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Pseudo-parabolic Category over Quaternionic Projective Plane

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Abstract
Quaternionic projective plane $\mathbb{H}P^2$ is the next simplest conjugacy class of a complex symplectic group with pseudo-Levi stabilizer subgroup after the sphere $\mathbb{S}^4 \simeq \mathbb{H}P^1$. Its quantization gives rise to a module category $\mathcal{O}_t(\mathbb{H}P^2)$ over finite-dimensional representations of the symplectic quantum group $U_q(\mathfrak{sp}(6))$, a full subcategory in the BGG category $\mathcal{O}$. We prove that $\mathcal{O}_t(\mathbb{H}P^2)$ is semi-simple and equivalent to a category of quantized equivariant vector bundles on $\mathbb{H}P^2$.

Keywords Quaternionic Grassmannians · Quantum symplectic group · Module category · Contravariant form · Vector bundles

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1 Introduction

With every point t of a maximal torus T of a simple complex algebraic group G one can associate a full subcategory $\mathcal{O}_t$ in the BGG category $\mathcal{O}$ of the corresponding quantum group, $U_q(\mathfrak{g})$. This subcategory is additive and stable under the tensor product with the category $\mathrm{Fin}_q(\mathfrak{g})$ of finite-dimensional (quasi-classical) $U_q(\mathfrak{g})$-modules. Its objects are submodules in tensor products of $V \in \mathrm{Fin}_q(\mathfrak{g})$ with a distinguished base module M of highest weight λ depending on t. In generic situation, the locally finite part of $\mathrm{End}(M)$ is an equivariant

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quantization $\mathcal{A}$ of the coordinate ring of $C_t = \mathrm{Ad}_G(t)$, the conjugacy class of t. If $\mathcal{O}_t$ is semi-simple, then its objects can be regarded as "representations" of quantum equivariant vector bundles on $\mathrm{Ad}_G(t)$. According to the famous Serre-Swan theorem [22, 24], global sections of vector bundles on an affine variety form finitely generated projective modules over its coordinate ring and *vice versa*. Finitely generated projective right $\mathcal{A}$-modules equivariant with respect to $U_q(\mathfrak{g})$ can be viewed as quantum equivariant vector bundles. They constitute a $\mathrm{Fin}_q(\mathfrak{g})$-module category, $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$.

Equivalence of $\mathrm{Fin}_q(\mathfrak{g})$-module categories $\mathcal{O}_t$ and $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ is established via functors acting on objects as $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g}) \ni \Gamma \mapsto \Gamma \otimes_{\mathcal{A}} M \in \mathcal{O}_t$ and $\mathcal{O}_t \ni N \mapsto \mathrm{Hom}^{\circ}_{\mathbb{C}}(M, N) \in \mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$, where the circle designates the locally finite part with respect to the $U_q(\mathfrak{g})$-action. The module M is absent in the classical picture as there is no faithful irreducible representation of a classical commutative coordinate ring.

Quantization of vector bundles is a natural extension of the deformation quantization programme for Poisson manifolds [1]. Vector bundles on non-commutative spaces are of interest in the K-theory [23], non-commutative geometry [4], and non-commutative quantum field theory [5]. There is one more area of their applications in connection with quantum symmetric pairs and universal K-matrices, [8, 12]. If the class C_t is a symmetric space, then there is a one-dimensional representation of $\mathcal{A}$ (a classical point on quantized C_t). It satisfies the reflection equation [9] defining a coideal subalgebra $U_q(\mathfrak{k}') \subset U_q(\mathfrak{g})$. Then $\mathcal{A}$ can be realized as the subalgebra of $U_q(\mathfrak{k}')$-invariants in the Hopf algebra of functions on the quantum group that is dual to $U_q(\mathfrak{g})$. In the classical limit, $U_q(\mathfrak{k}')$ turns into the centralizer $U(\mathfrak{k}')$ of a point $t' \in C_t$, which is conjugate to the centralizer $U(\mathfrak{k})$ of the point t.

The representation theory of $U_q(\mathfrak{k}')$ is a challenge since $t' \notin T$ (which is fixed for a quantum group) and the triangular decomposition of $U_q(\mathfrak{g})$ is not compatible with that of $U_q(\mathfrak{k}')$, [12, 13]. The category $\mathcal{O}_t$, if semi-simple, plays the role of a bridge between $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ and the category of finite-dimensional $U_q(\mathfrak{k}')$-modules via a chain of equivalences. This is discussed in detail in [18] for quantum spheres.

Remark that an associated vector bundle in the classical geometry is obtained via induction functor from a finite dimensional representation of the stabilizer subgroup, which is a relatively simple thing. In the non-commutative world the picture is quite opposite. It is surprisingly easier to construct an apparently more complex vector bundle, and arrive at the fiber via specialization at the (quantum) initial point, if any. This transition is demonstrated for projective spaces in [20].

In the present paper we study the category $\mathcal{O}_t$ for $G = SP(6)$ and $t \in T$ one of 6 points with the stabilizer $\simeq SP(4) \times SP(2)$ (they belong to two isomorphic conjugacy classes). In this case, C_t is the quaternionic projective plane $\mathbb{H}P^2$ which enters one of the two infinite series, $\mathbb{H}P^n$, of rank 1 non-Hermitian symmetric conjugacy classes. The other series comprises even spheres and has been studied in [18]. However, the approach of [18] (as well of the last section in [19]) is special for $\mathbb{S}^{2n}$ and cannot be extended any further. The method we demonstrate here on the example of $\mathbb{H}P^2$ works for any semi-simple conjugacy class comprising elements of finite order (e.g. symmetric conjugacy classes). This method reduces the question of semi-simplicity of $\mathcal{O}_t$ to simplicity of M.

We prove that the module M is irreducible in the case of $\mathbb{H}P^2$ and explicitly construct an orthonormal basis with respect to the contravariant form on it. Our approach is based on viewing M as a module over $U_q(\mathfrak{l}) \subset U_q(\mathfrak{g})$, where $\mathfrak{l} \simeq \mathfrak{gl}(2) \oplus \mathfrak{sp}(2)$ is the maximal reductive Lie subalgebra in $\mathfrak{k}$ such that $U(\mathfrak{l})$ is quantized as a Hopf subalgebra in $U_q(\mathfrak{g})$. This is the content of Section 2.

In Section 3, we prove semi-simplicity of the category $\mathcal{O}_t$. It is an illustration of a complete reducibility criterion for tensor products of highest weight modules based on a contravariant form and Zhelobenko extremal cocycle [17, 19, 25]. We show that for every finite-dimensional quasi-classical $U_q(\mathfrak{g})$-module V the tensor product $V \otimes M$ is completely reducible and its simple submodules are in a natural bijection with simple $\mathfrak{k}$-submodules in the classical $\mathfrak{g}$-module V. This way we establish equivalence of $\mathcal{O}_t$ and $\mathrm{Fin}(\mathfrak{k})$ as Abelian categories.

In Section 4 we present a classical point on quantum $\mathbb{H}P^2$, i.e. a one-dimensional representation of $\mathcal{A}$. It is a numerical solution of the reflection equation that satisfies other relations of quantized $\mathbb{C}[\mathbb{H}P^2]$. Therein we describe the coideal subalgebra $U_q(\mathfrak{k}')$.

In the last Section 5 we establish equivalence of the category $\mathcal{O}_t$ with the category $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$.

1.1 Quantum Group $U_q(\mathfrak{sp}(6))$ and Basic Conventions

In this paper, $\mathfrak{g} = \mathfrak{sp}(6)$, $\mathfrak{k} = \mathfrak{sp}(4) \oplus \mathfrak{sp}(2)$ and $\mathfrak{l} = \mathfrak{gl}(2) \oplus \mathfrak{sp}(2)$. There are inclusions $\mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{l}$ of Lie algebras, which we describe by inclusions of their root bases as follows. Both $\mathfrak{k}$ and $\mathfrak{l}$ are reductive subalgebras of maximal rank, i.e. they contain the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Fix the inner product on $\mathfrak{h}$ such that the long root has length 2. All positive roots of $\mathfrak{g}$ are expressed in an orthonormal basis of weights $\{\varepsilon_i\}_{i=1}^3 \in \mathfrak{h}^*$ as $\mathrm{R}^+_{\mathfrak{g}} = \{\varepsilon_i \pm \varepsilon_j\}_{i<j} \cup \{2\varepsilon_i\}_{i=1}^3$. Then $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2$, and $\alpha_3 = 2\varepsilon_3$ form the basis of simple roots $\Pi_{\mathfrak{g}} = \Pi$. The basis of simple roots of $\mathfrak{k}$ is $\Pi_{\mathfrak{k}} = \{\alpha_1, 2\alpha_2 + \alpha_3, \alpha_3\}$. Note that the root $2\alpha_2 + \alpha_3$ is not in $\Pi_{\mathfrak{g}}$. In contrast with $\mathfrak{l} \subset \mathfrak{k}$ whose basis of simple roots is $\Pi_{\mathfrak{l}} = \{\alpha_1, \alpha_3\}$.

For two elements x, y of an associative algebra and a scalar a we write $[x, y]_a = xy - ayx$. We say that x and y quasi-commute if $[x, y]_a = 0$ for some $a \in \mathbb{C}$, and call the algebra quasi-commutative if this holds for all pairs of its generators.

The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}$-algebra with unit parameterized by a complex number q, which is assumed not a root of unity, [2, 3]. It is generated by simple root vectors e_i, f_i (Chevalley generators), and invertible Cartan generators q^{h_i}, $i = 1, 2, 3$. The elements $q^{\pm h_i}$ generate a commutative subalgebra $U_q(\mathfrak{h})$ in $U_q(\mathfrak{g})$ isomorphic to the polynomial algebra on a torus. They obey the following commutation relations with e_i, f_i:

$$q^{h_i} e_j = q^{(\alpha_i,\alpha_j)} e_j q^{h_i} \quad q^{h_i} f_j = q^{-(\alpha_i,\alpha_j)} f_j q^{h_i} \quad i, j = 1, 2, 3.$$

Furthermore, $[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$ for all $i, j = 1, 2, 3$. Non-adjacent positive Chevalley generators commute while adjacent generators satisfy quantum Serre relations

$$[e_i, [e_i, e_j]_q]_{\bar q} = 0, \quad i, j = 1, 2, \ i \neq j, \quad [e_2, [e_2, [e_2, e_3]_{q^2}]]_{\bar q^2} = 0, \quad [e_3, [e_3, e_2]_{\bar q^2}]_{q^2} = 0,$$

where $\bar q = q^{-1}$. Similar relations hold for the negative Chevalley generators on replacement $f_i \to e_i$, which extends to an involutive algebra automorphism σ of $U_q(\mathfrak{g})$ with $\sigma(q^{h_i}) = q^{-h_i}$.

A comultiplication defined on the generators by

$$\Delta(f_i) = f_i \otimes 1 + q^{-h_i} \otimes f_i, \quad \Delta(q^{\pm h_i}) = q^{\pm h_i} \otimes q^{\pm h_i}, \quad \Delta(e_i) = e_i \otimes q^{h_i} + 1 \otimes e_i$$

makes $U_q(\mathfrak{g})$ a Hopf algebra. The assignment $q^{h_i} \mapsto 1$, $e_i \mapsto 0$, $f_i \mapsto 0$ extends to the counit homomorphism $U_q(\mathfrak{g}) \to \mathbb{C}$, then antipode γ acts on the generators by $q^{h_i} \mapsto q^{-h_i}$, $e_i \mapsto -e_i q^{-h_i}$, $f_i \mapsto -q^{h_i} f_i$. It is an anti-algebra and anti-coalgebra automorphism of $U_q(\mathfrak{g})$.

The composition $\omega = \sigma \circ \gamma$ is an involutive automorphism of $U_q(\mathfrak{g})$ that preserves comultiplication and flips multiplication.

The Serre relations are homogeneous with respect to the $U_q(\mathfrak{h})$-grading via its adjoint action on $U_q(\mathfrak{g})$. They are determined by the corresponding weight, so we refer to a particular relation by its weight in what follows.

We remind that a total ordering on the set of positive roots is called normal if any $\alpha \in \mathrm{R}^+$ presentable as a sum $\alpha = \mu + \nu$ with $\mu, \nu \in \mathrm{R}^+$ lies between μ and ν. A reductive Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}$ of maximal rank is called Levi if it has a basis $\Pi_{\mathfrak{l}}$ of simple roots which is a part of Π. Then there is a normal ordering such that every element of $\mathrm{R}^+_{\mathfrak{g}/\mathfrak{l}}$ is preceding all elements of $\mathrm{R}_{\mathfrak{l}}$. In this paper, $\mathfrak{l}$ designates the subalgebra $\mathfrak{gl}(2) \oplus \mathfrak{sp}(2)$ as agreed upon earlier.

With a normal ordering one can associate a system $\{\tilde{f}_\alpha\}_{\alpha \in \mathcal{R}^+} \subset U_q(\mathfrak{g}_-)$ of elements such that ordered monomials in $\tilde{f}_\alpha$ form a PBW-like basis in $U_q(\mathfrak{g}_-)$. In particular, the algebra $U_q(\mathfrak{g}_-)$ is freely generated over $U_q(\mathfrak{l}_-)$ by ordered monomials in $\tilde{f}_\alpha$ with $\alpha \in \mathrm{R}^+_{\mathfrak{g}/\mathfrak{l}}$. In the classical limit, the elements $\tilde{f}_\alpha$ form a basis of root vectors in $\mathfrak{g}_-$. For a detailed construction of such a basis, the reader is referred to [2].

