# Contravariant Finiteness and Iterated Strong Tilting

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## Abstract

Let $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ be the category of finitely generated left modules of finite projective dimension over a basic Artin algebra $\Lambda$. We develop a widely applicable criterion that reduces the test for contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in $\Lambda$-mod to corner algebras $e\Lambda e$ for suitable idempotents $e \in \Lambda$. The reduction substantially facilitates access to the numerous homological benefits entailed by contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. The consequences pursued here hinge on the fact that this finiteness condition is known to be equivalent to the existence of a strong tilting object in $\Lambda$-mod. We moreover characterize the situation in which the process of strongly tilting $\Lambda$-mod allows for unlimited iteration: This occurs precisely when, in the category mod-$\widetilde{\Lambda}$ of right modules over the strongly tilted algebra $\widetilde{\Lambda}$, the subcategory of modules of finite projective dimension is in turn contravariantly finite; the latter condition can, once again, be tested on suitable corners $e\Lambda e$ of the original algebra $\Lambda$. In the (frequently occurring) positive case, the sequence of consecutive strong tilts, $\widetilde{\Lambda}, \widetilde{\widetilde{\Lambda}}, \widetilde{\widetilde{\widetilde{\Lambda}}}, \ldots$, is shown to be periodic with period 2 (up to Morita equivalence); moreover, any two adjacent categories in the sequence $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$, $\mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-mod})$, $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\widetilde{\widetilde{\Lambda}}}), \ldots$, alternating between right and left modules, are dual via contravariant Hom-functors induced by tilting bimodules which are strong on both sides. Our methods rely on comparisons of right $\mathcal{P}^{<\infty}$-approximations in the categories $\Lambda$-mod, $e\Lambda e$-mod and the Giraud subcategory of $\Lambda$-mod determined by $e$; these interactions hold interest in their own right. In particular, they underlie our analysis of the indecomposable direct summands of strong tilting modules.

**Keywords** Projective dimension · Contravariantly finite subcategory · Finitistic dimension conjecture · Strong tilting module · Corner algebra · Giraud subcategoy · Morita context

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# 1 Introduction

*Co- and contravariant finiteness* of a subcategory $\mathcal{A}$ of the category $\Lambda$-mod of finitely generated modules over an Artin algebra $\Lambda$ were first considered by Auslander and Smalø in [2] and [3]: If $\mathcal{A} = \mathrm{add}(\mathcal{A})$ is closed under extensions, the combination of these two finiteness conditions implies the existence of internal almost split sequences in $\mathcal{A}$. Here we focus on the full subcategory $\mathcal{A} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ consisting of the modules of finite projective dimension; in this situation contravariant finiteness implies the dual property, covariant finiteness [14]. (See Section 2 for notation and definitions of the italicized terms.) Subsequently, it was found that contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ entails a plethora of additional homological benefits for $\Lambda$-mod and the unrestricted module category $\Lambda$-Mod. Namely, this finiteness condition not only validates the finitistic dimension conjectures for left $\Lambda$-modules, i.e., confirms that l.findim $(\Lambda) = $ l.Findim$(\Lambda) < \infty$ in that case, but gives rise to an intrinsic description of the $\Lambda$-modules of finite projective dimension; see [1] and [14].

Crucial to our present investigation are the following facts: $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod if and only if $\Lambda$-mod contains a *strong tilting module*, i.e., a tilting module $T$ which is relatively Ext-injective within the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Such a module $T$ is alternatively characterized by the condition that the contravariant functor $\mathrm{Hom}_\Lambda(-, T)$ induces a duality between $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and a specifiable resolving subcategory of $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$, where $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ (see [12] and [13]). In this situation we say that $\mathrm{mod}\text{-}\widetilde{\Lambda}$ results from $\Lambda$-mod via strong tilting. Finally, we recall that, up to isomorphism, $\Lambda$-mod has at most one *basic* strong tilting module; see [1].

In the present article we tackle the foremost obstacle that stands in the way of applying the theory we sketched: namely, the notoriously difficult task of deciding whether, for specific choices of $\Lambda$, the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. The driving goals of our investigation are to significantly reduce this difficulty and to explore the possibility and effect of iterating the process of strongly tilting $\Lambda$-mod. In both directions we make headway, next to retrieving known results in a simpler, more uniform format.

In more detail: In the first of our main results we address an Artin algebra $\Lambda$ for which $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is known to be contravariantly finite in $\Lambda$-mod. In general, $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$ need not inherit contravariant finiteness. However, whenever it does, the process of strongly tilting $\Lambda$-mod may be iterated in the following sense: $\Lambda$-mod *allows for unlimited iteration of strong tilting* if there exists an infinite sequence of basic algebras $(\Lambda_i)_{i\geq 0}$ with $\Lambda_0 = \Lambda$ such that $\mathrm{mod}\text{-}\Lambda_{i+1}$ results from $\Lambda_i$-mod via strong tilting if $i$ is even, and $\Lambda_{i+1}$-mod results from $\mathrm{mod}\text{-}\Lambda_i$ via strong tilting if $i$ is odd.

**Theorem A** *(For a more complete version, see Theorem 3.1 and Corollary 3.4.)* *Suppose that $\Lambda$ is a basic Artin algebra such that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. Let $_\Lambda T$ be the corresponding basic strong tilting module and $\widetilde{\Lambda} = End_\Lambda(T)^{op}$.*

*If $\mathcal{P}^{<\infty}(mod\text{-}\widetilde{\Lambda})$ is contravariantly finite in $mod\text{-}\widetilde{\Lambda}$, the category $\Lambda$-mod allows for unlimited iteration of strong tilting. The resulting sequence of basic Artin algebras $\Lambda = \Lambda_0$, $\Lambda_1$, $\Lambda_2$, $\Lambda_3$, ... is ultimately periodic with period 2: $\Lambda_i \cong \Lambda_j$ whenever $i$ and $j$ are positive integers with the same parity. Their categories of modules of finite projective dimension are linked by dualities*

$$\mathcal{P}^{<\infty}(mod\text{-}\Lambda_1) \leftrightarrow \mathcal{P}^{<\infty}(\Lambda_2\text{-}mod) \leftrightarrow \mathcal{P}^{<\infty}(mod\text{-}\Lambda_3) \leftrightarrow \cdots$$

*induced by the contravariant Hom-functors of the corresponding basic strong tilting bimodules $_{\Lambda_i}(T_i)_{\Lambda_{i+1}}$, resp. $_{\Lambda_{i+1}}(T_i)_{\Lambda_i}$.*

*In general, $\Lambda_0$ need not be isomorphic to $\Lambda_2$ however.*

To return to the problem of determining the contravariant finiteness status of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in the first place, we suppose that $e_1, \ldots, e_n$ form a complete set of orthogonal primitive idempotents of $\Lambda$ and that the subsum $e = e_1 + \cdots + e_m$ includes all idempotents $e_i$ that give rise to simple left modules of infinite projective dimension. Our second and third main results refer to this setting.

**Theorem B** *(For a complete version, see Theorem 4.6.) Adopt the notation of the preceding paragraph, and suppose that $e\Lambda(1 - e)$ has finite projective dimension as a left module over the corner algebra $e\Lambda e$. Then $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod if and only if $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$ is contravariantly finite in $e\Lambda e$-mod.*

Theorem 4.6 also spells out how, in case of existence, the *minimal (right)* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-*approximation* of $M \in \Lambda$-mod relates to the minimal $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$-approximation of $eM$. The ensuing possibility of shucking off primitive idempotents in checking for contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ not only provides effortless access to most of the cases in which this property has previously been established (formerly involving considerable effort), but yields contravariant finiteness in far more general situations.

Our argument relies on an exploration of the *torsion-torsionfree triple* $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ which is associated to the idempotent $e$, and on the *Giraud subcategory* $\mathcal{G}$ of $\Lambda$-Mod (in the sense of Gabriel [8]) corresponding to the hereditary torsion pair $(\mathcal{T}, \mathcal{F})$.

In order to extend Theorem B to an efficient test of whether the stronger conclusions of Theorem A hold for $\Lambda$, we once more assume that $\Lambda$ is a basic Artin algebra such that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite. As before, we let $e_1, \ldots, e_n \in \Lambda$ be a complete set of primitive idempotents, $T \in \Lambda$-mod the basic strong tilting module, and $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\operatorname{op}}$. By $S_i$ we denote the simple left $\Lambda$-module corresponding to $e_i$. It is well-known that the indecomposable direct summands $T_1, \ldots, T_n$ of $_\Lambda T$ coincide, up to isomorphism, with the distinct indecomposable direct summands of $\bigoplus_{1 \le i \le n} \mathcal{A}_i$, where $\mathcal{A}_i$ is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of the indecomposable injective left module with socle $S_i$. Our analysis of the $T_i$ will pave the road towards showing that the question of unlimited iterability of strong tilting of $\Lambda$-mod can in turn be played back to the corner algebra $e\Lambda e$ for any idempotent $e$ as specified in Theorem B.

