a single arrow, which we call a *boundary arrow* of α. If e connects one of the components of α' to α, we denote this component by α'_e. Let $\mathrm{ba}(\alpha)$ denote the set of boundary arrows of α. Thus we have the following:

$$\alpha' = \bigsqcup_{e \in \mathrm{ba}(\alpha)} \alpha'_e$$

Furthermore, we denote the subquiver of Ω which includes α'_e and e and both the source and target of e (one of which is already in α'_e) by α''_e.

If e is any arrow not contained in α, then α is contained entirely in one of the two components of Ω created by the removal of e, and therefore e either points towards it or away from it, i.e. α is either in front of e or behind it. Let the set of boundary arrows of α which point towards α be $\mathrm{iba}(\alpha)$ and the set pointing away be $\mathrm{oba}(\alpha)$.

Define a relation on the set $\mathcal{C}$ of connected subquivers of Ω by letting $\alpha > \beta$ if one can get from α to β by a sequence of the following moves:

1. We say that β is a *reduction* of α if there exists an arrow e in α such that β is the portion of α behind e (in terms of its orientation in Ω). Thus e is a boundary arrow of β pointing away.
2. We say that β is an *enhancement* of α if there exists an arrow e in β such that α is the portion of β in front of e. Thus e is a boundary arrow of α pointing towards it.

Proposition 3.1 *There is no infinite sequence $\alpha_1, \alpha_2, \ldots$ such that each α_{n+1} is a nontrivial reduction or enhancement of α_n.*

Proof Suppose for contradiction that such a sequence exists. For each $n \in \mathbb{N}$, let S_n be the set of arrows e not in α_n which point towards α_n and such that the journey from e to α_n goes through a boundary arrow of α_n that also points towards α_n. By Lemma 2.2, S_n is finite. Notice first that if α_{n+1} is a reduction or enhancement of α_n and $e \in S_{n+1}$, then $e \in S_n$, because in the case of reduction the same journey works, and in the case of enhancement the same journey works perhaps extended to go through the arrow of enhancement. So $S_{n+1} \subseteq S_n$, and in the case of an enhancement $S_{n+1} \subsetneq S_n$ because if we enhanced on boundary arrow e of α_n then $e \in S_n - S_{n+1}$. Therefore there can be only finitely many enhancements, and there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $\alpha_{n+1} < \alpha_n$ is a reduction. For each $n \geq N$ let e_{n-N} be the arrow in α_n along which the reduction $\alpha_{n+1} < \alpha_n$ occurs. Then we obtain a sequence of arrows $e_1, e_2, \ldots$ such that for all $m > n$, e_m is behind e_n.

Since Ω is finitely branched, it is the union of finitely many injective journeys, and each e_n must be on one of these journeys, so infinitely many $e_{n_1}, e_{n_2}, \ldots$ lie on a single journey. If any e_{n_i} is oriented consistently with the journey, then only finitely many other e_{n_j} can be behind it, which contradicts the fact that it was an infinite sequence. But if all the e_{n_i} are oriented inconsistently, this contradicts the fact that Ω was eventually outward. □

Corollary 3.1 *The relation $>$ on $\mathcal{C}$ defined above is a well-founded strict partial order.*

Proof Transitivity of the relation is trivially satisfied. To show asymmetry, if $\alpha < \beta < \alpha$ then we obtain an infinite decreasing sequence of moves, contradicting Proposition 3.1. Irreflexivity follows from asymmetry. Now that we know $<$ is a strict partial order, the aforementioned proposition also implies that it is well-founded as it shows there are no infinite strictly decreasing chains. □

Example 5 We now describe the partial order induced by reduction and enhancement on the mountain quiver Θ explicitly. For this quiver, a nonempty connected subquiver $\alpha \in \mathcal{C}$ can be described by specifying the first arrow $\ell(\alpha) \in \{-\infty, a_n \mid n \in \mathbb{Z} \setminus \{0\}\}$ to the left of α which is not contained in α and the first arrow $r(\alpha) \in \{+\infty, a_n \mid n \in \mathbb{Z} \setminus \{0\}\}$ to the right which is not in α. We define total orders $\leq_L$ on $\Theta_1 \cup \{-\infty\}$ and $\leq_R$ on $\Theta_1 \cup \{+\infty\}$ as follows.

$$a_{-1} <_L a_{-2} <_L \ldots <_L -\infty <_L a_1 <_L a_2 <_L \ldots$$
$$a_1 <_R a_2 <_R \ldots <_R +\infty <_R a_{-1} <_R a_{-2} <_R \ldots$$

Note that these total orders are clearly well-founded. Then $\alpha > \beta$ in the reduction/enhancement partial order if and only if $\ell(\alpha) >_L \ell(\beta)$ and $r(\alpha) >_R r(\beta)$, i.e. if and only if $(\ell(\alpha), r(\alpha)) >_{L\times R} (\ell(\beta), r(\beta))$ in the product order. And since the product of two well-founded orders is again well-founded, $\mathcal{C}$ is well-founded as well. We will extend this analysis to a general eventually outward $A_{\infty,\infty}$ quiver in Section 6.

We now break down the different connected full subquivers of Θ into five different types and describe how the reduction/enhancement order behaves with respect to this categorization. Every subquiver considered will be full and therefore can be specified by listing the vertices it contains.

- (Type 1): Let $n, m \in \mathbb{Z}_{\geq 0}$ and suppose without loss of generality that $m \leq n$. Let α consist of the vertices $x_{-n}, x_{-n+1}, \ldots, x_{m-1}, x_m$ so that $\ell(\alpha) = a_{-n-1}$ and $r(\alpha) = a_{m+1}$. The elements β of $\mathcal{C}$ with $\beta \leq \alpha$ are those with vertices $x_{-i}, \ldots, x_j$ for $i, j \in \mathbb{Z}_{\geq 0}$ and $i \leq n$, $j \leq m$.
- (Type 2): Let $n, m \in \mathbb{Z}_{\geq 0}$ be such that $0 < m \leq n$, and let α consist of the vertices $x_m, x_{m+1}, \ldots, x_n$ so that $\ell(\alpha) = a_m$ and $r(\alpha) = a_{n+1}$. The elements β of $\mathcal{C}$ with $\beta \leq \alpha$ are those with vertices $x_i, x_{i+1} \ldots, x_j$ for $-\infty < i \leq m$, $0 \leq j \leq n$, along with the infinite subquivers with vertices $\ldots, x_{j-1}, x_j$ for $0 \leq j \leq n$.
- (Type 3): Let $n \in \mathbb{Z}_{>0}$ and let α consist of the vertices $x_n, x_{n+1}, \ldots$ so that $\ell(\alpha) = a_n$ and $r(\alpha) = +\infty$. The elements β of $\mathcal{C}$ with $\beta \leq \alpha$ are those with vertices $x_i, x_{i+1} \ldots, x_j$ for $i \leq j$ and $0 \leq j < \infty$, the infinite subquivers with vertices $\ldots, x_{j-1}, x_j$ for $0 \leq j < \infty$, the infinite subquivers with vertices $x_i, x_{i+1}, \ldots$ for $i \leq n$, and the entire quiver Θ.
- (Type 4): Let $n \in \mathbb{Z}_{\geq 0}$ and let α be the full subquiver containing the vertices $x_{-n}, x_{-n+1}, \ldots, x_0, \ldots$ so that $\ell(\alpha) = a_{-n-1}$ and $r(\alpha) = +\infty$. The elements β of $\mathcal{C}$ with $\beta \leq \alpha$ are those with vertices $x_{-i}, \ldots, x_j$ where $i \leq n$ and $0 \leq j$, as well as the infinite subquivers with vertices $x_{-i}, x_{-i+1}, \ldots$ where $i \leq n$.
- (Type 5): Let $\alpha = \Theta$ so that $\ell(\alpha) = -\infty$ and $r(\alpha) = +\infty$. The elements β of $\mathcal{C}$ with $\beta \leq \alpha$ are those with vertices $x_{-i}, \ldots, x_j$ where $0 \leq i, j$, as well as the infinite subquivers with vertices $x_{-i}, x_{-i+1}, \ldots$ for $i \geq 0$ and those with vertices $\ldots, x_{j-1}, x_j$ for $j \geq 0$.