By $\Lambda_{\mathfrak{g}}$ we denote the weight lattice of $\mathfrak{g}$, i.e. a free Abelian group generated by fundamental weights relative to the fixed polarization of R. The semi-group of integral dominant weights is denoted by $\Lambda^+_{\mathfrak{g}}$. All $U_q(\mathfrak{g})$-modules are assumed diagonalizable over $U_q(\mathfrak{h})$. A non-zero vector v of a $U_q(\mathfrak{h})$-module V is said to be of weight $\mu \in \mathfrak{h}^*$ if $q^{h_\alpha} v = q^{(\alpha,\mu)} v$ for all $\alpha \in \Pi^+$. A weight vector is called singular if it is annihilated by all e_α, $\alpha \in \Pi^+$. Vectors of weight μ span a subspace in V denoted by $V[\mu]$. The set of weights of V is denoted by $\Lambda(V)$.

Infinitesimal character of a $U_q(\mathfrak{h})$-module is defined as a formal sum $\sum_{\mu \in \Lambda(V)} \dim V[\mu]_\mu e^\mu$. We write $\mathrm{ch}(V) \leqslant \mathrm{ch}(W)$ if $\dim V[\mu] \leqslant \dim W[\mu]$ for all μ and $\mathrm{ch}(V) < \mathrm{ch}(W)$ if this inequality is strict for some μ.

By all q we mean all not a root of unity; almost all q stands for all except for a finite set of values.

2 Base Module for $\mathbb{H}P^2$

In this section we study a $U_q(\mathfrak{g})$-module M that generates the category of our interest. We prove its irreducibility and construct an orthonormal basis with respect to a contravariant form on it.

Let ρ denote the half sum of positive roots of $\mathfrak{g}$ and κ the half-sum of the positive roots of $\mathfrak{k}$. Regard roots (more generally, integral weights) as characters of the maximal torus T of the group G (the torus has been fixed and its Lie algebra is $\mathfrak{h}$ participating in the construction of $U_q(\mathfrak{g})$). Define base weight $\lambda \in \mathfrak{h}^*$ as one featuring the property $q^{2(\lambda,\alpha)} = \alpha(t) q^{2(\kappa-\rho,\alpha)}$, for all $\alpha \in \Pi_{\mathfrak{g}}$, where $\alpha(t)$ is the value of root α on the initial point $t \in T$. It is the eigenvalue of the operator Ad_t on the corresponding root space in $\mathfrak{g}$ and, in particular, $\alpha(t) = 1$ once $\alpha \in \Pi_{\mathfrak{k}}$.

Remark that λ is evaluated on squared Cartan generators in the above equality. Therefore base weight is not uniquely determined by the point t but up to a choice of sign in $\pm\sqrt{\alpha(t)}$ for each $\alpha \in \Pi_{\mathfrak{g}}$. One can pick up any for λ, but we additionally assume $q^{(\lambda,\alpha)} = 1$ for all $\alpha \in \Pi_{\mathfrak{l}} = \Pi_{\mathfrak{g}} \cap \Pi_{\mathfrak{k}}$. This is consistent with the conditions on λ because $(\kappa, \alpha) = (\rho, \alpha) = 1$ for such α. The rational for this will be explained later.

We fix the initial point t by

$$\varepsilon_i(t) = \begin{cases} -1, & i = 1, 2, \\ 1, & i = 3, \end{cases}$$

so the base weight satisfies $q^{2(\lambda,\varepsilon_3)} = q^{(\lambda,\varepsilon_1-\varepsilon_2)} = 1$, $q^{2(\lambda,\varepsilon_1)} = q^{2(\lambda,\varepsilon_2)} = -q^{-2}$.

Set $\delta = 2\alpha_2 + \alpha_3$ and $f_\delta = f_2^2 f_3 - (q^2 + \bar{q}^2) f_2 f_3 f_2 + f_3 f_2^2$. It is easy to check that f_δ commutes with f_3 and e_3, cf. [21]. Let $\hat{M}_\lambda$ denote the Verma module with highest weight λ and define M as the quotient of $\hat{M}_\lambda$ by its submodule generated by vectors $f_1 1_\lambda$, $f_3 1_\lambda$, and $f_\delta 1_\lambda$. It is isomorphic to $U_q(\mathfrak{g}_-)/J$ as a $U_q(\mathfrak{g}_-)$-module, where $J \subset U_q(\mathfrak{g}_-)$ is the left ideal generated by f_1, f_3, f_δ.

The module M supports quantization of the conjugacy class $\mathbb{H}P^2$ in the sense that its quantized (deformed) coordinate ring $\mathbb{C}_q[\mathbb{H}P^2]$ can be represented as a $U_q(\mathfrak{g})$-invariant subalgebra in $\mathrm{End}(M)$. Its explicit formulation in terms of generators and relations is given in Section 4.

As $\mathfrak{l}$ is a Levi subalgebra in $\mathfrak{g}$, its universal enveloping algebra is quantized to a Hopf subalgebra $U_q(\mathfrak{l}) \subset U_q(\mathfrak{g})$. The module M is a quotient of the parabolic Verma module $M_\lambda^\natural$ of the same weight, by the submodule generated by (the image of) the vector $f_\delta 1_\lambda$, which is singular in $M_\lambda^\natural$. It follows that M is locally finite over $U_q(\mathfrak{l})$, [19]. We will study M regarding it as a $U_q(\mathfrak{l})$-module; then our additional requirements $(\lambda, \alpha) = 0$ for $\alpha \in \Pi_\mathfrak{l}$ will keep us within the category of quasi-classical $U_q(\mathfrak{l})$-modules (deformations of classical $U(\mathfrak{l})$-modules). However M is not quasi-classical for entire $U_q(\mathfrak{g})$.

Remark 2.1 Note that M contains a base module for the quantum 4-sphere, [21]. It is generated by the highest vector, over the natural quantum subgroup $U_q\big(\mathfrak{sp}(4)\big) \simeq U_q\big(\mathfrak{so}(5)\big)$ in $U_q\big(\mathfrak{sp}(6)\big)$. We will further refer to results on $\mathbb{S}^4$ in our study of higher pseudo-parabolic modules over $\mathbb{H}P^2$ in Section 3.

2.1 $U_q(\mathfrak{l})$-module Structure of M

It turns out that highest vectors of finite dimensional $U_q(\mathfrak{l})$-submodules in M belong to a subalgebra $\simeq U_q\big(\mathfrak{sl}(3)\big) \subset U_q(\mathfrak{g})$, which we describe next.

Set $\xi = \alpha_1 + \alpha_2 + \alpha_3$ and $\theta = \alpha_1 + 2\alpha_2 + \alpha_3$ and define root vectors

$$f_\xi = [[f_1, f_2]_{\bar{q}}, f_3]_{\bar{q}^2}, \quad f_\theta = [f_2, f_\xi]_q, \quad e_\xi = [e_3, [e_2, e_1]_q]_{q^2}, \quad e_\theta = [e_\xi, e_2]_{\bar{q}}.$$

Remark that e_ϕ is proportional to $\sigma(f_\phi)$ for $\phi = \xi, \theta$. The set $\{f_\xi, e_\xi, q^{\pm h_\xi}\}$ forms a quantum $\mathfrak{sl}(2)$-triple with $[e_\xi, f_\xi] = [2]_q [h_\xi]_q$.

Proposition 2.2 *The elements $e_2, f_2, q^{\pm h_2}, e_\xi, f_\xi, q^{\pm h_\xi}$ generate a subalgebra $U_q(\mathfrak{m}) \subset U_q(\mathfrak{g})$ isomorphic to $U_q\big(\mathfrak{sl}(3)\big)$, with the set of simple roots $\{\alpha_2, \xi\}$.*

Proof Observe that the set $\mathrm{R}_\mathfrak{m} = \{\pm\alpha_2, \pm\xi, \pm\theta\} \subset \mathfrak{h}^*$ is a root system of the $\mathfrak{sl}(3)$-type with

$$(\xi, \xi) = 2, \quad (\alpha_2, \alpha_2) = 2, \quad (\xi, \alpha_2) = -1,$$

so the commutation relations between the Cartan and simple root generators are correct. Furthermore, it is straightforward to check that $[e_2, f_\xi] = 0$ and $[e_\xi, f_2] = 0$. Finally, so long $f_\theta = [f_2, f_\xi]_q$, the Serre relations $[f_\theta, f_2]_q = 0 = [f_\xi, f_\theta]_q$ hold by (A.20) and

(A.23). This also yields the Serre relations $[e_\theta, e_2]_q = 0 = [e_\xi, e_\theta]_q$ via the involution σ. □

Remark that the subalgebra $U_q(\mathfrak{m})$ results from a Lusztig transformation of the subalgebra with the simple root basis α_1, α_2, see Appendix.

Proposition 2.3 *Vectors $\{f_2^k f_\theta^l 1_\lambda\}_{k,l\in\mathbb{Z}_+} \subset M$ are $U_q(\mathfrak{l})$-singular (killed by all e_α with $\alpha \in \Pi_\mathfrak{l}$).*

Proof Both e_1 and e_3 commute with f_2, so we check their interaction with f_θ. An easy calculation gives $[e_3, f_\theta] = 0$ and $[e_1, f_\theta] = f_\delta q^{h_1} \in JU_q(\mathfrak{h})$. Hence $f_2^k f_\theta^l 1_\lambda$ is annihilated by e_1 and e_3, by virtue of (A.21). □

Corollary 2.4 *The vector f_θ belongs to the normalizer of the left ideal J.*

Proof Indeed, $f_\delta f_\theta \in J$ by (A.21). Furthermore, $f_\theta 1_\lambda$ generates a finite-dimensional $U_q(\mathfrak{l})$-submodule in M. Since $(\lambda - \theta, \alpha_i) = 0$ for $i = 1, 3$, this submodule is trivial, hence $f_1 f_\theta$ and $f_3 f_\theta$ are in J. □

We denote by B the set $\{f_2^k f_\theta^l 1_\lambda\}_{k,l\in\mathbb{Z}_+} \subset M$. Our next objective is to show that B is a basis of the subspace of $U_q(\mathfrak{l}_+)$-invariants in M. Let $L_{k,l} \subset M$ be the $U_q(\mathfrak{l})$-submodule generated by $f_2^k f_\theta^l 1_\lambda$ and set $L = \oplus_{k,l=0}^\infty L_{k,l} \subset M$.

Introduce notation f_{ij} for $i \leqslant j$ by setting $f_{ii} = f_i$ and recursively $f_{i,j+1} = [f_{i,j}, f_{i+1}]_a$, where $a = q^{(\alpha_i+\ldots+\alpha_j,\alpha_{j+1})}$. Then Serre relations imply

$$f_1 f_2^k = [k]_q f_2^{k-1} f_{12} + q^{-k} f_2^k f_1, \quad f_3 f_2^k = -q^2 [k]_{q^2} f_2^{k-1} f_{23} + q^{2k} f_2^k f_3 \mod J \quad (2.1)$$

since f_δ commutes with f_2 and f_3. It will be also of use to write these formulas as

$$f_2^{k-1} f_{12} = \frac{1}{[k]_q} f_1 f_2^k \mod J, \quad f_2^{k-1} f_{23} = -\frac{1}{q^2 [k]_{q^2}} f_3 f_2^k \mod J. \quad (2.2)$$

Lemma 2.5 *For all $k \geqslant 2$, $f_1 f_3 f_2^k = [k]_{q^2} f_2^{k-2} \Big([k]_q f_2 f_3 f_1 f_2 - \frac{[k-1]_q [2]_q}{(1-\bar{q}^2)} f_\theta \Big) \mod J$.*

Proof Pushing f_3 and then f_1 to the right in $f_1 f_3 f_2^k$ we find it equal to

$$-q^2 f_1 [k]_{q^2} f_2^{k-1} f_{23} \mod J = -q^2 [k]_{q^2} [k-1]_q f_2^{k-2} f_{12} f_{23} - q^2 q^{-k+1} [k]_{q^2} f_2^{k-1} f_1 f_{23} \mod J,$$

where we have used (2.1). Expressing $f_{12} f_{23}$ and $f_2 f_1 f_{23}$ on the right through $f_2 f_\xi$ and f_θ modulo J we prove the lemma. □

Proposition 2.6 *L exhausts all of M.*

Proof It is sufficient to check that the $U_q(\mathfrak{l})$-submodule L is invariant under $U_q(\mathfrak{g}_-)$ as it contains 1_λ. That is so if and only it is f_2-invariant.

The elements f_{ij} with $i < j$ quasi-commute with f_k, $k = 1, 3$, unless $k = i - 1$ or $k = j + 1$. Therefore

$$f_2 L \subset U_q(\mathfrak{l}_-) f_{12} B + U_q(\mathfrak{l}_-) f_{23} B + U_q(\mathfrak{l}_-) f_\xi B + L.$$

Notice that f_{12} quasi-commutes with every power of f_2 while f_{23} quasi-commutes with it modulo J because $f_\delta \in J$ commutes with f_2 and f_3. Therefore we can further push them to the right until they hit f_θ-s and then apply (2.2). This way we prove $f_{12}B \subset L$ and $f_{23}B \subset L$, with the help of Corollary 2.4.