**Theorem C** *Let $m \le n$ be chosen such that $\operatorname{pdim}_\Lambda S_j < \infty$ for $j > m$. Set $e = e_1 + \cdots + e_m$ and assume that $\operatorname{pdim}_{e\Lambda e} e\Lambda(1 - e) < \infty$. Then:*

(1) (Proposition 5.2.) *The number of distinct indecomposable direct summands of $\bigoplus_{1 \le i \le m} \mathcal{A}_i$ is $m$. Denote them by $T_1, \ldots, T_m$ and set $T' = \bigoplus_{1 \le i \le m} T_i$.*
    *For $j \ge m + 1$, the approximation $\mathcal{A}_j$ of $E(S_j)$ decomposes in the form $\mathcal{A}_j = T_j \oplus U_j$, where $T_j$ is indecomposable with the property that $S_j$ is the only simple module of finite projective dimension in $\operatorname{soc} T_j$, and all indecomposable direct summands of $U_j$ occur as direct summands of $T'$.*

(2) (For more detail, see Theorem 5.4, and Corollaries 5.5, 5.6.) *$\Lambda$-mod allows for unlimited iteration of strong tilting if and only if the same is true for $e\Lambda e$-mod.*

Our proof of Theorem C is based on connections between $T \in \Lambda$-mod and the strong tilting objects in $e\Lambda e$-mod and in the Giraud subcategory $\mathcal{G}$.

Earlier results for truncated path algebras (i.e., for algebras of the form $KQ/I$, where $K$ is a field, $Q$ a quiver, and $I$ the ideal generated by the paths of length $L + 1$ for some $L \ge 1$), showcase the effects of iterated strong tilting and the associated dualities among

the $\mathcal{P}^{<\infty}$-categories encountered along the iterations [12]; these findings readily follow from the above theorems. In fact, the present results yield substantial generalizations of the picture that arose in the truncated case (Propositions 6.1, 6.2, Corollaries 6.3 and 6.4). From our reduction technique it also follows that, for any left serial algebra $\Lambda$, both $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$ are contravariantly finite in $\Lambda$-mod and mod-$\Lambda$, respectively (the former fact had been known; see [5]). Further applications address Artin algebras arising from Morita contexts, such as algebras of triangular matrix type (Theorem 6.6, Corollary 6.7, Examples 6.9), and the elimination of simple modules of low projective dimension in the test for contravariant finiteness (Proposition 5.7).

Section 2 assembles minimal conceptual background and builds the tools required for proving our main theorems. Section 3 provides a general characterization of the situation in which $\Lambda$-mod allows for unlimited iteration of strong tilting. The results targeting tests for contravariant finiteness of $\Lambda$-mod and for the availability of repeated strong tilts of $\Lambda$-mod are contained in Sections 4 and 5, respectively. In Section 6, we specify applications.

## 2 Notation, Background and Auxiliaries

Throughout, $\Lambda$ will be a basic Artin algebra, and $J$ its Jacobson radical. We point out that the restriction to basic algebras does not affect the generality of our investigation; we adopt it for increased transparency of the underlying ideas. $\Lambda$-Mod and $\Lambda$-mod stand for the categories of all (resp., all finitely generated) left $\Lambda$-modules. Further, $S_1, \ldots, S_n$ will be isomorphism representatives of the simple objects in $\Lambda$-Mod, and top $M$, soc $M$ will stand for the top and socle of $M \in \Lambda$-Mod, respectively. By $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ (resp., $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$) we denote the full subcategory of $\Lambda$-Mod (resp., $\Lambda$-mod) having as objects the modules of finite projective dimension. Moreover, for any finitely generated $\Lambda$-module $M$, we denote by add($M$) the full subcategory of $\Lambda$-mod consisting of the direct summands of finite direct sums of copies of $M$. The module $M$ is *basic* if it has no indecomposable direct summands of multiplicity $> 1$.

### 2.1 Contravariant Finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and Strong Tilting Modules

Following Miyashita [17], we call a left $\Lambda$-module $T$ a *tilting module* in case **(1)** $T$ belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, **(2)** $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for $i \geq 1$, and **(3)** there exists an exact sequence $0 \rightarrow {}_\Lambda\Lambda \rightarrow T_0 \rightarrow \cdots \rightarrow T_m \rightarrow 0$ with $T_j \in$ add($T$). It is well known that any basic tilting module has precisely $n = \mathrm{rank}\, K_0(\Lambda)$ indecomposable direct summands, and that any tilting module $T \in \Lambda$-mod gives rise to a left-right symmetric situation as follows: If $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$, then the right $\widetilde{\Lambda}$-module $T_{\widetilde{\Lambda}}$ is in turn a tilting module and $\mathrm{End}_{\widetilde{\Lambda}}(T)$ is canonically isomorphic to $\Lambda$. This justifies reference to a *tilting bimodule* ${}_\Lambda T_{\widetilde{\Lambda}}$. Strong tilting modules were first considered by Auslander and Reiten in [1]. We introduce them via a characterization equivalent to the original definition (see [1]): Namely, we call a tilting module $T$ *strong* if it satisfies the following relative injectivity condition in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$: **(4)** $\mathrm{Ext}^i(M, T) = 0$ for all $M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $i \geq 1$. It was shown by Auslander and Reiten [loc.cit.] that $\Lambda$-mod has a strong tilting module if and only if the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is *contravariantly finite in* $\Lambda$-mod, a property which we will recall next. Moreover, according to [loc.cit], in case of existence, there is precisely one basic strong tilting module in $\Lambda$-mod, up to isomorphism.

The concept of contravariant finiteness of subcategories of $\Lambda$-mod has its roots in work of Auslander and Smalø [2]: Namely, the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is *contravariantly finite in*

$\Lambda$-mod provided that for each object $M \in \Lambda$-mod, the functor $\mathrm{Hom}_\Lambda(-, M)_{|\mathcal{P}^{<\infty}(\Lambda\text{-mod})}$ is finitely generated in the category of additive contravariant functors $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \to \mathbf{Ab}$. This condition translates into the following requirement for arbitrary $M \in \Lambda$-mod: There is an object $A \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$, together with a map $\phi \in \mathrm{Hom}_\Lambda(A, M)$, such that each map in $\mathrm{Hom}_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), M)$ factors through $\phi$. Any such pair $(A, \phi)$ is called a (*right*) $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-*approximation of $M$*. Since we will only consider right approximations, we will frequently omit the qualifier "right". By a mild abuse of terminology, we will moreover refer to the domain $A$ of $\phi$ as a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $M$. Suppose $M$ has a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation, say $\phi : A \to M$. As was shown by Auslander-Smalø in [loc.cit.], up to isomorphism, there is only one approximation $\phi : \mathcal{A}(M) \to M$ such that $\mathcal{A}(M)$ has minimal length. It is alternatively characterized by the condition that any endomorphism $u$ of $\mathcal{A}(M)$ which satisfies $\phi \circ u = \phi$ is an automorphism; we say that the map $\phi$ is (right) *minimal* if this implication holds. Since it is unlikely to cause misunderstandings, we will also refer to $\mathcal{A}(M)$ as "the" minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $M$ whenever convenient.

The mentioned existence result by Auslander-Reiten will be crucial in the sequel. We state it for easy reference.

**Theorem 2.1** (1)   [1] *There exists a strong tilting module $T \in \Lambda$-mod if and only if $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. In the positive case, the basic strong tilting module is unique up to isomorphism: it is the direct sum of the distinct indecomposable modules $C \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ which satisfy $Ext^i_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), C) = 0$ for $i \geq 1$.*

(2)   [12, Supplement II in Section 2.A] *A more explicit description of the indecomposable direct summands of a strong tilting module $T$, when it exists, can be obtained from the following: Let $\mathcal{A}$ be the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of the minimal injective cogenerator of $\Lambda$-Mod. Then $add(T) = add(\mathcal{A})$.*

### 2.2 The TTF-triple Associated to an Idempotent $e$ in $\Lambda$

We fix an idempotent element $e \in \Lambda$. By Jans [16] (see also [19, Section VI.8]), $e$ defines a TTF triple $(\mathcal{C}_e, \mathcal{T}_e, \mathcal{F}_e)$ in the category $\Lambda$-Mod; this means that the pairs $(\mathcal{C}_e, \mathcal{T}_e)$ and $(\mathcal{T}_e, \mathcal{F}_e)$ are both torsion pairs. The torsion, resp. torsionfree, classes are as follows:

(a)   $\mathcal{C}_e$ consists of the $\Lambda$-modules $C$ generated by $\Lambda e$, i.e., the modules of the form $C = \Lambda e C$.

(b)   $\mathcal{T}_e$ consists of the left $\Lambda$-modules annihilated by $e$.

(c)   $\mathcal{F}_e$ consists of the $\Lambda$-modules $F$ with the property that the annihilator $\mathrm{ann}_F(e\Lambda)$ of $e\Lambda$ in $F$ is zero.

Observe that the torsion pair $(\mathcal{T}_e, \mathcal{F}_e)$ is hereditary, whence the corresponding torsion radical is left exact (see [19, Proposition VI.3.1]). On the other hand, $(\mathcal{C}_e, \mathcal{T}_e)$ fails to be hereditary in general.

**Further notation:** Since we will keep the idempotent $e$ fixed, we will more briefly write

$$(\mathcal{C}, \mathcal{T}, \mathcal{F}) \text{ for } (\mathcal{C}_e, \mathcal{T}_e, \mathcal{F}_e).$$

The torsion radicals $\nabla$ and $\Delta$ associated to the pairs $(\mathcal{C}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are the idempotent subfunctors of the identity functor on $\Lambda$-Mod given by:

$$\nabla(M) = \Lambda e M \quad \text{and} \quad \Delta(M) = \mathrm{ann}_M(e\Lambda) = \{x \in M \mid e\Lambda x = 0\},$$

respectively. Note that $\nabla(M)$ is the largest submodule of $M$ with the property that all simple modules in the top of $\nabla(M)$ belong to $\mathcal{F}$ (equivalently, $\mathrm{top}\big(\nabla(M)\big)$ belongs to $\mathrm{add}(\Lambda e/Je)$), whereas $\Delta(M)$ is the largest submodule of $M$ which is annihilated by $e$ (equivalently, all simple subfactors of $\Delta(M)$ belong to $\mathrm{add}\big(\Lambda(1-e)/J(1-e)\big)$).