4 Poset Filtrations of Subrepresentations

Let Ω be a quiver which is a tree, and V a representation of Ω. If e is any arrow of Ω, define two subrepresentations of V as follows. Define the subrepresentation $V^{e,+} \subseteq V$ by

$$V_i^{e,+} = \begin{cases} V_i & \text{if } i \text{ is behind } e \\ \text{transport of } V_{t(e)} & \text{else.} \end{cases}$$

Define $V^{e,-} \subseteq V$ by

$$V_i^{e,-} = \begin{cases} \{0\}_i \subseteq V_i & \text{if } i \text{ is in front of } e \\ \text{transport of } \{0\}_{t(e)} & \text{else.} \end{cases}$$

It is easy to check that each of these is indeed a subrepresentation.

Define a *virtual arrow* E to be an equivalence class of paths (oriented injective journeys) that are either infinite or end in a leaf. Two such paths are equivalent if they are *cofinal* in the sense that their intersection is again such a path. Thus if E ends in a leaf we may just think of it as the leaf itself (the trivial path consisting of that vertex). If $\alpha \in \mathcal{C}$ is a full connected subquiver and some representative of E lies entirely within α we say E is a *virtual boundary arrow* of α, and denote by $\mathrm{vba}(\alpha)$ the set of all virtual boundary arrows of α. Note that while boundary arrows of α **are not** contained in α, virtual boundary arrows of α **are** contained in α. We think of a virtual boundary arrow of Ω as oriented journey which cannot be extended past its tip. From now on, when referring to boundary arrows in the sense of Section 3 we'll call them *literal boundary arrows* to contrast with the virtual boundary arrows defined above.

Example 6

1. There are two virtual arrows of the mountain Θ, namely the equivalence classes containing the oriented injective journeys $x_0, x_1, \ldots$ and $x_0, x_{-1}, \ldots$. We call these E_R and E_L respectively. For every finite subquiver α of Θ, $\mathrm{vba}(\alpha)$ is empty, but if α is the full subquiver with vertices $x_n, x_{n+1}, \ldots$ for $n > 0$ then $\mathrm{vba}(\alpha) = \{E_R\}$ and similarly for $n < 0$.
2. The A_n quiver Λ in Fig. 2 has two virtual arrows, namely the equivalence class containing the leaf x_1 and the equivalence class containing the leaf x_n (notice that in the latter case this equivalence class also contains the journey $x_1, x_2, \ldots, x_n$. If α is a full connected subquiver of Λ then $\mathrm{vba}(\alpha)$ is empty unless α contains either x_1 or x_n. In the former case $\mathrm{vba}(\alpha)$ will contain the virtual boundary arrow defined by the journey x_1 and in the latter case it will contain the virtual boundary arrow defined by the journey x_n.

If E is a virtual arrow, define a subrepresentation $V^{E,+} \subseteq V$ by

$$V_i^{E,+} = \begin{cases} V_i & \text{if } i \in \omega \\ \text{transport of } V_j & \text{else} \end{cases}$$

where ω is any representative of E, and j is the closest vertex in ω to i, in the sense that the unique injective walk from j to i does not contain any other vertices of ω.

Remark 4.1 Notice that $V^{E,+}$ is a subrepresentation which does not depend on the representative ω of E, so it is in fact well-defined.

Example 7

1. Consider the representation M of the mountain quiver Θ from Example 3 (1). Let $n \in \mathbb{N}$. Then $M^{a_n,+}$ is simply M since all maps are surjective. Furthermore, if E denotes either

$$x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} x_n$$

Fig. 2 The Unidirectional A_n Quiver Λ

E_R or E_L then $M^{E,+} = M$ as well. On the other hand, $M^{a_n,-}(x_m)$ is the transport of $\{0\} \subseteq M(x_n)$ to x_m which is given in Example 4. Explicitly we have

$$M^{a_n,-}(x_m) = \begin{cases} 0 & \text{if } |m| \geq |n| \\ \mathrm{Span}(e_1, \ldots, e_{|n|-|m|}) & \text{if } |m| < |n| \end{cases}$$

For example $M^{a_2,-}$ is given below, where the nonzero maps send $e_1 \mapsto 0$ and $e_2 \mapsto e_1$.

$$\ldots \longleftarrow 0 \longleftarrow \mathrm{Span}(e_1) \longleftarrow \mathrm{Span}(e_1, e_2) \longrightarrow \mathrm{Span}(e_1) \longrightarrow 0 \longrightarrow \ldots$$

2. Now consider the representation N of the mountain quiver Θ from Example 3 (2). Let $n \in \mathbb{N}$. Then if x_m is in front of a_n we have that $N^{a_n,+}(x_m)$ is just the transport of $N(x_n)$ to x_m which is given in Example 4, and if x_m is behind a_n $N^{a_n,+}(x_m) = N(x_m)$. Furthermore $N^{E_R,+}(x_m)$ is the entire vector space $N(x_m)$ for $m \geq 0$, and is equal to $\mathrm{Span}(e_1, e_2) \subseteq N(x_m)$ for $m < 0$ (a similar result holds for E_L). On the other hand, $N^{a_n,-}$ is simply the zero representation since all maps are injective.

If $\alpha \in \mathcal{C}$ is a connected subquiver and $e \in \mathrm{ba}(\alpha)$, define $V^{e,\alpha}$ to be $V^{e,+}$ if $e \in \mathrm{iba}(\alpha)$ and $V^{e,-}$ if $e \in \mathrm{oba}(\alpha)$, and if $E \in \mathrm{vba}(\alpha)$ define $V^{E,\alpha} = V^{E,+}$. Finally, let $\mathrm{b}(\alpha) = \mathrm{ba}(\alpha) \cup \mathrm{vba}(\alpha)$.

Define a function $F : \mathcal{C} \to \mathrm{Sub}(V)$, denoted $\alpha \mapsto F^\alpha$, by

$$F^\alpha = \bigcap_{d \in \mathrm{b}(\alpha)} V^{d,\alpha}. \tag{1}$$

In general a *poset filtration* of an R module M consists of a partially ordered set $(P, \leq)$, and a function $F : P \to \mathrm{Sub}(M)$ which is order-preserving, meaning $p \leq q$ implies $F^p \subseteq F^q$.

Proposition 4.1 *F is a poset filtration of V, i.e. if $\beta \leq \alpha$ are connected subquivers then $F^\beta \subseteq F^\alpha$.*

Proof It suffices to check this if β is obtained from α by a single reduction or enhancement. First consider a reduction, with e an arrow contained in α and β the portion of α behind e. Then β and α share all the same literal boundary arrows except that β has one outward pointing literal boundary arrow e which α does not share and α has some number of literal boundary arrows which β does not share, all of which are in front of e. Furthermore, every virtual boundary arrow of β is also a virtual boundary arrow of α but α has some number of virtual boundary arrows which are not contained in β, and hence by Remark 4.1 we can assume are entirely contained in β'_e.

Therefore for any vertex i in Ω we have the following:

$$F_i^\alpha = \left[\bigcap_{d \in \mathrm{b}(\alpha) \cap \mathrm{b}(\beta)} V_i^{d,\alpha} \right] \cap \left[\bigcap_{d \in \mathrm{b}(\alpha) \setminus \mathrm{b}(\beta)} V_i^{d,\alpha} \right] \tag{2}$$

$$F_i^\beta = \left[\bigcap_{d \in \mathrm{b}(\alpha) \cap \mathrm{b}(\beta)} V_i^{d,\beta} \right] \cap V_i^{e,\beta}. \tag{3}$$

First notice that because $\beta \subseteq \alpha$, for any $d \in \mathrm{ba}(\alpha) \cap \mathrm{ba}(\beta)$, d is outward pointing with respect to α if and only if it is outward pointing with respect to β, and hence $V_i^{d,\alpha} = V_i^{d,\beta}$. Hence the first terms of the intersections in Eqs. 2 and 3 are equal. Now we consider two cases.

Case 1: If i is in front of e, then $V_i^{e,\beta} = \{0\}$ hence $F_i^\beta = \{0\}$ so trivially we have $F_i^\beta \subseteq F_i^\alpha$.