Furthermore, push f_ξ to the right in the third term until it hits f_θ-s, using $[f_2, f_\xi]_q = f_\theta$. Then for all $k, l \in \mathbb{Z}_+$ we get $f_\xi f_2^k f_\theta^l 1_\lambda = f_2^k f_\xi f_\theta^l 1_\lambda$ modulo L because f_θ quasi-commutes with f_2 by (A.20). But $f_\xi f_\theta^l 1_\lambda \propto f_1 f_3 f_2 f_\theta^l 1_\lambda$ because f_1 and f_3 are in J and kill $f_\theta^l 1_\lambda$ by Corollary 2.4. Applying Lemma 2.5 to $f_\xi f_2^k f_\theta^l 1_\lambda \propto f_2^k f_1 f_3 f_2 f_\theta^l 1_\lambda$ we prove $f_\xi B \subset L$. Then $f_2 L \subset U_q(\mathfrak{l}_-) f_\xi B + L \subset L$, as required. □

If follows from Proposition 2.2 that

$$[e_2, f_\theta^k] = [k]_q f_\xi f_\theta^{k-1} q^{-h_2}, \quad [e_\theta^k, f_\xi] = -q^{-(k-1)}[2]_q [k]_q e_\theta^{k-1} e_2 q^{-h_\xi}. \tag{2.3}$$

Setting $\lambda_i = (\alpha_i, \lambda)$ we get as a consequence that

$$e_2 f_2^l f_\theta^k 1_\lambda = [l]_q [\lambda_3 - l - k]_q f_2^{l-1} f_\theta^k 1_\lambda + [k]_q q^{-\lambda_2} f_2^l f_\xi f_\theta^{k-1} 1_\lambda. \tag{2.4}$$

Proposition 2.7 *The module M is irreducible.*

Proof It is sufficient to check that none of the $U_q(\mathfrak{l})$-singular vectors $f_2^l f_\theta^k 1_\lambda$ with $l+k > 0$ is killed by e_2. For $k = 0$ this is straightforward: $e_2^l f_2^l 1_\lambda = [l]_q \prod_{i=0}^{l-1} [\lambda_2 - i]_q 1_\lambda$. This never turns zero because $q^{2\lambda_2} = q^{2(\lambda, \varepsilon_2 - \varepsilon_3)} = q^{2(\lambda, \varepsilon_2)} = -q^{-2}$. For $k > 0$, the operator $e_1 e_3$ annihilates the first term in (2.4) and returns $f_2^{l+1} f_\theta^{k-1} 1_\lambda$, up to a non-zero scalar multiplier, on the second. Proceeding this way we obtain $(e_1 e_3 e_2)^k f_2^l f_\theta^k 1_\lambda \propto f_2^{k+l} 1_\lambda \neq 0$. Therefore $f_2^l f_\theta^k 1_\lambda \neq 0$ and $f_2^l f_\theta^k 1_\lambda \notin \ker(e_2)$ unless $l + k = 0$. Hence these vectors are highest for different $U_q(\mathfrak{l})$-submodules in M and none of them is singular for $U_q(\mathfrak{g})$. □

In summary, M is isomorphic to the natural $U_q\big(\mathfrak{gl}(2)\big) - U_q\big(\mathfrak{sl}(2)\big)$-bimodule $\mathbb{C}_q[\mathrm{End}(\mathbb{C}^2)]$. It is semi-simple and multiplicity free. In the classical limit, the subalgebra of $U(\mathfrak{l}_+)$-invariants in $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^2] \simeq \mathbb{C}[\mathrm{End}(\mathbb{C}^2)]$ is a polynomial algebra in two variables generated by the principal minors of the coordinate matrix, see e.g. [7]. In the quantum case, the space of $U_q(\mathfrak{l}_+)$-invariants in M is isomorphic to a polynomial algebra in quasi-commuting variables f_2, f_θ.

Corollary 2.8 *The infinitesimal character of the base module M equals $\prod_{\alpha \in \mathrm{R}^+ \setminus \mathrm{R}_{\mathfrak{k}}^+} (1 - e^{-\alpha})^{-1} e^\lambda$.*

Proof Readily follows from an isomorphism $\mathbb{C}[\mathrm{End}(\mathbb{C}^2)] \simeq U(\mathfrak{g}_-/\mathfrak{k}_-)$ of $U(\mathfrak{l})$-modules. □

2.2 Orthonormal Basis in M

A symmetric bilinear form $(.,.)$ on a $U_q(\mathfrak{g})$-module V is called contravariant if $(xv, w) = (v, \omega(x)w)$ for all $x \in U_q(\mathfrak{g})$ and all $v, w \in V$. Recall that every highest weight module over a reductive quantum group has a unique contravariant form with respect to the involution ω normalized to 1 on the highest vector. In this section we build an orthonormal basis in M, with the help of the subalgebras $U_q(\mathfrak{l})$ and $U_q(\mathfrak{m})$. It can be constructed as the Gelfand-Zeitlin basis in every $U_q(\mathfrak{l})$-submodule $L_{l,k} \subset M$, up to a common factor equal to the norm of the highest vector of $L_{l,k}$. Thus the problem essentially reduces to calculation of those norms. That is done within a $U_q(\mathfrak{m})$-submodule in M generated by 1_λ because the space of $U_q(\mathfrak{l}_+)$-invariants is in that submodule.

Proposition 2.9 *Set* $\lambda_\theta = (\lambda, \theta)$. *Then the assignment* $(l, k) \mapsto \tilde{c}_{l,k} = \langle 1_\lambda, e_\theta^k e_2^l f_2^l f_\theta^k 1_\lambda\rangle$ *is a unique function* $\mathbb{Z}_+^2 \to \mathbb{C}$ *satisfying*

$$\tilde{c}_{l,k} = -\tilde{c}_{l,k-1}[2]_q[k]_q^2 q^{-\lambda_\theta+l+1} + q^{-k}[l]_q[\lambda_2 - l + 1]_q\tilde{c}_{l-1,k}, \quad lk \neq 0,$$

and $$\tilde{c}_{l,0} = [l]_q!\prod_{i=0}^{l-1}[\lambda_2 - i]_q, \quad \tilde{c}_{0,k} = [k]_q![2]_q^k\prod_{i=0}^{k-1}[\lambda_\theta - i]_q.$$

Proof The boundary conditions easily follow from the basic relations of $U_q(\mathfrak{m})$. Uniqueness can be checked by an obvious induction on $l + k$. To prove the recurrence relation permute f_θ^k and f_2^l, then in the resulting matrix element $q^{-lk}\langle 1_\lambda, e_\theta^k e_2^l f_\theta^k f_2^l 1_\lambda\rangle$ push one copy of e_2 to the right:

$$\begin{aligned}\tilde{c}_{l,k} &= q^{-kl}\langle 1_\lambda, e_\theta^k f_\xi e_2^{l-1} f_\theta^{k-1} f_2^l 1_\lambda\rangle[k]_q q^{-\lambda_2+2l} + q^{-k}\langle 1_\lambda, e_\theta^k e_2^{l-1} f_2^{l-1} f_\theta^k 1_\lambda\rangle[l]_q[\lambda_2 - l+1]_q\\ &= -\tilde{c}_{l,k-1}[2]_q[k]_q^2 q^{-\lambda_\theta+l-1} + q^{-k}[l]_q[\lambda_2 - l + 1]_q\tilde{c}_{l-1,k}.\end{aligned}$$

This calculation is actually done in $U_q(\mathfrak{m})$. In particular, we used (2.3) and $[f_2, f_\theta]_{\bar{q}}=0$. □

Proposition 2.10 *The matrix element* $c_{l,k} = \langle f_2^l f_\theta^k 1_\lambda, f_2^l f_\theta^k 1_\lambda\rangle$ *equals* $(-1)^{l+k} q^{k(k-5)+lk+l(l-1)} \times q^{-l(\lambda,\alpha_2)}\tilde{c}_{l,k}$, *with*

$$\tilde{c}_{l,k} = [l]_q![k]_q![2]_q^k\prod_{i=0}^{l-1}[\lambda_2 - i]_q\frac{\prod_{i=0}^{l+k-1}[\lambda_\theta - i]_q}{\prod_{i=0}^{l-1}[\lambda_\theta - i]_q}. \tag{2.5}$$

Proof Let $\bar{f}_\theta \in U_q(\mathfrak{g}_-)$ be the vector obtained from f_θ by the substitution $q^{-1} \to q$. Using the formula (A.18), replace f_θ with $q^{-2}\bar{f}_\theta$ in the left argument. Then $c_{l,k}$ equals

$$\begin{aligned}\langle f_2^l f_\theta^k 1_\lambda, f_2^l f_\theta^k 1_\lambda\rangle &= (-1)^k q^{-2k}\langle f_2^l \bar{f}_\theta^k 1_\lambda, f_\theta^k f_2^l 1_\lambda\rangle\\ &= (-1)^l q^{-2k}\langle 1_\lambda, (q^{-h_\theta-4}e_\theta)^k(q^{-h_2}e_2)^l f_2^l f_\theta^k 1_\lambda\rangle\end{aligned}$$

since $\omega(\bar{f}_\theta) = -q^{-h_\theta-4}e_\theta$. One can express the right hand side through $\tilde{c}_{l,k} = \langle 1_\lambda, e_\theta^k e_2^l f_2^l f_\theta^k 1_\lambda\rangle$ and check that $\tilde{c}_{l,k}$ defined by (2.5) satisfies the conditions of Proposition (2.9). □

Note that λ_θ can be replaced with λ_2 because $q^{2\lambda_2} = -q^2 = q^{2\lambda_\theta}$.

Corollary 2.11 *The system* $y_{i,j}^{l,k} = \frac{1}{\sqrt{[2]_q^j d_{l,i} d_{l,j} c_{l,k}}} f_1^i f_3^j f_2^l f_\theta^k 1_\lambda$, *where* $l, k \in \mathbb{Z}_+$, $i, j \leqslant l$, *and* $d_{l,m} = (-1)^m q^{-m(l-m+1)}[m]_q[l-m+1]_q$, *is an orthonormal basis with respect to the contravariant form on* M.

3 Category $\mathcal{O}_t(\mathbb{H}P^2)$

While the base module M supports a representation of $\mathbb{C}_q[\mathbb{H}P^2]$, it generates a family of modules which may be regarded as "representations" of more general quantum vector bundles. This interpretation is only possible if all such modules are completely reducible: then they give rise to projective modules over $\mathbb{C}_q[\mathbb{H}P^2]$. They appear as submodules in tensor products $V \otimes M$ (representing a trivial vector bundle), for every V from the category $\mathrm{Fin}_q(\mathfrak{g})$ of finite-dimensional quasi-classical $U_q(\mathfrak{g})$-modules. Therefore the key issue is complete reducibility of tensor products $V \otimes M$. We solve this problem in the present section using a technique developed in [17, 19].

3.1 Complete Reducibility of Tensor Products

Suppose that V and Z are irreducible modules of highest weight. Each of them has a unique, upon a normalization, nondegenerate contravariant symmetric bilinear form, with respect to the involution $\omega\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$. Define a contravariant form on $V \otimes Z$ as the product of the forms on the factors. Then the module $V \otimes Z$ is completely reducible if and only if the form on $V \otimes Z$ is non-degenerate when restricted to the span of singular vectors $(V \otimes Z)^+$. Equivalently, if and only if every submodule of highest weight in $V \otimes Z$ is irreducible, [17].

For practical calculations, it is convenient to deal with the pullback of the form under an isomorphism of $(V \otimes Z)^+$ with a certain vector subspace in V (alternatively, in Z) which is defined as follows. Let $I_Z^- \subset U_q(\mathfrak{g}_-)$ be the left ideal annihilating the vector $1_\zeta \in Z$ of highest weight ζ, then $I_Z^+ = \sigma(I_Z^-)$ is a left ideal in $U_q(\mathfrak{g}_+)$. Denote by $V_Z^+ \subset V$ the kernel of I_Z^+, i.e. the subspace of vectors killed by I_Z^+. Since I_Z^+ is $U_q(\mathfrak{h})$-invariant, V_Z^+ is $U_q(\mathfrak{h})$-invariant too. There is a linear isomorphism between V_Z^+ and $(V \otimes Z)^+$ assigning a singular vector $u = v \otimes 1_\zeta + \ldots$ to any weight vector $v \in V_Z^+$, [17], Prop. 3.2. Here we suppressed the terms whose tensor Z-factors have lower weights than ζ. Note that the isomorphism $V_Z^+ \to (V \otimes Z)^+$ is "almost" $U_q(\mathfrak{h})$-equivariant: it shifts weights by ζ.

The pullback of the contravariant form under the map $V_Z^+ \to (V \otimes Z)^+$ can be expressed through the contravariant form $\langle -, - \rangle$ on V as $\langle \theta(v), w \rangle$, for a certain linear map θ on V_Z^+ with values in its dual space. We call it extremal twist defined by Z. In this paper, the contravariant form on V is always non-degenerate when restricted to V_Z^+, so we can write $\theta \in \mathrm{End}(V_Z^+)$. This operator is related with the extremal projector $p_\mathfrak{g}$, which is an element of a certain extension $\hat{U}_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$, [10]. It is constructed as follows.