A third functor $\Lambda\text{-Mod} \to \Lambda\text{-Mod}$ will serve to render the constructions in Sections 4 and 5 more transparent. It assigns to each $\Lambda$-module $M$ the following subfactor of $M$:

$$\mathfrak{core}(M) \;=\; \nabla(M)/\Delta\big(\nabla(M)\big) \;=\; \Lambda eM/\Delta(\Lambda eM).$$

If $M$ is finitely generated, the core of $M$ has maximal length among the subfactors $V/U$ of $M$ such that the top and socle of $V/U$ belong to $\mathcal{F}$, i.e., such that $\mathrm{top}(V/U), \mathrm{soc}(V/U) \in \mathrm{add}(\Lambda e/Je)$. In fact, it can easily be seen that this maximality property determines $\mathfrak{core}(M)$ up to isomorphism. Observe moreover, that $\mathcal{C} \cap \mathcal{F}$ consists of the $\Lambda$-modules which coincide with their cores.

*Example 2.2* To illustrate these endofunctors, we let $\Lambda = KQ/\langle R\rangle$, where $K$ is a field, $Q$ the quiver
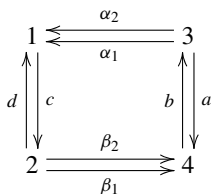


and $R = \{ab, cd, dc, \beta_1 c, \beta_2 c,\} \cup \{\alpha_k b\beta_l \colon k, l \in \{1, 2\}\}$.

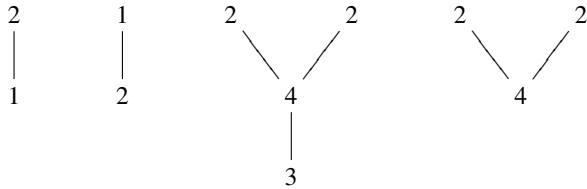We consider the TTF triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ associated to $e = e_1 + e_2$ and apply the functors $\nabla, \Delta, \mathfrak{core}$ to the indecomposable injective left $\Lambda$-modules shown below:
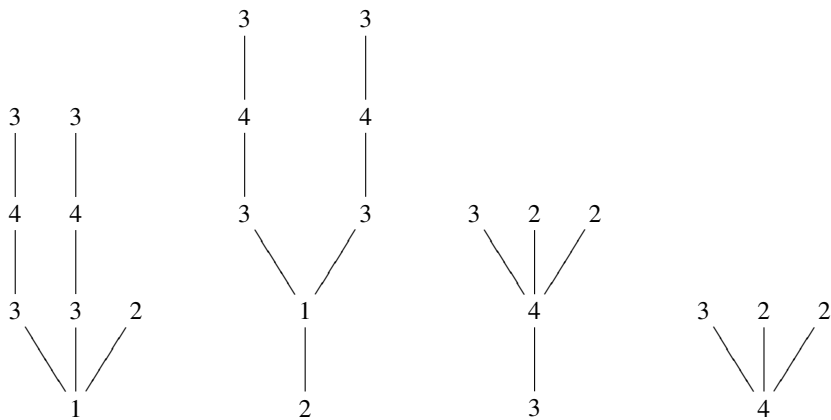


(1)   $\Delta(I_j) = 0$ (i.e. $I_j \in \mathcal{F}$) for $j = 1, 2$. The submodules $\Delta(I_3)$ of $I_3$ and $\Delta(I_4)$ of $I_4$ are depicted by the following diagrams:

(2) The submodules $\nabla(I_j)$, for $j = 1, ..., 4$, are depicted by the diagrams

$$
\begin{array}{ccccc}
2 & 1 & 2 \qquad 2 & 2 \qquad 2 \\
| & | & \diagdown \;\; \diagup & \diagdown \;\; \diagup \\
1 & 2 & 4 & 4 \\
& & | & \\
& & 3 &
\end{array}
$$

(3) Finally, $\mathfrak{core}(I_j) = \nabla(I_j)$, for $i = 1, 2$, and $\mathfrak{core}(I_3) \cong S_2 \oplus S_2 \cong \mathfrak{core}(I_4)$.

We point out that, in general, the core of a module may also have simple composition factors annihilated by $e$; for a broader spectrum of examples, see [12, 13].

### 2.3 The Adjoint Pair ($\Lambda e \otimes_{e\Lambda e} -, e$)

By $\mathbf{e}$ we denote the functor $\Lambda$-Mod $\to e\Lambda e$-Mod which sends $M$ to $eM$. The two functors of the title thus play the role of induction and restriction in the exchange of information between $e\Lambda e$-modules on one hand and $\Lambda$-modules on the other. Accordingly, $\Lambda e \otimes_{e\Lambda e} - :$ $e\Lambda e$-Mod $\to \Lambda$-Mod is left adjoint to $\mathbf{e}$.

The unit corresponding to this adjunction is the obvious natural transformation

$$\eta : 1_{e\Lambda e\text{-Mod}} \longrightarrow \mathbf{e} \circ (\Lambda e \otimes_{e\Lambda e} -), \quad U \longmapsto e(\Lambda e \otimes_{e\Lambda e} U).$$

Clearly, $\eta$ is an isomorphism of functors, whence $\Lambda e \otimes_{e\Lambda e} -$ is fully faithful (see [11, Proposition II.7.5]).

The counit $\epsilon : (\Lambda e \otimes_{e\Lambda e} -) \circ \mathbf{e} \to 1_{\Lambda\text{-Mod}}$ of the adjunction is given by the family $(\epsilon_M)_{M \in \Lambda\text{-Mod}}$, where $\epsilon_M : \Lambda e \otimes_{e\Lambda e} eM \to M$ sends $a \otimes x$ to $ax$. Note that the image $\mathrm{Im}(\epsilon_M)$ equals $\nabla(M)$.

We briefly explore the functor "restriction followed by induction", $\Lambda$-Mod $\to \Lambda$-Mod, which sends $M \in \Lambda$-Mod to $M^{\ddagger} := \Lambda e \otimes_{e\Lambda e} eM$. Evidently, this functor sees only the core of a $\Lambda$-module $M$; indeed, $M^{\ddagger}$ is naturally isomorphic to $\mathfrak{core}(M)^{\ddagger}$. On the other hand, it preserves this core, as the following lemma shows.

**Lemma 2.3** *The functor* $\mathfrak{core} : \Lambda$-Mod $\to \Lambda$-Mod *is naturally equivalent to the functor*

$$\big(\Lambda e \otimes_{e\Lambda e} e(-)\big)/\Delta\big(\Lambda e \otimes_{e\Lambda e} e(-)\big).$$

*More precisely, the counit $\epsilon$ of the adjunction induces a family of isomorphisms*

$$\bar{\epsilon}_M : \mathfrak{core}(M^{\ddagger}) = (\Lambda e \otimes_{e\Lambda e} eM)/\Delta(\Lambda e \otimes_{e\Lambda e} eM) \; \cong \; \mathfrak{core}(M), \quad M \in \Lambda\text{-Mod},$$

*which is natural in $M$.*

*Proof* Clearly, $\epsilon$ gives rise to a family of epimorphisms

$$\rho_M : \Lambda e \otimes_{e\Lambda e} eM \to \nabla(M)/\Delta(\nabla(M)) = \mathfrak{core}(M),$$

and $\mathrm{Ker}(\rho_M)$ contains $\Delta(\Lambda e \otimes_{e\Lambda e} eM)$. That, conversely, $\mathrm{Ker}(\rho_M)$ is contained in $\Delta(\Lambda e \otimes_{e\Lambda e} eM)$ follows from the facts that $\mathbf{e}(\rho_M) : e\Lambda e \otimes_{e\Lambda e} eM \to e\big(\mathfrak{core}(M)\big) = eM$ is an isomorphism in $e\Lambda e$-mod and $\Delta(\Lambda e \otimes_{e\Lambda e} eM)$ is the largest submodule of $\Lambda e \otimes_{e\Lambda e} eM$ which is annihilated by $e$. We conclude that the maps $\bar{\epsilon}_M$ are indeed isomorphisms. $\qquad\square$

## 2.4  The Giraud Subcategory of $\Lambda$-Mod **Corresponding to the Torsion Pair** $(\mathcal{T}, \mathcal{F})$

We apply well-known facts about localization with respect to a hereditary torsion class $\mathcal{T}$ to the specialized situation where $\mathcal{T} = \mathcal{T}_e$. We refer the reader to [9] and [19, Chapter IX] for detail.

Recall that the *Giraud subcategory* $\mathcal{G}$ of $\Lambda$-Mod relative to the hereditary torsion pair $(\mathcal{T}, \mathcal{F})$ is a realization, inside $\Lambda$-Mod, of the quotient category $\Lambda$-Mod$/\mathcal{T}$; here we identify the torsion class $\mathcal{T}$ with the (full) localizing subcategory of $\Lambda$-Mod that has object class $\mathcal{T}$. In particular, $\mathcal{G}$ is a Grothendieck category. Concretely, $\mathcal{G}$ is the full subcategory of $\Lambda$-Mod whose objects are the torsionfree $\Lambda$-modules $F$ with the following restricted injectivity property: $\mathrm{Ext}^1_\Lambda (X, F) = 0$ for all (cyclic) torsion modules $X \in \mathcal{T}$.

