Algebras and Representation Theory (2024) 27:429–460
https://doi.org/10.1007/s10468-023-10221-1

RESEARCH

Identities of Inverse Chevalley Type for the Graded Characters of Level-Zero Demazure Submodules over Quantum Affine Algebras of Type C

Takafumi Kouno[1] · **Satoshi Naito**[2] · **Daniel Orr**[3]

Received: 2 December 2022 / Accepted: 16 June 2023 / Published online: 15 August 2023

Abstract

We provide identities of inverse Chevalley type for the graded characters of level-zero Demazure submodules of extremal weight modules over a quantum affine algebra of type C. These identities express the product $e^{\mu}\mathrm{gch}\ V_x^-(\lambda)$ of the (one-dimensional) character e^{μ}, where μ is a (not necessarily dominant) minuscule weight, with the graded character $\mathrm{gch} V_x^-(\lambda)$ of the level-zero Demazure submodule $V_x^-(\lambda)$ over the quantum affine algebra $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ as an explicit finite linear combination of the graded characters of level-zero Demazure submodules. These identities immediately imply the corresponding inverse Chevalley formulas for the torus-equivariant K-group of the semi-infinite flag manifold $\mathbf{Q}_G$ associated to a connected, simply-connected and simple algebraic group G of type C. Also, we derive cancellation-free identities from the identities above of inverse Chevalley type in the case that μ is a standard basis element ε_k in the weight lattice P of G.

Keywords Level-zero Demazure module · Semi-infinite flag manifold · Inverse Chevalley formula · Quantum alcove model

Mathematics Subject Classification (2010) Primary 05E10 · Secondary 14N15 · 14M15 · 20G42 · 81R10

Presented by: Peter Littelmann

✉ Takafumi Kouno
t.kouno@kurenai.waseda.jp

Satoshi Naito
naito@math.titech.ac.jp

Daniel Orr
dorr@vt.edu

1 Department of Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

2 Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152-8551, Japan

3 Department of Mathematics (MC 0123), 460 McBryde Hall, Virginia Tech, 225 Stanger St., Blacksburg, VA 24061, USA

1 Introduction

The purpose of this paper is to prove identities (of inverse Chevalley type) for the graded characters of Demazure submodules (level-zero Demazure submodules) of extremal weight modules with level-zero extremal weight over a quantum affine algebra of type C.

Let $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ be the quantum affine algebra associated to the (untwisted) affine Lie algebra $\mathfrak{g}_{\mathrm{af}}$ whose underlying simple finite-dimensional Lie algebra is $\mathfrak{g}$. Let us denote by W_{af} (resp., W) the Weyl group, by $\mathfrak{h}_{\mathrm{af}}$ (resp., $\mathfrak{h}$) the Cartan subalgebra, and by P_{af} (resp., P) the weight lattice of $\mathfrak{g}_{\mathrm{af}}$ (resp., $\mathfrak{g}$), where $P = \sum_{i \in I} \mathbb{Z}\varpi_i$ and $P_{\mathrm{af}} = P + \mathbb{Z}\delta + \mathbb{Z}\Lambda_0$. For $x \in W_{\mathrm{af}}$ and $\lambda \in P^+$, with $P^+ \subset P$ the set of dominant weights for $\mathfrak{g}$, let $V_x^-(\lambda)$ denote the Demazure submodule (level-zero Demazure submodule) of the extremal weight module $V(\lambda)$ with extremal weight λ over $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$, where $\lambda \in P$ is regarded as an element of P_{af} in a canonical way. In recent years, the graded characters $\mathrm{gch}\, V_x^-(\lambda)$ of the level-zero Demazure submodules for $x \in W_{\mathrm{af}}$, $\lambda \in P$, have been studied in several works. Among them, Kato-Naito-Sagaki [7] obtained an explicit description of the expansion of the graded character $\mathrm{gch}\, V_x^-(\lambda+\mu)$ for $\lambda, \mu \in P^+$ as an infinite linear combination with coefficients in $\mathbb{Z}[q^{-1}][P]$ of graded characters $\mathrm{gch}\, V_y^-(\nu)$ for $y \in W_{\mathrm{af}}$ and $\nu \in P$. Also, Naito-Orr-Sagaki [14] obtained a similar description of the graded character $\mathrm{gch}\, V_x^-(\lambda-\mu)$ for $\lambda, \mu \in P^+$ such that $\lambda - \mu \in P^+$; note that in this case, the expansion is, in fact, a finite linear combination with coefficients in $\mathbb{Z}[q, q^{-1}][P]$. Recently, Kouno-Lenart-Naito [5] (cf. [13]) obtained an explicit description of the expansion, as an infinite linear combination with coefficients in $\mathbb{Z}[q, q^{-1}][P]$, of the graded character $\mathrm{gch}\, V_x^-(\lambda + \mu)$ for $\lambda \in P^+$ and an arbitrary $\mu \in P$ such that $\lambda + \mu \in P^+$.This identity is of the following form:

$$\mathrm{gch}\, V_x^-(\lambda + \mu) = \sum_{y \in W_{\mathrm{af}},\ \nu \in P} c_{x,\mu}^{y,\nu} e^{\nu}\, \mathrm{gch}\, V_y^-(\lambda), \tag{1.1}$$

where $c_{x,\mu}^{y,\nu} \in \mathbb{Z}[q, q^{-1}]$ for $y \in W_{\mathrm{af}}$ and $\nu \in P$, and e^{ν} for $\nu \in P$ denotes the (one-dimensional) character of H with weight ν. Here we should mention that the coefficients $c_{x,\mu}^{y,\nu}$ are independent of the weight $\lambda \in P$; also, for each $y \in W_{\mathrm{af}}$, the sum $\sum_{\nu \in P} c_{x,\mu}^{y,\nu} e^{\nu}$ is an element of $\mathbb{Z}[q, q^{-1}][P]$. This explicit identity is called *an identity of Chevalley type*.

Our main interest lies in an explicit description of the expansion of the product $e^{\nu}\, \mathrm{gch}\, V_x^-(\lambda)$ as a finite linear combination of the graded characters $\mathrm{gch}\, V_y^-(\lambda + \mu)$ for $y \in W_{\mathrm{af}}$ and $\mu \in P$; that is, an explicit description of the coefficients $d_{x,\nu}^{y,\mu}$ in the identity of the following form:

$$e^{\nu}\, \mathrm{gch}\, V_x^-(\lambda) = \sum_{y \in W_{\mathrm{af}},\ \mu \in P} d_{x,\nu}^{y,\mu}\, \mathrm{gch}\, V_y^-(\lambda + \mu), \tag{1.2}$$

where the coefficients $d_{x,\nu}^{y,\mu} \in \mathbb{Z}[q, q^{-1}]$ are independent of the weight $\lambda \in P$. In types A, D, E_6, E_7, Kouno-Naito-Orr-Sagaki [6] (for minuscule weights ν) and Lenart-Naito-Orr-Sagaki [12] (for arbitrary weights ν) gave an explicit description of the coefficients $d_{x,\nu}^{y,\mu}$ in the identity above; strictly speaking, the identities obtained in these works are ones in the equivariant K-group of the semi-infinite flag manifold $\mathbf{Q}_G$ associated to the connected, simply-connected and simple algebraic group G over $\mathbb{C}$ whose Lie algebra is $\mathfrak{g}$. In particular, these identities imply the following *finiteness* result: (i) the right-hand side of the identity (1.2) is a finite sum, and (ii) $d_{x,\nu}^{y,\mu} \in \mathbb{Z}[q, q^{-1}]$ for all $y \in W_{\mathrm{af}}$ and $\mu \in P$. Note that this finiteness result was obtained in simply-laced types by Orr [15], but the argument therein does not seem to work in non-simply-laced types. Since the identity (1.2) can be thought of as an “inverse expansion” of the identity (1.1), we call it *an identity of inverse Chevalley type*.

In this paper, we study identities of inverse Chevalley type in type C_n. We give an explicit description of the coefficients $d_{x,\nu}^{y,\mu}$ in the case that $\nu = v\varpi_1$ for $v \in W$, where ϖ_1 is the first fundamental weight. Note that the W-orbit of ϖ_1 is $\{\pm\varepsilon_k \mid k \in \{1, \dots, n\}\}$, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the standard basis of the weight lattice $P \cong \mathbb{Z}^n$; for any $v, w \in W$, there exists $m = 1, \dots, n$ such that $v\varpi_1 = w\varepsilon_m$ or $v\varpi_1 = -w\varepsilon_m$.

Now we are ready to state the main results of this paper; for the notation used in the following theorems, see Section 4.1. First, we state the “first half” of the desired identities of inverse Chevalley type.

Theorem 1.1 *(=Corollary 4.2) For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \dots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \dots, m$, there holds the following identity:*

$$
\begin{aligned}
&e^{w\varepsilon_m} \operatorname{gch} V_x^-(\lambda) \\
&= q^{\langle \varepsilon_m, \xi \rangle} \sum_{B \in \mathcal{A}(w, \Gamma_m(m))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\xi}}(\lambda + \varepsilon_m) \\
&\quad + \sum_{j=1}^{m-1} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{m,j}} \sum_{A_1 \in \mathcal{A}_w^{m,j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle \varepsilon_j, \mathrm{down}(A_1)+\cdots+\mathrm{down}(A_r)+\xi \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\mathrm{down}(A_1)+\cdots+\mathrm{down}(A_r)+\xi}}(\lambda + \varepsilon_j).
\end{aligned}
$$

Note that x and w in Theorem 1.1 are related as $x = wt_\xi$, but m is arbitrary. Next, we state the “second half” of the desired identities of inverse Chevalley type.

Theorem 1.2 *(=Corollary 4.4) For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \dots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \dots, n$ and $\lambda - \varepsilon_k \in P^+$ for $k = m+1, \dots, n$, there holds the following identity:*

$$
\begin{aligned}
&e^{-w\varepsilon_m} \operatorname{gch} V_x^-(\lambda) \\
&= q^{-\langle \varepsilon_m, \xi \rangle} \sum_{B \in \mathcal{A}(w, \Theta_m)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\xi}}(\lambda - \varepsilon_m) \\
&\quad + \sum_{j=m+1}^{n} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{\overline{m},\overline{j}}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle \varepsilon_j, \mathrm{down}(A_1)+\cdots+\mathrm{down}(A_r)+\xi \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r), \Theta_j)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\mathrm{down}(A_1)+\cdots+\mathrm{down}(A_r)+\xi}}(\lambda - \varepsilon_j) \\
&\quad + \sum_{j=1}^{n} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{\overline{m},j}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle \varepsilon_j, \mathrm{down}(A_1)+\cdots+\mathrm{down}(A_r)+\xi \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\mathrm{down}(A_1)+\cdots+\mathrm{down}(A_r)+\xi}}(\lambda + \varepsilon_j).
\end{aligned}
$$

Note that the proofs of Theorems 1.1 and 1.2 begin with auxiliary identities (Propositions 5.1 and 5.2) derived directly from a special case of the Chevalley formula given by Proposition 3.5. Also, observe that from the description of these identities, the finiteness result (i), (ii) mentioned above immediately follows, since every weight $\lambda \in P$ can be written as a $\mathbb{Z}$-linear combination of $\varepsilon_1, \dots, \varepsilon_n$.

Furthermore, we give cancellation-free identities of inverse Chevalley type in the "first-half" case, i.e., in the case $\nu = v\varpi_1 = w\varepsilon_1, \ldots, w\varepsilon_n$. The precise statement is as follows; in the following theorem, $\mathbf{p}_{m,j}(w)$ denotes a suitable directed path in the quantum Bruhat graph (for the definitions, see Section 4.2).

Theorem 1.3 *(= Corollary 4.6) For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, m$, there holds the following cancellation-free identity:*

$$\begin{aligned} & e^{w\varepsilon_m} \operatorname{gch} V_x^-(\lambda) \\ & = q^{\langle \varepsilon_m, \xi \rangle} \sum_{B \in \mathcal{A}(w, \Gamma_m(m))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\xi}}(\lambda + \varepsilon_m) \\ & \quad + \sum_{j=1}^{m-1} q^{\langle \varepsilon_j, \mathrm{wt}(\mathbf{p}_{m,j}(w)) + \xi \rangle} \sum_{B \in \mathcal{A}(\mathrm{end}(\mathbf{p}_{m,j}(w)), \Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \mathrm{wt}(\mathbf{p}_{m,j}(w)) + \xi}}(\lambda + \varepsilon_j). \end{aligned}$$

As for the "second-half" case, i.e., the case $\nu = v\varpi_1 = -w\varepsilon_1, \ldots, -w\varepsilon_n$, we provide conjectural cancellation-free identities of inverse Chevalley type in Section 4.3.

As an application of our identities of inverse Chevalley type, we can prove a formula for equivariant scalar multiplication (i.e., multiplication with the one-dimensional character e^ν, $\nu \in P$, of H) in the $(H \times \mathbb{C}^*)$-equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$ associated to G. To be more precise, let $\mathbf{Q}_G^{\mathrm{rat}}$ denote the semi-infinite flag manifold associated to G, that is, a reduced ind-scheme of infinite type whose set of $\mathbb{C}$-valued points is $G(\mathbb{C}((z)))/(H(\mathbb{C}) \cdot N(\mathbb{C}((z))))$ (see [8] for details), where $H \subset G$ is a maximal torus with Lie algebra $\mathfrak{h}$ and N is the unipotent radical of a Borel subgroup $B \supset H$. For $\lambda \in P$, there exists a line bundle on $\mathbf{Q}_G^{\mathrm{rat}}$ associated to λ; we denote by $\mathcal{O}(\lambda)$ the sheaf corresponding to this line bundle. Also, there exist semi-infinite Schubert varieties $\mathbf{Q}_G(x)$ for $x \in W_{\mathrm{af}}$, which are subvarieties of $\mathbf{Q}_G^{\mathrm{rat}}$; note that $\mathbf{Q}_G = \mathbf{Q}_G(e)$, with $e \in W_{\mathrm{af}}$ the identity element. The equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ is defined to be the $\mathbb{Z}[q, q^{-1}][P]$-submodule of (the Laurent series, in q^{-1}, extension of) the Iwahori-equivariant K-group $K'_{\mathbf{I} \rtimes \mathbb{C}^*}(\mathbf{Q}_G)$, introduced in [7], consisting of all "convergent" (possibly infinite) linear combinations with coefficients in $\mathbb{Z}[q, q^{-1}][P]$ of the semi-infinite Schubert classes $[\mathcal{O}_{\mathbf{Q}_G(x)}]$, $x \in W_{\mathrm{af}}^{\geq 0} := \{wt_\xi \in W_{\mathrm{af}} \mid w \in W, \xi \in Q^{\vee,+}\}$, where "convergence" holds in the sense of [7, Proposition 5.11]; here, $Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee$ denotes the positive part of the coroot lattice $Q^\vee = \sum_{i \in I} \mathbb{Z} \alpha_i^\vee$.