A normal order on R^+ defines an embedding $\iota_\alpha\colon U_q\big(\mathfrak{sl}(2)\big) \to U_q(\mathfrak{g})$ for each $\alpha \in \mathrm{R}^+$, [2]. It acts by the assignment

$$q \to q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}, \quad e \to \tilde{e}_\alpha, \quad f_\alpha \to \tilde{f}_\alpha, \quad q^h \to q^{h_\alpha},$$

where e, f and q^h are the standard generators of $U_q\big(\mathfrak{sl}(2)\big)$ and the twiddled elements are root vectors constructed via Lusztig automorphisms, [2]. For $\psi \in \mathfrak{h}^*$, set $p_\mathfrak{g}(\psi)$, to be an ordered product

$$p_\mathfrak{g}(\psi) = \prod_{\alpha \in \mathrm{R}^+}^{<} p_\alpha\big((\psi + \rho, \alpha^\vee)\big), \tag{3.6}$$

where $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ and $p_\alpha(z)$ is the ι_α-image of

$$p(z) = \sum_{k=0}^{\infty} f^k e^k \frac{(-1)^k q^{k(z-1)}}{[k]_q! \prod_{i=1}^{k} [h + z + i]_q} \in \hat{U}_q\big(\mathfrak{sl}(2)\big), \quad z \in \mathbb{C}. \tag{3.7}$$

For generic ψ, the operator $p_\mathfrak{g}(\psi)$ is well defined and invertible on every finite-dimensional $U_q(\mathfrak{g})$-module. The specialization $p_\mathfrak{g} = p_\mathfrak{g}(0)$ is an idempotent satisfying $e_\alpha p_\mathfrak{g} = 0 = p_\mathfrak{g} f_\alpha$ for all $\alpha \in \Pi$. This idempotent is called extremal projector.

The element $p_\mathfrak{g}(\psi)$ defines a rational trigonometric operator function of weight in every weight $U_q(\mathfrak{g})$-module that is locally nilpotent over $U_q(\mathfrak{g}_+)$. We say that it is well defined if possible singularities are removable. It the case of $\psi = 0$, we then also assume that the image of $p_\mathfrak{g}(0)$ is in the space of $U_q(\mathfrak{g}_+)$-invariants, cf. [19].

Theorem 3.1 [19] *Suppose that the map $p_\mathfrak{g}(0)\colon V_Z^+ \otimes 1_\zeta \to (V \otimes Z)^+$ is well defined. Then $p_\mathfrak{g}(\zeta)$ is well defined as an operator on V_Z^+. If $p_\mathfrak{g}(\zeta)$ invetible, then $\theta = p_\mathfrak{g}^{-1}(\zeta)$.*

In the case of our concern, $p_{\mathfrak{g}} = p_{\mathfrak{g}}(0)$ is well defined, cf. Proposition 3.2 below. However, the operator $p_{\mathfrak{g}}(\zeta)$ may have poles as a function of ζ. The above theorem claims that such poles are removable. In the special case of the fundamental module $V = \mathbb{C}^6$ all weights in V_Z^+ are multiplicity free. Then $\det(\theta) \propto \prod_{\alpha\in \mathrm{R}^+}\prod_{\mu\in\Lambda(V)}\theta_\mu^\alpha$ up to a non-zero factor, with

$$\theta_\mu^\alpha = \prod_{k=1}^{l_{\mu,\alpha}} \frac{[(\zeta+\rho+\mu,\alpha^\vee)+k]_{q_\alpha}}{[(\zeta+\rho,\alpha^\vee)-k]_{q_\alpha}}. \tag{3.8}$$

Here $l_{\mu,\alpha}$ is the maximal integer k such that $\tilde{e}_\alpha^k V^+[\mu] \neq \{0\}$ for $\tilde{e}_\alpha = \iota_\alpha(e)$. We compute θ in the next section.

3.2 Extremal Twist and Extremal Projector

In this section we calculate the determinant of the extremal twist defined by the base module M using its relation to extremal projector and show that it does not vanish at all q.

Denote simple positive roots of the Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ by $\beta_1 = \alpha_1$, $\beta_2 = \delta$, $\beta_3 = \alpha_3$. The corresponding fundamental weights of $\mathfrak{k}$ are $\mu_1 = \varepsilon_1$, $\mu_2 = \varepsilon_1 + \varepsilon_2$, $\mu_3 = \varepsilon_3$. Pick up an integral dominant (with respect to $\mathfrak{k}$) weight $\xi = \sum_{s=1}^3 i_s\mu_s$ with $\vec{i} = (i_s)_{s=1}^3 \in \mathbb{Z}_+^3$ and set $\zeta = \xi + \lambda$. The Verma module $\hat{M}_\zeta$ of highest weight ζ and highest vector 1_ζ has singular vectors $\bar{F}_s^{i_s+1}1_\zeta$, where $\bar{F}_s = f_s$, $s = 1, 3$, and

$$\bar{F}_2 = \bar{q}^2\Big(f_2^2 f_3 \frac{[h_2-1]_p}{[h_2+1]_q} - f_2 f_3 f_2 [2]_q \frac{[h_2-1]_q}{[h_2]_q} + f_3 f_2^2\Big) \in \hat{U}_q(\mathfrak{b}_-).$$

That is straightforward for $\bar{F}_1^{i_1+1}1_\zeta$ and for $\bar{F}_3^{i_3+1}1_\zeta$ and follows from [18], Proposition 2.7, since 1_ζ generates a Verma submodule over the quantum subgroup $U_q(\mathfrak{sp}(4))$, cf. Remark 2.1.

Denote by $\tilde{M}_{\vec{i}}$ the quotient of $\hat{M}_\zeta$ by the submodule generated by $\{\bar{F}_s^{i_s+1}1_\zeta\}_{s=1}^3$. The projection $\hat{M}_\zeta \to \tilde{M}_{\vec{i}}$ factors through a parabolic Verma module relative to $U_q(\mathfrak{l})$: it is the quotient of $\hat{M}_\zeta$ by the submodule generated by $\{f_s^{i_s+1}1_\zeta\}_{s=1,3}$. Therefore $\tilde{M}_{\vec{i}}$ is locally finite over $U_q(\mathfrak{l})$, [18]. We use the same notation 1_ζ for the highest vector in $\tilde{M}_{\vec{i}}$.

Denote by $F_s^{i_s+1} \in U_q(\mathfrak{g}_-)$ the Shapovalov elements, i.e. the images of singular vectors $\bar{F}_s^{i_s+1}1_\zeta$ under the natural isomorphisms $U_q(\mathfrak{g}_-) \simeq \hat{M}_\zeta$, and set $E_s^{i_s+1} = \sigma(F_s^{i_s+1}) \in U_q(\mathfrak{g}_+)$. The element F_2 equals f_δ modulo $U_q(\mathfrak{g}_-)f_3$. Note with care that, contrary to $\bar{F}_2^{i_2+1}$, the elements $F_2^{i_2+1}$ are not powers of F_2.

Let $\tilde{I}_{\vec{i}}^- \subset U_q(\mathfrak{g}_-)$ denote the left ideal annihilating the highest vector in $\tilde{M}_{\vec{i}}$. and put $\tilde{I}_{\vec{i}}^+ = \sigma(\tilde{I}_{\vec{i}}^-) \subset U_q(\mathfrak{g}_+)$. These ideals are generated by $\{F_s^{i_s+1}\}_{s=1}^3$ and $\{E_s^{i_s+1}\}_{s=1}^3$, respectively, For $i = 1, 3$, the generators are simply powers of simple root vectors.

From now to the end of the section we fix $V = \mathbb{C}^6$, the smallest fundamental module of $U_q(\mathfrak{g})$. Up to non-zero scalar factors, the action of $U_q(\mathfrak{g}_+)$ on V is described by a graph

$$v_{-1} \xrightarrow{e_1} v_{-2} \xrightarrow{e_2} v_{-3} \xrightarrow{e_3} v_3 \xrightarrow{e_2} v_2 \xrightarrow{e_1} v_1 \tag{3.9}$$

where the vectors $v_{\pm i}$ of weights $\pm\varepsilon_i$, $i = 1, 2, 3$, form an orthonormal basis with respect to the contravariant form. The diagram for the $U_q(\mathfrak{g}_-)$-action is obtained by reversing the arrows in (3.9). We find from the diagram that $\ker(E_s)$ equals

$$V \ominus \mathrm{Span}\{v_{-1}, v_2\},\ s = 1, \quad V \ominus \mathrm{Span}\{v_{-2}\},\ s = 2, \quad V \ominus \mathrm{Span}\{v_{-3}\},\ s = 3. \tag{3.10}$$

Furthermore, $\ker(E_s^i)$ is entire V if $i > 1$. We denote by $\tilde{V}_{\vec{i}}^+ = \cap_{s=1}^3 \ker(E_s^{i_s+1})$ the kernel of the left ideal $\tilde{I}_{\vec{i}}^+$ in V.

Proposition 3.2 *The extremal projector $p_{\mathfrak{g}}\colon \tilde{V}_{\vec{i}}^+ \otimes 1_\zeta \to (V \otimes \tilde{M}_{\vec{i}})^+$ is well defined.*

Proof It is argued in [19] that the factors $p_\alpha(z)$ in (3.6) for $\alpha \in \mathrm{R}_{\mathfrak{l}}^+$ are regular on $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$ at $z = (\rho, \alpha^\vee)$ because all weights in $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$ are $\mathfrak{k}$- and therefore $\mathfrak{l}$-dominant.

Suppose that $\alpha \in \mathrm{R}_{\mathfrak{g}}^+ \backslash \mathrm{R}_{\mathfrak{l}}^+$ and evaluate the denominators in $p_\alpha(z)$ at $z = (\rho, \alpha^\vee)$ on a tensor of weight $\eta = \mu + \zeta$, $\mu \in \Lambda(\tilde{V}_{\vec{i}}^+)$. They contain $[z + (\eta, \alpha^\vee) + k]_{q_\alpha}$ with $k \in \mathbb{N}$. For $\alpha \in \mathrm{R}_{\mathfrak{g}}^+ \backslash \mathrm{R}_{\mathfrak{k}}^+$, such a factor is proportional to $q^x + q^{-x}$ for some $x \in \mathbb{Q}$ and does not vanish because q is not a root of unity. Therefore all factors $p_\alpha(t)$ for such α are regular at $z = (\rho, \alpha^\vee)$. Moreover, the extremal projector of the subalgebra $U_q(\mathfrak{g}^{\alpha_2})$ is well defined on $V \otimes 1_\zeta$ taking it to $\ker e_2$.

Now suppose that $\alpha \in \mathrm{R}_{\mathfrak{k}}^+ \backslash \mathrm{R}_{\mathfrak{l}}^+$. With $\xi = 0$ (i.e. $\zeta = \lambda$), the factor $[(\eta + \rho, \alpha^\vee) + k]_q$ entering $p_\alpha(t)$ is equal, upon evaluation of h_α on the subspace of weight $\eta = \mu + \zeta$ in $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$, to

$$[(\mu, \alpha^\vee) + 2 + k]_{q^2}, \quad [(\mu, \alpha^\vee) + 1 + k]_{q^2}, \quad [(\mu, \alpha^\vee) + 3 + k]_q,$$

for $\alpha = 2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 + \varepsilon_2$, respectively. They are not zero since $k > 0$ and $(\mu, \alpha^\vee) \in \{-1, 0, 1\}$ for $\mu \in \Lambda(\tilde{V}_{\vec{i}}^+)$. That is *a fortiori* true when $\xi \neq 0$ because $(\xi, \alpha^\vee) \in \mathbb{Z}_+$. Therefore such $p_\alpha(t)$ are also regular on $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$ at $t = (\rho, \alpha^\vee)$.

Thus all root factors in $p_{\mathfrak{g}}(\psi)$ are regular on $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$ at $\psi = 0$, so $p_{\mathfrak{g}}(0)$ is independent of normal ordering. For a simple root α choose an order with α on the left. Then $e_\alpha p_{\mathfrak{g}}(0) = 0$ on $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$. We already saw that for $\alpha = \alpha_2$; for $\alpha = \alpha_1, \alpha_3$ this is true because $V \otimes M_{\vec{i}}$ is locally finite over $U_q(\mathfrak{l})$ and all weights in $\tilde{V}_{\vec{i}}^+ \otimes 1_\zeta$ are dominant with respect to $\mathfrak{l}$, cf. [19], Proposition 3.6. This completes the proof. □

Thus the first condition of Theorem 3.1 is satisfied. The second condition will be secured by the following calculation.