It is well known that the fully faithful inclusion functor $\iota : \mathcal{G} \to \Lambda$-Mod has an exact left adjoint $\sigma : \Lambda$-Mod $\to \mathcal{G}$, which is identifiable with the quotient functor $\Lambda$-Mod $\to \Lambda$-Mod$/\mathcal{T}$; in particular, the pair $(\mathcal{G}, \sigma)$ has the universal property of such a quotient. We will write $M_\sigma$ for $\sigma(M)$. In parallel, $f_\sigma = \sigma(f)$ is the map $M_\sigma \to N_\sigma$ induced by $f \in \mathrm{Hom}_\Lambda (M, N)$.

To describe $M_\sigma$ up to isomorphism, we abbreviate $M/\Delta(M)$ by $\overline{M}$ and let $E(\overline{M})$ be an injective envelope of $\overline{M}$. Then $M_\sigma$ is the preimage under the canonical map $E(\overline{M}) \to E(\overline{M})/\overline{M}$ of the torsion submodule $\Delta\big(E(\overline{M})/\overline{M}\big)$ of the quotient. In particular, $\sigma$ is the identity on the objects of $\mathcal{G}$.

The explicit description of $\sigma : \Lambda$-Mod $\to \Lambda$-Mod reveals that this functor, in turn, sees only the core of a $\Lambda$-module M, i.e., $M_\sigma = \mathfrak{core}(M)_\sigma$, and that it preserves cores, meaning that $\mathfrak{core}(M_\sigma)$ is canonically isomorphic to $\mathfrak{core}(M)$; in fact, $M_\sigma$ is an essential extension of $\mathfrak{core}(M)$ which is maximal relative to the requirement that this core be preserved. The following alternative incarnations of the category $\mathcal{G}$ will be helpful in Sections 4 and 5.

**Lemma 2.4** *Suppose* $e = e_1 + \cdots + e_m$ *is a decomposition of* $e$ *into primitive idempotents.*

1. [19, Proposition XI.8.6] *The categories* $\mathcal{G}$ *and* $e\Lambda e$-Mod *are equivalent. Quasi-inverse equivalences send* $F \in \mathcal{G}$ *to* $eF$ *in one direction, and send* $X \in e\Lambda e$-Mod *to* $(\Lambda e \otimes_{e\Lambda e} X)_\sigma$ *in the other.*

   *The indecomposable projective objects of* $\mathcal{G}$ *are* $(\Lambda e_i)_\sigma$ *for* $1 \le i \le m$, *and the indecomposable injectives are* $\big(E(S_i)\big)_\sigma = E(S_i)$ *for* $1 \le i \le m$.
2. [8] *The subcategory* $\mathcal{C} \cap \mathcal{F}$ *of* $\Lambda$-Mod *is in turn equivalent to* $\mathcal{G}$. *Quasi-inverse equivalences send* $M = \mathfrak{core}(M)$ *to* $M_\sigma$; *in reverse,* $F \in \mathcal{G}$ *is sent to* $\mathfrak{core}(F) = \nabla(F)$. *In particular, the functors* $\sigma$ *and* $\sigma \circ \mathfrak{core}$ *from* $\Lambda$-Mod *to* $\Lambda$-Mod *are naturally isomorphic.*

   *The indecomposable projective objects of* $\mathcal{C} \cap \mathcal{F}$ *are* $\mathfrak{core}(\Lambda e_i) = \Lambda e_i / \Delta(\Lambda e_i)$ *for* $1 \le i \le m$, *and the indecomposable injectives are* $\mathfrak{core}\big(E(S_i)\big) = \nabla\big(E(S_i)\big)$ *for* $1 \le i \le m$.

We add some notation: The unit of the adjunction $(\sigma, \iota)$ is the natural transformation

$$\mu = (\mu_M) : 1_{\Lambda\text{-Mod}} \longrightarrow \iota \circ \sigma, \quad \text{with } \mu_M \in \mathrm{Hom}_\Lambda (M, M_\sigma) \text{ canonical.}$$

If $M$ is torsionfree, we identify $\mu_M$ with the inclusion map $M \hookrightarrow M_\sigma \subseteq E(M)$. Clearly, $\mu_M$ is an isomorphism precisely when $M$ belongs to $\mathcal{G}$.

*Remark 2.5* All the mentioned functors linking the subcategories of $\Lambda - \text{Mod}$ (resp. of $e\Lambda e - \text{Mod}$) introduced in Sections 2.2–2.4 restrict to functors connecting the pertinent intersections of these subcategories with $\Lambda - \text{mod}$ (resp. $e\Lambda e - \text{mod}$).

*Example 2.6* (Return to Example 2.2.) In this instance, $(I_j)_\sigma = I_j$ for $j = 1, 2$, because $I_1$ and $I_2$ are injective objects of $\mathcal{F}$, and consequently of $\mathcal{G}$. Now let $j \in \{3, 4\}$. In either case, $I_j/\Delta(I_j) \cong S_2 \oplus S_2$, whence $E_j := E(I_j/\Delta(I_j)) \cong I_2 \oplus I_2$. Since the torsion submodule of $(I_2 \oplus I_2)/(S_2 \oplus S_2)$ is zero, we obtain $(I_j)_\sigma = S_2 \oplus S_2$, and the map $\mu_{I_j} : I_j \longrightarrow (I_j)_\sigma = S_2 \oplus S_2$ is the obvious projection.

## 2.5 The Poset of Essential $\Delta$-extensions of a Morphism

In constructing $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of $\Lambda$-modules $M$ from $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$-approximations of $eM$, passage to maximal extensions of the type described in this subsection will be crucial. Throughout we refer to the torsion theory $(\mathcal{T}, \mathcal{F})$ introduced in 2.2.

**Definition** Given a morphism $f : M \longrightarrow Y$ in $\Lambda$-mod, we consider the following *eligible extensions of $f$*: These are the pairs $(L, g)$, where $L$ is an essential extension of $M$ with the additional property that $L/M \in \mathcal{T}$ and $g \in \text{Hom}_\Lambda(L, Y)$ satisfies $g_{|M} = f$.

We say that two eligible pairs $(L, g)$ and $(L', g')$ are *isomorphic* if there is an isomorphism $\psi : L \to L'$ such that $g' \circ \psi = g$. An *essential $\Delta$-extension of $f$* is the isomorphism class $[(L, g)]$ of an eligible pair $(L, g)$.

The set $\mathcal{E}_f$ of all essential $\Delta$-extensions $[(L, g)]$ of $f$ is a poset under the following partial order: $[(L, g)] \preceq [(L', g')]$ in case there is a monomorphism $\psi : L \longrightarrow L'$ such that $g' \circ \psi = g$.

We comment on the legitimacy of the final definition: Welldefinedness of the relation $\preceq$ is clear. To check that it is antisymmetric, it suffices to observe that the existence of monomorphisms $L \longrightarrow L'$ and $L' \longrightarrow L$ forces the finitely generated $\Lambda$-modules $L$ and $L'$ to have the same length, whence monomorphisms between them are isomorphisms.

**Proposition 2.7** *Suppose that $M$ and $Y$ are finitely generated torsionfree left $\Lambda$-modules, and let $f \in Hom_\Lambda(M, Y)$. Then the poset $(\mathcal{E}_f, \preceq)$ has a maximum.*

- *In case $Y \in \mathcal{G}$, this maximum is $[(M_\sigma, f_\sigma : M_\sigma \longrightarrow Y_\sigma = Y)]$.*
- *For a general torsionfree module $Y$, the maximum of $\mathcal{E}_f$ is $[(N, \phi)]$, where $\phi : N \longrightarrow Y$ results from the pullback of $(f_\sigma, \mu_Y)$:*

$$
\begin{array}{ccc}
N & \xrightarrow{\phi} & Y \\
\chi \downarrow & & \downarrow \mu_Y \\
M_\sigma & \xrightarrow{f_\sigma} & Y_\sigma
\end{array}
$$

*Proof* In light of the hypothesis $\Delta(M) = 0$, we have $M \subseteq M_\sigma \subseteq E(M)$ for an injective envelope $E(M)$ of $M$, where the first inclusion coincides with the map $\mu_M : M \hookrightarrow M_\sigma$ of 2.3. Due to the equality $M_\sigma/M = \Delta(E(M)/M)$, we find $M_\sigma$ to be the largest among the submodules $M'$ of $E(M)$ which contain $M$ and satisfy $M'/M \in \mathcal{T}$. Consequently, all

elements of $\mathcal{E}_f$ are represented by pairs $(L, g)$ with $M \subseteq L \subseteq M_\sigma$, and we may restrict our focus to such representatives. Conversely, whenever $h : N \to Y$ is a homomorphism with $M \subseteq N \subseteq M_\sigma$ and $h_{|M} = f$, the pair $(N, h)$ represents a class in $\mathcal{E}_f$.