Now, following [14, Sect. 9], we recall how the graded characters of level-zero Demazure submodules over the quantum affine algebra $U_\mathsf{q}(\mathfrak{g}_{\mathrm{af}})$ are related to the equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$. Let us define $\mathbb{C}[q, q^{-1}][P]$-modules $\mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P])$, $\mathrm{Fun}_P^{\mathrm{neg}}(\mathbb{C}((q^{-1}))[P])$, and $\mathrm{Fun}_P^{\mathrm{ess}}(\mathbb{C}((q^{-1}))[P])$ by

$$\begin{aligned} \mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P]) &:= \{f : P \to \mathbb{C}((q^{-1}))[P]\}, \\ \mathrm{Fun}_P^{\mathrm{neg}}(\mathbb{C}((q^{-1}))[P]) &:= \left\{ f \in \mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P]) \;\middle|\; \begin{array}{l} \text{there exists } \gamma \in P \text{ such} \\ \text{that } f(\mu) = 0 \text{ for all } \mu \in \\ \gamma + P^+ \end{array} \right\}, \\ \mathrm{Fun}_P^{\mathrm{ess}}(\mathbb{C}((q^{-1}))[P]) &:= \mathrm{Fun}_P(\mathbb{C}((q^{-1}))[P]) / \mathrm{Fun}_P^{\mathrm{neg}}(\mathbb{C}((q^{-1}))[P]). \end{aligned}$$

Then there exists an injective $\mathbb{Z}[q, q^{-1}][P]$-module homomorphism $\Phi : K_{H \times \mathbb{C}^*}(\mathbf{Q}_G) \to \mathrm{Fun}_P^{\mathrm{ess}}(\mathbb{C}((q^{-1}))[P])$ such that for the class $[\mathcal{E}] \in K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$ of a certain quasi-coherent sheaf $\mathcal{E}$ on $\mathbf{Q}_G$, the element $\Phi([\mathcal{E}]) \in \mathrm{Fun}_P^{\mathrm{ess}}(\mathbb{C}(\!(q^{-1})\!)[P])$ is given as:

$$P \to \mathbb{C}(\!(q^{-1})\!)[P],\ \lambda \mapsto \sum_{i=0}^{\infty} (-1)^i \ \mathrm{gch}\ H^i(\mathbf{Q}_G, \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{Q}_G}} \mathcal{O}(\lambda));$$

here, gch $H^i(\mathbf{Q}_G, \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{Q}_G}} \mathcal{O}(\lambda))$ for $i \geq 0$ is the graded character of the i-th cohomology group $H^i(\mathbf{Q}_G, \mathcal{E} \otimes_{\mathcal{O}_{\mathbf{Q}_G}} \mathcal{O}(\lambda))$, which is regarded as an $(H \times \mathbb{C}^*)$-module. Also, it is proved in [7] that we can take $\mathcal{E} = \mathcal{O}_{\mathbf{Q}_G(x)}$ for $x \in W_{\mathrm{af}}^{\geq 0}$, and that

$$\mathrm{gch}\ H^i(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G(x)} \otimes_{\mathcal{O}_{\mathbf{Q}_G}} \mathcal{O}(\lambda)) = \begin{cases} \mathrm{gch}\ V_x^-(-w_\circ\lambda) & \text{if } \lambda \in P^+ \text{ and } i = 0, \\ 0 & \text{otherwise;} \end{cases}$$

where $w_\circ$ denotes the longest element of W. By making use of these results, we can translate an identity for graded characters of level-zero Demazure submodules into one for the $(H \times \mathbb{C}^*)$-equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$. Namely, if we have a finite sum of the form (1.2), then we obtain the following identity in $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$:

$$e^\nu \cdot [\mathcal{O}_{\mathbf{Q}_G(x)}] = \sum_{y \in W_{\mathrm{af}},\ \mu \in P} d_{x,\nu}^{y,\mu} [\mathcal{O}_{\mathbf{Q}_G(y)} \otimes_{\mathcal{O}_{\mathbf{Q}_G}} \mathcal{O}(-w_\circ\mu)].$$

In particular, our identities for graded characters of inverse Chevalley type yield explicit identities for the $(H \times \mathbb{C}^*)$-equivariant K-group $K_{H \times \mathbb{C}^*}(\mathbf{Q}_G)$, which we call *inverse Chevalley forumlas*.

In addition, by the specialization at $q = 1$ (of the coefficients $d_{x,\nu}^{y,\mu}$), we obtain corresponding inverse Chevalley formulas for equivariant scalar multiplication in the H-equivariant K-group $K_H(\mathbf{Q}_G)$ of the semi-infinite flag manifold $\mathbf{Q}_G$. Here we mention that in [4], Kato established a $\mathbb{Z}[P]$-module isomorphism from $K_H(\mathbf{Q}_G)$ onto the (small) H-equivariant quantum K-theory $QK_H(G/B) = K_H(G/B) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[P][\![Q^{\vee,+}]\!]$ of the finite-dimensional flag manifold G/B which sends each semi-infinite Schubert class to the corresponding (opposite) Schubert class, where $\mathbb{Z}[P][\![Q^{\vee,+}]\!]$ denotes the ring of formal power series with coefficients in $\mathbb{Z}[P]$ in the Novikov variables $Q_i = Q^{\alpha_i^\vee}$, $i \in I$. Through this $\mathbb{Z}[P]$-module isomorphism, we obtain inverse Chevalley formulas for equivariant scalar multiplication in $QK_H(G/B)$.

This paper is organized as follows. In Section 2, we fix our basic notation, and recall the definitions of the quantum Bruhat graph and quantum alcove model. In Section 3, we briefly recall the definition of level-zero Demazure submodules, and review identities of Chevalley type for their graded characters. In Section 4, we state identities of inverse Chevalley type. Also, we give the cancellation-free form of the first half of these identities. In Section 5, we prove our identities of inverse Chevalley type. In Section 6, we derive the cancellation-free form of our identities of inverse Chevalley type in the first-half case.

2 Basic setting

In this section, we fix basic notation, and review the definitions of the quantum Bruhat graph and quantum alcove model.

2.1 Lie algebras and root systems

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be the root system of $\mathfrak{g}$, $\Delta^+ \subset \Delta$ the set of positive roots, and $\{\alpha_i\}_{i \in I} \subset \Delta^+$ the simple roots. We denote by $\langle \cdot, \cdot \rangle$ the canonical pairing $\mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$. For $\alpha \in \Delta$, we define $\mathrm{sgn}(\alpha) \in \{1, -1\}$ as

$$\mathrm{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \Delta^+, \\ -1 & \text{if } \alpha \in -\Delta^+, \end{cases}$$

and set $|\alpha| := \mathrm{sgn}(\alpha)\alpha \in \Delta^+$.

For $\alpha \in \Delta$, we denote by $\alpha^\vee \in \mathfrak{h}$ the coroot corresponding to α, and define the fundamental weights ϖ_i, $i \in I$, by $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for $i, j \in I$. Let $P := \sum_{i \in I} \mathbb{Z}\varpi_i$ be the weight lattice, $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ the root lattice, and $Q^\vee := \sum_{i \in I} \mathbb{Z}\alpha_i^\vee$ the coroot lattice. Elements of $P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i (\subset P)$ are called dominant weights. We denote by $\mathbb{Z}[P] := \sum_{\lambda \in P} \mathbb{Z}e^\lambda$ the group algebra of P, where $\{e^\lambda \mid \lambda \in P\}$ is a formal basis with relations $e^\lambda e^\mu = e^{\lambda+\mu}$. Note that if G is the connected, simply-connected and simple algebraic group over $\mathbb{C}$ whose Lie algebra is $\mathfrak{g}$, then the element e^λ for $\lambda \in P$ also denotes the one-dimensional representation (character) of the maximal torus H of G of weight λ. In particular, $\mathbb{Z}[P]$ is isomorphic to the representation ring $R(H)$ of the torus H.

For $\alpha \in \Delta$, we define the reflection $s_\alpha \in GL(\mathfrak{h}^*)$ by $s_\alpha(\lambda) := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, $\lambda \in \mathfrak{h}^*$. In particular, the reflection $s_i := s_{\alpha_i}$ for $i \in I$ is called a simple reflection. The Weyl group W is defined as the subgroup of $GL(\mathfrak{h}^*)$ generated by $\{s_i\}_{i \in I}$, i.e., $W = \langle s_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$.

2.2 Type *C* root system

We review the standard realization of the root system of type C. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathbb{R}^n$. Then, the set

$$\Delta = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\} \sqcup \{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \leq i < j \leq n\} \sqcup \{\pm 2\varepsilon_k \mid 1 \leq k \leq n\}$$

forms the root system of type C_n, and

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\} \sqcup \{\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n\} \sqcup \{2\varepsilon_k \mid 1 \leq k \leq n\}$$

is the set of all positive roots. In particular, α_i, $i = 1, \ldots, n$, defined by

$$\alpha_i := \varepsilon_i - \varepsilon_{i+1}, \ 1 \leq i \leq n-1, \quad \alpha_n := 2\varepsilon_n$$

are the simple roots.

For $1 \leq i < j \leq n$, we set

$$(i, j) := \varepsilon_i - \varepsilon_j, \quad (i, \overline{j}) := \varepsilon_i + \varepsilon_j, \quad (i, \overline{i}) := 2\varepsilon_i.$$

The Weyl group W of type C_n is realized as a subgroup of the group of permutations of the set $[\overline{n}] := \{1, 2, \ldots, n, \overline{n}, \overline{n-1}, \ldots, \overline{1}\}$ by identifying simple reflections $s_1, \ldots, s_{n-1}, s_n$ with transpositions $(1\ 2), \ldots, (n-1\ n), (n\ \overline{n})$, respectively.

2.3 The quantum Bruhat graph

The quantum Bruhat graph is a labeled directed graph on the Weyl group W, introduced by Brenti-Fomin-Postnikov [1].

Definition 2.1 ([1, Definition 6.1]) The *quantum Bruhat graph* QBG(W) is the Δ^+-labeled directed graph whose vertex set is W, and whose edges are given as follows. For $x, y \in W$ and $\alpha \in \Delta^+$, we have a directed edge $x \xrightarrow{\alpha} y$ if $y = xs_\alpha$, and either of the following holds: (B) $\ell(y) = \ell(x) + 1$, or (Q) $\ell(y) = \ell(x) - 2\langle \rho, \alpha^\vee \rangle + 1$, where $\rho := (1/2)\sum_{\alpha \in \Delta^+} \alpha$. If the condition (B) (resp., (Q)) holds, then the corresponding edge $x \xrightarrow{\alpha} y$ is called a *Bruhat edge* (resp., *quantum edge*).

For a directed path $\mathbf{p} : w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} w_r$ in QBG(W), we define $\mathrm{wt}(\mathbf{p}) \in Q^\vee$ by

$$\mathrm{wt}(\mathbf{p}) := \sum_{\substack{1 \le k \le r \\ w_{k-1} \xrightarrow{\gamma_k} w_k \text{ is a quantum edge}}} \gamma_k^\vee .$$

2.4 The quantum alcove model

We briefly review the theory of quantum alcove model, first introduced by Lenart-Lubovsky [10], and then generalized by [13]. We set $\mathfrak{h}_\mathbb{R}^* := P \otimes_\mathbb{Z} \mathbb{R}$. For $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we define a hyperplane $H_{\alpha,k}$ in $\mathfrak{h}_\mathbb{R}^*$ by

$$H_{\alpha,k} := \{\xi \in \mathfrak{h}_\mathbb{R}^* \mid \langle \xi, \alpha^\vee \rangle = k\};$$

we denote by $s_{\alpha,k}$ the reflection for the hyperplane $H_{\alpha,k}$. Connected components of the space $\mathfrak{h}_\mathbb{R}^* \setminus \bigcup_{\alpha \in \Delta,\ k \in \mathbb{Z}} H_{\alpha,k}$ are called *alcoves*. Two alcoves A, B are said to be *adjacent* if the closures of A and B have an intersection, called a *common wall*.

Definition 2.2 ([11, Definition 5.2]) A sequence $(A_0, A_1, \dots, A_r)$ of alcoves $A_0, \dots, A_r$ is called an *alcove path* if A_{i-1} and A_i are adjacent for each $i = 1, \dots, r$. An alcove path $\Gamma = (A_0, \dots, A_r)$ is called *reduced* if Γ has a minimal length r among all alcove paths from A_0 to A_r.

For adjacent alcoves A, B, and a root $\alpha \in \Delta$, we write $A \xrightarrow{\alpha} B$ if the common wall of A and B is contained in the hyperplane $H_{\alpha,k}$ for some $k \in \mathbb{Z}$, and α points in a direction from A to B (as a direction vector). We take a special alcove $A_\circ$, called the *fundamental alcove*, defined by

$$A_\circ := \{\xi \in \mathfrak{h}_\mathbb{R}^* \mid 0 < \langle \xi, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+\}.$$

For $\lambda \in P$, we define A_λ by

$$A_\lambda := A_\circ + \lambda = \{\xi + \lambda \mid \xi \in A_\circ\}.$$

Definition 2.3 ([11, Definition 5.4]) Let $\lambda \in P$. A sequence $\Gamma = (\gamma_1, \dots, \gamma_r)$ of roots $\gamma_1, \dots, \gamma_r \in \Delta$ is called a *λ-chain* if there exists an alcove path $(A_\circ = A_0, A_1, \dots, A_r = A_{-\lambda})$ such that

$$A_\circ = A_0 \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} \cdots \xrightarrow{-\gamma_r} A_r = A_{-\lambda}.$$

We say that Γ is *reduced* if the corresponding alcove path $(A_0, \dots, A_r)$ is reduced.

Let Γ be a sequence of roots, i.e., $\Gamma = (\gamma_1, \dots, \gamma_r)$, with $\gamma_k \in \Delta$, $k = 1, \dots, r$.

Definition 2.4 ([13, Definition 17]) Let $w \in W$. A subset $A = \{i_1 < \cdots < i_s\} \subset \{1, \dots, r\}$ is said to be *w-admissible* if

$$w = w_0 \xrightarrow{|\gamma_{i_1}|} w_1 \xrightarrow{|\gamma_{i_2}|} \cdots \xrightarrow{|\gamma_{i_s}|} w_s$$

is a directed path in QBG(W). In this case, we define end(A) by end$(A) := w_s$. Also, we set

$$A^- := \{k \in A \mid \text{the edge } w_{k-1} \xrightarrow{|\gamma_k|} w_k \text{ is a quantum edge}\},$$

and then define down(A) by

$$\mathrm{down}(A) := \sum_{k \in A^-} |\gamma_k|^\vee.$$

Also, we set

$$n(A) := \#\{k \in A \mid \gamma_k \in -\Delta^+\}.$$

We denote by $\mathcal{A}(w, \Gamma)$ the set of all w-admissible subsets.

Let $\Gamma_1, \dots, \Gamma_r$ be sequences of roots, and $w \in W$. For a tuple $(A_1, A_2 \dots, A_r)$ of admissible subsets $A_1 \in \mathcal{A}(w, \Gamma_1)$, $A_2 \in \mathcal{A}(\mathrm{end}(A_1), \Gamma_2), \dots, A_r \in \mathcal{A}(\mathrm{end}(A_{r-1}), \Gamma_r)$, we set

$$\mathrm{down}(A_1, A_2, \dots, A_r) := \mathrm{down}(A_1) + \mathrm{down}(A_2) + \cdots + \mathrm{down}(A_r).$$

If Γ is a λ-chain for some $\lambda \in P$, then we can consider additional statistics denoted by wt and height. For a λ-chain $\Gamma = (\gamma_1, \dots, \gamma_r)$ with $\lambda \in P$, let $(A_\circ = A_0, \dots, A_r = A_{-\lambda})$ be the alcove path corresponding to Γ, and take integers $l_k \in \mathbb{Z}$, $k = 1, \dots, r$, such that the common wall of adjacent alcoves A_{k-1} and A_k is contained in the hyperplane $H_{\gamma_k, -l_k}$. Then, we define wt(A) and height(A) for $A = \{i_1 < \cdots < i_s\}$ by

$$\mathrm{wt}(A) := -w s_{\gamma_{i_1}, -l_{i_1}} \cdots s_{\gamma_{i_s}, -l_{i_s}}(-\lambda), \quad \mathrm{height}(A) := \sum_{k \in A^-} \mathrm{sgn}(\gamma_k)(\langle \lambda, \gamma_k^\vee \rangle - l_k).$$

2.5 Specific chains of roots

In this subsection, we deal with the root system of type C_n. We choose specific $(-\varpi_{k-1}+\varpi_k)$-chain and $(\varpi_{k-1} - \varpi_k)$-chain, which will play a crucial role in this paper; we understand that $\varpi_0 = 0$. Note that $-\varpi_{k-1} + \varpi_k = \varepsilon_k$. We set

$$\begin{aligned}
\Gamma_k(k) := (&-(1, \overline{k}), \dots, -(k-1, \overline{k}), \\
&-(k, \overline{k+1}), \dots, -(k, \overline{n}), \\
&-(k, \overline{k}), \\
&-(k, n), \dots, -(k, k+1)), \\
\Gamma_k^*(k) := (&(k, k+1), \dots, (k, n), \\
&(k, \overline{k}), \\
&(k, \overline{n}), \dots, (k, \overline{k+1}), \\
&(k-1, \overline{k}), \dots, (1, \overline{k})), \\
\Theta_k := (&-(1, k), \dots, -(k-1, k)), \\
\Theta_k^* := (&(k-1, k), \dots, (1, k)).
\end{aligned}$$

For sequences $\Gamma = (\gamma_1, \dots, \gamma_r)$, $\Xi = (\xi_1, \dots, \xi_s)$ of roots, we denote by $\Gamma * \Xi$ the concatenation of Γ and Ξ, i.e., $\Gamma * \Xi := (\gamma_1, \dots, \gamma_r, \xi_1, \dots, \xi_s)$.