Proposition 3.3 *For all $\xi = \sum_{s=1}^3 i_s \mu_s$ with $\vec{i} \in \mathbb{Z}_+^3$, the operator $p_{\mathfrak{g}}(\xi + \lambda)$ is invertible on $\tilde{V}_{\vec{i}}^+$.*

Proof Let us calculate θ_μ^α, which are inverse eigenvalues of the root factors constituting $p_{\mathfrak{g}}(\zeta)$, up to a non-zero factor. From (3.9) we conclude that all integers $l_{\mu,\alpha}$ in (3.8) are at most 1. Put $\zeta = \lambda + \xi$, then (3.8) reduces to $\theta_\mu^\alpha = 1$ for $l_{\alpha,\mu} = 0$ and to $\theta_\mu^\alpha = \frac{[(\zeta+\rho+\mu,\alpha^\vee)+1]_{q_\alpha}}{[(\zeta+\rho,\alpha^\vee)-1]_{q_\alpha}}$ for $l_{\alpha,\mu} = 1$. Observe that

$$\theta_{-\varepsilon_1}^{\varepsilon_1-\varepsilon_3}, \quad \theta_{\varepsilon_3}^{\varepsilon_1-\varepsilon_3}, \quad \theta_{-\varepsilon_2}^{\varepsilon_2-\varepsilon_3}, \quad \theta_{\varepsilon_3}^{\varepsilon_2-\varepsilon_3}, \quad \theta_{-\varepsilon_2}^{\varepsilon_2+\varepsilon_3}, \quad \theta_{-\varepsilon_3}^{\varepsilon_2+\varepsilon_3}, \quad \theta_{-\varepsilon_1}^{\varepsilon_1+\varepsilon_3}, \quad \theta_{-\varepsilon_3}^{\varepsilon_1+\varepsilon_3}.$$

are all of the form $\frac{\{m_1\}_q}{\{m_2\}_q}$ for some integers m_1, m_2, where $\{x\}_q = \frac{q^x+q^{-x}}{q+q^{-1}}$. They cannot turn zero as q is not a root of unity. The remaining non-trivial factors θ_μ^α are

$$\theta_{-\varepsilon_1}^{2\varepsilon_1} = \frac{[i_1 + i_2 + 2]_{q^2}}{[i_1 + i_2 + 1]_{q^2}}, \quad \theta_{-\varepsilon_2}^{2\varepsilon_2} = \frac{[i_2 + 1]_{q^2}}{[i_2]_{q^2}}, \quad \theta_{-\varepsilon_3}^{2\varepsilon_3} = \frac{[i_3 + 1]_{q^2}}{[i_3]_{q^2}},$$

$$\theta^{\varepsilon_1-\varepsilon_2}_{-\varepsilon_1}=\frac{[i_1+1]_q}{[i_1]_q}=\theta^{\varepsilon_1-\varepsilon_2}_{\varepsilon_2},\quad \theta^{\varepsilon_1+\varepsilon_2}_{-\varepsilon_1}=\frac{[i_1+2i_2+3]_q}{[i_1+2i_2+2]_q}=\theta^{\varepsilon_1+\varepsilon_2}_{-\varepsilon_3}.$$

Notice that the denominator in θ^α_μ may vanish only for $\alpha\in\Pi_{\mathfrak{k}}$. That happens if $i_s=0$ for some $s=1,2,3$. However, such μ do not belong to $\Lambda(\tilde V^+_{\vec i})$, as seen from (3.10). Since q is not a root of unity, all θ^α_μ never turn zero. Therefore $p_{\mathfrak{g}}(\zeta)$ is invertible, and $\theta=p_{\mathfrak{g}}(\zeta)^{-1}$. □

In the next section we shall see that the kernels $\tilde V^+_{\vec i}$ parameterise irreducible decompositions in a pseudo-parabolic category associated with $\mathbb{H}P^2$.

3.3 Pseudo-parabolic Category $\mathcal{O}_t(\mathbb{H}P^2)$ and its Structure

In this section we define the pseudo-parabolic category over $\mathbb{H}P^2$, prove its semi-simplicity and describe simple objects, based on the results of the previous section.

Denote by $\mathcal{O}_t(\mathbb{H}P^2)$ a full subcategory in the category $\mathcal{O}$ whose objects are submodules in $W\otimes M$, where $W\in \mathrm{Fin}_q(\mathfrak{g})$ is a quasi-classical finite-dimensional module over $U_q(\mathfrak{g})$. It is a module category over $\mathrm{Fin}_q(\mathfrak{g})$ because for every submodule $N\subset W\otimes M$ and $U\in \mathrm{Fin}_q(\mathfrak{g})$, the module $U\otimes N$ is in $U\otimes W\otimes M$.

We denote by $\mathrm{Fin}(\mathfrak{k})$ the tensor category of finite-dimensional $\mathfrak{k}$-modules. It is a module category over $\mathrm{Fin}(\mathfrak{g})$ via the restriction functor.

Let $M_{\vec i}$ denote the irreducible quotient of $\tilde M_{\vec i}$ (we will later prove that they coincide at almost all q). We call it pseudo-parabolic Verma module of the corresponding highest weight.

We define $V^+_{\vec i}$ as the kernel of the left ideal $I^+_{\vec i}=\sigma(I^-_{\vec i})$, where $I^-_{\vec i}$ is the annihilator of the highest vector in $M_{\vec i}$. Obviously $V^+_{\vec i}\subseteq \tilde V^+_{\vec i}$ because $\tilde I^+_{\vec i}\subseteq I^+_{\vec i}$. The subspace $V^+_{\vec i}$ is isomorphic to the span of singular vectors in $V\otimes M_{\vec i}$, in compliance with discussion of Section 3.1. In principle, $\tilde V^+_{\vec i}$ might be bigger than $V^+_{\vec i}$ but we shall see that they coincide for almost all q (for all if $\dim V=6$).

From now until Corollary 3.9 we assume that $V=\mathbb{C}^6$. Let $X_{\vec i}\in\mathrm{Fin}(\mathfrak{k})$, with $\vec i\in\mathbb{Z}^3_+$, denote the finite-dimensional $\mathfrak{k}$-module of highest weight $\xi=\sum_{s=1}^3 i_s\mu_s$. For each $\vec i\in\mathbb{Z}^3_+$, introduce a set of triples $\tilde I(\vec i)\subset\mathbb{Z}^3_+$:

$$\tilde I(\vec i)=\big\{(i_1\pm1,i_2,i_3),\ (i_1,i_2,i_3\pm1),\ (i_1\pm1,i_2\mp1,i_3)\big\},\tag{3.11}$$

where those with negative coordinates are excluded. Elements of $\tilde I(\vec i)$ parameterize irreducible $\mathfrak{k}$-submodules in $V\otimes X_{\vec i}$: their components are coordinates of highest weights in the basis of fundamental weights $\{\mu_s\}_{s=1}^3$.

Let $\mathrm{Fin}(\mathfrak{g}\downarrow\mathfrak{k})$ denote the subcategory of $\mathfrak{k}$-modules that are submodules in modules from $\mathrm{Fin}(\mathfrak{g})$.

Proposition 3.4 $\mathrm{Fin}(\mathfrak{g}\downarrow\mathfrak{k})\sim\mathrm{Fin}(\mathfrak{k})$.

Proof Since $\mathrm{Fin}(\mathfrak{g})$ is generated by V as a tensor category, it is sufficient to prove that for each $\vec i\in\mathbb{Z}^3_+$ the $\mathfrak{k}$-module $X_{\vec i}$ is in some tensor power of V. We do it by induction on $|\vec i|=i_1+i_2+i_3$. For $|\vec i|=0$, $X_{\vec i}$ is the trivial module $\mathbb{C}$, which is in $\mathrm{Fin}(\mathfrak{g}\downarrow\mathfrak{k})$. Suppose that the statement is proved for all $X_{\vec i}$ with $|\vec i|=m\geqslant 0$. Fix an index $\vec i$ with $|\vec i|=m+1$ and let ℓ be the minimal $s\in\{1,2,3\}$ such that $i_s>0$. We will separately consider two cases depending on the value of ℓ.

For $\ell = 1, 3$ we define $\vec{j}^\ell \in \mathbb{Z}_+^3$ by setting $j_s^\ell = i_s - \delta_{s\ell}$. Then $\vec{i} \in \tilde{I}(\vec{j}^\ell)$, as follows from (3.11). Since $|\vec{j}^\ell| = m$ by assumption, $X_{\vec{i}}$ is in $\mathrm{Fin}(\mathfrak{g} \downarrow \mathfrak{k})$.

In the case of $\ell = 2$ we consider a pair of vectors $\vec{j}, \vec{k} \in \mathbb{Z}_+^3$ via $j_s = i_s - \delta_{2s}$ and $k_s = i_s + \delta_{1s} - \delta_{2s}$ for $s = 1, 2, 3$. Since $|\vec{j}| = m$, the module $X_{\vec{j}}$ is in $\mathrm{Fin}(\mathfrak{g} \downarrow \mathfrak{k})$ by the induction assumption. Now observe from (3.11) that $\vec{k} \in \tilde{I}(\vec{j})$ and $\vec{i} \in \tilde{I}(\vec{k})$. Therefore $X_{\vec{k}}$ and $X_{\vec{i}}$ are in $\mathrm{Fin}(\mathfrak{g} \downarrow \mathfrak{k})$. This completes the proof. □

Let us denote by $f_{\beta_s}, e_{\beta_s} \in \mathfrak{k}$, $s = 1, 2, 3$, its negative and positive (classical) Chevalley generators.

Lemma 3.5 *For all $i \in \mathbb{Z}_+$, there are isomorphisms* $\ker(F_s^i) \simeq \ker(f_{\beta_s}^i)$ *and* $\ker(E_s^i) \simeq \ker(e_{\beta_s}^i)$*, $s = 1, 2, 3$, in V.*

Proof Notice that the case of $i > 1$ is easy because the kernels coincide with the whole V. The case $i = 1$ is an elementary calculation based on the diagram (3.9). □

Corollary 3.6 *The vector space $\tilde{V}_{\vec{i}}^+$ is isomorphic to $(V \otimes X_{\vec{i}})^{\mathfrak{k}_+}$.*

Proof First of all observe that $\cap_{s=1}^3 \ker(e_{\beta_s}^{i_s+1}) \simeq \cap_{s=1}^3 \ker(E_{\beta_s}^{i_s+1})$ because all weights in V are multiplicity free. Then the statement is due to the isomorphism $\tilde{V}_{\vec{i}}^+ \simeq \cap_{s=1}^3 \ker(e_{\beta_s}^{i_s+1})$ because the right-hand side is in bijection with the span of singular vectors in the $\mathfrak{k}$-module $V \otimes X_{\vec{i}}$. □

Proposition 3.7 *All modules in $\mathcal{O}_t(\mathbb{H}P^2)$ are semi-simple, and its simple objects are $U_q(\mathfrak{g})$-modules of highest weights $\lambda + \xi$, $\xi \in \Lambda_{\mathfrak{k}}^+$.*

Proof Since $V_{\vec{i}}^+ \subseteq \tilde{V}_{\vec{i}}^+$ and $M_{\vec{i}}$ is a quotient of $\tilde{M}_{\vec{i}}$, the extremal projector $p_{\mathfrak{g}}\colon V_{\vec{i}}^+ \otimes 1_\zeta \to (V \otimes M_{\vec{i}})^+$ is well defined, by Proposition 3.2. The operator $p_{\mathfrak{g}}(\zeta)$ is invertible on $V_{\vec{i}}^+$ by Proposition 3.3. Then the tensor product $V \otimes M_{\vec{i}}$ is completely reducible, for each $\vec{i} \in \mathbb{Z}_+^3$, thanks to Theorem 3.1. The highest weights of irreducible submodules are from $\lambda + \Lambda(\tilde{V}_{\vec{i}}^+ \otimes 1_{\vec{i}}) \subset \lambda + \Lambda_{\mathfrak{k}}^+$.

Now observe that every module from $\mathrm{Fin}_q(\mathfrak{g})$ can be realized as a submodule in a tensor power of V. Applying induction on $m \in \mathbb{Z}_+$ such that $M_{\vec{i}} \subset V^{\otimes m} \otimes M$ (for $m = 0$ the statement is obvious) we prove that all modules from $\mathcal{O}_t(\mathbb{H}P^2)$ are completely reducible and the weights of irreducible components are as stated. □

Since $M_{\vec{i}}$ is a quotient of $\tilde{M}_{\vec{i}}$, singular vectors in $V \otimes M_{\vec{i}}$ may have only weights $\sum_{s=1}^3 j_s \mu_s + \lambda$ with $\vec{j} \in \tilde{I}(\vec{i})$, by Proposition 3.7. Let $I(\vec{i}) \subseteq \tilde{I}(\vec{i})$ denote the subset of such triples. We aim to prove that $I(\vec{i}) = \tilde{I}(\vec{i})$.

Proposition 3.8 *For each $\vec{i} \in \mathbb{Z}_+^3$:*

1. $\mathrm{ch}(M_{\vec{i}}) = \mathrm{ch}(X_{\vec{i}})\mathrm{ch}(M)$ *for all q,*
2. *all $M_{\vec{i}}$ with $\vec{i} \in \mathbb{Z}_+^3$ are in $\mathcal{O}_t(\mathbb{H}P^2)$.*

Proof Consider $\tilde{M}_{\vec{i}}$ as a $U_q(\mathfrak{g}_-)$-module, $\tilde{M}_{\vec{i}} \simeq U_q(\mathfrak{g}_-)/\tilde{I}^-_{\vec{i}}$, which makes sense at $q = 1$ too.[1] It the classical limit $q \to 1$, it goes to a quotient of $U(\mathfrak{g}_-)$ by the left ideal generated by $f_{\beta_s}^{i_s+1}$, $s = 1, 2, 3$. Therefore

$$\mathrm{ch}(M_{\vec{i}}) \leqslant \mathrm{ch}(\tilde{M}_{\vec{i}}) \leqslant \mathrm{ch}(X_{\vec{i}})\mathrm{ch}(\mathfrak{g}_-/\mathfrak{k}_-)e^{\lambda} = \mathrm{ch}(X_{\vec{i}})\mathrm{ch}(M),$$

at generic q. That is, the inequalities hold for dimensions of subspaces of the same weight for almost all q. The set of q-s where they are violated may depend on the weight.