To address our first claim, we note that the lengths of increasing sequences in $\mathcal{E}_f$ are bounded from above by the composition length of $E(M)$. In particular, each element of $\mathcal{E}_f$ is majorized by a maximal element in this set. Therefore, we only need to prove that any two elements $[(L, g)]$ and $[(N, h)]$ of $\mathcal{E}_f$ have a common upper bound. Consider the restrictions $g_{|X}, h_{|X} : X \longrightarrow Y$ to the intersection $X = L \cap N$. Since the difference $g_{|X} - h_{|X}$ vanishes on $M$, it induces a homomorphism $X/M \longrightarrow Y$, which in turn vanishes because $X/M \in \mathcal{T}$ and $Y \in \mathcal{F}$. The resulting equality $g_{|X} = h_{|X}$ guarantees that the assignment $\rho : L + N \longrightarrow Y$ with $\rho(l+m) = g(l) + h(m)$ is a well-defined map in $\mathrm{Hom}_\Lambda(L + N, Y)$. In light of $L + N \subseteq M_\sigma$, the pair $(L + N, \rho)$ represents an element in $\mathcal{E}_f$ which majorizes both $[(L, g)]$ and $[(N, h)]$.

Now suppose that $Y \in \mathcal{G}$. Then $f_\sigma : M_\sigma \to Y_\sigma \cong Y$ gives rise to an essential $\Delta$-extension $[(M_\sigma, f_\sigma)]$. That this extension is maximal in $\mathcal{E}_f$ is immediate from the description of $M_\sigma$.

To prove the final claim, we note that the map $\chi$ in the pullback diagram is an injection since $\mu_Y$ is. Thus the submodule $\mu_M^{-1}(\chi(N))$ of $M_\sigma$ is isomorphic to $N$. On replacing $N$ by this copy and adjusting $\phi$ and $\chi$ accordingly, we obtain another pullback diagram for $(f_\sigma, \mu_Y)$ with $M \subseteq N \subseteq M_\sigma$, where $\chi$ is now replaced by the inclusion map. This shows that $(N, \phi)$ is an eligible pair in the sense of the above definition. To check maximality of $[(N, \phi)]$ in $\mathcal{E}_f$, suppose that this $\Delta$-extension is majorized by $[(N', \phi')]$, where $M \subseteq N' \subseteq M_\sigma$; let $\psi : N \hookrightarrow N'$ be an embedding with $\phi = \phi' \circ \psi$. Since $\psi$ extends to an automorphism of $E(M)$, we do not lose generality in viewing $\psi$ as a set inclusion. We thus obtain a commutative diagram, in which $\kappa$ denotes the inclusion map $N' \hookrightarrow M_\sigma$:
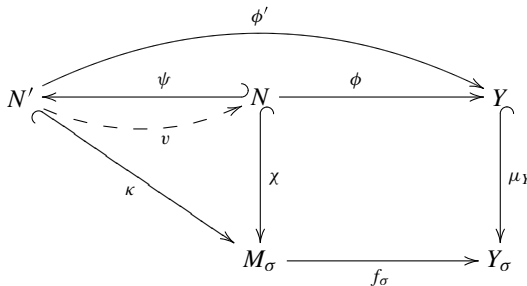


Then $f_\sigma \circ \kappa \circ \psi = \mu_Y \circ \phi' \circ \psi$, whence the restrictions of $f_\sigma \circ \kappa$ and $\mu_Y \circ \phi'$ to the domain $N$ of $\psi$ coincide. Since the epimorphic image $N'/N$ of $N'/M$ is a torsion module, while $Y_\sigma$ is torsionfree, the argumentation backing the first claim shows that $f_\sigma \circ \kappa = \mu_Y \circ \phi'$. The universal property of the pullback therefore yields a map $v \in \mathrm{Hom}_\Lambda(N', N)$ with $\phi' = \phi \circ v$ and $\kappa = \chi \circ v$. In particular, $v$ is a monomorphism. We infer that $N$ and $N'$ have the same length and conclude that $\psi$ is an isomorphism. This proves $[(N', \phi')] = [(N, \phi)]$ as required.                                                                                                         $\square$

## 3 Iteration of Strong Tilting

When can the process of strongly tilting $\Lambda$-mod to a category of right modules, mod-$\widetilde{\Lambda}$, be iterated? In the positive case, how do the resulting sequences of strongly tilted module categories behave? The main purpose of this section is to answer these questions. More specifically, we will show that, whenever $\Lambda$-mod can be strongly tilted twice, the process can be iterated arbitrarily and turns periodic after the initial step. Roughly speaking: In case $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is in turn contravariantly finite in mod-$\widetilde{\Lambda}$, the initial transition from $\Lambda$ to $\widetilde{\Lambda}$ increases the homological symmetry by increasing the number of simple modules of finite projective dimension. This symmetrization makes the subsequent sequence mod-$\widetilde{\Lambda} \rightsquigarrow \widetilde{\widetilde{\Lambda}}$-mod $\rightsquigarrow \cdots$ periodic with period 2.

The statement of the following theorem is based on part (1) of Theorem 2.1.

**Theorem 3.1** *Let $\Lambda$ be a basic Artin algebra such that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. Moreover, let $_\Lambda T$ be the corresponding basic strong tilting module, and $\widetilde{\Lambda} = End_\Lambda(T)^{op}$. Suppose that $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is in turn contravariantly finite in mod-$\widetilde{\Lambda}$, thus giving rise to a basic tilting bimodule $_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$ which is strong in mod-$\widetilde{\Lambda}$. Then $\widetilde{T}$ is strong on both sides.*

*In particular, the process of strongly tilting $\Lambda$-mod then allows for unlimited iteration, yielding a sequence of basic Artin algebras $\Lambda = \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \ldots$ with the property that $\Lambda_i$ and $\Lambda_j$ are isomorphic whenever $i$ and $j$ are positive integers with the same parity.*

*Moreover, the algebras $\Lambda_0$ and $\Lambda_2$ are isomorphic precisely when the tilting bimodule $_\Lambda T_{\widetilde{\Lambda}}$ is strong on both sides, i.e., when $T = \widetilde{T}$.*

*Comment 3.2* If, in the hypotheses of Theorem 3.1, the qualifiers "basic" are dropped, one obtains Morita equivalent pairs of algebras $(\Lambda_i, \Lambda_{i+2})$ for all $i \geq 1$.

*Proof* By a result of Auslander and Green (see [1, Proposition 6.5] and [6, Proposition 7.1] for a short argument), a tilting bimodule $_A U_B$ over Artin algebras $A$ and $B$ is strong in $A$-mod if and only if all simple right $B$-modules embed into $U_B$. In light of the hypothesis that $_\Lambda T$ is a strong tilting module, we thus find that all simple right $\widetilde{\Lambda}$-modules are contained in soc $T_{\widetilde{\Lambda}}$, up to isomorphism. Moreover, we deduce that strongness of $_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}$ as a tilting object in $\widetilde{\widetilde{\Lambda}}$-mod will follow if we can show that all simple right $\widetilde{\Lambda}$-modules embed into $\widetilde{T}_{\widetilde{\Lambda}}$.

To realize such an embedding, let $(\widetilde{\Lambda}/\widetilde{J})_{\widetilde{\Lambda}} = \widetilde{S}_1 \oplus \cdots \oplus \widetilde{S}_n$, where the $\widetilde{S}_i$ are simple. Further denote by $E(\widetilde{S}_1 \oplus \cdots \oplus \widetilde{S}_n)$ an injective envelope, and let $\widetilde{\mathcal{A}}$ be a minimal $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation of the latter. In light of our hypothesis that $\widetilde{T}_{\widetilde{\Lambda}}$ is a strong tilting object in mod-$\widetilde{\Lambda}$, the categories add($\widetilde{T}_{\widetilde{\Lambda}}$) and add($\widetilde{\mathcal{A}}$) coincide; see Theorem 2.1(2). Embed $T_{\widetilde{\Lambda}}$ into an injective right $\widetilde{\Lambda}$-module, say $\iota : T_{\widetilde{\Lambda}} \hookrightarrow \bigoplus_{1 \leq i \leq n} E(\widetilde{S}_i)^{m_i}$. For a suitable exponent $m$, we obtain a $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$-approximation $\phi : \widetilde{\mathcal{A}}^m \to \bigoplus_{1 \leq i \leq n} E(\widetilde{S}_i)^{m_i}$. Since $T_{\widetilde{\Lambda}}$ belongs to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$, the embedding $\iota$ factors through $\phi$, say $\iota = \phi \circ f$ for some $f \in \text{Hom}_{\widetilde{\Lambda}}(T, \widetilde{\mathcal{A}}^m)$. Then $f$ is injective, and we deduce that all simple right $\widetilde{\Lambda}$-modules embed into $\widetilde{\mathcal{A}}^m$. Clearly, this implies that all simples in mod-$\widetilde{\Lambda}$ embed into soc $\widetilde{\mathcal{A}}$. Invoking the fact that add($\widetilde{\mathcal{A}}$) = add($\widetilde{T}_{\widetilde{\Lambda}}$), we conclude that all $\widetilde{S}_i$ occur in soc $\widetilde{T}_{\widetilde{\Lambda}}$ as well, which proves two-sided strongness of $_{\widetilde{\widetilde{\Lambda}}}\widetilde{T}_{\widetilde{\Lambda}}$.

The final assertion follows from the preceding argument. □

The result is sharp in the sense that the existence of a strong tilting module in $\Lambda$-mod does not, by itself, provide sufficient homological symmetry to allow for iteration of strong tilting.