Lemma 2.5 *The concatenation $\Gamma_{k-1,k} := \Gamma_k^*(k) * \Theta_k$ is a reduced $(-\varpi_{k-1} + \varpi_k)$-chain.*

Proof We set $x := s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1$, $y := s_1 \cdots s_{k-1}$, and $\mu := \varpi_1$. Then, x is a minimal-length representative for the coset xW_μ, where $W_\mu := \{w \in W \mid w\mu = \mu\}$, yx is the minimal-length representative for the coset $\{w \in W \mid w\mu = w_\circ \mu\}$, and $x\mu = -(-\varpi_{k-1} + \varpi_k)$. Now, following [12, Lemma 4.1], we define Γ as follows. Let us write $x = s_{j_a} \cdots s_{j_1}$, $y = s_{i_1} \cdots s_{i_b}$, and set

$$\beta_c := s_{j_a} \cdots s_{j_{c+1}} \alpha_{j_c}, \quad 1 \le c \le a,$$
$$\zeta_d := s_{i_b} \cdots s_{i_{d+1}} \alpha_{i_d}, \quad 1 \le d \le b.$$

Then we define Γ as $\Gamma := (\beta_1, \dots, \beta_a, -\gamma_1, \dots, -\gamma_b)$; note that the convention for the sign of roots in alcove paths in this paper is different from that of [12]. By direct calculation, we see that this Γ is identical to $\Gamma_{k-1,k}$. Since μ is a minuscule fundamental weight, the argument in the proof of [12, Lemma 4.1] still works in our setting of the type C root system, and hence we obtain the following reduced alcove path Π from $A_\circ$ to $A_\circ + x\mu = A_\circ - (-\varpi_{k-1} + \varpi_k) = A_{-(-\varpi_{k-1}+\varpi_k)}$:

$$\Pi : A_\circ = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_a} A_a = B_0$$
$$\xrightarrow{\zeta_1} B_1 \xrightarrow{\zeta_2} \cdots \xrightarrow{\zeta_b} B_b = A_{-(-\varpi_{k-1}+\varpi_k)}.$$

Thus we have shown that $\Gamma_{k-1,k}$ is a reduced $(-\varpi_{k-1} + \varpi_k)$-chain corresponding to Π. This proves the lemma.

Remark 2.6 The proof of [12, Lemma 4.1] also shows that for $t = 1, \dots, a$, the common wall of the adjacent alcoves A_{t-1} and A_t in the above path Π is contained in the hyperplane $H_{\beta_t,0}$, while for $t = 1, \dots, b$, the common wall of the adjacent alcoves B_{t-1} and B_t is contained in the hyperplane $H_{\zeta_t,1}$.

By reversing the order of roots in Γ and negating all roots, we obtain a specific $(\varpi_{k-1} - \varpi_k)$-chain.

Corollary 2.7 *The concatenation* $\Gamma^*_{k-1,k} := \Theta^*_k * \Gamma_k(k)$ *is a reduced* $(\varpi_{k-1} - \varpi_k)$*-chain.*

3 Level-zero Demazure submodules over quantum affine algebras

We recall the definition of level-zero Demazure submodules over quantum affine algebras and their graded characters.

3.1 Notation for affine Lie algebras and quantum affine algebras

Let $\mathfrak{g}_{\mathrm{af}} := (\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the (untwisted) affine Lie algebra associated to $\mathfrak{g}$, where c is the canonical central element and d is the scaling element. We denote by $\mathfrak{h}_{\mathrm{af}}$ its Cartan subalgebra. Let $\langle \cdot, \cdot \rangle$ be the canonical pairing $\mathfrak{h}^*_{\mathrm{af}} \times \mathfrak{h}_{\mathrm{af}} \to \mathbb{C}$, where $\mathfrak{h}^*_{\mathrm{af}} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}_{\mathrm{af}}, \mathbb{C})$. We set $I_{\mathrm{af}} := I \sqcup \{0\}$. Then, the simple roots α_i, $i \in I \subsetneq I_{\mathrm{af}}$, of $\mathfrak{g}$ can be regarded as simple roots of $\mathfrak{g}_{\mathrm{af}}$. Let s_i, $i \in I_{\mathrm{af}}$, be the simple reflection corresponding to α_i. Let $W_{\mathrm{af}} := \langle s_i \mid i \in I_{\mathrm{af}} \rangle$ denote the (affine) Weyl group of $\mathfrak{g}_{\mathrm{af}}$. We know that $W_{\mathrm{af}} = \{w t_\xi \mid w \in W,\ \xi \in Q^\vee\} \simeq W \ltimes Q^\vee$, where t_ξ, $\xi \in Q^\vee$, is the translation element [2, Chapter 6].

Let $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ be the quantum affine algebra associated to $\mathfrak{g}_{\mathrm{af}}$, and denote by E_i, F_i, $i \in I_{\mathrm{af}} = I \sqcup \{0\}$, the Chevalley generators of $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$. Then, we define $U^-_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ as the subalgebra of $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ generated by $\{F_i \mid i \in I_{\mathrm{af}}\}$, i.e., $U^-_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}}) = \langle F_i \mid i \in I_{\mathrm{af}} \rangle$.

3.2 Extremal weight submodules and level-zero Demazure submodules

Definition 3.1 ([3, Definition 8.1.1]) Let M be an integrable $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$-module, and $\lambda \in P_{\mathrm{af}}$. An element $v \in M$ is called an *extremal weight vector of weight* λ if v is a weight vector of λ, and there exists a family $\{v_x \mid x \in W_{\mathrm{af}}\} \subset M$ of vectors such that

(1) $v_e = v$,
(2) for $i \in I_{\mathrm{af}}$ and $x \in W_{\mathrm{af}}$, if $\langle x\lambda, \alpha_i^\vee \rangle \geq 0$, then $E_i v_x = 0$ and $F_i^{(\langle x\lambda, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$, and
(3) for $i \in I_{\mathrm{af}}$ and $x \in W_{\mathrm{af}}$, if $\langle x\lambda, \alpha_i^\vee \rangle \leq 0$, then $F_i v_x = 0$ and $E_i^{(-\langle x\lambda, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$,

where $F_i^{(k)}$ and $E_i^{(k)}$, $k \geq 0$, denote the divided powers.

For $\lambda \in P_{\mathrm{af}}$, the *extremal weight module* of weight λ, denoted by $V(\lambda)$, is the integrable weight module over $U_{\mathsf{q}}(\mathfrak{g}_{\mathrm{af}})$ whose generator is a single element v_λ, and whose defining relation is that "v_λ is an extremal weight vector of weight λ"; for the precise definition of extremal weight modules, see [3, Proposition 8.2.2].

Let $\lambda \in P^+ \subset P_{\mathrm{af}}$, and $x \in W_{\mathrm{af}}$. By the definition of extremal weight vectors, there exists a family $\{v_x \mid x \in W_{\mathrm{af}}\} \subset V(\lambda)$ of vectors satisfying the conditions in Definition 3.1, with $v_e = v_\lambda$. The *level-zero Demazure submodule* $V_x^-(\lambda)$ for $x \in W_{\mathrm{af}}$ is a $U_{\mathsf{q}}^-(\mathfrak{g}_{\mathrm{af}})$-submodule of $V(\lambda)$ generated by v_x, i.e., $V_x^-(\lambda) = U_{\mathsf{q}}^-(\mathfrak{g}_{\mathrm{af}}) v_x$. For $\nu \in P_{\mathrm{af}}$, we denote by $V_x^-(\lambda)_\nu$ the weight space of $V_x^-(\lambda)$ of weight $\nu \in P_{\mathrm{af}}$. Then we have the following weight space decomposition with respect to $\mathfrak{h}_{\mathrm{af}}$:

$$V_x^-(\lambda) = \bigoplus_{\gamma \in Q,\ k \in \mathbb{Z}} V_x^-(\lambda)_{\lambda+\gamma+k\delta},$$

where each weight space $V_x^-(\lambda)_{\lambda+\gamma+k\delta}$, $\gamma \in Q$, $k \in \mathbb{Z}$, is a finite-dimensional $\mathbb{C}(\mathsf{q})$-vector space; here, δ denotes the (primitive) null root of $\mathfrak{g}_{\mathrm{af}}$. Now we define the *graded character* of $V_x^-(\lambda)$ by

$$\mathrm{gch}\, V_x^-(\lambda) := \sum_{\gamma \in Q,\ k \in \mathbb{Z}} \dim(V_x^-(\lambda)_{\lambda+\gamma+k\delta}) q^k e^{\lambda+\gamma} \in \mathbb{Z}[P](\!(q^{-1})\!),$$

where q is an indeterminate (not to be confused with q).

The following identity is useful to compute the graded characters of level-zero Demazure submodules.

Proposition 3.2 *([7, Proposition D.1]) Let $x \in W_{\mathrm{af}}$ and $\lambda \in P^+$. For $\xi \in Q^\vee$, we have*

$$\mathrm{gch}\, V_{xt_\xi}^-(\lambda) = q^{-\langle \lambda, \xi \rangle}\, \mathrm{gch}\, V_x^-(\lambda).$$

3.3 Identities of Chevalley type

Let $\lambda \in P^+$ and $x \in W_{\mathrm{af}}$. We consider the graded character $\mathrm{gch}\, V_x^-(\lambda + \mu)$, where $\mu \in P$ is such that $\lambda + \mu \in P^+$. For this, we need to introduce more notation. A *partition* is a weakly decreasing sequence $\chi = (\chi_1 \geq \cdots \geq \chi_l)$ of positive integers $\chi_1, \ldots, \chi_l \in \mathbb{Z}_{>0}$; we call l the *length* of χ. Also, we set $|\chi| := \chi_1 + \cdots + \chi_l$, the *size* of χ. If $\chi = \emptyset$, the empty partition, then we set $\ell(\chi) := 0$, and $|\chi| := 0$. Let $\mu \in P$, and write $\mu = \sum_{i \in I} m_i \varpi_i$. We define a set $\overline{\mathrm{Par}}(\mu)$ as follows:

$$\overline{\mathrm{Par}}(\mu) := \{\boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \mid \chi^{(i)},\ i \in I, \text{ are partitions such that } \ell(\chi^{(i)}) \leq \max\{m_i, 0\}\}.$$

For $\boldsymbol{\chi} = (\chi^{(i)})_{i \in I} \in \overline{\mathrm{Par}}(\mu)$, we write $\chi^{(i)} = (\chi_1^{(i)} \geq \cdots \geq \chi_{l_i}^{(i)})$, where $\chi_1^{(i)}, \ldots, \chi_{l_i}^{(i)} \in \mathbb{Z}_{>0}$, with $l_i = \ell(\chi^{(i)})$. We set

$$|\boldsymbol{\chi}| := \sum_{i \in I} |\chi^{(i)}|, \quad \iota(\boldsymbol{\chi}) := \sum_{i \in I} \chi_1^{(i)} \alpha_i^\vee \in Q^{\vee,+},$$

where, if $\chi^{(i)} = \emptyset$, then we set $\chi_1^{(i)} := 0$.

Lenart-Naito-Sagaki [13] and Kouno-Lenart-Naito [5] proved the following identity, called *the identity of Chevalley type*.

Theorem 3.3 *([13, Theorem 33] and [5, Theorem 5.16]) Let $\lambda \in P^+$, $\mu \in P$, and $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$. Assume that $\lambda + \mu \in P^+$. Take a reduced μ-chain Γ. Then, there holds the following identity:*

$$\begin{aligned} &\mathrm{gch}\, V_x^-(\lambda + \mu) \\ &= \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\mu)} (-1)^{n(A)} q^{-\mathrm{height}(A) - \langle \lambda, \xi \rangle - |\boldsymbol{\chi}|} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\xi + \mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\lambda). \end{aligned} \tag{3.1}$$

Remark 3.4 Strictly speaking, Lenart-Naito-Sagaki proved an identity, called the Chevalley formula, in the equivariant K-group of semi-infinite flag manifolds, which is essentially equivalent to (3.1).

Now, we consider the root system of type C_n, and apply the identity of Chevalley type above to the case that $\mu = -\varpi_{k-1} + \varpi_k = \varepsilon_k$, $k = 1, \ldots, n$, to obtain the following.

Proposition 3.5 *Let $2 \leq k \leq n$ and $\mu := \varepsilon_k = -\varpi_{k-1} + \varpi_k$. Take an arbitrary reduced μ-chain Γ. Let $w \in W$. For $\lambda \in P^+$ such that $\lambda + \mu \in P^+$, we have*

$$\mathrm{gch}\, V_w^-(\lambda + \mu) = \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\mathrm{height}(A)} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda).$$

Proof Since $\mu = -\varpi_{k-1} + \varpi_k$, we have

$$\overline{\mathrm{Par}}(\mu) = \{\boldsymbol{i}_k := (\emptyset, \ldots, \emptyset, \underset{k}{(i)}, \emptyset, \ldots, \emptyset) \mid i \geq 0\},$$

where $\emptyset$ denotes the empty partition (of length 0), which is also regard as (0). For $i \geq 0$, we have $|\boldsymbol{i}_k| = i$, and $\iota(\boldsymbol{i}_k) = i\alpha_k^\vee$. Therefore, by Theorem 3.3, we compute:

$$\begin{aligned} \mathrm{gch}\, V_w^-(\lambda + \mu) &= \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{\boldsymbol{\chi} \in \overline{\mathrm{Par}}(\mu)} (-1)^{n(A)} q^{-\mathrm{height}(A) - |\boldsymbol{\chi}|} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A) + \iota(\boldsymbol{\chi})}}(\lambda) \\ &= \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{i=0}^{\infty} (-1)^{n(A)} q^{-\mathrm{height}(A) - i} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A) + i\alpha_k^\vee}}(\lambda) \\ &= \sum_{A \in \mathcal{A}(w,\Gamma)} \sum_{i=0}^{\infty} (-1)^{n(A)} q^{-\mathrm{height}(A) - i} e^{\mathrm{wt}(A)} q^{-\langle \lambda, i\alpha_k^\vee \rangle}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda) \\ &= \sum_{i=0}^{\infty} q^{-i - \langle \lambda, i\alpha_k^\vee \rangle} \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\mathrm{height}(A)} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda) \\ &= \sum_{i=0}^{\infty} q^{-i \langle \lambda + \varpi_k, \alpha_k^\vee \rangle} \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\mathrm{height}(A)} e^{\mathrm{wt}(A)}\, \mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda) \end{aligned}$$

$$= \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{A \in \mathcal{A}(w,\Gamma)} (-1)^{n(A)} q^{-\operatorname{height}(A)} e^{\operatorname{wt}(A)} \operatorname{gch} V^-_{\operatorname{end}(A) t_{\operatorname{down}(A)}}(\lambda),$$

as desired; for the third equality, we have used Proposition 3.2. This proves the proposition.

4 Main Results

In this section, we give the precise statements of our identities of inverse Chevalley type in type C_n. First, we give identities in which some terms may cancel. Next, we describe the cancellations in the "first half" of these identities to obtain cancellation-free ones. Also, we propose a conjecture for the cancellations in the "second half" of these identities. In the rest of this paper, we assume that $\mathfrak{g}$ is of type C_n.