Suppose that $\mathrm{ch}(M_{\vec{i}}) = \mathrm{ch}(X_{\vec{i}})\mathrm{ch}(M)$ for each $M_{\vec{i}} \subset V^m \otimes M$, $m \geqslant 0$, at all q. That holds trivially for $m = 0$. The direct sum decomposition $V \otimes M_{\vec{i}} \simeq \oplus_{\vec{j}\in I(\vec{i})} M_{\vec{j}}$ implies

$$\begin{aligned}\mathrm{ch}(V)\mathrm{ch}(M_{\vec{i}}) &= \sum_{\vec{j}\in I(\vec{i})} \mathrm{ch}(M_{\vec{j}}) \leqslant \sum_{\vec{j}\in \tilde{I}(\vec{i})} \mathrm{ch}(\tilde{M}_{\vec{j}}) \leqslant \\ &\leqslant \sum_{\vec{j}\in \tilde{I}(\vec{i})} \mathrm{ch}(X_{\vec{j}})\mathrm{ch}(M) = \mathrm{ch}(V)\mathrm{ch}(X_{\vec{i}})\mathrm{ch}(M)\end{aligned} \tag{3.12}$$

for generic q, because $\oplus_{\vec{j}\in \tilde{I}(\vec{i})} X_{\vec{j}} = V \otimes X_{\vec{i}}$. We conclude that the inequalities in (3.12) are all equalities (for generic q), and, secondly, $I(\vec{i}) = \tilde{I}(\vec{i})$. In particular, for each $\vec{j}$ and each weight μ we have

$$\dim M_{\vec{j}}[\mu] = \dim \tilde{M}_{\vec{j}}[\mu] = \dim(X_{\vec{j}} \otimes M)[\mu] \tag{3.13}$$

at all q in a punctured neighbourhood of 1 (that might depend on $\vec{i}$ and μ). Then $\mathrm{ch}(M_{\vec{j}}) = \mathrm{ch}(X_{\vec{j}})\mathrm{ch}(M)$ at all q as $M_{\vec{j}}$ is simultaneously a quotient of a Verma module and is a submodule in $V^{\otimes(m+1)} \otimes M$, which are both flat at all q including $q = 1$. Induction on m proves 1) for all $M_{\vec{j}}$.

To prove 2), we use the equality $\tilde{I}(\vec{i}) = I(\vec{i})$ we have already established. That is, for each weight η of a singular vector in the $\mathfrak{k}$-module $V \otimes X_{\vec{i}}$ the pseudo-parabolic module of highest weights $\lambda+\eta$ does appear in $V \otimes M_{\vec{i}}$ (uniquely since all weights in V are multiplicity free). Again induction on m such that $V^{\otimes m} \otimes M \supset M_{\vec{i}}$ along with Proposition 3.4 secures 2). □

Corollary 3.9 *For every $V \in \mathrm{Fin}_q(\mathfrak{g})$ and for all $\vec{i}, \vec{j} \in \mathbb{Z}^3_+$, there is an isomorphism*

$$\mathrm{Hom}_{U_q(\mathfrak{g})}(M_{\vec{j}}, V \otimes M_{\vec{i}}) \simeq \mathrm{Hom}_{\mathfrak{k}}(X_{\vec{j}}, V \otimes X_{\vec{i}}).$$

Proof The equality $\mathrm{ch}(V \otimes M_{\vec{i}}) = \sum_{\vec{j}\in I} \mathrm{ch}(M_{\vec{j}})$, where the summation is over an irreducible decomposition of $V \otimes M_{\vec{i}}$, implies $\mathrm{ch}(V \otimes X_{\vec{i}}) = \sum_{\vec{j}\in I} \mathrm{ch}(X_{\vec{j}})$, thanks to Proposition 3.8. Therefore the $\mathfrak{k}$-module $\oplus_{\vec{j}\in I} X_{\vec{j}}$ is isomorphic to $V \otimes X_{\vec{i}}$ and the assertion follows. □

Now we summarise the main result of the paper.

Theorem 3.10 1. *$\mathcal{O}_t(\mathbb{H}P^2)$ is semi-simple for all q.*
2. *For all q, $\mathcal{O}_t(\mathbb{H}P^2)$ is equivalent to the category* $\mathrm{Fin}(\mathfrak{k})$.
3. *Simple objects in $\mathcal{O}_t(\mathbb{H}P^2)$ are exactly pseudo-parabolic Verma modules, for almost all q.*

[1] Although the action of $U_q(\mathfrak{g})$ on these modules does not extend to the classical point $q = 1$, they are quasi-classical as modules over $U_q(\mathfrak{g}_-)$ and equipped with an obvious grading by weights.

Proof The category $\mathcal{O}_t(\mathbb{H}P^2)$ is clearly additive. To prove the first statement, observe that a module V from $\mathrm{Fin}_q(\mathfrak{g})$ can be realized as a submodule in a tensor power of $\mathbb{C}^6$. Then apply Propositions 3.4 and 3.7.

Equivalence $\mathcal{O}_t(\mathbb{H}P^2) \sim \mathrm{Fin}(\mathfrak{k})$ as Abelian categories can be proved similarly to [18], Proposition 3.8 (cf. also Corollary 3.9 above).

We know from Propositions 3.7 and 3.8 that simple objects of $\mathcal{O}_t(\mathbb{H}P^2)$ are exactly $M_{\vec{i}}$, $\vec{i} \in \mathbb{Z}_+^3$. Let us prove that for $M_{\vec{i}} \simeq \tilde{M}_{\vec{i}}$ for all but a finite number of values of q.

Indeed, a module of highest weight is irreducible if and only if its contravariant form is non-degenerate or, alternatively, it has no singular vectors. Weights of singular vectors may be only in the orbit of the highest weight under the shifted action of the Weyl group. Let $\tilde{W} \subset \tilde{M}_{\vec{i}}$ and $W \subset M_{\vec{i}}$ denote the sums of weight spaces whose weights are in that orbit. It is sufficient to check non-degeneracy of the form only on $\tilde{W}$. Since $\tilde{W}$ is finite dimensional, there is an alternative: either the form is degenerate for all q or it is not at some and therefore almost all q. From (3.13) we see that $\tilde{W} \simeq W$ in an open neighbourhood of 1. Therefore the form is non-degenerate on $\tilde{W}$ and hence on $\tilde{M}_{\vec{i}}$ for almost all q as required. □

Note that the set of exceptional q where $M_{\vec{i}} \not\simeq \tilde{M}_{\vec{i}}$ may depend on a module. We nevertheless conjecture that it is empty for all $\vec{i}$, as is the case for the base module.

4 The Algebra $\mathbb{C}_q[\mathbb{H}P^2]$ and Reflection Equation

In this section we give a more detailed description of the quantized polynomial ring $\mathcal{A} = \mathbb{C}_q[\mathbb{H}P^2]$ and its one-dimensional representation. This is a special case of a general construction, and the reader is referred to [16, 20] for details.

Let π be the representation homomorphisms of $U_q(\mathfrak{g})$ to $\mathrm{End}(V)$, $V \simeq \mathbb{C}^6$. Pick up a basis $\{v_i\}_{i=1}^6 \subset V$ as in Section 3.2. Let ν_i denote the weight of v_i, then $\nu_i = -\nu_{i'}$, where $i' = 7 - i$. Denote $\varsigma_i = 1$ and $\varsigma_{i'} = -1$ for $i = 1, 2, 3$.

Let $\mathcal{R}$ be a universal R-matrix of $U_q(\mathfrak{g})$ such that $(\pi \otimes \mathrm{id})(\mathcal{R}) \in \mathrm{End}(\mathbb{C}^6) \otimes U_q(\mathfrak{b}_+)$ and set $\mathcal{Q} = \mathcal{R}_{21}\mathcal{R}$. It commutes with the coproduct of every element in $U_q(\mathfrak{g})$. Denote by P the flip of the tensor factors in $\mathbb{C}^6 \otimes \mathbb{C}^6$ and fix a $U_q(\mathfrak{g})$-invariant braid matrix $S \in \mathrm{End}(\mathbb{C}^6) \otimes \mathrm{End}(\mathbb{C}^6)$. Note that $R = PS$ needs not to be image of the particular $\mathcal{R}$ entering $\mathcal{Q}$: e.g. one can take $R = (\pi \otimes \pi)(\mathcal{R}_{21}^{-1})$. One can choose π and R as in [6].

It is known that $\mathbb{C}_q[G]$ can be realized as the locally finite part of the adjoint $U_q(\mathfrak{g})$-module. It is a subalgebra in $U_q(\mathfrak{g})$ generated by entries of the matrix $(\pi \otimes \mathrm{id})(\mathcal{Q})$. The image of $\mathbb{C}_q[G]$ in $\mathrm{End}(M)$ is a flat deformation of a quotient of $\mathbb{C}[G]$ by the defining ideal of $\mathbb{H}P^2$. That is a maximal proper invariant ideal in $\mathbb{C}[G]$, whence the image is a quantization of $\mathbb{C}[\mathbb{H}P^2]$, see [16] for details.

Let $\varpi \propto \sum_{i,j=1}^6 q^{\rho_i-\rho_j}\varsigma_i\varsigma_j e_{i'j}e_{ij'}$ be the invariant projector onto the trivial one-dimensional submodule in $\mathbb{C}^6 \otimes \mathbb{C}^6$. Here $\rho_i = (\rho, \nu_i) = -(\rho, \nu_{i'})$; in particular, $\rho_i = 4 - i$ for $i = 1, 2, 3$.

Let $\pi_{\vec{i}}$, $\vec{i} \in \mathbb{Z}_+^3$, denote the representation homomorphism $U_q(\mathfrak{g}) \to \mathrm{End}(M_{\vec{i}})$. The operator $(\pi \otimes \pi_{\vec{i}})(\mathcal{Q})$ has eigenvalues

$$x_\nu = q^{2(\lambda+\xi+\rho,\nu)-2(\rho,\varepsilon_1)}, \quad \nu \in \Lambda(V_{\vec{i}}^+), \tag{4.14}$$

where $\lambda + \xi$ is the highest weight of $M_{\vec{i}}$. In particular, the matrix $Q = (\pi \otimes \pi_{\vec{0}})(\mathcal{Q})$ has two eigenvalues $q^{2(\lambda+\rho,\varepsilon_1)-2(\rho,\varepsilon_1)}$ and $q^{2(\lambda+\rho,\varepsilon_3)-2(\rho,\varepsilon_1)}$ on $\mathbb{C}^6 \otimes M$ corresponding to irreducible submodules of highest weights $\varepsilon_1 + \lambda$ and $\varepsilon_3 + \lambda$. The value of its q-trace

$\mathrm{Tr}_q(Q) = \mathrm{Tr}\big(\pi(q^{2h_\rho})Q\big)$ on M can be found by the formula $\mathrm{Tr}_q(Q) = \mathrm{Tr}\big(\pi(q^{2h_\rho+2h_\lambda})\big)$, cf. [15].

The algebra $\mathcal{A}$ is generated by the entries $\{Q_{ij}\}_{i,j=1}^6$, which satisfy

$$S_{12}Q_2S_{12}Q_2 = Q_2S_{12}Q_2S_{12}, \quad Q_2S_{12}Q_2\varpi_{12} = q^{-7}\varpi_{12} = \varpi_{12}Q_2S_{12}Q_2,$$
$$(Q+q^{-2})(Q-q^{-4}) = 0, \quad \mathrm{Tr}_q(Q) = -(q^4+q^{-4}).$$

Equations of the first line are understood in $\mathrm{End}(\mathbb{C}^6)\otimes\mathrm{End}(\mathbb{C}^6)\otimes\mathrm{End}(M)$ and the subscripts label the $\mathrm{End}(\mathbb{C}^6)$-factors. They are equations of $\mathbb{C}_q[G]$, a deformation of $\mathbb{C}[G]$ that is equivariant under the conjugation action of G on itself. The last two equations fix the quantized conjugacy class $\mathbb{H}P^2$. This is the full set of relations defining $\mathcal{A}$, [16].

There is a one-dimensional representation $\chi: \mathcal{A}\to\mathbb{C}$, $Q_{ij}\mapsto A_{ij}$, where

$$A = -q^{-3}\begin{pmatrix} q-\bar q & 0 & 0 & 0 & 1 & 0\\ 0 & q-\bar q & 0 & 0 & 0 & -1\\ 0 & 0 & q & 0 & 0 & 0\\ 0 & 0 & 0 & q & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\end{pmatrix}.$$

In the classical limit, the matrix A goes over to a point $t'\in\mathbb{H}P^2$ where the Poisson bracket vanishes.

The matrix A defines an embedding of $\mathcal{A}$ in the restricted Hopf dual to $U_q(\mathfrak{g})$ that we denote by $\mathcal{T}$. A description of the algebra $\mathcal{T}$ can be extracted from [6]. Let $T = (T_{ij})_{i,j=1}^6$ denote its matrix of generators. This matrix is invertible with $(T^{-1})_{ij} = \gamma(T_{ij})$, where γ is the antipode of $\mathcal{T}$. One has two commuting left and right translation actions of $U_q(\mathfrak{g})$ on $\mathcal{T}$ expressed through the Hopf paring and the comultiplication in $\mathcal{T}$ by

$$h\triangleright a = a^{(1)}(h, a^{(2)}), \quad a\triangleleft h = (a^{(1)}, h)a^{(2)}, \quad a\in\mathcal{T}, \quad h\in U_q(\mathfrak{g}).$$

They are compatible with multiplication on $\mathcal{T}$ making it a $U_q(\mathfrak{g})$-bimodule algebra.