*Example 3.3* This example is due to Igusa-Smalø-Todorov [15]. Let $\Lambda = KQ/I$, where $Q$ is the quiver
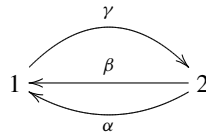


and $I \subseteq KQ$ is the ideal generated by $\beta\gamma, \gamma\alpha, \gamma\beta$. Clearly the right socle of $\Lambda$ contains both simple right $\Lambda$-modules, whence l.findim $(\Lambda) = 0$ (see [4]). Hence the basic strong tilting object in $\Lambda$-mod is $T = {}_\Lambda\Lambda$, and $\widetilde{\Lambda}$ is isomorphic to $\Lambda$. However, the category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) = \mathcal{P}^{<\infty}(\text{mod-}\Lambda)$ fails to be contravariantly finite in mod-$\Lambda$ (see [15]).

For a first application, we combine Theorem 3.1 with the fact that strong tilting modules induce contravariant equivalences of categories of modules of finite projective dimension (see [12, Reference Theorem III and Theorem 1] or [13, Theorems 7,8]). This yields

**Corollary 3.4** *Retain the notation and hypotheses of Theorem 3.1. Then the pairs of functors $\left(Hom_\Lambda(-, T), Hom_{\widetilde{\Lambda}}(-, T)\right)$ and $\left(Hom_{\widetilde{\Lambda}}(-, \widetilde{T}), Hom_{\widetilde{\widetilde{\Lambda}}}(-, \widetilde{T})\right)$ induce dualities*

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^\perp(T_{\widetilde{\Lambda}}) \subseteq \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \longleftrightarrow \mathcal{P}^{<\infty}(\widetilde{\widetilde{\Lambda}}\text{-}mod),$$

*where ${}^\perp(T_{\widetilde{\Lambda}})$ is the full subcategory of mod-$\widetilde{\Lambda}$ whose objects are the modules $\widetilde{M}$ with $Ext^i_{\widetilde{\Lambda}}(\widetilde{M}, T) = 0$ for $i \geq 1$. Moreover, ${}^\perp(T_{\widetilde{\Lambda}})$ contains $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ if and only if ${}_\Lambda T_{\widetilde{\Lambda}}$ is two-sided strong, i.e., precisely when $T \cong \widetilde{T}$.*

In case the process of strongly tilting $\Lambda$-mod permits for iteration, periodicity of the sequence $\Lambda$-mod $\rightsquigarrow$ mod-$\widetilde{\Lambda}$ $\rightsquigarrow \cdots$ may set in with delay. In fact, if $\Lambda$ is a truncated path algebra one always obtains a sequence of basic strong tilts, $\Lambda$-mod $\rightsquigarrow$ mod-$\widetilde{\Lambda}$ $\rightsquigarrow \cdots$ (see [12, Theorem 19]); for this class of algebras, $\Lambda$-mod $\approx \widetilde{\widetilde{\Lambda}}$-mod if and only if $Q$ does not have a precyclic source (see [6, Corollary 7.2]). We refer to [12] and [13] for examples based on quivers with precyclic sources where the transition from the tilting module $T_{\widetilde{\Lambda}}$ to the basic strong tilting module $\widetilde{T}_{\widetilde{\Lambda}}$ in mod-$\widetilde{\Lambda}$ is displayed.

We conclude the section with a variant of the criterion of Auslander and Green for a strong tilting module $T \in \Lambda$-mod to be strong also as a tilting module over $\text{End}_\Lambda(T)^{\text{op}}$; it was implicitly used in the proof of Theorem 3.1. The upcoming version of the criterion does not rely on structural information regarding ${}_\Lambda T$ and provides additional motivation for the detection of twosided strongness.

*Observation 3.5* Suppose $\Lambda$ is an Artin algebra such that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. Then the following statements are equivalent:

(1) The basic strong tilting module $T \in \Lambda$-mod is strong also in mod-$\widetilde{\Lambda}$, where $\widetilde{\Lambda} = \text{End}_\Lambda(T)^{\text{op}}$.
(2) Every simple left $\Lambda$-module of infinite projective dimension embeds into a $\Lambda$-module of finite projective dimension.
(3) There exists a duality $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

If (1)–(3) are satisfied, then $\text{Hom}_\Lambda(-, T)$ and $\text{Hom}_{\widetilde{\Lambda}}(-, T)$ induce (unique up to isomorphism) quasi-inverse dualities $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

*Proof* "(1) $\Longrightarrow$ (2)" is immediate from the criterion of Auslander and Green quoted in the proof of Theorem 3.1. "(2) $\Longrightarrow$ (1)": By (2), every simple left $\Lambda$-module $S$ embeds into a module $M = M(S)$ of finite projective dimension. Consider an embedding $\iota : M \hookrightarrow E$ of $M$ into an injective module $E$, and let $\phi : \mathcal{A} \to E$ be a minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation. Then $\iota$ factors through $\phi$, which shows $M$ to be contained in $\mathcal{A}$ up to isomorphism. Hence so is $S$. Since add($\mathcal{A}$) $\subseteq$ add($T$) by hypothesis (cf. the proof of Theorem 3.1), we conclude that $S$ embeds into $T$. Thus an application of [1, Proposition 6.5] yields (1).

The equivalence "(1) $\Longleftrightarrow$ (3)" follows from [12, Reference Theorem III and Theorem 1] or [13, Theorems 7,8]. For the supplementary statement, see [13, Corollary 9]. $\qquad\square$

## 4 Testing for Contravariant Finiteness of the Category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$

Our main objective in this section is to show that, in exploring whether the subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod, a comparatively mild hypothesis (Setting 4.1(ii) below) permits us to eliminate primitive idempotents which correspond to simple left $\Lambda$-modules of finite projective dimension. This reduces the problem to a more manageable corner $e\Lambda e$ of $\Lambda$.

### 4.1 The Setting

*Setting 4.1* (Blanket hypotheses) Throughout this section, we will assume that the idempotent $e \in \Lambda$ satisfies the following two conditions:

 (i)  The semisimple left $\Lambda$-module $\Lambda(1 - e)/J(1 - e)$ has finite projective dimension.
 (ii) $e\Lambda(1 - e)$ has finite projective dimension as a left $e\Lambda e$-module.

*Remarks 4.2* • Suppose that $e = e_1 + \cdots + e_m$ and $1 - e = e_{m+1} + \cdots + e_n$, where the $e_i$ are orthogonal primitive idempotents of $\Lambda$. Then the conditions under Section 4.1 amount to finiteness of the projective dimensions of the simple $\Lambda$-modules $S_i = \Lambda e_i/J e_i$ for $i \geq m+1$ and those of the $e\Lambda e$-modules $e\Lambda e_i$ for $i \geq m+1$. In particular, Section 4.1 (i) implies that, for all $M \in \Lambda$-mod,

$$\operatorname{pdim}_\Lambda M < \infty \iff \operatorname{pdim}_\Lambda \operatorname{core}(M) < \infty \iff \operatorname{pdim}_\Lambda M_\sigma < \infty.$$

With regard to Section 4.1 (ii), we mention that, in case $\Lambda$ is a path algebra modulo relations, the quiver and relations of $e\Lambda e$ are available from those of $\Lambda$, making this condition computationally accessible.
• Suppose that $S_1, \ldots, S_m$ are precisely the simple left $\Lambda$-modules of infinite projective dimension; in particular, this means that Section 4.1 (i) is satisfied. Even over monomial algebras, condition Section 4.1 (ii) is not automatic in this situation, as is witnessed by an example of Fuller and Saorín [7, Example 4.2]. However, if all of the supplementary simples $S_{m+1}, \ldots, S_n$ have projective dimension at most 1, then Section 4.1 (ii) does always hold in this scenario [loc.cit.].

In the present setting, the adjoint pair $(\Lambda e \otimes_{e\Lambda e} -, \mathbf{e})$ is particularly useful towards the transfer of homological information between the categories $\Lambda$-mod and $e\Lambda e$-mod. Namely:

**Lemma 4.3** *Suppose $\Lambda$ satisfies the conditions under 4.1. Then the functors $\mathbf{e}$ : $\Lambda$-mod $\to$ $e\Lambda e$-mod and its left adjoint $\Lambda e \otimes_{e\Lambda e} -$ : $e\Lambda e$-mod $\to$ $\Lambda$-mod preserve and reflect finite projective dimension.*

*Proof* By [7, Lemma 1.2(b)], the functor $\mathbf{e}$ : $\Lambda$-mod $\to$ $e\Lambda e$-mod preserves finite projective dimension. In light of the canonical isomorphism $\mathbf{e} \circ (\Lambda e \otimes_{e\Lambda e} -) \cong \mathrm{id}_{e\Lambda e}$-mod this shows that $\Lambda e \otimes_{e\Lambda e} -$ reflects finite projective dimension.