4.1 Identities of inverse Chevalley type

To give precise statements of our main results, we prepare additional notation. Let us define a total order $<$ on the set $[\overline{n}]$ by $1 < 2 < \cdots < n < \overline{n} < \overline{n-1} < \cdots < \overline{1}$. For $j, m \in [\overline{n}]$ with $j < m$, we define $\mathcal{S}_{m,j}$ to be the set of all strictly decreasing sequences of integers starting from m and ending at j, that is,

$$\mathcal{S}_{m,j} := \{(j_1, \ldots, j_r) \mid r \geq 1,\ j_1, \ldots, j_r \in [\overline{n}],\ m > j_1 > \cdots > j_r = j\}.$$

For $w \in W$ and $1 \leq l < k \leq n$, we set

$$\mathcal{A}_w^{k,l} := \{A \in \mathcal{A}(w, \Theta_k) \setminus \{\emptyset\} \mid \operatorname{end}(A)^{-1} w\varepsilon_k = \varepsilon_l\}.$$

Also, for $w \in W$, $k \in \{1, \ldots, n\}$, and $l < \overline{k}$, we set

$$\mathcal{A}_w^{\overline{k},l} := \{A \in \mathcal{A}(w, \Gamma_k(k)) \setminus \{\emptyset\} \mid \operatorname{end}(A)^{-1} w(-\varepsilon_k) = \varepsilon_l\},$$

where for simplicity, we write $\varepsilon_{\overline{m}} = -\varepsilon_m$ for $m \in \{1, \ldots, n\}$.

The following is the "first half" of identities of inverse Chevalley type (in which cancellations may occur).

Theorem 4.1 *For $w \in W$, $\lambda \in P^+$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, m$, there holds the following identity:*

$$\begin{aligned}
& e^{w\varepsilon_m} \operatorname{gch} V_w^-(\lambda) \\
&= \sum_{B \in \mathcal{A}(w, \Gamma_m(m))} (-1)^{|B|} \operatorname{gch} V^-_{\operatorname{end}(B) t_{\operatorname{down}(B)}}(\lambda + \varepsilon_m) \\
&\quad + \sum_{j=1}^{m-1} \sum_{(j_1,\ldots,j_r) \in \mathcal{S}_{m,j}} \sum_{A_1 \in \mathcal{A}_w^{m,j_1}} \cdots \sum_{A_r \in \mathcal{A}_{\operatorname{end}(A_{r-1})}^{j_{r-1},j_r}} (-1)^{|A_1| + \cdots + |A_r| - r} q^{\langle \varepsilon_j, \operatorname{down}(A_1,\ldots,A_r) \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\operatorname{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\operatorname{end}(B) t_{\operatorname{down}(A_1,\ldots,A_r,B)}}(\lambda + \varepsilon_j).
\end{aligned} \tag{4.1}$$

We give a proof of Theorem 4.1 in Section 5. By Proposition 3.2, we obtain the following identities for an arbitrary $x \in W_{\mathrm{af}}$ (not only for $x \in W$).

Corollary 4.2 *For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \dots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \dots, m$, there holds the following identity:*

$$\begin{aligned}
&e^{w\varepsilon_m} \operatorname{gch} V_x^-(\lambda) \\
&= q^{\langle \varepsilon_m, \xi \rangle} \sum_{B \in \mathcal{A}(w, \Gamma_m(m))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\xi}}(\lambda + \varepsilon_m) \\
&\quad + \sum_{j=1}^{m-1} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{m,j}} \sum_{A_1 \in \mathcal{A}_w^{m,j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle \varepsilon_j, \mathrm{down}(A_1,\dots,A_r)+\xi \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(A_1,\dots,A_r,B)+\xi}}(\lambda + \varepsilon_j).
\end{aligned}$$

The following theorem is the "second half" of identities of inverse Chevalley type.

Theorem 4.3 *For $w \in W$, $m = 1, \dots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \dots, n$ and $\lambda - \varepsilon_k \in P^+$ for $k = m+1, \dots, n$, there holds the following identity:*

$$\begin{aligned}
&e^{-w\varepsilon_m} \operatorname{gch} V_w^-(\lambda) \\
&= \sum_{B \in \mathcal{A}(w, \Theta_m)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)}}(\lambda - \varepsilon_m) \\
&\quad + \sum_{j=m+1}^{n} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{\overline{m},\overline{j}}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle \varepsilon_j, \mathrm{down}(A_1,\dots,A_r) \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r), \Theta_j)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda - \varepsilon_j) \\
&\quad + \sum_{j=1}^{n} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{\overline{m},j}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle \varepsilon_j, \mathrm{down}(A_1,\dots,A_r) \rangle} \\
&\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda + \varepsilon_j).
\end{aligned} \tag{4.2}$$

We give a proof of Theorem 4.3 in Section 5. Again, by Proposition 3.2, we obtain the following identities for an arbitrary $x \in W_{\mathrm{af}}$ (not only for $x \in W$).

Corollary 4.4 *For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \dots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \dots, n$ and $\lambda - \varepsilon_k \in P^+$ for $k = m+1, \dots, n$, there holds the following identity:*

$$\begin{aligned}
&e^{-w\varepsilon_m} \operatorname{gch} V_x^-(\lambda) \\
&= q^{-\langle \varepsilon_m, \xi \rangle} \sum_{B \in \mathcal{A}(w, \Theta_m)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\xi}}(\lambda - \varepsilon_m) \\
&\quad + \sum_{j=m+1}^{n} \sum_{(j_1,\dots,j_r) \in \mathcal{S}_{\overline{m},\overline{j}}} \sum_{A_1 \in \mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle \varepsilon_j, \mathrm{down}(A_1,\dots,A_r)+\xi \rangle}
\end{aligned}$$

$$\times \sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Theta_j)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)+\xi}}(\lambda-\varepsilon_j)$$

$$+\sum_{j=1}^{n} \sum_{(j_1,\dots,j_r)\in\mathcal{S}_{\overline{m},j}} \sum_{A_1\in\mathcal{A}_w^{\overline{m},j_1}} \cdots \sum_{A_r\in\mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)+\xi\rangle}$$

$$\times \sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)+\xi}}(\lambda+\varepsilon_j).$$

Here we should mention that all the sums on the right-hand side of Theorems 4.1 and 4.3, together with Corollaries 4.2 and 4.4, are indeed finite sums.

4.2 Cancellation-free identities of inverse Chevalley type in the first-half case

We consider cancellations of terms in the first-half identities of inverse Chevalley type. Let $w \in W$. Take $l, m \in \{1,\dots,n\}$ such that $l > m$. We define a directed path $\mathbf{p}_{l,m}(w)$ in $\mathrm{QBG}(W)$ inductively as follows:

(1) if $l-m=1$, then $\mathbf{p}_{l,m}(w) : w \xrightarrow{(l-1,l)} ws_{l-1}$;
(2) if $l-m>1$, then assume that $\mathbf{p}_{l',m'}(v)$ is defined for $v\in W$ and $l', m' \in \{1,\dots,n\}$ such that $0<l'-m'<l-m$. Take minimal $k\in\{m,\dots,l-1\}$ such that $w \xrightarrow{(k,l)} ws_{(k,l)}$. Then, since $k-m<l-m$, $\mathbf{p}_{k,m}(ws_{(k,l)})$ is defined. Let $\mathbf{p}_{l,m}(w)$ be the directed path obtained as the concatenation of the edge $w \xrightarrow{(k,l)} ws_{(k,l)}$ with the directed path $\mathbf{p}_{k,m}(ws_{(k,l)})$.

The following theorem gives the cancellation-free identities of inverse Chevalley type in the first-half case.

Theorem 4.5 *For $w\in W$, $m=1,\dots,n$, and $\lambda\in P^+$ such that $\lambda+\varepsilon_k\in P^+$ for all $k=1,\dots,m$, there holds the following cancellation-free identity:*

$$e^{w\varepsilon_m} \operatorname{gch} V_w^-(\lambda)$$
$$= \sum_{B\in\mathcal{A}(w,\Gamma_m(m))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_m)$$
$$+\sum_{j=1}^{m-1} q^{\langle\varepsilon_j,\mathrm{wt}(\mathbf{p}_{m,j}(w))\rangle} \sum_{B\in\mathcal{A}(\mathrm{end}(\mathbf{p}_{m,j}(w)),\Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{wt}(\mathbf{p}_{m,j}(w))}}(\lambda+\varepsilon_j).$$

We give a proof of this theorem in Section 6. Again, by using Proposition 3.2, we obtain the following cancellation-free identities for an arbitrary $x\in W_{\mathrm{af}}$ (not only for $x\in W$).

Corollary 4.6 *For $x = wt_\xi \in W_{\mathrm{af}}$ with $w\in W$ and $\xi\in Q^\vee$, $m=1,\dots,n$, and $\lambda\in P^+$ such that $\lambda+\varepsilon_k\in P^+$ for all $k=1,\dots,m$, there holds the following cancellation-free identity:*

$$e^{w\varepsilon_m} \operatorname{gch} V_x^-(\lambda)$$
$$= q^{\langle\varepsilon_m,\xi\rangle} \sum_{B\in\mathcal{A}(w,\Gamma_m(m))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\xi}}(\lambda+\varepsilon_m)$$
$$+\sum_{j=1}^{m-1} q^{\langle\varepsilon_j,\mathrm{wt}(\mathbf{p}_{m,j}(w))+\xi\rangle} \sum_{B\in\mathcal{A}(\mathrm{end}(\mathbf{p}_{m,j}(w)),\Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{wt}(\mathbf{p}_{m,j}(w))+\xi}}(\lambda+\varepsilon_j).$$

4.3 Conjectural cancellation-free identities of inverse Chevalley type in the second-half case

We propose a conjecture for the cancellations of terms in the second-half identities of inverse Chevalley type. We define a distance function $d(\cdot,\cdot)$ on $[\overline{n}]$ as follows:

(1) for $k \in [\overline{n}]$, set $d(k,k) := 0$;
(2) for $k, l \in [\overline{n}]$ with $k > l$, set

$$d(k,l) := \begin{cases} k-l & \text{if } 1 \le l < k \le n, \\ (2n+1-p)-l & \text{if } l = \overline{p} \text{ for some } 1 \le p \le n \text{ and } 1 \le l \le n, \\ q-p & \text{if } k = \overline{p} \text{ and } l = \overline{q} \text{ for some } 1 \le p \le q \le n; \end{cases}$$

(3) for $k, l \in [\overline{n}]$ with $k < l$, set $d(k,l) := d(l,k)$.

Also, for $1 \le l \le n$ and $1 \le k < \bar{l}$, we define $\gamma_{\bar{l},k} \in \Delta^+$ by

$$\gamma_{\bar{l},k} := \begin{cases} (k,\bar{l}) & \text{if } k = 1,\dots,l, \\ (l,\bar{k}) & \text{if } k = l+1,\dots,n, \\ (l,p) & \text{if } k = \overline{p} \text{ for some } p = l+1,\dots,n. \end{cases}$$

Now, we define a directed path $\mathbf{p}_{\bar{l},m}(w)$ in $\mathrm{QBG}(W)$ for $w \in W$, $1 \le l \le n$, and $1 \le m \le \overline{l+1}$ by induction on $d(\bar{l},m)$ as follows; we understand that $\overline{n+1} := n$.

(1) If $d(\bar{l},m) = 1$ (in this case, $\gamma_{\bar{l},m}$ is a simple root), then $\mathbf{p}_{\bar{l},m}(w) : w \xrightarrow{\gamma_{\bar{l},m}} ws_{\gamma_{\bar{l},m}}$.
(2) If $d(\bar{l},m) > 1$, then assume that $\mathbf{p}_{q,m'}(v)$ is defined for $v \in W$ and $q, m' \in [\overline{n}]$, with $q > m'$, such that $0 < d(q,m') < d(\bar{l},m)$; note that for $1 \le q \le n$, $\mathbf{p}_{q,m'}(v)$ is already defined in Section 4.2. Take minimal $k \in [\overline{n}]$, with $m \le k \le \overline{l+1}$, such that $w \xrightarrow{\gamma_{\bar{l},k}} ws_{\gamma_{\bar{l},k}}$. Then, since $0 < d(k,m) < d(\bar{l},m)$, $\mathbf{p}_{k,m}(ws_{\gamma_{\bar{l},k}})$ is defined. Let $\mathbf{p}_{\bar{l},m}(w)$ be the directed path obtained as the concatenation of the edge $w \xrightarrow{\gamma_{\bar{l},k}} ws_{\gamma_{\bar{l},k}}$ with the directed path $\mathbf{p}_{k,m}(ws_{\gamma_{\bar{l},k}})$.

Our conjectural cancellation-free identities in the second-half case are as follows.

Conjecture 4.7 For $w \in W$, $m = 1,\dots,n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1,\dots,n$ and such that $\lambda - \varepsilon_k \in P^+$ for $k = m+1,\dots,n$, there exists some $m \le l \le n$ for which the following cancellation-free identity holds:

$$\begin{aligned} &e^{-w\varepsilon_m} \operatorname{gch} V_w^-(\lambda) \\ &= \sum_{B \in \mathcal{A}(w,\Theta_m)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_m) \\ &\quad + \sum_{k=m+1}^{n} q^{-\langle \varepsilon_k, \mathrm{down}(\mathbf{p}_{\overline{m},\overline{k}}(w))\rangle} \\ &\quad \times \sum_{B \in \mathcal{A}(\mathrm{end}(\mathbf{p}_{\overline{m},\overline{k}}(w)),\Theta_k)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{down}(\mathbf{p}_{\overline{m},\overline{k}}(w))}}(\lambda - \varepsilon_k) \end{aligned}$$

$$+\sum_{k=1}^{l} q^{\langle \varepsilon_k, \mathrm{down}(\mathbf{p}_{\overline{m},k}(w))\rangle}$$
$$\times \sum_{B\in\mathcal{A}(\mathrm{end}(\mathbf{p}_{\overline{m},k}(w)),\Gamma_k(k))} (-1)^{|B|} \,\mathrm{gch}\, V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{down}(\mathbf{p}_{\overline{m},k}(w))}}(\lambda+\varepsilon_k).$$

Conjecture 4.8 For $x = wt_\xi \in W_{\mathrm{af}}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, n$ and such that $\lambda - \varepsilon_k \in P^+$ for $k = m+1, \ldots, n$, there exists some $m \le l \le n$ for which the following cancellation-free identity holds:

$$e^{-w\varepsilon_m}\,\mathrm{gch}\, V_x^-(\lambda)$$
$$= q^{-\langle\varepsilon_m,\xi\rangle} \sum_{B\in\mathcal{A}(w,\Theta_m)} (-1)^{|B|}\,\mathrm{gch}\, V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\xi}}(\lambda-\varepsilon_m)$$
$$+ \sum_{k=m+1}^{n} q^{-\langle\varepsilon_k, \mathrm{down}(\mathbf{p}_{\overline{m},\overline{k}}(w))+\xi\rangle}$$
$$\times \sum_{B\in\mathcal{A}(\mathrm{end}(\mathbf{p}_{\overline{m},\overline{k}}(w)),\Theta_k)} (-1)^{|B|}\,\mathrm{gch}\, V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{down}(\mathbf{p}_{\overline{m},\overline{k}}(w))+\xi}}(\lambda-\varepsilon_k)$$
$$+ \sum_{k=1}^{l} q^{\langle\varepsilon_k, \mathrm{down}(\mathbf{p}_{\overline{m},k}(w))+\xi\rangle}$$
$$\times \sum_{B\in\mathcal{A}(\mathrm{end}(\mathbf{p}_{\overline{m},k}(w)),\Gamma_k(k))} (-1)^{|B|}\,\mathrm{gch}\, V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{down}(\mathbf{p}_{\overline{m},k}(w))+\xi}}(\lambda+\varepsilon_k).$$

We expect that the l in the above conjectures is either m or n.

4.4 Examples of identities of inverse Chevalley type

We give an example of the identities given by Theorem 4.5, and also give two examples of the conjectural identities proposed by Conjecture 4.7; in these examples, we assume that $n = 3$. We used SageMath [17] for calculations in these examples.