The assignment $Q_{ij}\mapsto (T^{-1}AT)_{ij}$ defines an equivariant homomorphism $\mathcal{A}\to\mathcal{T}$, where $\mathcal{T}$ is viewed as a $U_q(\mathfrak{g})$-module under the left translation action. It is an embedding by similar deformation arguments as with the case of $\mathcal{A}\subset\mathrm{End}(M)$. The character χ factors through the composition $\mathcal{A}\to\mathcal{T}\to\mathbb{C}$, where the right arrow is the counit ϵ.

The entries of the matrix

$$\mathcal{K} = (\mathrm{id}\otimes\pi)(\mathcal{R}_{12})A_2(\mathrm{id}\otimes\pi)(\mathcal{R}_{21})\in U_q(\mathfrak{g})\otimes\mathrm{End}(\mathbb{C}^6)$$

generate a left coideal subalgebra $U_q(\mathfrak{k}')\subset U_q(\mathfrak{g})$. It is a deformation of $U(\mathfrak{k}')$ with $\mathfrak{k}'\simeq\mathfrak{k}$ being the Lie algebra of the centralizer of t'.

One can check that $a\triangleleft b = \epsilon(b)a$ for all $b\in U_q(\mathfrak{k}')$ and $a\in\mathcal{A}$. We argue that $\mathcal{A}$ exhausts all of the subalgebra of $U_q(\mathfrak{k}')$-invariants, for generic q. Indeed, the latter is $\cap_{i,j=1}^6\ker\mathcal{K}'_{ij}$ where $\mathcal{K}'_{ij} = \mathcal{K}_{ij}-\epsilon(\mathcal{K}_{ij})\in\mathfrak{k}' \mod (q-1)$. Restricted to every isotypic component of the Peter-Weyl decomposition of $\mathcal{T}$, the kernel cannot increase in deformation.

5 Quantization of Equivariant Vector Bundles on $\mathbb{H}P^2$

In this section, we will interpret $\mathcal{O}_t(\mathbb{H}P^2)$ as a category of "representations" for quantum vector bundles on $\mathbb{H}P^2$.

In the classical algebraic geometry, global sections of vector bundles on a variety are finitely generated projective modules over its coordinate ring. If a group G acts on the bundle coherently with the base, the vector bundle is called equivariant. Algebraically it means that G acts on global functions by automorphisms, G acts on global sections, and the multiplication between functions and sections is equivariant.

In the case of homogeneous space G/K, a vector bundle $\Gamma(G/K, X)$ is characterized by a finite dimensional K-module X over the initial point. It can be realized as the space of K-invariants in $\mathbb{C}[G] \otimes X$ under right translations. The group G acts on $\Gamma(G/K, X) \simeq (\mathbb{C}[G] \otimes X)^K$ by left translations.

For a reductive pair $G \supset K$, the Peter-Weyl decomposition $\mathbb{C}[G] = \sum_{[V]} V \otimes V^*$ gives the isotypic component of an irreducible module V in $\Gamma(G/K, X)$; it is $\simeq V \otimes \mathrm{Hom}_K(X, V)$. This is the classical input that we are going to mimic in our approach to quantization.

We have already argued that the base module M supports a faithful representation of $\mathcal{A}$ as a subalgebra in the locally finite part $\mathrm{End}^\circ(M)$ of linear operators on M. Similarly we claim that the locally finite part $\mathrm{Hom}^\circ(M, M_{\vec{i}})$ of the $U_q(\mathfrak{g})$-module of linear maps from M to $M_{\vec{i}}$ is a quantization of the vector bundle $\Gamma(\mathbb{H}P^2, X_{\vec{i}})$ with fiber $X_{\vec{i}}$. Note that $\mathrm{Hom}^\circ(M, M_{\vec{i}})$ is a natural equivariant right $\mathrm{End}^\circ(M)$-module via the composition of linear maps.

Proposition 5.1 *As a $U_q(\mathfrak{g})$-module,* $\mathrm{Hom}^\circ(M, M_{\vec{i}})$ *is a deformation of* $\Gamma(\mathbb{H}P^2, X_{\vec{i}})$.

Proof Since M and $M_{\vec{i}}$ are irreducible along with their dual modules of lowest weight, equivariant maps from V to $\mathrm{Hom}(M, M_{\vec{i}})$ are in bijection with equivariant maps from $\mathrm{Hom}(M_{\vec{i}}^*, M^*)$ to V^*, for every $V \in \mathrm{Fin}_q(\mathfrak{g})$. We have a version of Corollary 3.9 for dual modules and we can write

$$\begin{aligned}\mathrm{Hom}_{U_q(\mathfrak{g})}\big(\mathrm{Hom}(M_{\vec{i}}^*, M^*), V^*\big) &\simeq \mathrm{Hom}_{U_q(\mathfrak{g})}\big(M_{\vec{i}}^*, V^* \otimes M^*\big) \simeq \mathrm{Hom}_{\mathfrak{k}}(X_{\vec{i}}^*, V^*)\\ &\simeq \mathrm{Hom}_{\mathfrak{k}}(V, X_{\vec{i}}).\end{aligned}$$

The rightmost term is isomorphic to $\mathrm{Hom}_{\mathfrak{k}}(X_{\vec{i}}, V)$ as V is completely reducible over $\mathfrak{k}$. Thus the isotypic component of V in $\mathrm{Hom}^\circ(M, M_{\vec{i}})$ is a deformation of the isotypic component of its classical counterpart in $\Gamma(\mathbb{H}P^2, X_{\vec{i}})$. $\square$

In particular, setting $M_{\vec{i}} = M$ we conclude that $\mathrm{End}^\circ(M)$ has the same module structure as $\mathcal{A}$. This implies that, for $q \neq 1$, the algebra $\mathcal{A}$ exhausts all of $\mathrm{End}^\circ(M)$. We will give a recipe for construction of $\mathrm{Hom}^\circ(M, M_{\vec{i}})$ in what follows. For each $V \in \mathrm{Fin}_q(\mathfrak{g})$ an invariant projector from $\mathrm{End}(V) \otimes \mathrm{End}(M)$ is in $\mathrm{End}(V) \otimes \mathrm{End}^\circ(M)$ and therefore in $\mathrm{End}(V) \otimes \mathcal{A}$. Such projectors can be constructed with the help of the invariant element $\mathcal{Q}$.

Lemma 5.2 *For each $\vec{i} \in \mathbb{Z}_+^3$, the operator $\mathcal{Q}$ separates irreducible components in $\mathbb{C}^6 \otimes M_{\vec{i}}$.*

Proof Let $\lambda + \xi$, $\xi \in \Lambda_{\mathfrak{k}}^+$, be the highest weight of $M_{\vec{i}}$. We will calculate eigenvalues ratio $x_\mu x_\nu^{-1}$ with $\mu \neq \nu$ using the formula (4.14). By definition of the base weight, we find

$$x_\mu x_\nu^{-1} = q^{2(\lambda+\xi+\rho, \mu-\nu)} = \alpha(t) q^{2(\kappa+\xi,\alpha)}, \quad \alpha = \mu - \nu \in \mathrm{R}_{\mathfrak{g}}.$$

Here we used the fact that all non-zero weight differences in $\mathbb{C}^6$ are roots. The right-hand side cannot turn 1 if $\alpha \in \mathrm{R}_{\mathfrak{g}} \backslash \mathrm{R}_{\mathfrak{k}}$, because it has the form $-q^{\mathbb{Z}}$, and q is not a root of unity. On the other hand, if $\alpha \in \mathrm{R}_{\mathfrak{k}}^+$, then $x_\mu x_\nu^{-1} = q^{2(\kappa+\xi,\alpha)} \neq 1$ either because $(\kappa, \alpha) > 0$ and $(\xi, \alpha) \geqslant 0$. $\square$

It turns out that the matrix Q together with intertwiners from $\mathrm{Fin}_q(\mathfrak{g})$ are enough to get all morphisms in $\mathcal{O}_t(\mathbb{H}P^2)$. The braid matrix S from the previous section produces a family of $U_q(\mathfrak{g})$-invariant operators on the tensor algebra $T(V)$ of the module $V = \mathbb{C}^6$ in the standard way, see e.g. [6]. Denote by $\mathcal{I}$ the algebra of invariant operators on $T(V) \otimes M$ generated by the matrix $Q \in \mathrm{End}(V \otimes M)$ and all invariant operators on $T(V)$.

Proposition 5.3 *$\mathcal{I}$ exhausts all of the algebra of invariant operators on $T(V) \otimes M$.*

Proof We need to show that $\mathcal{I}$ separates submodules in $V^{\otimes m} \otimes M$ for all $m \geqslant 0$. We do it by induction on m.

The assertion is true for $m = 0$ because M is irreducible. For $m = 1$ it is true because the Q separates two irreducible submodules in $V \otimes M$. Suppose that is proved for some $m \geqslant 1$ and pick up $M_{\vec{i}} \subset V^{\otimes m} \otimes M$ with the representation $\pi_{\vec{i}}\colon U_q(\mathfrak{g}) \to \mathrm{End}(M_{\vec{i}})$. Choose an invariant projector $P_{\vec{i}}\colon V^{\otimes m} \otimes M \to M_{\vec{i}}$. By induction assumption, $P_{\vec{i}}$ belongs to $\mathcal{I}$.

Observe that the image of the operator $(\mathrm{id} \otimes \Delta^m)(\mathcal{Q})$ in $\mathrm{End}(V^{\otimes m}) \otimes \mathrm{End}(M)$ belongs to $\mathcal{I}$ for all m. This readily follows from the identity $(\mathrm{id} \otimes \Delta)(\mathcal{Q}) = \mathcal{R}_{12}^{-1}\mathcal{Q}_{13}\mathcal{R}_{12}\mathcal{Q}_{23}$, which reduces $(\mathrm{id} \otimes \Delta^m)(\mathcal{Q})$ to a product of S-matrices and Q, [11]. Therefore the operator

$$\big(\pi^{\otimes(m+1)} \otimes \pi_{\vec{0}}\big)(\mathrm{id} \otimes \Delta^{m+1})(\mathcal{Q}) \times (\mathrm{id} \otimes P_{\vec{i}}) \in \mathcal{I}$$

separates irreducible submodules in $V \otimes M_{\vec{i}}$, by Lemma 5.2. This is true for each summand in the decomposition $V \otimes V^{\otimes m} \otimes M = \oplus_{\vec{i}} V \otimes M_{\vec{i}}$. Induction on m is completed. □

By construction, $\mathcal{I}$ is a subalgebra in $T\big(\mathrm{End}(V)\big) \otimes \mathcal{A}$. Applying χ to the right factor one obtains a subalgebra $\mathcal{I}_A$ of $U_q(\mathfrak{k}')$-invariant operators in $T(V)$. It is generated by the matrix A over the subalgebra of $U_q(\mathfrak{g})$-invariant operators on $T(V)$, cf. [20], Proposition 4.5. It follows from Proposition 5.3 above that $\mathcal{I}_A$ is exactly the commutant of $U_q(\mathfrak{k}')$ in $T\big(\mathrm{End}(V)\big)$.

In the remaining part of the section we prove equivalence of $\mathcal{O}_t(\mathbb{H}P^2)$ and a category $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ of $U_q(\mathfrak{g})$-equivariant projective right $\mathcal{A}$-modules. Objects in $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ are direct summands in $\mathcal{A}$-modules freely generated by $U_q(\mathfrak{g})$-modules from $\mathrm{Fin}_q(\mathfrak{g})$. Arrows in $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ are $U_q(\mathfrak{g})$-equivariant $\mathcal{A}$-module homomorphisms.

Every module N from $\mathcal{O}_t(\mathbb{H}P^2)$ is a direct summand in $V \otimes M$ for some $V \in \mathrm{Fin}_q(\mathfrak{g})$, therefore $\mathrm{Hom}^\circ(M, N)$ is a direct summand in a free equivariant $\mathcal{A}$-module $\mathrm{Hom}^\circ(M, V \otimes M) \simeq V \otimes \mathrm{Hom}^\circ(M, M) \simeq V \otimes \mathcal{A}$. The assignment $N \mapsto \mathrm{Hom}^\circ(M, N)$ is a covariant functor from $\mathcal{O}_t(\mathbb{H}P^2)$ to $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$, which we denote by $\mathfrak{H}$. It is obviously additive and respects tensor multiplication by modules from $\mathrm{Fin}_q(\mathfrak{g})$.

Lemma 5.4 *For every $N \in \mathcal{O}_t(\mathbb{H}P^2)$, the evaluation map* $\mathrm{Hom}^\circ(M, N) \otimes M \to N$, *$\phi \otimes m \mapsto \phi(m)$ factors through an isomorphism* $\mathrm{Ev}\colon \mathrm{Hom}^\circ(M, N) \otimes_{\mathcal{A}} M \to N$.