Case 2: If i is behind e, then $V_i^{e,\beta}$ is the transport of $\{0\}_{t(e)}$ to i. But notice that for $d \in \mathrm{b}(\alpha) \setminus \mathrm{b}(\beta)$, $V_i^{d,\alpha}$ is the transport of $V_{t(e)}^{d,\alpha}$ to i. Since $\{0\}_{t(e)} \subseteq V_{t(e)}^{d,\alpha}$ and transport of subspaces preserves inclusion, it follows that $V_i^{e,\beta} \subseteq V_i^{d,\alpha}$ for all $d \in \mathrm{b}(\alpha) \setminus \mathrm{b}(\beta)$, and therefore the second term Eq. 3 is contained in the second term of Eq. 2.

Now consider an enhancement from α to β along an inward literal boundary arrow e of α contained in β. Then α and β share all the same literal/ virtual boundary arrows except that α has one arrow e pointing towards it which β does not share and β has literal/virtual boundary arrows all behind e which α does not share. Therefore for a vertex i in Ω we have the following:

$$F_i^\beta = \left[\bigcap_{d \in \mathrm{b}(\alpha)\cap \mathrm{b}(\beta)} V_i^{d,\beta}\right] \cap \left[\bigcap_{d \in \mathrm{b}(\beta)\setminus \mathrm{b}(\alpha)} V_i^{d,\beta}\right] \tag{4}$$

$$F_i^\alpha = \left[\bigcap_{d \in \mathrm{b}(\alpha)\cap \mathrm{b}(\beta)} V_i^{d,\alpha}\right] \cap V_i^{e,\alpha} \tag{5}$$

As before, the first terms of the intersections in Eqs. 4 and 5 are equal and the third terms in Eqs. 4 and 5 are equal. Now we consider two cases.

Case 1: If i is behind e, then $V_i^{e,\alpha} = V_i$, hence the second term of Eq. 4 is trivially contained in the second term of Eq. 5, yielding $F_i^\beta \subseteq F_i^\alpha$.

Case 2: If i is in front of e, then $V_i^{e,\beta}$ is the transport of $V_{t(e)}$ to i. But notice that for $d \in \mathrm{b}(\beta) \setminus \mathrm{b}(\alpha)$, $V_i^{d,\beta}$ is the transport of $V_{t(e)}^{d,\beta}$ to i. Since $V_{t(e)}^{d,\beta} \subseteq V_{t(e)}$ and transport of subspaces preserves inclusion, it follows that $V_i^{e,\beta} \subseteq V_i^{d,\alpha}$ for all $d \in \mathrm{b}(\alpha) \setminus \mathrm{b}(\beta)$, and therefore the second term in Eq. 4 is contained in the second term of Eq. 5. So $F_i^\beta \subseteq F_i^\alpha$. □

Example 8 Let V be any representation of the mountain quiver Θ. We compute F^α for the different types of $\alpha \in \mathcal{C}$ discussed in Example 5, and then specialize to the representations M and N from Example 3 using Example 7.

- (Type 1): For this type, $\mathrm{oba}(\alpha) = \{a_{-n-1}, a_{m+1}\}$, and $\mathrm{iba}(\alpha) = \mathrm{vba}(\alpha) = \emptyset$. Hence $F^\alpha = V^{a_{-n-1},\alpha} \cap V^{a_{m+1},\alpha} = V^{a_{-n-1},-} \cap V^{a_{m+1},-}$. Therefore if $V = M$ we have $F^\alpha = M^{a_{m+1},-}$ and if $V = N$ we have $F^\alpha = 0$.
- (Type 2): For this type, $\mathrm{oba}(\alpha) = \{a_{n+1}\}$, $\mathrm{iba}(\alpha) = \{a_m\}$, and $\mathrm{vba}(\alpha) = \emptyset$. Hence $F^\alpha = V^{a_{n+1},\alpha} \cap V^{a_m,\alpha} = V^{a_{n+1},-} \cap V^{a_m,+}$. Hence if $V = M$ we have $F^\alpha = M^{a_{n+1},-}$ and if $V = N$ we have $F^\alpha = 0$.
- (Type 3): For this type, $\mathrm{oba}(\alpha) = \emptyset$, $\mathrm{iba}(\alpha) = \{a_n\}$, and $\mathrm{vba}(\alpha) = \{E_R\}$. Hence $F^\alpha = V^{a_n,\alpha} \cap V^{E_R,\alpha} = V^{a_n,+} \cap V^{E_R,+}$. On the one hand, if $V = M$ we have $F^\alpha = M^{a_n,+} \cap M^{E,+} = M$. On the other hand, if $V = N$ we have that $N^{a_n,+}$ is the entire vector space $N(x_m)$ for x_m behind a_n and $\mathrm{Span}(e_1, \ldots, e_{n+2})$ for $m \geq n$. Additionally, $N^{E_R,\alpha} = N^{E_R,+}$ which is equal to the entire vector space $N(x_m)$ for $m \geq 0$, and is equal to $\mathrm{Span}(e_1, e_2) \subseteq N(x_m)$ for $m < 0$. Therefore $F^\alpha = N^{a_n,+} \cap N^{E_R,+}$ is given by the following formula:

$$F^\alpha(x_m) = \begin{cases} \mathrm{Span}(e_1, e_2) & \text{if} \quad m \leq 0 \\ \mathrm{Span}(e_1, \ldots, e_{m+2}) & \text{if} \ 0 \leq m < n \\ \mathrm{Span}(e_1, \ldots, e_{n+2}) & \text{if} \quad n \leq m \end{cases}$$

- (Type 4): Here $\mathrm{oba}(\alpha) = \{a_{-n-1}\}$, $\mathrm{iba}(\alpha) = \emptyset$, and $\mathrm{vba}(\alpha) = \{E_R\}$. Hence $F^\alpha = V^{a_{-n-1},\alpha} \cap V^{E,\alpha} = V^{a_{-n-1},-} \cap V^{E,+}$, so if $V = M$ we have $F^\alpha = M^{a_{-n-1},-}$ and if $V = N$ we have $F^\alpha = 0$.
- (Type 5): For this case, $\mathrm{oba}(\alpha) = \mathrm{iba}(\alpha) = \emptyset$, and $\mathrm{vba}(\alpha) = \{E_R, E_L\}$. We therefore have that $F^\alpha = V^{E_R,\alpha} \cap V^{E_L,\alpha} = V^{E_R,+} \cap V^{E_L,+}$. If $V = M$ we have $F^\alpha = M$ and if $V = N$ we have:

$$N^{E_R,+}(x_i) = \begin{cases} \mathrm{Span}(e_1, e_2) & \text{if } i \le 0 \\ \mathrm{Span}(e_1, \ldots, e_{i+2}) & \text{if } 0 \le i \end{cases}$$

$$N^{E_L,+}(x_i) = \begin{cases} \mathrm{Span}(e_1, e_2) & \text{if } 0 \le i \\ \mathrm{Span}(e_1, \ldots, e_{|i|+2}) & \text{if } i \le 0 \end{cases}$$

Hence $F^\alpha(x_i) = \mathrm{Span}(e_1, e_2)$ for all i.

Lemma 4.1 *If e is an arrow in α with exactly one literal or virtual boundary arrow of α behind it, then f_e is onto when restricted to F^α.*

Proof Let b be the boundary behind e, and let $X = V^{b,\alpha}_{s(e)}$. For each literal or virtual boundary b_i of α other than b, let $Y_i = V^{b_i,\alpha}_{t(e)}$. Then we have

$$F^\alpha_{s(e)} = X \cap \bigcap_i f_e^{-1}[Y_i] \qquad F^\alpha_{t(e)} = f_e[X] \cap \bigcap_i Y_i.$$

If $y \in f_e[X] \cap \bigcap_i Y_i$ then $y = f_e(x)$ where $x \in X$, and since $y \in \bigcap_i Y_i$ necessarily $x \in \bigcap_i f_e^{-1}[Y_i]$. So f_e is onto on F^α. ◻

5 Quotient and Lift

For each connected subgraph $\alpha \in \mathcal{C}$ define

$$W^\alpha = F^\alpha / \sum_{\beta < \alpha} F^\beta. \tag{6}$$

Example 9 Consider the representations M and N of the mountain quiver Θ from Example 3. We compute $F^\alpha / F^{<\alpha}$ for the different types of connected full subquivers α from Example 5.