Example 4.9 We consider the product $e^{w\varepsilon_3}\,\mathrm{gch}\, V_w^-(\lambda)$. Let $w = s_1s_2s_1$, and take $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for $k = 1, 2, 3$. We see that

$$\mathcal{S}_{3,1} = \{(1), (2, 1)\}, \quad \mathcal{S}_{3,2} = \{(2)\},$$

and that the admissible subsets which appear on the right-hand side of (4.1) are as follows:

$$\mathcal{A}^{3,1}_{s_1s_2s_1} = \{\underbrace{\{1\}}_{=:A_1}, \underbrace{\{1,2\}}_{=:A_2}\}, \quad \mathcal{A}^{3,2}_{s_1s_2s_1} = \{\underbrace{\{2\}}_{=:A_3}\}, \quad \mathcal{A}^{2,1}_{s_2s_1} = \{\underbrace{\{1\}}_{=:A_4}\}.$$

Table 1 is the list of end$(\cdot)$ and down$(\cdot)$ for admisible subsets A_1, A_2, A_3, A_4.

Therefore, by Theorem 4.1, we compute:

$$e^{s_1s_2s_1\varepsilon_3}\,\mathrm{gch}\, V^-_{s_1s_2s_1}(\lambda)$$
$$= \sum_{B\in\mathcal{A}(s_1s_2s_1,\Gamma_3(3))} (-1)^{|B|}\,\mathrm{gch}\, V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_3)$$

Table 1 The list of end(A) and down(A) for $A = A_1, A_2, A_3, A_4$

Admissible subsets	end$(\cdot)$	down$(\cdot)$
A_1	e	$\alpha_1^\vee + \alpha_2^\vee$
A_2	s_2	$\alpha_1^\vee + \alpha_2^\vee$
A_3	$s_2 s_1$	$\alpha_2^\vee$
A_4	s_2	$\alpha_1^\vee$

$$
\begin{aligned}
&\underbrace{+\, q^{\langle \varepsilon_2, \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(s_2 s_1, \Gamma_2(2))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_2^\vee}}(\lambda + \varepsilon_2)}_{A_3} \\
&\underbrace{+\, q^{\langle \varepsilon_1, \alpha_1^\vee + \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(e, \Gamma_1(1))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_1^\vee + \alpha_2^\vee}}(\lambda + \varepsilon_1)}_{A_1} \\
&\underbrace{-\, q^{\langle \varepsilon_1, \alpha_1^\vee + \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(s_2, \Gamma_1(1))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_1^\vee + \alpha_2^\vee}}(\lambda + \varepsilon_1)}_{A_2} \\
&\underbrace{+\, q^{\langle \varepsilon_1, \alpha_1^\vee + \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(s_2, \Gamma_1(1))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_1^\vee + \alpha_2^\vee}}(\lambda + \varepsilon_1)}_{(A_3, A_4)} \\
= &\sum_{B \in \mathcal{A}(s_1 s_2 s_1, \Gamma_3(3))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)}}(\lambda + \varepsilon_3) \\
&+ q^{\langle \varepsilon_2, \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(s_2 s_1, \Gamma_2(2))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_2^\vee}}(\lambda + \varepsilon_2) \\
&+ q^{\langle \varepsilon_1, \alpha_1^\vee + \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(e, \Gamma_1(1))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B) + \alpha_1^\vee + \alpha_2^\vee}}(\lambda + \varepsilon_1).
\end{aligned}
$$

The above result agrees with Theorem 4.5, since we have

$$
\mathbf{p}_{3,2}(s_1 s_2 s_1) : s_1 s_2 s_1 \xrightarrow{(2,3)} s_2 s_1, \quad \mathbf{p}_{3,1}(s_1 s_2 s_1) : s_1 s_2 s_1 \xrightarrow{(1,3)} e.
$$

Example 4.10 We consider the product $e^{-w\varepsilon_2} \operatorname{gch} V_w^-(\lambda)$. Let $w = s_3 s_2$, and take $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for $k = 1, 2, 3$ and such that $\lambda - \varepsilon_k \in P^+$ for $k = 2, 3$. We see that the sets $\mathcal{S}_{\overline{2},k}$ for $k = 1, 2, 3, \overline{3}$ are:

$$
\begin{aligned}
\mathcal{S}_{\overline{2},1} &= \{(1), (2, 1), (3, 1), (\overline{3}, 1), (3, 2, 1), (\overline{3}, 2, 1), (\overline{3}, 3, 1), (\overline{3}, 3, 2, 1)\}, \\
\mathcal{S}_{\overline{2},2} &= \{(2), (3, 2), (\overline{3}, 2), (\overline{3}, 3, 2)\}, \\
\mathcal{S}_{\overline{2},3} &= \{(3), (\overline{3}, 3)\}, \\
\mathcal{S}_{\overline{2},\overline{3}} &= \{(\overline{3})\},
\end{aligned}
$$

and that the admissible subsets which appear on the right-hand side of (4.2) are as follows:

$$\mathcal{A}^{\overline{2},1}_{s_3s_2} = \emptyset, \qquad \mathcal{A}^{\overline{2},2}_{s_3s_2} = \{\underbrace{\{2,4\}}_{=:A_1}, \underbrace{\{2,3,4\}}_{=:A_2}\}, \qquad \mathcal{A}^{\overline{2},3}_{s_3s_2} = \{\underbrace{\{2\}}_{=:A_3}, \underbrace{\{2,3\}}_{=:A_4}\}, \qquad \mathcal{A}^{\overline{2},\overline{3}}_{s_3s_2} = \{\underbrace{\{4\}}_{=:A_5}\},$$

$$\mathcal{A}^{\overline{3},1}_{s_3} = \emptyset, \qquad \mathcal{A}^{\overline{3},2}_{s_3} = \{\underbrace{\{2\}}_{=:A_6}, \underbrace{\{2,3\}}_{=:A_7}\}, \qquad \mathcal{A}^{\overline{3},3}_{s_3} = \{\underbrace{\{3\}}_{=:A_8}\},$$

$$\mathcal{A}^{3,1}_{e} = \emptyset, \qquad \mathcal{A}^{3,2}_{e} = \{\underbrace{\{2\}}_{=:A_9}\},$$

$$\mathcal{A}^{3,1}_{s_2s_3s_2} = \{\underbrace{\{1\}}_{=:A_{10}}, \underbrace{\{1,2\}}_{=:A_{11}}\}, \qquad \mathcal{A}^{3,2}_{s_2s_3s_2} = \{\underbrace{\{2\}}_{=:A_{12}}\},$$

$$\mathcal{A}^{2,1}_{s_2s_3} = \{\underbrace{\{1\}}_{=:A_{13}}\},$$

$$\mathcal{A}^{2,1}_{s_2} = \{\underbrace{\{1\}}_{=:A_{14}}\}.$$

Table 2 is the list of end$(\cdot)$ and down$(\cdot)$ for admissible subsets given above.

Therefore, by Theorem 4.3, we see that

$$\begin{aligned} &e^{-s_3s_2\varepsilon_2} \operatorname{gch} V^-_{s_3s_2}(\lambda) \\ &= \underbrace{\sum_{B \in \mathcal{A}(s_3s_2,\Theta_2)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_2) \\ &\quad + q^{-\langle \varepsilon_3, \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(s_3,\Theta_3)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\alpha_2^\vee}}(\lambda - \varepsilon_3)}_{A_5} \end{aligned}$$

Table 2 The list of end(A) and down(A) for $A = A_1, \dots, A_{14}$

Admissible subset	end$(\cdot)$	down$(\cdot)$
A_1	s_2s_3	$\alpha_2^\vee$
A_2	s_2	$\alpha_2^\vee + \alpha_3^\vee$
A_3	$s_2s_3s_2$	0
A_4	e	$\alpha_2^\vee + \alpha_3^\vee$
A_5	s_3	$\alpha_2^\vee$
A_6	s_2s_3	0
A_7	s_2	$\alpha_3^\vee$
A_8	e	$\alpha_3^\vee$
A_9	s_2	0
A_{10}	$s_2s_3s_1s_2$	0
A_{11}	$s_2s_3s_1$	$\alpha_2^\vee$
A_{12}	s_2s_3	$\alpha_2^\vee$
A_{13}	$s_2s_3s_1$	0
A_{14}	s_2s_1	0

$$\underbrace{+ \sum_{B \in \mathcal{A}(s_2 s_3 s_2, \Gamma_3(3))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)}}(\lambda + \varepsilon_3)}_{A_3}$$

$$\underbrace{+ q^{\langle \varepsilon_2, \alpha_2^\vee \rangle} \sum_{B \in \mathcal{A}(s_2 s_3, \Gamma_2(2))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)+\alpha_2^\vee}}(\lambda + \varepsilon_2)}_{(A_3, A_{12})}$$

$$\underbrace{+ \sum_{B \in \mathcal{A}(s_2 s_3 s_1 s_2, \Gamma_1(1))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)}}(\lambda + \varepsilon_1);}_{(A_3, A_{10})}$$

here, the other terms cancel out. The above result agrees with Conjecture 4.7 if we take $l = 3$, since we have

$$\mathbf{p}_{\overline{2},\overline{3}}(s_3 s_2) : s_3 s_2 \xrightarrow{(2,3)} s_3, \qquad \mathbf{p}_{\overline{2},3}(s_3 s_2) : s_3 s_2 \xrightarrow{(2,\overline{3})} s_2 s_3 s_2,$$

$$\mathbf{p}_{\overline{2},2}(s_3 s_2) : s_3 s_2 \xrightarrow{(2,\overline{3})} s_2 s_3 s_2 \xrightarrow{(2,3)} s_2 s_3, \quad \mathbf{p}_{\overline{2},1}(s_3 s_2) : s_3 s_2 \xrightarrow{(2,\overline{3})} s_2 s_3 s_2 \xrightarrow{(1,3)} s_2 s_3 s_1 s_2.$$

Example 4.11 We consider the product $e^{-w\varepsilon_1} \operatorname{gch} V_w^-(\lambda)$. Let $w = s_1 s_2 s_3 s_2 s_1$, and take $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for $k = 1, 2, 3$ and such that $\lambda - \varepsilon_k \in P^+$ for $k = 1, 2, 3$. We see that the sets $\mathcal{S}_{\overline{1},k}$ for $k = 1, 2, 3, \overline{3}, \overline{2}$ are:

$$\begin{aligned}
\mathcal{S}_{\overline{1},1} &= \{(1), (2, 1), (3, 1), (\overline{3}, 1), (\overline{2}, 1), (3, 2, 1), (\overline{3}, 2, 1), (\overline{2}, 2, 1), (\overline{3}, 3, 1), (\overline{2}, 3, 1), (\overline{2}, \overline{3}, 1),\\
&\qquad (\overline{3}, 3, 2, 1), (\overline{2}, 3, 2, 1), (\overline{2}, \overline{3}, 2, 1), (\overline{2}, \overline{3}, 3, 1), (\overline{2}, \overline{3}, 3, 2, 1)\},\\
\mathcal{S}_{\overline{1},2} &= \{(2), (3, 2), (\overline{3}, 2), (\overline{2}, 2), (\overline{3}, 3, 2), (\overline{2}, 3, 2), (\overline{2}, \overline{3}, 2), (\overline{2}, \overline{3}, 3, 2)\},\\
\mathcal{S}_{\overline{1},3} &= \{(3), (\overline{3}, 3), (\overline{2}, 3), (\overline{2}, \overline{3}, 3)\},\\
\mathcal{S}_{\overline{1},\overline{3}} &= \{(\overline{3}), (\overline{2}, \overline{3})\},\\
\mathcal{S}_{\overline{1},\overline{2}} &= \{(\overline{2})\},
\end{aligned}$$

and that the admissible subsets which appear on the right-hand side of (4.2) are as follows:

$$\mathcal{A}^{\overline{1},1}_{s_1 s_2 s_3 s_2 s_1} = \{\underbrace{\{3\}}_{=:A_1}\}, \quad \mathcal{A}^{\overline{1},2}_{s_1 s_2 s_3 s_2 s_1} = \{\underbrace{\{3, 5\}}_{=:A_2}\}, \quad \mathcal{A}^{\overline{1},3}_{s_1 s_2 s_3 s_2 s_1} = \emptyset,$$

$$\mathcal{A}^{\overline{1},\overline{3}}_{s_1 s_2 s_3 s_2 s_1} = \emptyset, \quad \mathcal{A}^{\overline{1},\overline{2}}_{s_1 s_2 s_3 s_2 s_1} = \{\underbrace{\{5\}}_{=:A_3}\},$$

$$\mathcal{A}^{\overline{2},1}_{s_1 s_2 s_3 s_2} = \emptyset, \quad \mathcal{A}^{\overline{2},2}_{s_1 s_2 s_3 s_2} = \{\underbrace{\{3\}}_{=:A_4}\}, \quad \mathcal{A}^{\overline{2},3}_{s_1 s_2 s_3 s_2} = \{\underbrace{\{3, 4\}}_{=:A_5}\},$$

$$\mathcal{A}^{\overline{2},\overline{3}}_{s_1 s_2 s_3 s_2} = \{\underbrace{\{4\}}_{=:A_6}\},$$

$$\mathcal{A}^{\overline{3},1}_{s_1 s_2 s_3} = \emptyset, \quad \mathcal{A}^{\overline{3},2}_{s_1 s_2 s_3} = \emptyset, \quad \mathcal{A}^{\overline{3},3}_{s_1 s_2 s_3} = \{\underbrace{\{3\}}_{=:A_7}\},$$

$$\mathcal{A}^{3,1}_{s_1s_2} = \emptyset, \qquad \mathcal{A}^{3,2}_{s_1s_2} = \{\underbrace{\{2\}}_{=:A_8}\},$$

$$\mathcal{A}^{2,1}_{s_1} = \{\underbrace{\{1\}}_{=:A_9}\}.$$

Table 3 is the list of $\mathrm{end}(\cdot)$ and $\mathrm{down}(\cdot)$ for admissible subsets given above.

Therefore, by Theorem 4.3, we see that

$$\begin{aligned}
& e^{-s_1s_2s_3s_2s_1\varepsilon_1}\,\mathrm{gch}\,V^-_{s_1s_2s_3s_2s_1}(\lambda) \\
&= \sum_{B\in\mathcal{A}(s_1s_2s_3s_2s_1,\Theta_1)} (-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_1) \\
&\quad \underbrace{+\,q^{-\langle\varepsilon_2,\alpha_1^\vee\rangle} \sum_{B\in\mathcal{A}(s_1s_2s_3s_2,\Theta_2)} (-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\alpha_1^\vee}}(\lambda-\varepsilon_2)}_{A_3} \\
&\quad \underbrace{+\,q^{-\langle\varepsilon_3,\alpha_1^\vee+\alpha_2^\vee\rangle} \sum_{B\in\mathcal{A}(s_1s_2s_3,\Theta_3)} (-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\alpha_1^\vee+\alpha_2^\vee}}(\lambda-\varepsilon_3)}_{(A_3,A_6)} \\
&\quad \underbrace{+\,q^{\langle\varepsilon_1,\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee\rangle} \sum_{B\in\mathcal{A}(e,\Gamma_1(1))} (-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee}}(\lambda+\varepsilon_1)}_{A_1};
\end{aligned}$$

here, the other terms cancel out. The above result agrees with Conjecture 4.7 if we take $l = 1$, since we have

$$\begin{aligned}
\mathbf{p}_{\bar{1},\bar{2}}(s_1s_2s_3s_2s_1) &: s_1s_2s_3s_2s_1 \xrightarrow{(1,2)} s_1s_2s_3s_2, \\
\mathbf{p}_{\bar{1},\bar{3}}(s_1s_2s_3s_2s_1) &: s_1s_2s_3s_2s_1 \xrightarrow{(1,2)} s_1s_2s_3s_2 \xrightarrow{(2,3)} s_1s_2s_3, \\
\mathbf{p}_{\bar{1},1}(s_1s_2s_3s_2s_1) &: s_1s_2s_3s_2s_1 \xrightarrow{(1,\bar{1})} e.
\end{aligned}$$

Table 3 The list of $\mathrm{end}(A)$ and $\mathrm{down}(A)$ for $A = A_1, \ldots, A_9$

A	$\mathrm{end}(A)$	$\mathrm{down}(A)$
A_1	e	$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$
A_2	s_1	$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$
A_3	$s_1s_2s_3s_2$	$\alpha_1^\vee$
A_4	s_1	$\alpha_2^\vee + \alpha_3^\vee$
A_5	s_1s_2	$\alpha_2^\vee + \alpha_3^\vee$
A_6	$s_1s_2s_3$	$\alpha_2^\vee$
A_7	s_1s_2	$\alpha_3^\vee$
A_8	s_1	$\alpha_2^\vee$
A_9	e	$\alpha_1^\vee$

5 Proofs of Theorems 4.1 and 4.3

We give proofs of our identities of inverse Chevalley type.