Proof First suppose that $N \neq \{0\}$ is irreducible. As the map is equivariant, its image is a submodule in N and hence coincides with N because $\mathrm{Hom}^\circ(M, N) \neq \{0\}$, by Proposition 5.1. In general, N is a direct sum of irreducibles, $N = \oplus_i N_i$. Then $\mathrm{Hom}^\circ(M, N) = \oplus_i \mathrm{Hom}^\circ(M, N_i)$, so Ev is an epimorphism. In particular, that holds true for $N = V \otimes M$ for any $V \in \mathrm{Fin}_q(\mathfrak{g})$.

On the other hand, any N can be embedded in $V \otimes M$ for some finite dimensional V. Since $\mathrm{Hom}^\circ(M, V \otimes M) \simeq V \otimes \mathcal{A}$, we conclude that $\mathrm{Hom}^\circ(M, V \otimes M) \otimes_{\mathcal{A}} M$ is isomorphic to $V \otimes M$. Thus Ev is a surjective endomorphism of completely reducible $V \otimes M$. Therefore it is injective as well as its restriction to $N \subset V \otimes M$. □

Define a functor $\mathfrak{T}$ from $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ to $\mathcal{O}_t(\mathbb{H}P^2)$ setting $\mathfrak{T}\colon \Gamma \mapsto \Gamma \otimes_{\mathcal{A}} M$ on objects and $\mathfrak{T}\colon f \mapsto f \otimes_{\mathcal{A}} \mathrm{id}_M$ on morphisms.

Proposition 5.5 *The functor $\mathfrak{T}$ is a left inverse to $\mathfrak{H}$.*

Proof Let N be a module from $\mathcal{O}_t(\mathbb{H}P^2)$ and P be an invariant projector $V \otimes M \to N$ for some $V \in \mathrm{Fin}_q(\mathfrak{g})$. As we commented after Proposition 5.1, $P \in \mathrm{End}(V) \otimes \mathcal{A}$. Then $\mathrm{Hom}^\circ(M, N)$ is isomorphic to $P(V \otimes \mathcal{A})$ and $P(V \otimes \mathcal{A}) \otimes_{\mathcal{A}} M = N$ because $\mathcal{A}M = M$.

If $f : N_1 \to N_2$ is a $U_q(\mathfrak{g})$-homomorphism and $\phi \in \mathrm{Hom}^\circ(M, N_1)$, then $\mathfrak{H}(f)(\phi) = f \circ \phi$ is a map from $\mathrm{Hom}^\circ(M, N_2)$. Then

$$\mathrm{Ev} \circ \big(\mathfrak{H}(f) \otimes_{\mathcal{A}} \mathrm{id}_M\big)(\phi \otimes_{\mathcal{A}} m) = \mathrm{Ev}\big((f \circ \phi) \otimes_{\mathcal{A}} m\big) = f\big(\phi(m)\big) = f \circ \mathrm{Ev}(\phi \otimes_{\mathcal{A}} m)$$

for all $\phi \in \mathrm{Hom}^\circ(M, N_1)$ and all $m \in M$. That is, the isomorphism Ev takes $(\mathfrak{T} \circ \mathfrak{H})(f)$ to f. □

The functor $\mathfrak{H}$ is surjective on objects up to an isomorphism. If $V \in \mathrm{Fin}_q(\mathfrak{g})$ and $P(V \otimes \mathcal{A})$ is an $\mathcal{A}$-module from $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ determined by an invariant idempotent $P \in \mathrm{End}(V) \otimes \mathcal{A}$, then $P(V \otimes \mathcal{A})$ is isomorphic to $\mathrm{Hom}^\circ(M, N)$ with $N = P(V \otimes M) \in \mathcal{O}_t(\mathbb{H}P^2)$ because $\mathcal{A} \simeq \mathrm{End}^\circ(M)$. Every module from $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ can be presented this way.

Theorem 5.6 *The* $\mathrm{Fin}_q(\mathfrak{g})$*-module categories* $\mathcal{O}_t(\mathbb{H}P^2)$ *and* $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ *are equivalent.*

Proof We have seen that $\mathfrak{H}$ is surjective on objects and injective on morphisms. By [14], Theorem IV.4.1, we are left to check that it is surjective on morphisms.

Suppose that $G : \Gamma_1 \to \Gamma_2$ is a morphism in $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$. We can assume that $\Gamma_i = \mathfrak{H}(N_i)$ for some $N_i \in \mathcal{O}_t(\mathbb{H}P^2)$, $i = 1, 2$. Denote by $\jmath_i : N_i \to V_i \otimes M$ and by $\wp_i : V_i \otimes M \to N_i$ their embeddings and projections, respectively, such that $\wp_i \circ \jmath_i = \mathrm{id}_{N_i}$. They give rise to embeddings and projections $\mathfrak{H}(\jmath_i) : \Gamma_i \to V_i \otimes \mathcal{A}$ and $\mathfrak{H}(\wp_i) : V_i \otimes \mathcal{A} \to \Gamma_i$, satisfying $\mathfrak{H}(\wp_i) \circ \mathfrak{H}(\imath_i) = \mathrm{id}_{\Gamma_i}$, for $i = 1, 2$.

Consider a morphism $F = \mathfrak{H}(\jmath_2) \circ G \circ \mathfrak{H}(\wp_1)$ from $V_1 \otimes \mathcal{A}$ to $V_2 \otimes \mathcal{A}$. It implies that $G = \mathfrak{H}(\wp_2) \circ F \circ \mathfrak{H}(\jmath_1)$. An equivariant map $F : (V_1 \otimes 1_{\mathcal{A}}) \to V_2 \otimes \mathcal{A}$ gives rise to an equivariant map $f \in V_1 \otimes M \to V_2 \otimes M$ because $\mathcal{A} \simeq \mathrm{End}^\circ(M)$. Then $F = \mathfrak{H}(f)$, and $G = \mathfrak{H}(\wp_2 \circ f \circ \jmath_1)$, hence $\mathfrak{H}$ is bijective on morphisms. This completes the proof. □

Note that the category of general projective $\mathcal{A}$-modules is not semi-simple because the quotient of two projectives is not necessarily so.

The presence of a one-dimensional representation $\chi : \mathcal{A} \to \mathbb{C}$ from the previous section enables a realization of $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ via quantized functions on the group G. This construction is a deformation of the classical realization of associated vector bundles. Define $\mathrm{Fin}_q(\mathfrak{k}')$ as the category of modules that are submodules of modules from $\mathrm{Fin}_q(\mathfrak{g})$. It is a $\mathrm{Fin}_q(\mathfrak{g})$-module category as $U_q(\mathfrak{k}')$ is a coideal subalgebra in $U_q(\mathfrak{g})$.

Given $X \in \mathrm{Fin}_q(\mathfrak{k}')$ define the associated bundle with fiber X as the subspace of $U_q(\mathfrak{k}')$-invariants in $\mathcal{T} \otimes X$. It is in $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$ because for all $V \in \mathrm{Fin}_q(\mathfrak{g})$ there is a natural bijection between $U_q(\mathfrak{g})$-invariant idempotents in $\mathrm{End}(V) \otimes \mathcal{A}$ and $U_q(\mathfrak{k}')$-invariant projectors on V, cf. [20]. The inverse functor acts by $\Gamma \mapsto \Gamma \otimes_{\mathcal{A}} \mathbb{C}$ for $\Gamma \in \mathrm{Pr}_q(\mathcal{A}, \mathfrak{g})$. This yields an equivalence between $\mathrm{Fin}_q(\mathfrak{k}')$ and $\mathrm{Pr}_q(\mathcal{A}, \mathfrak{g}) \sim \mathcal{O}_t(\mathbb{H}P^2)$, which obviously respects the action of $\mathrm{Fin}_q(\mathfrak{g})$.

Appendix

In this technical section, we derive some identities in the algebra $U_q(\mathfrak{g}_-)$ which are needed for this exposition.

Lemma A.1 *Define* $\bar f_\theta$ *obtained from* f_θ *by replacement* $q\to\bar q$. *Then*

$$f_\theta = [f_2,[[f_1,f_2]_{\bar q},f_3]_{\bar q^2}]_q = \bar q[[f_1,f_2]_{\bar q},[f_2,f_3]_{q^2}]_{\bar q}, \tag{A.15}$$

$$\bar f_\theta = [f_2,[[f_1,f_2]_{q},f_3]_{q^2}]_{\bar q} = q[[f_1,f_2]_{q},[f_2,f_3]_{\bar q^2}]_{q}. \tag{A.16}$$

Proof We will use a modified Jacobi identity

$$[x,[y,z]_a]_b = [[x,y]_c,z]_{\frac{ab}{c}} + c[y,[x,z]_{\frac{b}{c}}]_{\frac{a}{c}}, \tag{A.17}$$

which holds true for any elements x,y,z of an associative algebra and any scalars a,b,c with invertible c. This can be verified by a direct calculation.

Now let us prove the right equality in (A.15). Apply (A.17) to $[f_2,[[f_1,f_2]_{\bar q},f_3]_{\bar q^2}]_q$ choosing $c=\bar q$:

$$[f_2,[[f_1,f_2]_{\bar q},f_3]_{\bar q^2}]_q = [[f_2,[f_1,f_2]_{\bar q}]_{\bar q},f_3] + \bar q[[f_1,f_2]_{\bar q},[f_2,f_3]_{q^2}]_{\bar q}.$$

The first summand vanishes thanks to the Serre relation of weight $-(2\alpha_2+\alpha_1)$ whence (A.15) follows. Then (A.16) follows from (A.15) by replacement $q\to\bar q$. □

Lemma A.2 *One has*

$$qf_\theta + \bar q\bar f_\theta = [f_1,f_\delta]\in J, \tag{A.18}$$

Proof Apply (A.17) to $[f_1,f_\delta]=[f_1,[f_2,[f_2,f_3]_{q^2}]_{\bar q^2}]$ choosing $c=\bar q$. Then

$$[f_1,f_\delta] = [[f_1,f_2]_{\bar q},[f_2,f_3]_{q^2}]_{\bar q} + \bar q[f_2,[f_1,[f_2,f_3]_{q^2}]_q]_{\bar q}.$$

The first summand is qf_θ from (A.15). In the second summand, replace $[f_1,[f_2,f_3]_{q^2}]_q$ with $[[f_1,f_2]_q,f_3]_{q^2}$, then it becomes $\bar q\bar f_\theta$ from (A.18). □

Other identities of interest can be also derived from the Serre relations a with the use of the modified Jacobi identity (A.17). We will give another proof based on Lusztig's braid group automorphisms of $U_q(\mathfrak g)$, [2].

Proposition A.3 *The following relations hold true in* $U_q(\mathfrak g_-)$*:*

$$[f_3,f_\theta]=0=[f_3,\bar f_\theta], \tag{A.19}$$

$$f_2f_\theta=\bar q f_\theta f_2,\quad f_2\bar f_\theta = q\bar f_\theta f_2, \tag{A.20}$$

$$f_\delta f_\theta = \bar q^2 f_\theta f_\delta, \tag{A.21}$$

$$f_\nu f_\theta = qf_\theta f_\nu, \tag{A.22}$$

$$f_\xi f_\theta = qf_\theta f_\xi, \tag{A.23}$$

where $f_\nu=[f_1,f_2]_{\bar q}$.

Proof Let T_i be Lusztig automorphisms of $U_q(\mathfrak g)$ corresponding to simple reflections $\sigma_i: \mathrm{R}\to\mathrm{R}$ relative the simple roots α_i, as in [2]. They satisfy braid group relations, of which we will need only

$$T_2T_3T_2T_3 = T_3T_2T_3T_2.$$

In particular, $f_\nu = T_2^{-1}(f_1)$ and $T_3^{-1}(f_2)=[f_2,f_3]_{\bar q^2}$ which implies

$$T_3^{-1}T_2^{-1}T_3^{-1}(f_1) = T_3^{-1}(f_\nu) = f_\xi,$$

because $T_3^{-1}(f_1) = f_1$. Set $w = T_3^{-1}T_2^{-1}T_3^{-1}$, then

$$w(f_1) \propto f_\xi, \quad w(f_2) = f_2, \quad wT_2^{-1}(f_1) \propto w(f_\nu) \propto f_\theta, \quad wT_2^{-1}(e_3) \propto q^{-h_3} f_3.$$

The first equality has been checked. The second equality is fulfilled because $\sigma_3\sigma_2\sigma_3(\alpha_2) = \alpha_2$. The third formula follows from the first two as w is an algebra automorphism. The last one readily follows from the equality $T_2^{-1}T_3^{-1}T_2^{-1}(e_3) = e_3$ as a result of $T_3^{-1}(e_3)$, cf. [2].

Applying wT_2^{-1} to a commuting pair (e_3, f_1) one gets the left equality (A.19) because $(\theta, \alpha_3) = 0$. Applying w to a quasi-commuting pair (f_2, f_ν), one gets the left equality in (A.20). The right equalities in (A.19) and (A.20) result from replacement $q \to q^{-1}$. Then (A.20) follows since f_δ comprises two f_2-factors and one f_3-factor. To prove (A.22), apply $T_2^{-1}T_3^{-1}$ to a quasi-commuting pair of f_1 and $f_\nu \simeq T_2^{-1}T_3^{-1}(f_1)$, using the equality $T_3^{-1}(f_1) = f_1$ and the braid relation. The formula (A.23) is obtained by applying w to quasi-commuting f_2 and f_ν. □

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