To see that $\Lambda e \otimes_{e\Lambda e} -$ preserves finite projective dimension, let $X \in \mathcal{P}^{<\infty}(e\Lambda e$-mod$)$, and let

$$\mathbb{X}: \ 0 \to Q_m \xrightarrow{f_m} \cdots \xrightarrow{f_2} Q_1 \xrightarrow{f_1} Q_0 \xrightarrow{f_0} X \to 0$$

be a projective resolution in $e\Lambda e$-mod. Then all terms $\Lambda e \otimes_{e\Lambda e} Q_j$ of the complex $\Lambda e \otimes_{e\Lambda e} \mathbb{X}$ are projective over $\Lambda$; indeed, the $Q_j$ belong to add$(_{e\Lambda e}e\Lambda e)$, whence the modules $\Lambda e \otimes_{e\Lambda e} Q_j$ belong to the category add$(\Lambda e \otimes_{e\Lambda e} e\Lambda e) = $ add$(_\Lambda \Lambda e)$. Moreover, all homology modules of $\Lambda e \otimes_{e\Lambda e} \mathbb{X}$ have finite projective dimension in $\Lambda$-mod; indeed, due to the natural isomorphism $\mathbf{e}(\Lambda e \otimes_{e\Lambda e} \mathbb{X}) \cong \mathbb{X}$, they are all annihilated by $\mathbf{e}$, whence their simple composition factors are direct summands of $\Lambda(1-e)/J(1-e)$. Writing $(F_i)$ for the differential $(\Lambda e \otimes_{e\Lambda e} f_i)$ of $\Lambda e \otimes_{e\Lambda e} \mathbb{X}$, we find that $\mathrm{Ker}(F_m)$ has finite projective dimension. In view of projectivity of $\Lambda e \otimes_{e\Lambda e} Q_m$, so does $\mathrm{Im}(F_m)$, whence we obtain finiteness of $\mathrm{pdim}_\Lambda \mathrm{Ker}(F_{m-1})$ from that of $\mathrm{pdim}_\Lambda \mathrm{Ker}(F_{m-1})/\mathrm{Im}(F_m)$. An obvious induction thus shows that $\mathrm{Im}(F_0) \cong \Lambda \otimes_{e\Lambda e} X$ has finite projective dimension over $\Lambda$.

That the functor $\mathbf{e}$ reflects finite projective dimension is now seen as follows: Given $M \in \Lambda$-mod such that $\mathbf{e}(M)$ has finite projective dimension over $e\Lambda e$, we apply the conclusion of the preceding paragraph and Lemma 2.3 to deduce that $\mathrm{pdim}_\Lambda \mathfrak{core}(M) < \infty$. Once more invoking Section 4.1 (i), we conclude that $\mathrm{pdim}_\Lambda M < \infty$. $\qquad\square$

The following consequences of Lemma 4.3 will facilitate applications of the main result of this section.

First we combine Lemma 4.3 with the equivalences $e\Lambda e$-Mod $\approx \mathcal{G} \approx \mathcal{C} \cap \mathcal{F}$ of Lemma 2.4, to find that finiteness of projective dimensions transfers smoothly among the categories singled out in Section 2.4. Namely: If $M_\sigma \in \Lambda$-Mod is an object of the Giraud subcategory $\mathcal{G}$ of $\Lambda$-Mod, then $M_\sigma$ has finite projective dimension in $\mathcal{G}$ if and only if $\mathrm{pdim}_{e\Lambda e} eM_\sigma < \infty$, if and only if $\mathrm{pdim}_\Lambda M_\sigma < \infty$. Analogously: If $M = \mathfrak{core}(M)$ is an object of $\mathcal{C} \cap \mathcal{F}$, then $M$ has finite projective dimension in this subcategory if and only if $\mathrm{pdim}_\Lambda M < \infty$. This leads to

**Lemma 4.4** *Let $M \in \Lambda$-mod, suppose that $eM$ has a $\mathcal{P}^{<\infty}(e\Lambda e$-mod$)$-approximation, and let $q : X \to eM$ be a minimal such approximation. Then $(\Lambda e \otimes_{e\Lambda e} q)_\sigma : (\Lambda e \otimes_{e\Lambda e} X)_\sigma \to (\Lambda e \otimes_{e\Lambda e} eM)_\sigma \cong M_\sigma$ is a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod$)$-approximation of $M_\sigma$.*

*Proof* Referring to the explicit category equivalences of Lemma 2.4, we see that $(\Lambda e \otimes_{e\Lambda e} q)_\sigma$ is a minimal approximation of $M_\sigma$ with respect to the subcategory $\mathcal{P}^{<\infty}(\mathcal{G} \cap (\Lambda$-mod$))$ of $\mathcal{G} \cap (\Lambda$-mod$)$. In light of the preceding remarks, the domain of this map has finite projective dimension also in $\Lambda$-mod. Right minimality of $(\Lambda e \otimes_{e\Lambda e} q)_\sigma$ in $\Lambda$-mod is clear. To see that $(\Lambda e \otimes_{e\Lambda e} q)_\sigma$ is actually a $\mathcal{P}^{<\infty}(\Lambda$-mod$)$-approximation of $M_\sigma$, let $U \in \mathcal{P}^{<\infty}(\Lambda$-mod$)$ and $f \in \mathrm{Hom}_\Lambda(U, M_\sigma)$. Since $M_\sigma$ is torsionfree and $U/\Delta(U)$ again belongs to $\mathcal{P}^{<\infty}(\Lambda$-mod$)$, we do not lose generality in assuming that $U$ is torsionfree. Consequently, $\mu_U : U \to U_\sigma$ is an embedding with the property that $U_\sigma/U \in \mathcal{T}$. Now the restricted injectivity property of $M_\sigma$ yields an extension of $f$ to $f^* : U_\sigma \to M_\sigma$, i.e., $f = f^* \circ \mu_U$. Since $U_\sigma$ has finite projective dimension in $\mathcal{G} \cap (\Lambda$-mod$)$, we further obtain a factorization $f^* = (\Lambda e \otimes_{e\Lambda e}$

$q)_\sigma \circ g$ for some $g \in \operatorname{Hom}_\Lambda(U_\sigma, (\Lambda e \otimes_{e\Lambda e} X)_\sigma)$. This clearly yields a factorization of $f$ through $(\Lambda e \otimes_{e\Lambda e} q)_\sigma$. $\qquad\square$

The next observation depends only on Condition 4.1(i).

**Lemma 4.5** *Let $M \in \Lambda$-mod, and let $p : \mathcal{A}(M) \to M$ be a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation. Suppose $p$ factors in the form $p = \tau \circ \rho$ with $\rho \in \operatorname{Hom}_\Lambda(\mathcal{A}(M), N)$ and $\tau \in \operatorname{Hom}_\Lambda(N, M)$.*

*If pdim $N < \infty$ and $\rho$ is an epimorphism, then $\rho$ is an isomorphism.*
*In particular: If $M \in \mathcal{F}$, then also $\mathcal{A}(M) \in \mathcal{F}$.*

*Proof* For the first implication it suffices to observe that, under the given premise, $\tau : N \to M$ is in turn a $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation of $M$. Minimality of $p$ thus shows that length$(\mathcal{A}(M)) \leq$ length$(N)$. Therefore the epimorphism $\rho$ is an isomorphism.

To derive the final implication, assume that $M$ is torsionfree, and apply the preceding implication to the canonical map $\rho : \mathcal{A}(M) \to \mathcal{A}(M)/\Delta\big(\mathcal{A}(M)\big)$. $\qquad\square$

### 4.2 The Main Theorem

The upcoming theorem shows that, in the situation of Section 4.1, contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda$-mod) in $\Lambda$-mod is equivalent to contravariant finiteness of $\mathcal{P}^{<\infty}(e\Lambda e$-mod) in $e\Lambda e$-mod. In fact, we obtain sharper information, relating minimal $\mathcal{P}^{<\infty}$-approximations in $\Lambda$-mod to minimal $\mathcal{P}^{<\infty}$-approximations in $e\Lambda e$-mod.

As before, $(\mathcal{T}, \mathcal{F}) = (\mathcal{T}_e, \mathcal{F}_e)$ is the torsion pair of Section 2.2. Recall that the class $\mathcal{T}$ consists of the $\Lambda$-modules all of whose composition factors belong to add$\big(\Lambda(1-e)/J(1-e)\big)$, while $\mathcal{F}$ consists of the modules $F$ with soc $F \in$ add$(\Lambda e/Je)$.

**Theorem 4.6** *Let $\Lambda$ be a basic Artin algebra, and let $e \in \Lambda$ be an idempotent such that the left $\Lambda$-module $\Lambda(1-e)/J(1-e)$ has finite projective dimension and $e\Lambda(1-e)$ has finite projective dimension as a left $e\Lambda e$-module.*

*Then $\mathcal{P}^{<\infty}(\Lambda$-mod) is contravariantly finite in $\Lambda$-mod if and only if $\mathcal{P}^{<\infty}(e\Lambda e$-mod) is contravariantly finite in $e\Lambda e$-mod.*

*In more detail: The following two implications hold for every finitely generated left $\Lambda$-module $F \in \mathcal{F}$.*

(1) *If $p : M \longrightarrow F$ is a minimal right $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation of $F$, then $p_{|eM} : eM \longrightarrow eF$ is a minimal right $\mathcal{P}^{<\infty}(e\Lambda e$-mod)-approximation of $eF$.*

(2) *Suppose that $q : X \longrightarrow eF$ is a minimal right $\mathcal{P}^{<\infty}(e\Lambda e$- mod)-approximation of $eF$, and let $f$ be the composition*

$$\frac{\Lambda e \otimes_{e\Lambda e} X}{\Delta(\Lambda e \otimes_{e\Lambda e} X)} \xrightarrow{\ \overline{1 \otimes q}\ } \frac{\Lambda e \otimes_{e\Lambda e} eF}{\Delta(\Lambda e \otimes_{e\Lambda e} eF)} \xrightarrow{\ \overline{\epsilon}_F\ } F,$$

*where $\overline{\epsilon}_F$ is induced by the natural map $\epsilon_F : \Lambda e \otimes_{e\Lambda e} eF \to F$. Then any representative of the unique maximal element of the poset $\mathcal{E}_f$ of essential $\Delta$-extensions of $f$ (cf. Proposition 2.7 for existence) is a minimal right $\mathcal{P}^{<\infty}(\Lambda$- mod)-approximation of $F$.*

*Remark 4.7* By Lemma 2.3, the map $\overline{1 \otimes q}$ in statement (2) of the theorem is naturally isomorphic to the map $\mathfrak{core}(1 \otimes q) : \mathfrak{core}(\Lambda e \otimes_{e\Lambda e} X) \to \mathfrak{core}(F)$ and, modulo the canonical isomorphism $\mathfrak{core}(F) \cong \frac{\Lambda e \otimes_{e\Lambda e} eF}{\Delta(\Lambda e \otimes_{e\Lambda e} eF)}$, the map $\overline{\epsilon}_F : \mathfrak{core}(F) = \nabla(F) \to F$ is the embedding of $\mathfrak{core}(F)$ into $F$.