5.1 First-half case

First, we consider the first-half case. The following proposition is a key to the proof of the identities.

Proposition 5.1 *Let $w \in W$ and $\lambda \in P^+$ be such that $\lambda + \varepsilon_k \in P^+$. Then we have*

$$\sum_{B \in \mathcal{A}(w, \Gamma_k(k))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)}}(\lambda + \varepsilon_k) = \sum_{A \in \mathcal{A}(w, \Theta_k)} (-1)^{|A|} e^{w\varepsilon_k} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda). \tag{5.1}$$

Proof In this proof,

(1) the sequence $\Gamma_k(k)$ is of the form:

$$\Gamma_k(k) = (\beta_{1,\overline{k}}, \ldots, \beta_{k-1,\overline{k}}, \beta_{k,\overline{k+1}}, \ldots, \beta_{k,\overline{n}}, \beta_{k,\overline{k}}, \beta_{k,n}, \ldots, \beta_{k,k+1}),$$

(2) admissible subsets $B \in \mathcal{A}(v, \Gamma_k(k))$ for $v \in W$ are subsets of the (totally ordered) index set

$$I_1 := \{(1, \overline{k}) \lhd \cdots \lhd (k-1, \overline{k}) \lhd (k, \overline{k+1}) \lhd \cdots \lhd (k, \overline{n}) \lhd (k, \overline{k}) \lhd (k, n) \lhd \cdots \lhd (k, k+1)\};$$

here, $\lhd$ defines a total order.

Similarly,

(1) the sequence $\Gamma_{k-1,k}$ is of the form:

$$\Gamma_{k-1,k} = (\gamma_{k,k+1}, \ldots, \gamma_{k,n}, \gamma_{k,\overline{k}}, \gamma_{k,\overline{n}}, \ldots, \gamma_{k,\overline{k+1}}, \gamma_{k-1,\overline{k}}, \ldots, \gamma_{1,\overline{k}}, \gamma_{1,k}, \ldots, \gamma_{k-1,k}),$$

(2) admissible subsets $A \in \Gamma(v, \Gamma_{k-1,k})$ for $v \in W$ are subsets of the (totally ordered) index set

$$I_2 := \{(k, k+1) \prec \cdots \prec (k, n) \prec (k, \overline{k}) \prec (k, \overline{n}) \prec \cdots \prec (k, \overline{k+1}) \prec (k-1, \overline{k}) \prec \cdots \prec (1, \overline{k}) \prec (1, k) \prec \cdots \prec (k-1, k)\};$$

here, $\prec$ defines a total order. Note that $I_1 \subset I_2$, and that if $\beta \lhd \gamma$, then $\gamma \prec \beta$ for $\beta, \gamma \in I_1$.

By Proposition 3.5 and Lemma 2.5, we deduce that

$$\begin{aligned}
&(\text{LHS of } (5.1)) \\
&= \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B) t_{\mathrm{down}(B)}}(\lambda + \varepsilon_k) \\
&= \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) \rangle} \operatorname{gch} V^-_{\mathrm{end}(B)}(\lambda + \varepsilon_k) \\
&= \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) \rangle} \\
&\quad \times \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{A \in \mathcal{A}(\mathrm{end}(B), \Gamma_{k-1,k})} (-1)^{n(A)} q^{-\operatorname{height}(A)} e^{\mathrm{wt}(A)} \operatorname{gch} V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda).
\end{aligned} \tag{5.2}$$

Let us take the alcove path

$$(A_\circ = \underbrace{A_0, A_1, \ldots, A_a}_{\Gamma_k^*(k)} = \underbrace{B_0, B_1, \ldots, B_b}_{\Theta_k} = A_{-\varepsilon_k})$$

corresponding to $\Gamma_{k-1,k} = \Gamma_k^*(k) * \Theta_k$. Then, the hyperplane containing the common wall of A_{t-1} and A_t, $t = 1, \ldots, a$, is of the form $H_{\beta,0}$ with $\beta \in \Delta^+$. Also, the hyperplane containing the common wall of B_{t-1} and B_t, $t = 1, \ldots, b$, is of the form $H_{\beta,1}$ with $\beta \in \Delta^+$ (see Remark 2.6). This implies that if we divide $A \in \mathcal{A}(v, \Gamma_{k-1,k})$ into the two parts: $A^{(1)} := A \cap \{(k, k+1), \ldots, (1, \overline{k})\}$ $(\in \mathcal{A}(v, \Gamma_k^*(k)))$ and $A^{(2)} := A \cap \{(1, k), \ldots, (k-1, k)\}$ $(\in \mathcal{A}(\mathrm{end}(A^{(1)}), \Theta_k))$, then we have

$$\begin{aligned}
\mathrm{height}(A) &= \sum_{a \in A^- \cap A^{(1)}} \langle \varepsilon_k, \gamma_a^\vee \rangle \\
&= \left\langle \varepsilon_k, \sum_{a \in A^- \cap A^{(1)}} \gamma_a^\vee \right\rangle \\
&= \langle \varepsilon_k, \mathrm{down}(A^{(1)}) \rangle.
\end{aligned}$$

In addition, we have $\mathrm{wt}(A) = \mathrm{end}(A^{(1)})\varepsilon_k$. Also, since all the roots in $\Gamma_k^*(k)$ are positive roots and those in Θ_k are negative roots, it follows that $n(A) = |A^{(2)}|$. Therefore, we see that

$$\begin{aligned}
&\overset{(5.2)}{=} \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) \rangle} \\
&\quad \times \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{A \in \mathcal{A}(\mathrm{end}(B), \Gamma_{k-1,k})} (-1)^{n(A)} q^{-\langle \varepsilon_k, \mathrm{down}(A^{(1)}) \rangle} e^{\mathrm{end}(A^{(1)})\varepsilon_k} \,\mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A)}}(\lambda) \\
&= \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} \sum_{A \in \mathcal{A}(\mathrm{end}(B), \Gamma_{k-1,k})} (-1)^{|B|} (-1)^{|A^{(2)}|} \\
&\quad \times q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) \rangle} e^{\mathrm{end}(A^{(1)})\varepsilon_k} q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(A^{(1)}) \rangle} \,\mathrm{gch}\, V^-_{\mathrm{end}(A) t_{\mathrm{down}(A^{(2)})}}(\lambda) \\
&= \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} \sum_{A \in \mathcal{A}(\mathrm{end}(B), \Gamma_{k-1,k})} (-1)^{|B|} (-1)^{|A^{(2)}|} \\
&\quad \times q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) + \mathrm{down}(A^{(1)}) \rangle} e^{\mathrm{end}(A^{(1)})\varepsilon_k} \,\mathrm{gch}\, V^-_{\mathrm{end}(A^{(2)}) t_{\mathrm{down}(A^{(2)})}}(\lambda) \\
&= \frac{1}{1 - q^{-\langle \lambda + \varpi_k, \alpha_k^\vee \rangle}} \sum_{B \in \mathcal{A}(w, \Gamma_k(k))} \sum_{A^{(1)} \in \mathcal{A}(\mathrm{end}(B), \Gamma_k^*(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) + \mathrm{down}(A^{(1)}) \rangle} \\
&\quad \times e^{\mathrm{end}(A^{(1)})\varepsilon_k} \sum_{A^{(2)} \in \mathcal{A}(\mathrm{end}(A^{(1)}), \Theta_k)} (-1)^{|A^{(2)}|} \,\mathrm{gch}\, V^-_{\mathrm{end}(A^{(2)}) t_{\mathrm{down}(A^{(2)})}}(\lambda).
\end{aligned} \tag{5.3}$$

Now, we define an involution on the set

$$\mathcal{P} := \{(B, A^{(1)}) \mid B \in \mathcal{A}(w, \Gamma_k(k)),\ A^{(1)} \in \mathcal{A}(\mathrm{end}(B), \Gamma_k^*(k))\}.$$

There are the following six cases:

(1) $[\max B \prec \min A^{(1)}]$ or $[B \neq \emptyset$ and $A^{(1)} = \emptyset]$,
(2) $[\max B \succ \min A^{(1)}]$ or $[B = \emptyset$ and $A^{(1)} \neq \emptyset]$,
(3) $\max B = \min A^{(1)} = (k, k+1)$, and one of the following holds:

- $\max(B \setminus \{(k, k+1)\}) \prec \min(A^{(1)} \setminus \{(k, k+1)\})$ or
- $B \setminus \{(k, k+1)\} \neq \emptyset$ and $A^{(1)} = \{(k, k+1)\}$,

(4) $\max B = \min A^{(1)} = (k, k+1)$, and one of the following holds:

- $\max(B \setminus \{(k, k+1)\}) \succ \min(A^{(1)} \setminus \{(k, k+1)\})$ or
- $B = \{(k, k+1)\}$ and $A^{(1)} \setminus \{(k, k+1)\} \neq \emptyset$,

(5) $B = A^{(1)} = \{(k, k+1)\}$,
(6) $B = A^{(1)} = \emptyset$.

Here we remark that, if we have a directed path $v \xrightarrow{\alpha} vs_\alpha \xrightarrow{\alpha} v$ in $\mathrm{QBG}(W)$ for $v \in W$ and $\alpha \in \Delta^+$, then α must be a simple root. Conversely, for $v \in W$ and a simple root α, $v \xrightarrow{\alpha} vs_\alpha \xrightarrow{\alpha} v$ is a directed path in $\mathrm{QBG}(W)$. For $(B, A^{(1)}) \in \mathcal{P}$, we define $\iota(B, A^{(1)}) = (B', A'^{(1)}) \in \mathcal{P}$ as follows:

- if $(B, A^{(1)})$ satisfies (1) above, then set
$$B' := B \setminus \{\max B\}, \quad A'^{(1)} := A^{(1)} \sqcup \{\max B\};$$
- if $(B, A^{(1)})$ satisfies (2) above, then set
$$B' := B \sqcup \{\min A^{(1)}\}, \quad A'^{(1)} := A^{(1)} \setminus \{\min A^{(1)}\};$$
- if $(B, A^{(1)})$ satisfies (3) above, then set
$$B' := B \setminus \{\max(B \setminus \{(k, k+1)\})\}, \quad A'^{(1)} := A' \sqcup \{\max(B \setminus \{(k, k+1)\};$$
- if $(B, A^{(1)})$ satisfies (4) above, then set
$$B' := B \sqcup \{\min(A'^{(1)} \setminus \{(k, k+1)\}, \quad A'^{(1)} := A' \setminus \{\min(A'^{(1)} \setminus \{(k, k+1)\})\};$$
- if $(B, A^{(1)})$ satisfies (5) or (6) above, then set
$$B' := B, \quad A'^{(1)} := A^{(1)}.$$

It is clear that ι defines an involution on $\mathcal{P}$. Moreover, in cases (1) and (3) (resp., (2) and (4)), we have

- $|B'| = |B| - 1$ (resp., $|B'| = |B| + 1$),
- $\mathrm{down}(B') + \mathrm{down}(A'^{(1)}) = \mathrm{down}(B) + \mathrm{down}(A^{(1)})$, and
- $\mathrm{end}(A'^{(1)}) = \mathrm{end}(A^{(1)})$.

This implies that

$$\sum_{\substack{(B, A^{(1)}) \in \mathcal{P} \\ (B, A^{(1)}) \text{ satisfies one of (1)–(4)}}} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \mathrm{down}(B) + \mathrm{down}(A^{(1)}) \rangle} e^{\mathrm{end}(A^{(1)}) \varepsilon_k}$$
$$\times \sum_{A^{(2)} \in \mathcal{A}(\mathrm{end}(A^{(1)}), \Theta_k)} (-1)^{|A^{(2)}|} \operatorname{gch} V^-_{\mathrm{end}(A^{(2)}) t_{\mathrm{down}(A^{(2)})}}(\lambda) = 0.$$

Therefore, we conclude that

$$(5.3) = \frac{1}{1-q^{-\langle\lambda+\varpi_k,\alpha_k^\vee\rangle}} \left(\underbrace{-q^{-\langle\lambda+\varepsilon_k,\alpha_k^\vee\rangle} e^{w\varepsilon_k} \sum_{A^{(2)}\in\mathcal{A}(w,\Theta_k)} (-1)^{|A^{(2)}|} \operatorname{gch} V^-_{\mathrm{end}(A^{(2)})t_{\mathrm{down}(A^{(2)})}}(\lambda)}_{B=A^{(1)}=\{(k,k+1)\}} \right.$$

$$\left. + \underbrace{e^{w\varepsilon_k} \sum_{A^{(2)}\in\mathcal{A}(w,\Theta_k)} (-1)^{|A^{(2)}|} \operatorname{gch} V^-_{\mathrm{end}(A^{(2)})t_{\mathrm{down}(A^{(2)})}}(\lambda)}_{B=A^{(1)}=\emptyset} \right)$$

$$= \frac{1-q^{-\langle\lambda+\varepsilon_k,\alpha_k^\vee\rangle}}{1-q^{-\langle\lambda+\varpi_k,\alpha_k^\vee\rangle}} e^{w\varepsilon_k} \sum_{A^{(2)}\in\mathcal{A}(w,\Theta_k)} (-1)^{|A^{(2)}|} \operatorname{gch} V^-_{\mathrm{end}(A^{(2)})t_{\mathrm{down}(A^{(2)})}}(\lambda)$$

$$= e^{w\varepsilon_k} \sum_{A^{(2)}\in\mathcal{A}(w,\Theta_k)} (-1)^{|A^{(2)}|} \operatorname{gch} V^-_{\mathrm{end}(A^{(2)})t_{\mathrm{down}(A^{(2)})}}(\lambda)$$

$$= (\text{RHS of } (5.1)),$$

as desired; for the third equality, we have used that $\langle\varepsilon_k,\alpha_k^\vee\rangle = \langle\varpi_k,\alpha_k^\vee\rangle = 1$. This completes the proof of the proposition.