- (Type 1): Suppose $V = M$. If $m < n$ and we define β to be the element of $\mathcal{C}$ with vertices $x_{-n+1}, \ldots, x_{m-1}, x_m$, then $F^\beta = M^{a_{m+1},-} = F^\alpha$, hence $F^\alpha / F^{<\alpha} = 0$. However if $m = n$ then $F^{<\alpha} = M^{a_m,-}$ and so $F^\alpha / F^{<\alpha}$ has dimension 1 at x_i for $-m \le i \le m$ and is zero everywhere else.
  On the other hand, if $V = N$, since $F^\alpha = 0$ we have $F^\alpha / F^{<\alpha} = 0$ also.
- (Type 2): For $V = M$, if β has vertices $x_{-n}, \ldots, x_n$, then $F^\beta = M^{a_{n+1},-} = F^\alpha$ so $F^\alpha / F^{<\alpha} = 0$.
  For $V = N$ we have $F^\alpha = 0$ which implies $F^\alpha / F^{<\alpha} = 0$.
- (Type 3): If $V = M$, let β be the connected full subquiver containing every vertex, i.e. Θ. Then $\beta < \alpha$ but $F^\beta = M = F^\alpha$, hence $F^\alpha / F^{<\alpha} = 0$.
  For $V = N$, if β is the full subquvier with vertices $x_{n-1}, x_n, \ldots$, then we see that $F^{<\alpha} = F^\beta$, and hence $F^\alpha / F^{<\alpha}$ has dimension 1 on the support of α and dimension 0 elsewhere.

- (Type 4): For $V = M$, if β is the full subquiver with vertices $x_{-n}, \ldots, x_{n+1}$ then $\beta < \alpha$ and $F^{\beta} = M^{a_{-n-1},-} = F^{\alpha}$ so that $F^{\alpha}/F^{<\alpha} = 0$.
  For $V = N$ we have $F^{\alpha} = 0$ which implies $F^{\alpha}/F^{<\alpha} = 0$.
- (Type 5): For $V = M$, if β is the full subquiver with vertices $x_0, x_1, \ldots$, then $\beta < \alpha$ but $F^{\beta} = M = F^{\alpha}$, hence $F^{\alpha}/F^{<\alpha} = 0$.
  Suppose that $V = N$. Notice that for every β with $\beta < \alpha$ we have $F^{\beta} = 0$. Therefore $F^{\alpha}/F^{<\alpha} = F^{\alpha}$ which is 2-dimensional at each vertex.

Proposition 5.1 *For each arrow e in α, if there is exactly one virtual boundary arrow of Ω behind e, then f_e is bijective on W^{α}. For each vertex i not in α, $W_i^{\alpha} = \{0\}$.*

Proof Suppose e is in α and there is exactly one virtual boundary arrow of Ω behind it. Then either that virtual boundary is a virtual boundary of α as well, or there is a literal boundary arrow of α behind e, so by Lemma 4.1, f_e is onto when restricted to F^{α}, and similarly when restricted to $\sum_{\beta<\alpha} F^{\beta}$. Thus it suffices to show that the kernel of f_e in F^{α} is contained in $\sum_{\beta<\alpha} F^{\beta}$ (because that shows that the image of f_e on the quotient is injective). Let β be the reduction of α by the arrow e. This is nonempty, and $F^{\beta}_{s(e)} = f_e^{-1}[0] \cap F^{\alpha}_{s(e)}$. This implies the first sentence.

Suppose $i \in \alpha'_e$ for some boundary e. If α is behind e then $F_i^{\alpha} = V_i^{e,-} = 0$ and the second sentence follows. If α is in front of e then F_i^{α} is $F^{\alpha}_{t(e)}$ transported to i. We will enhance α as follows. Take β to contain all the vertices of α, and since there is a unique injective walk from i to $t(e)$, add all the vertices in that path to β. For any vertex you add to β, if it is the source of an arrow, add the target to β. The resulting β is an enhancement of α by e, and has the following properties. 1) It contains i. 2) All the literal boundary arrows of β which are not in α are inward. 3) The injective journey from i to any of those literal boundary arrows, or to the virtual boundary arrow in β but not α agree in orientation with all its component arrows. The intersection of all the representations associated to literal and virtual boundary that β shares with α at the vertex i will be F_i^{α}. The representation associated to a new boundary e' or a new virtual boundary E will be $V^{e',+}$ or $V^{E,+}$, and because there is a consistently oriented injective journey from i to this boundary the value of that representation at i is V_i. Thus $F_i^{\beta} = F_i^{\alpha}$, and thus $\sum_{\beta<\alpha} F_i^{\beta} = F_i^{\alpha}$, and the quotient is trivial. □

Example 10 The following diagram shows an example of a quiver Ω, an edge e of Ω, and a full subquiver α containing the vertices $s(e)$ and $t(e)$ which satisfy the conditions of Proposition 5.1.

$$\begin{tikzcd}
& \bullet \arrow[d] & & & \\
\bullet & \bullet \arrow[l, "e"'] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \cdots
\end{tikzcd}$$

In general we define an *almost gradation* of a poset filtration (F, P) of an R module M to be a function $C : P \to \mathrm{Sub}(M)$ satisfying the condition that for all $p \in P$, $F^p = F^{<p} \oplus C^p$, where we define $F^{<p} = \sum_{q<p} F^q$.

Proposition 5.2 *For each connected subquiver $\alpha \in \mathcal{C}$, if every edge e in α has exactly one virtual boundary of Ω behind it, the map $F^{\alpha} \to W^{\alpha}$ lifts, i.e. there exists an almost gradation of the poset filtration F.*

Proof We need to lift each W_i^{α} to F_i^{α} consistently with the maps f_e. Since all F_i^{α} outside of α are 0, this is trivial except for vertices and arrows in α. So it suffices to choose lifts for each i in α, in such a way that for each arrow e in α between i and j the following diagram commutes (using Lemma 4.1 and Proposition 5.1)

$$\begin{tikzcd}
0 \arrow[r] & X_j^\alpha \arrow[r, hook] & F_j^\alpha \arrow[r, two heads] & W_j^\alpha \arrow[r] \arrow[l, dashed, bend left] & 0 \\
0 \arrow[r] & X_i^\alpha \arrow[r, hook] & F_i^\alpha \arrow[r, two heads] \arrow[u, two heads, "f_e"'] & W_i^\alpha \arrow[r] \arrow[l, dashed, bend left] \arrow[u, equal, "f_e"'] & 0
\end{tikzcd} \tag{7}$$

where $X^\alpha = \sum_{\beta<\alpha} F^\beta$. In the category of vector spaces we can lift one particular W_i^α to F_i^α. We will extend this lift recursively to each other vertex across each arrow as follows. If the lift $l_i : W_i^\alpha \to F_i^\alpha$ is chosen in the above diagram, then on W_j^α define the lift $f_e \circ l_i \circ f_e^{-1}$ (the inverse exists by Proposition 5.1). If on the other hand the lift $l_j : W_j^\alpha \to F_j^\alpha$ is chosen in the above diagram, the lift in i is $f_e^{-1} \circ l_j \circ f_e$, where f_e^{-1} is any lift of f_e. □

In general we say that an almost gradation $C : P \to \mathrm{Sub}(M)$ of a poset filtration $F : P \to \mathrm{Sub}(M)$ of an R module M *spans* if $M = \sum_{p \in P} C^p$.

Proposition 5.3 *The sum of the subrepresentations F^α equals V.*

Proof Let i be a vertex. Define α recursively by including i in α, and including the target of every arrow whose source is in α. Then α has the property that every literal boundary arrow necessarily has α in front of it and for every vertex $j \in \alpha$, there is a path from i to j. Then F_i^α is the intersection of the transport of V_j for various j in α, which is then V_i. Thus $F_i^\alpha = V_i$. □

Proposition 5.4 *For each connected subquiver $\alpha \in \mathcal{C}$, if every edge e in α has exactly one virtual boundary of Ω behind it, then W_α is an isotypic Ω-representation.*

Proof If α satisfies the assumptions of the proposition then by Proposition 5.1 every f_e is an isomorphism if e is in α and 0 otherwise. Choose a vertex i for which W_i is nonempty, and for each j the sequence of isomorphisms f_e associated to a path from i to j induces an isomorphism $W_i \cong W_j$. Choose a basis S of W_i, and for each s and each j let W_j^s be the image in W_j under this isomorphism of the span of s in W_i. Then for each arrow e the isomorphism f_e from j to k restricts to a rank 1 isomorphism f_e^s from W_j^s to W_k^s. Then the collection W^s of these restricted isomorphisms and subspaces is a subrepresentation of W, clearly indecomposable, and it is immediate that $W = \bigoplus_{s \in S} W^s$. Finally, note that the vector space isomorphism W_i^s to W_i^t extends to a representation isomorphism W^s to W^t, and thus W is isotypic. □

Example 11

1. Consider the representation M of the mountain quiver Θ from Example 3 (1). According to Example 9, $F^\alpha/F^{<\alpha} = 0$ except if α is a full subquiver containing $x_{-n}, \ldots, x_n$ for $n \geq 0$. If we define $W^\alpha \subseteq M$ to be $\mathrm{Span}(e_{n+1})$ at all vertices in the support of α and zero elsewhere, then $F^\alpha = W^\alpha \oplus F^{<\alpha}$.
2. Consider the representation N of the mountain quiver Θ from Example 3 (2). According to Example 9, $F^\alpha/F^{<\alpha} = 0$ except if α is either a subquiver with vertices $x_n, x_{n+1}, \ldots$ for $n > 0$, a subquiver with vertices $\ldots, x_{-n-1}, x_{-n}$ for $n > 0$, or $\alpha = \Theta$. In the first two cases we define $W^\alpha(x_i)$ to be $\mathrm{Span}(e_{n+2})$ for all x_i in the support of α and zero elsewhere, and in the third case we define $W^\alpha(x_i)$ to be $\mathrm{Span}(e_1, e_2)$ for all i. Note that in the latter case, W^α is non-trivially isotypic, i.e. it is a direct sum of two isomorphic subrepresentations each of which has a single dimension at each vertex.
For all such α we have $F^\alpha = W^\alpha \oplus F^{<\alpha}$.

$$\cdots \overset{a_{-3}}{\text{———}} x_{-2} \overset{a_{-2}}{\text{———}} x_{-1} \overset{a_{-1}}{\text{———}} x_0 \overset{a_0}{\text{———}} x_1 \overset{a_1}{\text{———}} x_2 \overset{a_2}{\text{———}} \cdots$$

Fig. 3 The Graph $A_{\infty,\infty}$

In either case, $\alpha \mapsto W^\alpha$ is an almost gradation of the poset filtration F. The existence of W^α is guaranteed by Propositions 5.2 and 5.3 implies that $M = \sum_{\alpha\in\mathcal{C}} F^\alpha = \sum_{\alpha\in\mathcal{C}} W^\alpha$.

6 Complete Decomposition of $A_{\infty,\infty}$

Let $A_{\infty,\infty}$ be the graph shown in Fig. 3 with a vertex for each integer i and an edge between each two adjacent integers, and let Ω be an eventually outward quiver with underlying graph $A_{\infty,\infty}$.

Notice that every connected subquiver α of Ω has exactly two literal/virtual boundary arrows, a left boundary which is either the literal left boundary arrow or the left virtual arrow of Ω, and a right boundary which is either the literal right boundary arrow or the right virtual arrow of Ω.

We will define two total orderings, one $<_L$ on the set of all arrows of $A_{\infty,\infty}$ together with a symbol $-\infty$ (representing the left virtual boundary of Ω), and the other $<_R$ on the set of all arrows in $A_{\infty,\infty}$ together with the symbol ∞ representing the right virtual arrow. If e and e' are two arrows oriented the same way, then $e <_L e'$ and $e <_R e'$ if e is behind e'. If they are oriented oppositely then $e <_L -\infty <_L e'$ and $e' <_R \infty <_R e$ if e is pointing to the left and e' to the right. Note that the restriction of either $<_R$ or $<_L$ to the set of arrows Ω_1 is a total order extension of the partial order $\prec$ from Section 2.4, in the sense that for any pair of edges e, e' if $e \prec e'$ then we have that $e <_R e'$ and $e <_L e'$.

Recall that if $(P, \leq_P)$ and $(Q, \leq_Q)$ are two partially ordered sets, their *product* $(P, \leq_P) \times (Q, \leq_Q)$ is the partially ordered set $(P \times Q, \leq_{P\times Q})$ where $(p, q) \leq_{P\times Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$.

Lemma 6.1 *There is an order-embedding ι (i.e. order-preserving, injective, and order-reflecting) of the poset $\mathcal{C}$ of connected subquivers of Ω with the reduction/enhancement partial order into the product poset $((\Omega_1 \cup \{-\infty\}) \times (\Omega_1 \cup \{\infty\}), <_{L\times R})$ given by sending a connected subquiver α to the ordered pair consisting of its left and right literal or virtual boundary arrows respectively.*

Proof The map is clearly injective since a connected subgraph of $A_{\infty,\infty}$ is uniquely determined by its literal/virtual boundary arrows. To prove it is order-preserving, it suffices to show that if $\alpha, \beta \in \mathcal{C}$ are connected components and $\alpha > \beta$ with β a reduction or enhancement of α then the left/right literal or virtual boundary arrow of α is larger than the left/right literal or virtual boundary arrow of β.

If β is a reduction of α along an arrow e in α then α and β agree behind e, so each has a literal or virtual boundary arrow behind e and these arrows are equal. The remaining boundary arrow a of α is in front of e which is the remaining boundary arrow of β. If e is pointing to the right, then $e <_R a$, while if e is pointing to the left, then $e <_L a$. The desired result follows. The case when β is an enhancement of α is similar.

To see that it is order-reflecting, assume $\iota(\alpha) >_{L\times R} \iota(\beta)$ and show $\alpha > \beta$. If $\iota(\alpha) = (i, j)$ and $\iota(\beta) = (k, \ell)$, then $k \leq_L i$ and $\ell \leq_R j$. Let γ be the unique connected subset such that $\iota(\gamma) = (k, j)$. Then by the definition of $<_L$, γ is an enhancement of α if k is to the left of i

and an reduction if k is to the right, in either case $\gamma < \alpha$ (or equal if $k = i$). Similarly we see by the definition of $<_R$ that $\beta < \gamma$. □

Associate to any arrow e a subrepresentation of V called L^e which is $V^{e,+}$ if e points to the right and $V^{e,-}$ if e points to the left, and a subrepresentation of V called R^e which is $V^{e,+}$ if e points to the left and $V^{e,-}$ if e points to the right. Associate to $-\infty$ a subrepresentation $L^{-\infty} = V^{E,+}$ where E is the left virtual arrow of $A_{\infty,\infty}$ and associate to ∞ a subrepresentation $R^{\infty} = V^{E,+}$ where E is the right virtual arrow of $A_{\infty,\infty}$.

Lemma 6.2 *L is a poset filtration of V with respect to the $<_L$ order and R is a poset filtration of V with respect to the $<_R$ order. For each connected subquiver α the subrepresentation F^α is the intersection of L^e and $R^{e'}$, where e and e' are respectively the left and right virtual or literal boundary arrows of α.*

Proof We check the first sentence only for the L, the argument for R is the same. If e is a literal arrow oriented to the right then $L^e = V^{e,+}$, and thus L_i^e is V_i to its left and a transport of $V_{t(e)}$ to its right. $L_i^{-\infty}$ is the transport of V_j for j sufficiently far to the left of it. If e is oriented to the left, $L^e = V^{e,-}$, hence L_i^e is 0 to its left and a transport of $\{0\}_{t(e)}$ to its right. Thus if e is oriented to the right, L_i^e is bigger than any $L_i^{e'}$ to its left (behind it), including the left virtual arrow $L^{-\infty}$. Similarly, if e is oriented to the left, L_i^e is smaller than any arrow to its left (in front of it), and smaller than $L^{-\infty}$.

That F^α is the intersection of L and R is just a restatement of the definition of F^α in the case when there are always two literal or virtual boundary arrows. □

Let M be an R module. We say that a poset filtration (F, P) of M is *distributive* if for all finite subsets $Q \subseteq P$ and all $p \in Q$ which is maximal in Q we have $F_p \cap \sum_{q\in Q\smallsetminus\{p\}} F_q \subseteq F_{<p}$.

Furthermore, we say that an almost gradation $C : P \to \mathrm{Sub}(M)$ of a poset filtration $F : P \to \mathrm{Sub}(M)$ of an R module M is *independent* if the family of submodules $(C^p \mid p \in P)$ is independent, in the sense that whenever we have $c_{p_1} + \ldots + c_{p_n} = 0$ for $c_{p_i} \in C^{p_i}$ and $p_1, \ldots, p_n$ distinct elements of P, then $c_{p_i} = 0$ for all $1 \le i \le n$.