Proof of Theorem 4.1 We prove the assertion of the theorem by induction on $m = 1,\dots,n$. If $m = 1$, then the assertion immediately follows from Proposition 5.1 since $\mathcal{A}(w,\Theta_1) = \{\emptyset\}$. Let $1 < l \le n$, and assume the assertion for $m = 1,\dots,l-1$. We will prove the assertion for $m = l$. Note that for $A \in \mathcal{A}(w,\Theta_l)\setminus\{\emptyset\}$, if the index k satisfies $\mathrm{end}(A)^{-1}w\varepsilon_l = \varepsilon_k$, then we have $1 \le k \le l-1$. Therefore, by Proposition 5.1, we see that

$$\begin{aligned}
& e^{w\varepsilon_l} \operatorname{gch} V_w^-(\lambda) \\
&= \sum_{B\in\mathcal{A}(w,\Gamma_l(l))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_l) \\
&\quad - \sum_{A\in\mathcal{A}(w,\Theta_l)\setminus\{\emptyset\}} (-1)^{|A|} e^{w\varepsilon_l} \operatorname{gch} V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda) \\
&= \sum_{B\in\mathcal{A}(w,\Gamma_l(l))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_l) \\
&\quad - \sum_{A\in\mathcal{A}(w,\Theta_l)\setminus\{\emptyset\}} (-1)^{|A|} q^{-\langle\lambda,\mathrm{down}(A)\rangle} e^{w\varepsilon_l} \operatorname{gch} V^-_{\mathrm{end}(A)}(\lambda) \\
&= \sum_{B\in\mathcal{A}(w,\Gamma_l(l))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_l) \\
&\quad + \sum_{k=1}^{l-1} \sum_{\substack{A\in\mathcal{A}(w,\Theta_l)\setminus\{\emptyset\} \\ \mathrm{end}(A)^{-1}w\varepsilon_l=\varepsilon_k}} (-1)^{|A|-1} q^{-\langle\lambda,\mathrm{down}(A)\rangle} \underbrace{e^{w\varepsilon_l} \operatorname{gch} V^-_{\mathrm{end}(A)}(\lambda)}_{\text{induction hypothesis}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{B\in\mathcal{A}(w,\Gamma_l(l))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_l) \\
&\quad + \sum_{k=1}^{l-1} \sum_{A\in\mathcal{A}_w^{l,k}} (-1)^{|A|-1} q^{-\langle\lambda,\mathrm{down}(A)\rangle} \\
&\quad \times \Bigg(\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma_k(k))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_k) \\
&\qquad + \sum_{j=1}^{k-1} \sum_{(j_1,\dots,j_r)\in\mathcal{S}_{k,j}} \sum_{A_1\in\mathcal{A}^{k,j_1}_{\mathrm{end}(A)}} \cdots \sum_{A_r\in\mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle} \\
&\qquad \times \sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j) \Bigg) \\
&= \sum_{B\in\mathcal{A}(w,\Gamma_l(l))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_l) \\
&\quad + \sum_{k=1}^{l-1} \sum_{A\in\mathcal{A}_w^{l,k}} (-1)^{|A|-1} \\
&\quad \times \Bigg(\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma_k(k))} (-1)^{|B|} q^{\langle\varepsilon_k,\mathrm{down}(A)\rangle} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{down}(A)}}(\lambda+\varepsilon_k) \\
&\qquad + \sum_{j=1}^{k-1} \sum_{(j_1,\dots,j_r)\in\mathcal{S}_{k,j}} \sum_{A_1\in\mathcal{A}^{k,j_1}_{\mathrm{end}(A)}} \cdots \sum_{A_r\in\mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle} \\
&\qquad \times \sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))} (-1)^{|B|} q^{\langle\varepsilon_j,\mathrm{down}(A)\rangle} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(A,A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j) \Bigg) \\
&= \sum_{B\in\mathcal{A}(w,\Gamma_l(l))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda+\varepsilon_l) \\
&\quad + \sum_{k=1}^{l-1} \sum_{(j_1,\dots,j_r)\in\mathcal{S}_{l,j}} \sum_{A_1\in\mathcal{A}^{l,j_1}_{w}} \cdots \sum_{A_r\in\mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle} \\
&\quad \times \sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j),
\end{aligned}
$$

as desired. Thus, the assertion also holds for $m = l$. This proves the theorem.

5.2 Second-half case

The following proposition is a key to the proof of the second half of our identities of inverse Chevalley type.

Proposition 5.2 *Let $w \in W$, $\lambda \in P^+$ such that $\lambda - \varepsilon_k \in P^+$. Then we have*

$$\sum_{B \in \mathcal{A}(w,\Theta_k)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_k) = \sum_{A \in \mathcal{A}(w,\Gamma_k(k))} (-1)^{|A|} e^{-w\varepsilon_k} \operatorname{gch} V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda).$$

Proof By replacing λ in equation (5.1) with $\lambda - \varepsilon_k$ and multiplying both sides of equation (5.1) by $e^{-w\varepsilon_k}$, we obtain the desired identity.

Proof of Theorem 4.3 We prove the assertion of the theorem by downward induction on $m = n, n-1, \ldots, 1$. First, assume that $m = n$. Then, by Proposition 5.2 and Corollary 4.2, we see that

$$\begin{aligned}
&e^{-w\varepsilon_m} \operatorname{gch} V^-_w(\lambda) \\
&= \sum_{B \in \mathcal{A}(w,\Theta_n)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_n) \\
&\quad - \sum_{A \in \mathcal{A}(w,\Gamma_n(n))\setminus\{\emptyset\}} (-1)^{|A|} e^{-w\varepsilon_n} \operatorname{gch} V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda) \\
&= \sum_{B \in \mathcal{A}(w,\Theta_n)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_n) \\
&\quad + \sum_{k=1}^{n} \sum_{A \in \mathcal{A}_w^{\overline{n},k}} (-1)^{|A|-1} e^{-w\varepsilon_n} \operatorname{gch} V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda) \\
&= \sum_{B \in \mathcal{A}(w,\Theta_n)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_n) \\
&\quad + \sum_{k=1}^{n} \sum_{A \in \mathcal{A}_w^{\overline{n},k}} (-1)^{|A|-1} e^{\mathrm{end}(A)\varepsilon_k} \operatorname{gch} V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda) \\
&= \sum_{B \in \mathcal{A}(w,\Theta_n)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_n) \\
&\quad + \sum_{k=1}^{n} \sum_{A \in \mathcal{A}_w^{\overline{n},k}} (-1)^{|A|-1} \Bigg(q^{\langle \varepsilon_k, \mathrm{down}(A)\rangle} \sum_{B \in \mathcal{A}(\mathrm{end}(A),\Gamma_k(k))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda + \varepsilon_k) \\
&\quad + \sum_{j=1}^{k-1} \sum_{(j_1,\ldots,j_r) \in \mathcal{S}_{k,j}} \sum_{A_1 \in \mathcal{A}^{k,j_1}_{\mathrm{end}(A)}} \cdots \sum_{A_r \in \mathcal{A}^{j_{r-1},j_r}_{\mathrm{end}(A_{r-1})}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{\langle \varepsilon_j, \mathrm{down}(A,A_1,\ldots,A_r)\rangle} \\
&\qquad \times \sum_{B \in \mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(A,A_1,\ldots,A_r,B)}}(\lambda + \varepsilon_j) \Bigg) \\
&= \sum_{B \in \mathcal{A}(w,\Theta_n)} (-1)^{|B|} \operatorname{gch} V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda - \varepsilon_n)
\end{aligned}$$

$$+\sum_{j=1}^{n}\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{\overline{n},j}}\sum_{A_1\in\mathcal{A}_w^{\overline{n},j_1}}\cdots\sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}(-1)^{|A_1|+\cdots+|A_r|-r}q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle}$$
$$\times\sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j),$$

as desired. Let $1\le l<n$, and assume the assertion for $m=n,n-1,\dots,l+1$. We prove the assertion for $m=l$. For $A\in\mathcal{A}(w,\Gamma_l(l))\setminus\{\emptyset\}$, if the index k satisfies $\mathrm{end}(A)^{-1}w(-\varepsilon_l)=\varepsilon_k$, then we have $k=1,\dots,n-1,n,\overline{n},\overline{n-1},\dots,\overline{l+1}$. Therefore, by Proposition 5.2 (and Proposition 3.2), we compute:

$$\begin{aligned}
&e^{-w\varepsilon_l}\,\mathrm{gch}\,V_w^-(\lambda)\\
&=\sum_{B\in\mathcal{A}(w,\Theta_l)}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_l)\\
&\quad-\sum_{A\in\mathcal{A}(w,\Gamma_l(l))\setminus\{\emptyset\}}(-1)^{|A|}e^{-w\varepsilon_l}\,\mathrm{gch}\,V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda)\\
&=\sum_{B\in\mathcal{A}(w,\Theta_l)}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_l)\\
&\quad+\sum_{k=l+1}^{n}\sum_{A\in\mathcal{A}_w^{\overline{l},\overline{k}}}(-1)^{|A|-1}e^{-w\varepsilon_l}\,\mathrm{gch}\,V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda)\\
&\quad+\sum_{k=1}^{n}\sum_{A\in\mathcal{A}_w^{\overline{l},k}}(-1)^{|A|-1}e^{-w\varepsilon_l}\,\mathrm{gch}\,V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda)\\
&=\sum_{B\in\mathcal{A}(w,\Theta_l)}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_l)\\
&\quad+\sum_{k=l+1}^{n}\sum_{A\in\mathcal{A}_w^{\overline{l},\overline{k}}}(-1)^{|A|-1}q^{-\langle\lambda,\mathrm{down}(A)\rangle}\underbrace{e^{-\mathrm{end}(A)\varepsilon_k}\,\mathrm{gch}\,V^-_{\mathrm{end}(A)}(\lambda)}_{\text{induction hypothesis}}\\
&\quad+\sum_{k=1}^{n}\sum_{A\in\mathcal{A}_w^{\overline{l},k}}(-1)^{|A|-1}\underbrace{e^{\mathrm{end}(A)\varepsilon_k}\,\mathrm{gch}\,V^-_{\mathrm{end}(A)t_{\mathrm{down}(A)}}(\lambda)}_{\text{Corollary 4.2}}\\
&=\sum_{B\in\mathcal{A}(w,\Theta_l)}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_l)\\
&\quad+\sum_{k=l+1}^{n}\sum_{A\in\mathcal{A}_w^{\overline{l},\overline{k}}}(-1)^{|A|-1}q^{-\langle\lambda,\mathrm{down}(A)\rangle}\Bigg(\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Theta_k)}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_k)\\
&\quad+\sum_{j=k+1}^{n}\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{\overline{k},\overline{j}}}\sum_{A_1\in\mathcal{A}_{\mathrm{end}(A)}^{\overline{k},j_1}}\cdots\sum_{A\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}(-1)^{|A_1|+\cdots+|A_r|-r}q^{-\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle}\\
&\quad\times\sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Theta_j)}(-1)^{|B|}\,\mathrm{gch}\,V^-_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda-\varepsilon_j)
\end{aligned}$$

$$+\sum_{j=1}^{n}\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{\overline{k},j}}\sum_{A_1\in\mathcal{A}_{\mathrm{end}(A)}^{\overline{k},j_1}}\cdots\sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}(-1)^{|A_1|+\cdots+|A_r|-r}q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle}$$

$$\left.\times\sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))}(-1)^{|B|}\,\mathrm{gch}\,V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j)\right)$$

$$+\sum_{k=1}^{n}\sum_{A\in\mathcal{A}_w^{\overline{l},k}}(-1)^{|A|-1}\left(q^{\langle\varepsilon_k,\mathrm{down}(A)\rangle}\sum_{B\in\mathcal{A}(\mathrm{end}(A),\Gamma_k(k))}(-1)^{|B|}\,\mathrm{gch}\,V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(B)+\mathrm{down}(A)}}(\lambda+\varepsilon_k)\right.$$

$$+\sum_{j=1}^{k-1}\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{k,j}}\sum_{A_1\in\mathcal{A}_{\mathrm{end}(A)}^{k,j_1}}\cdots\sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}(-1)^{|A_1|+\cdots+|A_r|-r}q^{\langle\varepsilon_j,\mathrm{down}(A,A_1,\dots,A_r)\rangle}$$

$$\left.\times\sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))}(-1)^{|B|}\,\mathrm{gch}\,V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(A,A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j)\right)$$

$$=\sum_{B\in\mathcal{A}(w,\Theta_l)}(-1)^{|B|}\,\mathrm{gch}\,V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(B)}}(\lambda-\varepsilon_l)$$

$$+\sum_{j=l+1}^{n}\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{\overline{l},\overline{j}}}\sum_{A_1\in\mathcal{A}_w^{\overline{l},j_1}}\cdots\sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}(-1)^{|A_1|+\cdots+|A_r|-r}q^{-\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle}$$

$$\times\sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Theta_j)}(-1)^{|B|}\,\mathrm{gch}\,V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda-\varepsilon_j)$$

$$+\sum_{j=1}^{n}\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{\overline{l},j}}\sum_{A_1\in\mathcal{A}_w^{\overline{l},j_1}}\cdots\sum_{A_{r-1}\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}(-1)^{|A_1|+\cdots+|A_r|-r}q^{\langle\varepsilon_j,\mathrm{down}(A_1,\dots,A_r)\rangle}$$

$$\times\sum_{B\in\mathcal{A}(\mathrm{end}(A_r),\Gamma_j(j))}(-1)^{|B|}\,\mathrm{gch}\,V^{-}_{\mathrm{end}(B)t_{\mathrm{down}(A_1,\dots,A_r,B)}}(\lambda+\varepsilon_j),$$

as desired. By downward induction, this completes the proof of the theorem.

6 Proof of Theorem 4.5

We will derive the cancellation-free form of the first-half identities of inverse Chevalley type (Theorem 4.5). For this purpose, we need the following lemmas on edges of the quantum Bruhat graph. We continue to assume that $\mathfrak{g}$ is of type C_n. Recall that a total order $<$ on $[\overline{n}]$ is defined by: $1<2<\cdots<n<\overline{n}<\overline{n-1}<\cdots<\overline{1}$; for each $1\le k\le n$, we define an order $\prec_k$ (resp., $\prec_{\overline{k}}$) on $[\overline{n}]$ by: $k\prec_k k+1\prec_k\cdots\prec_k n\prec_k\overline{n}\prec_k\overline{n-1}\prec_k\cdots\prec_k\overline{1}\prec_k 1\prec_k 2\prec_k\cdots\prec_k k-1$ (resp., $\overline{k}\prec_{\overline{k}}\overline{k-1}\prec_{\overline{k}}\cdots\prec_{\overline{k}}\overline{1}\prec_{\overline{k}}1\prec_{\overline{k}}2\prec_{\overline{k}}\cdots\prec_{\overline{k}}n\prec_{\overline{k}}\overline{n}\prec_{\overline{k}}\overline{n-1}\prec_{\overline{k}}\cdots\prec_{\overline{k}}\overline{k+1}$). For $a_1,\dots,a_r\in[\overline{n}]$ with $r\ge 2$, we write $a_1\prec a_2\prec\cdots\prec a_r$ if $a_1\prec_{a_1}a_2\prec_{a_1}\cdots\prec_{a_1}a_r$ (the order $\prec$ is different from that introduced in the proof of Proposition 5.1). Also, on the set $[\overline{n}]$, we define the sign function $\mathrm{sgn}(\cdot)$: for $a\in[\overline{n}]$, we set

$$\mathrm{sgn}(a):=\begin{cases}1 & \text{if } a=1,2,\dots,n,\\ -1 & \text{if } a=\overline{n},\overline{n-1},\dots,\overline{1}.\end{cases}$$

We know the following useful criterion.

Lemma 6.1 *([9, Proposition 5.7]) Let $w \in W$.*

(1) *Let $1 \le k < l \le n$. Then, $w \xrightarrow{(k,l)} ws_{(k,l)}$ is an edge in $\mathrm{QBG}(W)$ if and only if there does not exist $k < j < l$ such that $w(k) \prec w(j) \prec w(l)$.*

(2) *Let $1 \le k < l \le n$. Then, $w \xrightarrow{(k,\bar{l})} ws_{(k,\bar{l})}$ is an edge in $\mathrm{QBG}(W)$ if and only if the following hold:*

- *$w(k) < w(\bar{l})$;*
- *$\mathrm{sgn}(w(k)) = \mathrm{sgn}(w(l))$; and*
- *there does not exist $k < j < \bar{l}$ such that $w(k) < w(j) < w(\bar{l})$.*

(3) *Let $1 \le k \le n$. Then, $w \xrightarrow{(k,\bar{k})} ws_{(k,\bar{k})}$ is an edge in $\mathrm{QBG}(W)$ if and only if there does not exist $k < j < \bar{k}$ such that $w(k) \prec w(j) \prec w(\bar{k})$.*

By using this criterion, we can show the following three lemmas.