Proposition 6.1 *Let R be a ring and M an R module. If (F, P) is a distributive poset filtration of M then every almost gradation of F is independent.*

Proof Let $C : P \to \mathrm{Sub}(M)$ be an almost gradation of F, and suppose that $Q \subseteq P$ is a finite subset and for each $q \in Q$ we have $c_q \in C_q$ such that $\sum_{q\in Q} c_q = 0$. Since Q is finite, there exists a maximal element $p \in Q$ of Q (i.e. if $p \le q$ for some $q \in Q$ then $p = q$). Then we write

$$c_p = -\sum_{q\in Q\smallsetminus\{p\}} c_q.$$

Note that the LHS is contained in F_p, and the RHS is contained in $\sum_{q\in Q-\{p\}} F_q$, so both sides are contained in $F_p \cap \sum_{q\in Q\smallsetminus\{p\}} F_q \subseteq F_{<p}$ by distributivity. Thus $c_p \in F_{<p}$ and since $c_p \in C_p$ and $V_p = C_p \oplus F_{<p}$, it follows $c_p = 0$, and then by induction on the size of Q, all $c_q = 0$ as desired. □

We define the *intersection* of two poset filtrations (E, P), and (F, Q) of an R-module M, to be the poset filtration $(E \cap F, P \times Q)$ (where $P \times Q$ is given the product order) defined by $[E \cap F]_{(p,q)} = E_p \cap F_q$. One can easily check that $E \cap F : P \times Q \to \mathrm{Sub}(M)$ is an order-preserving function, so this does indeed define a poset filtration of M.

Proposition 6.2 *If (E, I), and (F, J) are linear filtrations of M then for each $(i, j) \in I \times J$, we have that*

$$[E \cap F]_{<(i,j)} = E_i \cap F_{<j} + E_{<i} \cap F_j. \tag{8}$$

and therefore their intersection $(E \cap F, I \times J)$ is distributive.

Proof By definition, we have that $[E \cap F]_{<(i,j)} = \sum_{(k,\ell)<(i,j)} E_k \cap F_\ell$. We now show mutual inclusion of the desired equality.

For the containment ($\subseteq$), suppose that $(k, \ell) < (i, j)$, and thus either $k < i$ and $\ell \le j$, or $k \le i$ and $\ell < j$. Without loss of generality, suppose that $k < i$ and $\ell \le j$. Then we have that $E_k \subseteq E_{<i}$, and $F_\ell \subseteq F_j$. Therefore $E_k \cap F_\ell \subseteq E_{<i} \cap F_j$.

To show the containment ($\supseteq$), we prove only that $E_i \cap F_{<j} \subseteq [E \cap F]_{<(i,j)}$, the other case being similar. Consider $v \in E_i \cap F_{<j} \subseteq \sum_{\ell<j} F_\ell$. In particular we have that $v \in F_{<j} = \sum_{\ell<j} F_\ell$. Since $(J, \le)$ is totally ordered and the sum is manifestly nonempty, $\sum_{\ell<j} F_\ell = \bigcup_{\ell<j} F_\ell$. Therefore there exists some $\ell' < j$ such that $v \in F_{\ell'}$. Hence $v \in E_i \cap F_{\ell'} \subseteq [E \cap F]_{<(i,j)}$.

We now check that $(E \cap F, I \times J)$ is distributive. Let $Q \subseteq I \times J$ be finite and let $p = (i, j) \in Q$ be maximal. If $x \in [E \cap F]_p \cap \sum_{Q \smallsetminus \{p\}} [E \cap F]_q$, then we can write $x = \sum_{(\ell,m)\in Q\smallsetminus\{p\}} x_{\ell,m}$ where $x_{\ell,m} \in E_\ell \cap F_m$. Because p is maximal either $\ell < i$ or $m < j$. Since each of the latter lie in $F_{<j}$ we have

$$x - \sum_{\substack{(\ell,m)\in Q \\ \ell<i}} x_{\ell,m} \in F_{<j}.$$

But of course every term in the left hand side is in E_i so

$$x - \sum_{\substack{(\ell,m)\in Q \\ \ell<i}} x_{\ell,m} \in E_i \cap F_{<j}.$$

On the other hand each $x_{\ell,m} \in E_{<i}$, so

$$x \in E_{<i} + E_i \cap F_{<j}.$$

and noting that $x \in F_j$ and $E_i \cap F_{<j} \subseteq F_j$ gives

$$x \in E_{<i} \cap F_j + E_i \cap F_{<j} \subseteq [E \cap F]_{<(i,j)}$$

by Eq. 8 □

A *morphism of poset filtrations* $\varphi : (E, P) \to (F, Q)$ of an R module V is an order-preserving map $\varphi : P \to Q$ such that $F \circ \varphi = E$.

Lemma 6.3 *Suppose that $\varphi : (E, P) \to (F, Q)$ is a morphism of poset filtrations which is an order-embedding, that (F, Q) is distributive, and that for all $p \in P$ we have that $F_{<\varphi(p)} \subseteq E_{<p}$. Then (E, P) is distributive as well.*

Proof Let $S \subseteq P$ be a finite subset and $p \in S$ be maximal. Then $\varphi(S) \subseteq Q$ is finite and $\varphi(p) \in \varphi(S)$ is maximal because φ is order-reflecting. Then

$$E_p \cap \sum_{s\in S\smallsetminus\{p\}} E_s \overset{(1)}{=} F_{\varphi(p)} \cap \sum_{s\in S\smallsetminus\{p\}} F_{\varphi(s)} \overset{(2)}{=} F_{\varphi(p)} \cap \sum_{q\in\varphi(S)\smallsetminus\{\varphi(p)\}} F_q \overset{(3)}{\subseteq} F_{<\varphi(p)} \overset{(4)}{\subseteq} E_{<p}$$

where (1) follows because φ is a poset filtration morphism, (2) follows because order-embeddings are injective, (3) follows because (F, Q) is distributive, and (4) follows by hypothesis. □

Lemma 6.4 *The order-embedding* $\iota : \mathcal{C} \to (\Omega_1 \cup \{-\infty\}, <_L) \times (\Omega_1 \cup \{\infty\}, <_R)$ *defined in* Lemma 6.1 *satisfies the condition that for all* $\alpha \in \mathcal{C}$, $[L \cap R]_{<\iota(\alpha)} \subseteq F_{<\alpha}$.

Proof Suppose that $\iota(\alpha) = (e_L, e_R)$, and by Proposition 6.2, $[L \cap R]_{<\iota(\alpha)} = L_{e_L} \cap R_{<e_R} + L_{<e_L} \cap R_{e_R}$. It suffices to show that $L_{e_L} \cap R_{<e_R}$ and $L_{<e_L} \cap R_{e_R}$ are contained in $F_{<\alpha}$, and we only show the former, the latter being similar.

Note that $R_{<e_R} = \sum_{e<_R e_R} R_e = \bigcup_{e<_R e_R} R_e$, the last equality holding because $\Omega_1 \cup \{\infty\}$ is linearly ordered and some R_e is nonempty. Therefore $L_{e_L} \cap R_{<e_R} = L_{e_L} \cap \bigcup_{e<_R e_R} R_e = \bigcup_{e<_R e_R} L_{e_L} \cap R_e$. Define $\beta = \iota^{-1}(e_L, e)$ since ι is injective and note $\beta < \alpha$ since $\iota(\beta) < \iota(\alpha)$ and ι is order-reflecting. So $F_\beta \subseteq F_{<\alpha}$ and hence $L_{e_L} \cap R_{<e_R} \subseteq F_{<\alpha}$. □

Corollary 6.1 *Let V be an $A_{\infty,\infty}$-representation. Every almost gradation of the poset filtration $(F, \mathcal{C})$ is independent.*

Proof By Lemmas 6.1 and 6.2 $\iota : (F, \mathcal{C}) \to (L \cap R, \Omega_1 \cup \{-\infty\} \times \Omega_1 \cup \{\infty\})$ is an order-embedding morphism. By Proposition 6.2 $(L \cap R, \Omega_1 \cup \{-\infty\} \times \Omega_1 \cup \{\infty\})$ is distributive. By Lemma 6.4, ι satisfies the necessary conditions to apply Lemma 6.3, which implies that $(F, \mathcal{C})$ is also distributive. Hence Proposition 6.1 gives the desired result. □

Theorem 1 *The category of representations of an eventually outward $A_{\infty,\infty}$ quiver is infinite Krull Schmidt.*

Proof We first show that a representation V of an eventually outward $A_{\infty,\infty}$ quiver can be written as a direct sum of representations, each of which is isomorphic to W^α for some connected subquiver α, and each W^α is an isotypic.

By Proposition 4.1, $(F, \mathcal{C})$ is a poset filtration of subrepresentations of V. By Proposition 5.2 W^α is an almost gradation of $(F, \mathcal{C})$, by Corollary 6.1 W^α is independent, and by Proposition 5.3, W^α spans V. Therefore $V = \bigoplus_\alpha W^\alpha$. By Proposition 5.4 each W^α is an isotypic.

To show this decomposition is unique up to reordering and isomorphism, assume that we have another decomposition $V = \bigoplus_\alpha X^\alpha$ where X^α is a direct sum of representations all isomorphic the the thin representation with support exactly α. We will show inductively that for each α if F^α is the subrepresentation of V defined in Eq. 1, then $F^\alpha = \bigoplus_{\beta \le \alpha} X^\beta$. If this is true then the quotient map $X^\alpha \to F^\alpha / \sum_{\beta<\alpha} = W^\alpha$ is an isomorphism and the sum of all these isomorphisms gives an isomorphism of the two direct sum decompositions.

To see that isomorphism, note first from the definition of F^α we have that $X^\alpha \subset F^\alpha$ and therefore $\sum_{\beta \le \alpha} X^\alpha \subset F^\alpha$. But the injection of F^α into V projects onto a map $F^\alpha \to X^\beta$ for each β, and this map is clearly 0 unless $\beta \le \alpha$, so $F^\alpha \subset \sum_{\beta \le \alpha} X^\alpha$. □

Example 12 Let V be one of the representations M of N of the mountain quiver Θ from Example 3. According to Example 11, $\alpha \mapsto W^\alpha$ is an almost gradation of the poset filtration F, and because Θ has type $A_{\infty,\infty}$, Corollary 6.1 implies that this almost gradation is independent too. Whence we have $V = \bigoplus_{\alpha \in \mathcal{C}} W^\alpha$ in accordance with Theorem 1.

7 Description of the Indecomposables

In this section we continue to restrict to the case where Ω is eventually outward and has underlying graph $A_{\infty,\infty}$.

Proposition 7.1 *For each full connected subquiver $\alpha \in \mathcal{C}$ the isomorphism class of representations X^α with X_i^α being zero dimensional if i is not in α and 1-dimensional if i is in α and with f_e a bijection for each arrow in α is indecomposable, and thus every representation of an eventually outward $A_{\infty,\infty}$ quiver is a direct sum of indecomposables X^α.*

Proof Each X^α is clearly indecomposable, because if it were a sum of two subrepresentations then each is either one or zero dimensional at each vertex, and thus each is the sum of the X_i^α for some subset of the vertices and the other is the sum for the complementary set of vertices. If each is a nonempty set, since α is connected there must be a pair of vertices connected by an arrow from different subrepresentations. Since the morphism associated to that edge is a bijection, this is impossible.

By Propositions 5.1 and 5.4, each W^α is a sum of copies of X^α, and by Theorem 1 any representation is a sum of X^α. □

Note that the indecomposables in Proposition 7.1 are exactly the indecomposable representations found by Bautista et al. [3] in their Prop. 5.9, although they deal only with locally finite representations.

There is a useful interpretation of this with the Euler form. Let us define the *root space* of the quiver to be the space of functions from Ω_0 to the real numbers which are zero on all but finitely many vertices. If n and m are two such functions define the Euler form to be

$$\langle n, m\rangle_\Omega = \sum_{i\in\Omega_0} n_i m_i - \frac{1}{2}\sum_{e\in\Omega_1}\left(n_{s(e)}m_{t(e)} + n_{t(e)}m_{s(e)}\right) \tag{9}$$

where of course n_i is the value of n at i.

Observe that the Euler form is positive definite because the corresponding Tits quadratic form satisfies

$$\langle n, n\rangle_\Omega = \frac{1}{2}\sum_{i\in\Omega_0}(n_i + n_{i+1})^2 > 0$$

unless every $n_i = -n_{i+1}$, which contradicts the finiteness. The number $\langle n, n\rangle_\Omega$ is called the *length* of the vector n. We define a *root* to be an element of the root space which has integer entries and Tits form 1. Define also a *positive root* to be a root with nonnegative entries.

Define the *weight space* of the quiver to be the space of functions from Ω_0 to the real numbers, with no restriction on the values. Observe that if n is in the weight space and m is in the root space then Eq. 9 still makes sense and thus the weight space can be thought of as functionals on the root space. Define a uniform topology on the weight space with an entourage for each finite subset S of the vertices, the open neighborhood of each n being the weights that assign the same value as n to every vertex in S. Equivalently, Cauchy sequences are those that for each such S are eventually constant on each vertex of S. It is easy to check that the weight space is the completion of the root space in this topology and that each element of the root space gives through the Euler form a continuous map from the weight space to the real numbers (the reals are given the discrete topology).

Observe that every representation of a quiver which is locally finite-dimensional (that is the vector space at each vertex is finite-dimensional) determines a natural number-valued weight called its dimension vector that assigns to each vertex the dimension of the associated vector space.

Proposition 7.2 *For the quiver $A_{\infty,\infty}$ a weight valued in the natural numbers is the dimension of an indecomposable representation if and only if it is in the closure of the set of all positive roots.*

Proof By Proposition 7.1 the representations X^α are the only indecomposables. One checks that if α consists of finitely many vertices its dimension is of length 1. It is also clear that if α is infinite it is in the closure of the set of all finite connected subquivers α' contained in it.

Suppose first n is in the root space and valued in the natural numbers, and the support α (vertices i with $n_i > 0$) of n is connected. Then if i and j are its left and right endpoints

$$\langle n, n\rangle_\Omega = \frac{1}{2}(n_i^2 + n_j^2) + \sum_{e\in\alpha} \frac{1}{2}\left(n_{s(e)} - n_{t(e)}\right)^2.$$

The first term is at least 1 and only 1 if $n_i = n_j = 1$. The second term is at least 0 and only 0 if all n_k values in between are equal. Therefore $\langle n, n\rangle_\Omega = 1$ if n is all 1s on α and $\langle n, n\rangle_\Omega \geq 2$ otherwise. If n has disconnected support $\langle n, n\rangle_\Omega$ is the sum of the values of each connected component, and therefore at least 2. So if n is in the root space and valued in $\mathbb{N}$ it is a positive root only when it corresponds to α.

If n is in the closure of the set of all positive roots, it cannot assign a number bigger than 1 to any vertex, because an open set around it based on a finite set of vertices containing that vertex would include no positive roots. So it is the indicator function of a set of vertices. If that set were not connected it has a finite gap, and a finite set of vertices containing that gap and the points on either side would give an open set that includes no positive roots. So every weight in the closure is of the desired form.

The dimensions of course determine the indecomposable representation, because if a representation has the same dimensions as X^α but is not isomorphic, then for one edge $e \in \alpha$ we must have $f_e = 0$. In that case the representation decomposes as the sum of the subrepresentations supported respectively behind and in front of e. □

8 A Representation which is not Krull-Schmidt

In this section we give an example of a representation of a type $A_{\infty,\infty}$ quiver which is not infinite Krull-Schmidt. It follows from Theorem 1 that this quiver is not eventually outward.

Let Ω be the $A_{\infty,\infty}$ quiver in Fig. 4 with all arrows pointing towards the vertex x_0, and note that this quiver is not eventually outward. Let V be the representation of Ω with $V_n = 0$ for all $n < 0$ and for $n \geq 0$, V_n is the vector space of all sequences $(b_0, b_1, \ldots)$ with entries in $\mathbb{F}$ and $b_m = 0$ for all $m < n$. Let $f_n : V_{n+1} \to V_n$ be the inclusion map for all $n \geq 0$.

Suppose that V is infinite Krull-Schmidt, i.e. that $V = \bigoplus_{i\in I} V^i$ where for each $i \in I$, V^i is an indecomposable representation of Ω.