Lemma 6.2 *Let $w \in W$, and $1 \le k < l < m \le n$. Then, the following are equivalent:*

(1) *$w \xrightarrow{(k,m)} ws_{(k,m)}$ and $w \xrightarrow{(l,m)} ws_{(l,m)} \xrightarrow{(k,l)} ws_{(l,m)}s_{(k,l)}$;*
(2) *$w \xrightarrow{(k,m)} ws_{(k,m)}$ and $w \xrightarrow{(l,m)} ws_{(l,m)}$;*
(3) *$w \xrightarrow{(k,m)} ws_{(k,m)} \xrightarrow{(l,m)} ws_{(k,m)}s_{(l,m)}$.*

Lemma 6.3 *Let $w \in W$. Take $1 \le k_1 < l_1 \le n$ and $1 \le k_2 < l_2 \le n$ such that $\{k_1, l_1\} \cap \{k_2, l_2\} = \emptyset$. Then, the following are equivalent:*

(1) *we have the directed path $w \xrightarrow{(k_1,l_1)} ws_{(k_1,l_1)} \xrightarrow{(k_2,l_2)} ws_{(k_1,l_1)}s_{(k_2,l_2)}$;*
(2) *we have the directed path $w \xrightarrow{(k_2,l_2)} ws_{(k_2,l_2)} \xrightarrow{(k_1,l_1)} ws_{(k_2,l_2)}s_{(k_1,l_1)}$.*

Lemma 6.4 *Let $w \in W$, $m = 1, \dots, n$, and take $a_1, \dots, a_s \in \{k \in [1, m-1] \mid w \xrightarrow{(k,m)} ws_{(k,m)}\}$ such that $a_1 < \cdots < a_s$; by Lemma 6.2, we have the directed path*

$$w = y_0 \xrightarrow{(a_1,m)} y_1 \xrightarrow{(a_2,m)} \cdots \xrightarrow{(a_s,m)} y_s$$

in $\mathrm{QBG}(W)$. Let us take $c < a_1$ such that

- *$y_s \xrightarrow{(c,a_1)} y_s s_{(c,a_1)} =: z$ is an edge in $\mathrm{QBG}(W)$, and*
- *$w \xrightarrow{(c,m)} ws_{(c,m)}$ is an edge in $\mathrm{QBG}(W)$.*

For $p < a_1$, if $w \xrightarrow{(p,m)} ws_{(p,m)}$ is an edge in $\mathrm{QBG}(W)$, then we have $p < c$.

Corollary 6.5 *Let $w \in W$, $m = 1, \dots, n$, and let $\{a_1 < \cdots < a_s\} = \{k \in [1, m-1] \mid w \xrightarrow{(k,m)} ws_{(k,m)}\}$ with $s \ge 2$; by Lemma 6.4, for $2 = b_1 < \cdots < b_u \le s$, we have the directed path*

$$w = z_0 \xrightarrow{(a_{b_1},m)} z_1 \xrightarrow{(a_{b_2},m)} \cdots \xrightarrow{(a_{b_u},m)} z_u$$

in $\mathrm{QBG}(W)$. Then, a_1 is equal to the minimal c with $1 \le c < a_{b_1}$ for which $z_u \xrightarrow{(c,a_{b_1})} z_u s_{(c,a_{b_1})}$ is an edge in $\mathrm{QBG}(W)$.

Proof Let us take the minimal c for which $z_u \xrightarrow{(c,a_{b_1})} z_u s_{(c,a_{b_1})}$ is an edge in $\mathrm{QBG}(W)$. Note that such a c exists since $z_u \xrightarrow{(a_1,a_{b_1})} z_u s_{(a_1,a_{b_1})}$ is an edge in $\mathrm{QBG}(W)$ by Lemmas 6.2 and 6.3. This also implies that $c \le a_1$. Assume, for a contradiction, that $c < a_1$. Then, since a_1 is the minimum of the set $\{k \in [1, m-1] \mid w \xrightarrow{(k,m)} ws_{(k,m)}\}$, $w \xrightarrow{(c,m)} ws_{(c,m)}$ is not an edge in $\mathrm{QBG}(W)$. Also, we see that $a_1 < a_2 \le a_{b_1}$ and that $w \xrightarrow{(a_1,m)} ws_{(a_1,m)}$ is an edge in $\mathrm{QBG}(W)$. Therefore, by Lemma 6.4, we obtain $a_1 < c$, which is a contradiction. Hence we conclude that $c = a_1$, as desired. This proves the corollary.

Theorem 4.5 follows immediately from Theorem 4.1 and the following key proposition. Let $\mathbb{Z}[q^{-1}][W]$ denote the group algebra of W with coefficients in $\mathbb{Z}[q^{-1}]$; the elements of $\mathbb{Z}[q^{-1}][W]$ are of the form $\sum_{v\in W} c_v(q^{-1})v$, with $c_v(q^{-1}) \in \mathbb{Z}[q^{-1}]$.

Proposition 6.6 *Let $w \in W$, $m = 1, \dots, n$, and $j = 1, \dots, m-1$. Then, there holds the following equality in $\mathbb{Z}[q^{-1}][W]$:*

$$\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{m,j}} \sum_{A_1\in\mathcal{A}_w^{m,j_1}} \cdots \sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle\lambda,\mathrm{down}(A_1,\dots,A_r)\rangle}\,\mathrm{end}(A_r)$$
$$= q^{-\langle\lambda,\mathrm{wt}(\mathbf{p}_{m,j}(w))\rangle}\,\mathrm{end}(\mathbf{p}_{m,j}(w)).$$

Proof We prove the assertion of the proposition by induction on $m-j$. If $m-j=1$, then the assertion is clear since $\mathcal{S}_{m,m-1} = \{(m-1)\}$ and $\mathcal{A}_w^{m,m-1} = \{\{m-1\}\}$. Assume that $m-j>1$. By Lemma 6.2, we can verify that if $\{a_1 < \cdots < a_s\} = \{k \in [1, m-1] \mid w \xrightarrow{(k,m)} ws_{(k,m)}\}$ (note that $a_s = m-1$), then

$$\mathcal{A}_w^{m,j} = \begin{cases} \{\{j, a_{c_1}, \dots, a_{c_u}\} \mid l < c_1 < \cdots < c_u \le s\} & \text{if } j = a_l \text{ for some } l = 1, \dots, s, \\ \emptyset & \text{if } j \ne a_1, \dots, a_s. \end{cases}$$

Therefore, we compute:

$$\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{m,j}} \sum_{A_1\in\mathcal{A}_w^{m,j_1}} \cdots \sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle\lambda,\mathrm{down}(A_1,\dots,A_r)\rangle}\,\mathrm{end}(A_r)$$
$$= \sum_{k=1}^{s} \sum_{(j_1,\dots,j_r)\in\mathcal{S}_{a_k,j}} \sum_{A\in\mathcal{A}_w^{m,a_k}} \sum_{A_1\in\mathcal{A}_{\mathrm{end}(A)}^{a_k,j_1}} \cdots \sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}}$$
$$\times (-1)^{|A|+|A_1|+\cdots+|A_r|-(r+1)} q^{-\langle\lambda,\mathrm{down}(A,A_1,\dots,A_r)\rangle}\,\mathrm{end}(A_r)$$
$$= \sum_{k=1}^{s} \sum_{A\in\mathcal{A}_w^{m,a_k}} (-1)^{|A|-1} q^{-\langle\lambda,\mathrm{down}(A)\rangle}$$
$$\times \underbrace{\sum_{(j_1,\dots,j_r)\in\mathcal{S}_{a_k,j}} \sum_{A_1\in\mathcal{A}_{\mathrm{end}(A)}^{a_k,j_1}} \cdots \sum_{A_r\in\mathcal{A}_{\mathrm{end}(A_{r-1})}^{j_{r-1},j_r}} (-1)^{|A_1|+\cdots+|A_r|-r} q^{-\langle\lambda,\mathrm{down}(A_1,\dots,A_r)\rangle}\,\mathrm{end}(A_r)}_{\text{induction hypothesis}}$$
$$= \sum_{k=1}^{s} \sum_{A\in\mathcal{A}_w^{m,a_k}} (-1)^{|A|-1} q^{-\langle\lambda,\mathrm{down}(A)+\mathrm{wt}(\mathbf{p}_{a_k,j}(\mathrm{end}(A)))\rangle}\,\mathrm{end}(\mathbf{p}_{a_k,j}(\mathrm{end}(A))). \tag{6.1}$$

If $s = 1$, then the assertion is clear since $\mathcal{A}_w^{m,a_1} = \{\{m-1\}\}$. Now, assume that $s \geq 2$. For $A \in \mathcal{A} := (\mathcal{A}_w^{m,a_1} \setminus \{\{a_1\}\}) \sqcup (\bigsqcup_{k=2}^{s} \mathcal{A}_w^{m,a_k})$, we define $\iota(A)$ by

$$A \in \bigsqcup_{k=2}^{s} \mathcal{A}_w^{m,a_k} \quad \mapsto \quad \iota(A) := A \sqcup \{a_1\} \in \mathcal{A}_w^{m,a_1} \setminus \{\{a_1\}\},$$

$$A \in \mathcal{A}_w^{m,a_1} \setminus \{\{a_1\}\} \quad \mapsto \quad \iota(A) := A \setminus \{a_1\} \in \mathcal{A}_w^{m,\min(A\setminus\{a_1\})} \subset \bigsqcup_{k=2}^{s} \mathcal{A}_w^{m,a_k}.$$

We see that this ι defines an involution on the set $\mathcal{A}$ such that $|\iota(A)| = |A| \pm 1$ for $A \in \mathcal{A}$. For $A \in \mathcal{A}_w^{m,a_k}$ with $k = 2, \dots, s$, it follows from Corollary 6.5 that the first edge in the directed path $\mathbf{p}_{a_k,j}(\mathrm{end}(A))$ in $\mathrm{QBG}(W)$ is $\mathrm{end}(A) \xrightarrow{(a_1,a_k)} \mathrm{end}(A)s_{(a_1,a_k)} = \mathrm{end}(\iota(A))$. Hence we have $\mathrm{end}(\mathbf{p}_{a_k,j}(\mathrm{end}(A))) = \mathrm{end}(\mathbf{p}_{a_1,j}(\mathrm{end}(\iota(A))))$. Also, the directed path $w \to \cdots \to \mathrm{end}(\iota(A))$ in $\mathrm{QBG}(W)$ corresponding to $\iota(A)$ is a shortest one of length $|\iota(A)| = |A|+1$, since the order $\prec$ given by $(1, m) \prec \cdots \prec (m-1, m)$ forms a part of a reflection order on the set Δ^+ of positive roots. Hence, the concatenation $w \to \cdots \to \mathrm{end}(A) \xrightarrow{(a_1,a_k)} \mathrm{end}(A)s_{(a_1,a_k)}$ of the directed path corresponding to A with the edge $\mathrm{end}(A) \xrightarrow{(a_1,a_k)} \mathrm{end}(A)s_{(a_1,a_k)}$ is also a shortest one. Here we know that for any $v, u \in W$, all shortest directed paths from v to u in $\mathrm{QBG}(W)$ have the same weight $\mathrm{wt}(\cdot)$ (see [16, Lemma 1 (2)]). It follows that

$$\mathrm{down}(A) + \mathrm{wt}(\mathbf{p}_{a_k,j}(\mathrm{end}(A))) = \mathrm{down}(\iota(A)) + \mathrm{wt}(\mathbf{p}_{a_1,j}(\mathrm{end}(\iota(A)))).$$

Therefore, for $A \in \mathcal{A}_w^{m,a_k}$ with $k = 2, \dots, s$, we deduce that

$$(-1)^{|A|-1} q^{-\langle \lambda, \mathrm{down}(A)+\mathrm{wt}(\mathbf{p}_{a_k,j}(\mathrm{end}(A)))\rangle} \mathrm{end}(\mathbf{p}_{a_k,j}(\mathrm{end}(A)))$$
$$+ (-1)^{|\iota(A)|+1} q^{-\langle \lambda, \mathrm{down}(\iota(A))+\mathrm{wt}(\mathbf{p}_{a_1,j}(\mathrm{end}(\iota(A))))\rangle} \mathrm{end}(\mathbf{p}_{a_1,j}(\mathrm{end}(\iota(A)))) = 0.$$

This implies that

$$(6.1) = \underbrace{q^{-\langle \lambda, \mathrm{down}(\{a_1\})+\mathrm{wt}(\mathbf{p}_{a_1,j}(\mathrm{end}(\{a_1\})))\rangle} \mathrm{end}(\mathbf{p}_{a_1,j}(\mathrm{end}(\{a_1\})))}_{k=1 \text{ and } A=\{a_1\}\in\mathcal{A}_w^{m,a_1}} = q^{-\langle \lambda, \mathbf{p}_{m,j}(w)\rangle} \mathrm{end}(\mathbf{p}_{m,j}(w)),$$

as desired. This proves the proposition.

Author Contributions All authors jointly wrote the main manuscript text. Also, all authors reviewed the manuscript.

Funding T.K. was partly supported by JSPS Grant-in-Aid for Scientific Research 20J12058 and 22J00874. S.N. was partly supported by JSPS Grant-in-Aid for Scientific Research (C) 21K03198. D.O. was partly supported by a Collaboration Grant for Mathematicians from the Simons Foundation 638577.

Availability of data and materials Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflicts of interest The authors declare that they have no competing interest.

Ethical Approval Not applicable.

References

1. Brenti, F., Fomin, S., Postnikov, A.: Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. Int. Math. Res. Not. **1999**(8), 419–441 (1999)
2. Kac, V.G.: Infinite Dimensional Lie Algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
3. Kashiwara, M.: Crystal bases of modified quantized enveloping algebra. Duke Math. J. **73**(2), 383–413 (1994)
4. Kato, S.: Loop structure on equivariant K-theory of semi-infinite flag manifolds, arXiv:1805.01718
5. Kouno, T., Lenart, C., Naito, S.: New structure on the quantum alcove model with applications to representation theory and Schubert calculus, to appear in J. Comb. Algebra, arXiv:2105.02546
6. Kouno, T., Naito, S., Orr, D., Sagaki, D.: Inverse K-Chevalley formulas for semi-infinite flag manifolds, I: minuscule weights in ADE type, Forum Math. Sigma **9**, Paper No. e51 (2021)
7. Kato, S., Naito, S., Sagaki, D.: Equivariant K-theory of semi-infinite flag manifolds and the Pieri-Chevalley formula. Duke Math. J. **169**, 2421–2500 (2020)
8. Kato, S.: Frobenius splitting of Schubert varieties of semi-infinite flag manifolds. Forum Math., Pi **9**, Paper No. e56 (2021)
9. Lenart, C.: From Macdonald polynomials to a charge statistic beyond type A. J. Combin. Theory Ser. A **119**, 683–712 (2012)
10. Lenart, C., Lubovsky, A.: A generalization of the alcove model and its applications. J. Algebr. Comb. **41**(3), 751–783 (2015)
11. Lenart, C., Postnikov, A.: Affine Weyl groups in K-theory and representation theory, Int. Math. Res. Not. **2007**(12), Art. ID rnm038, 65pp. (2007)
12. Lenart, C., Naito, S., Orr, D., Sagaki, D.: Inverse K-Chevalley formulas for semi-infinite flag manifolds, II: arbitrary weights in ADE type. Adv. Math. **423**, Paper No. 109037 (2023)
13. Lenart, C., Naito, S., Sagaki, D.: A general Chevalley formula for semi-infinite flag manifolds and quantum K-theory, arXiv:2010.06143
14. Naito, S., Orr, D., Sagaki, D.: Chevalley formula for anti-dominant weights in the equivariant K-theory of semi-infinite flag manifolds, Adv. Math. **387**, Paper No. 107828 (2021)
15. Orr, D.: Equivariant K-theory of the semi-infinite flag manifold as a nil-DAHA module. Selecta Math. (N.S.) **29**(3), Paper No. 45, 26pp (2023)
16. Postnikov, A.: Quantum Bruhat graph and Schubert polynomials. Proc. Amer. Math. Soc. **133**(3), 699–709 (2005)
17. The Sage Developers, SageMath, the Sage Mathematics Software System (Version 9.1), https://www.sagemath.org (2022)