Lemma 8.1 *For all $k \geq 0$ there exists $i_k \in I$ and nonzero $x \in V_k^{i_k}$ such that $x \in V_k \setminus f_k(V_{k+1})$.*

Proof Consider the standard basis vector $e_{k+1} = (0, \ldots, 0, 1, 0, \ldots) \in V_k \setminus f_k(V_{k+1})$. Since $V = \bigoplus_{i\in I} V^i$, we can write $e_{k+1} = v_1 + \ldots + v_n$ where $v_m \in V_k^{i_m}$ and $i_m \in I$. It is not possible that $v_m \in f_k(V_{k+1})$ for all $1 \leq m \leq n$ because then $e_{k+1} \in f_k(V_{k+1})$, which is false. Hence there must be some $v_m \in V_k \setminus f_k(V_{k+1})$, and taking $x = v_m$ and $i_k = i_m$ yields the desired result. □

$$\cdots \xrightarrow{a_{-3}} x_{-2} \xrightarrow{a_{-2}} x_{-1} \xrightarrow{a_{-1}} x_0 \xleftarrow{a_0} x_1 \xleftarrow{a_1} x_2 \xleftarrow{a_2} \cdots$$

Fig. 4 The Quiver Ω

Lemma 8.2 *If there exists $i \in I$ and nonzero $x \in V_k^i$ such that $x \in V_k \setminus f_k(V_{k+1})$ then $V_m^i = 0$ if $m < 0$ or $m > k$ and $V_m^i = \mathrm{Span}(x)$ for all $0 \le m \le k$.*

Proof We prove this by induction on k. Suppose $k \geq 0$ and assume the result holds for all $0 \leq n < k$.

First prove that given $n < k$, $f_n : V_{n+1}^i \to V_n^i$ is surjective. Given $v \in V_n^i$, if $v \in V_n \setminus f_n(V_{n+1})$, then by induction V_m^i is 0 for $m > n$ and in particular $V_k^i = 0$ contradicting that $x \neq 0$. Therefore it must be that $v \in f_n(V_{n+1})$, and we call its preimage v'. Then by our direct sum decomposition of V we can write $v' = \sum_{t=1}^k v_{i_t}$ where $v_{i_t} \in V_{n+1}^{i_t}$. But then $v = f_n(v') = \sum_{t=1}^k f_n(v_{i_t})$ and $f_n(v_{i_t}) \in V_n^{i_t}$. Since $v \in V_n^i$ it must be that there is some s such that $i_s = i$ and $v = f_n(v_{i_s})$ but $f_n(v_{i_t}) = 0$ for all $t \neq s$. Since f_n is injective, it follows that $v' = v_{i_s} \in V_{n+1}^i$ and $v_{i_t} = 0$ for all $t \neq s$, hence $v \in f_n(V_{n+1}^i)$ as desired.

Given $x \in V_k^i$ such that $x \in V_k \setminus f_k(V_{k+1})$ let C and D be the subrepresentations of V given respectively below.

$$\ldots \to 0 \to f\,(\mathrm{Span}(x)) \xleftarrow{f_0} \ldots \xleftarrow{f_{k-1}} \mathrm{Span}(x) \xleftarrow{f_k} 0 \ldots$$

$$\ldots \to 0 \to f f_k(V_{k+1}) \xleftarrow{f_0} \ldots \xleftarrow{f_{k-1}} f_k(V_{k+1}) \xleftarrow{f_k} V_{k+1} \xleftarrow{f_{k+1}} \ldots$$

Here $f = f_0 \circ \ldots \circ f_{k-1}$. Note that C is a subrepresentation of V^i and thus $C \oplus [D \cap V^i] \subseteq V_i$ (since $x \notin f_k(V_{k+1})$, it follows that C_k and $D_k \cap V_k^i$ are independent). In fact $V^i = C \oplus [D \cap V^i]$. For $n > k$ it is obvious that $V_n^i = C_n \oplus [D \cap V^i]_n$. If $v \in V_k^i \subseteq V_k$ then $v = \alpha x + w$ where $\alpha \in \mathbb{F}$ and $w \in f_{k+1}\,(V_{k+1})$, but in fact $w = v - \alpha x \in V^i$ and thus $w \in V_k^i \cap f_{k+1}\,(V_{k+1})$, and so $v \in C \oplus (D \cap V^i)$. Thus $V_k^i = C_k \oplus (D_k \cap V_k^i)$. Since f_n is injective and surjective on V^i for $0 \le n < k$ the same is true for every $n \geq 0$, and both sides of the last equation are 0 for $n < 0$, so $V^i = C \oplus (D \cap V^i)$.

Since V^i is indecomposable, this implies that $D \cap V^i$ is zero and $V^i = C$. □

Corollary 8.1 *All V^i are of the form*

$$\ldots \to 0 \to \mathrm{Span}(x) \xleftarrow{f_0} \ldots \xleftarrow{f_{k-1}} \mathrm{Span}(x) \xleftarrow{f_k} 0 \ldots$$

for some $k \geq 0$.

Proof Given $i \in I$, since V^i is indecomposable, it is nonzero. Therefore there exists a nonzero vector x in V_ℓ^i for some $\ell, \geq 0$. If $x \in V_\ell \setminus f_\ell(V_{\ell+1})$ then by Lemma 8.2 we are done. If $x = f_\ell(y)$ for $y \in V_{\ell+1}$ write $y = \sum y_t$ for $y_t \in V^{i_t}$ and thus $f_\ell(y_t) \in V^{i_t}$ and by independence of the direct summands one i_t must equal i and $y \in V^i$. Repeating this process we must come to a $z \in V_k^i \setminus f_k\,(V_k)$, since every nonzero sequence has a first nonzero entry. □

Lemma 8.3 *For each $k \geq 0$ there is at most one $i \in I$ such that $V_k^i \cap [V_k \setminus f_k(V_{k+1})]$ is nonempty.*

Proof Suppose that $i, j \in I$ are such that there exists $x \in V_k^i$, $y \in V_k^j$ and $x, y \in V_k \setminus f_k(V_{k+1})$. By Lemma 8.2, we have that V^i and V^j are equal, respectively, to the following subrepresentations.

$$\ldots \to 0 \to \mathrm{Span}(x) \xleftarrow{f_0} \ldots \xleftarrow{f_{k-1}} \mathrm{Span}(x) \xleftarrow{f_k} 0 \ldots$$

$$\ldots \to 0 \to \mathrm{Span}(y) \xleftarrow{f_0} \ldots \xleftarrow{f_{k-1}} \mathrm{Span}(y) \xleftarrow{f_k} 0 \ldots$$

Now by hypothesis, $V_{k+1} = \bigoplus_{\ell \in I} V_{k+1}^{\ell}$, and then because f_k is injective, we have $f_k(V_{k+1}) = \bigoplus_{\ell \in I} f_k(V_{k+1}^{\ell})$. By Corollary 8.1, either $V_{k+1}^{\ell} = 0$ or f_k is an isomorphism when restricted to V_{k+1}^{ℓ}, hence we can write $f_k(V_{k+1}) = \bigoplus_{\ell \in L} V_{k+1}^{\ell}$ with $L = \{\ell \in I : V_{k+1}^{\ell} \neq 0\}$.

Again by hypothesis, $V_k = \bigoplus_{\ell \in I} V_k^{\ell}$ and hence the family $(V_k^i, V_k^j, V_k^{\ell} : \ell \in L)$ of V_k is independent. But this contradicts the fact that V_k / V_{k+1} is 1-dimensional. ◻

Corollary 8.2 *V is not infinite Krull-Schmidt.*

Proof If V were a direct sum of indecomposable subrepresentations, say $V = \bigoplus_{i \in I} V^i$, then by Corollary 8.1, each of the indecomposables is of the form described in said corollary, which means for all $i \in I$ there exists $k \geq 0$ such that $V_k^i \cap [V_k \setminus f_k(V_{k+1})]$ is nonempty. But by Lemma 8.3 there is at most one such i for each k, hence I must be countable. However V_0^i is one-dimensional for each i and V_0 has uncountable dimension, yielding a contradiction. ◻

Remark 8.1 The example above seems closely related to Example 2.5.1 in [6], which is a locally countable-dimensional representation that cannot be written as a sum of thin indecomposables. It seems likely that similar arguments as the above example would prove that it cannot in fact be written as a sum of indecomposables at all.

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