*Proof* We start by showing that the first assertion follows from (1) and (2). Indeed, the correspondence $S \leftrightarrow eS$ is a bijection between the isomorphism classes of simple $\Lambda$-modules in add$(\Lambda e/Je)$ and those of the simple left $e\Lambda e$-modules. In light of [1, Proposition 3.7] and the fact that the simples in add$(\Lambda(1 - e)/J(1 - e))$ have finite projective dimension, it thus suffices to show that any simple module $S \in$ add$(\Lambda e/Je)$ has a $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation precisely when $eS$ has a $\mathcal{P}^{<\infty}(e\Lambda e$-mod)-approximation. Clearly, the simple summands $S$ of $\Lambda e/Je$ satisfy $S = \Lambda eS \in \mathcal{F}$, and thus the required equivalence arises from (1) and (2) as a special case.

(1)   We assume that $p : M \to F$ is a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation of $F$, and set $p' = p_{|eM} : eM \longrightarrow eF$. Moreover, we abbreviate $\Lambda e \otimes_{e\Lambda e} Y$ to $Y^{\dagger}$ for $Y \in e\Lambda e$-mod and $\Lambda e \otimes_{e\Lambda e} eN$ to $N^{\ddagger}$ for $N \in \Lambda$-Mod; thus $(-)^{\dagger}$ and $(-)^{\ddagger}$ are functors $e\Lambda e$-Mod $\to \Lambda$-Mod and $\Lambda$-Mod $\to \Lambda$-Mod, respectively, such that $N^{\ddagger} = (eN)^{\dagger}$. Let $\alpha \in$ Hom$_{e\Lambda e}(Z, eF)$ for some $Z \in \mathcal{P}^{<\infty}(e\Lambda e$-mod). By Lemma 4.3, $Z^{\dagger} \in \mathcal{P}^{<\infty}(\Lambda$-mod), and hence the composition $\epsilon_F \circ (1 \otimes \alpha) : \Lambda e \otimes_{e\Lambda e} Z = Z^{\dagger} \to F$ belongs to Hom$_{\Lambda}(\mathcal{P}^{<\infty}(\Lambda$-mod), F). By hypothesis, we thus obtain a map $\beta \in$ Hom$_{\Lambda}(Z^{\dagger}, M)$ such that $\epsilon_F \circ (1 \otimes \alpha) = p \circ \beta$. Since the functor $\mathbf{e} \circ (\Lambda e \otimes_{e\Lambda e} -) = \mathbf{e} \circ (-)^{\dagger}$ is naturally equivalent to $1_{\mathbf{e}\Lambda e\text{-Mod}}$, application of $\mathbf{e}$ to this equality shows that $\beta$ factors through $p'$. Therefore $p'$ is a $\mathcal{P}^{<\infty}(e\Lambda e$-mod)-approximation of $eF$.

To prove that $p'$ is (right) minimal, it suffices to show that no nonzero direct summand of $eM$ is contained in the kernel of $p'$. So suppose $eM = X \oplus Y$ with $p'(Y) = 0$, and let $\pi : eM \to X$ be the projection onto $X$ along $Y$. The latter gives rise to a corresponding projection pr $: M^{\ddagger}/\Delta(M^{\ddagger}) \to X^{\dagger}/\Delta(X^{\dagger})$. Bearing in mind that $M \in \mathcal{F}$ by Lemma 4.5, we obtain the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\ddagger}/\Delta(M^{\ddagger}) & \overset{\overline{\epsilon}_M}{\longrightarrow} & M & \longrightarrow & \text{Coker}(\overline{\epsilon}_M) & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle\text{pr}} & & \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\cong} & & \\
0 & \longrightarrow & X^{\dagger}/\Delta(X^{\dagger}) & \overset{u}{\longrightarrow} & N & \longrightarrow & \text{Coker}(u) & \longrightarrow & 0
\end{array}
$$

where the left-hand square is the pushout of $\overline{\epsilon}_M$ and pr. In particular, $\rho$ is an epimorphism and $u$ is a monomorphism. Both of the flanking terms of the bottom exact sequence have finite projective dimension: For $X^{\dagger}/\Delta(X^{\dagger})$ this follows from Lemma 4.3, for the righthand term it follows from Coker$(u) \cong$ Coker$(\overline{\epsilon}_M) \cong M/\nabla(M)$. Consequently, also $N$ has finite projective dimension.

On defining $q$ to be the restriction of $\overline{\epsilon}_F \circ (\overline{1 \otimes p'})$ to $X^{\dagger}/\Delta(X^{\dagger})$, one readily checks that the following diagram commutes:

$$
\begin{array}{ccc}
M^{\ddagger}/\Delta(M^{\ddagger}) & \overset{\overline{\epsilon}_M}{\longrightarrow} & M \\
\Big\downarrow{\scriptstyle\text{pr}} & & \Big\downarrow{\scriptstyle p} \\
X^{\dagger}/\Delta(X^{\dagger}) & \overset{q}{\longrightarrow} & F
\end{array}
$$

The universal property of the pushout therefore yields a map $v \in$ Hom$_{\Lambda}(N, F)$ such that $v \circ \rho = p$. By Lemma 4.5, $\rho$ is an isomorphism, whence so is pr, meaning that $Y^{\dagger}/\Delta(Y^{\dagger}) = 0$. Thus $Y^{\dagger} \in \mathcal{T}$, and we conclude that $Y \cong \mathbf{e}(Y^{\dagger}) = 0$. This confirms minimality of $p'$.

(2) Let $q : X \longrightarrow eF$ and $f$ be as in the assertion, and suppose that the pair $(M, p)$ represents the maximum of the set $\mathcal{E}_f$. To check that $p : M \to F$ is a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $F$, let $h \in \mathrm{Hom}_\Lambda(N, F)$ with $N \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$. Since $\Delta(F) = 0$ and $N/\Delta(N)$ again belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$, we may assume that $\Delta(N) = 0$. In light of Lemma 4.3, the restriction $h' = h|_{eN} : eN \to eF$ is a morphism in $\mathrm{Hom}_{e\Lambda e}\big(\mathcal{P}^{<\infty}(e\Lambda e\text{-mod}), eF\big)$, whence it factors through $q$, say $h' = q \circ \eta$ with $\eta \in \mathrm{Hom}_{e\Lambda e}(eN, X)$. Again referring to Lemma 4.3, we moreover see that the map $\overline{1 \otimes \eta} : N^\ddagger/\Delta(N^\ddagger) \to X^\dagger/\Delta(X^\dagger)$ induced by $\eta$ is a morphism in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. In order to suitably extend $\overline{1 \otimes \eta}$ to a homomorphism with domain $N$ via the monomorphism $\overline{\epsilon}_N : N^\ddagger/\Delta(N^\ddagger) \to N$, we consider the pushout diagram of the pair $(\overline{\epsilon}_N, \overline{1 \otimes \eta})$, as shown in the diagram below:
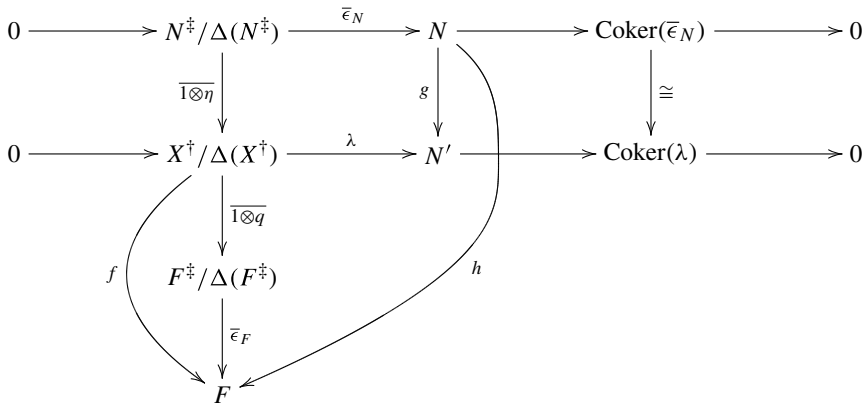


Since the image of $\overline{\epsilon}_N$ is $\nabla(N)$, we may use the argumentation in the proof of (1) to infer that $N' \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$. By the naturality of $\overline{\epsilon}$ (see Lemma 2.3), $\overline{\epsilon}_F \circ \overline{1 \otimes h'}$ coincides with $h \circ \overline{\epsilon}_N$, which yields $f \circ (\overline{1 \otimes \eta}) = h \circ \overline{\epsilon}_N$; in other words, the above diagram fully commutes. Thus the universal property of the pushout provides us with a map $\phi \in \mathrm{Hom}_\Lambda(N', F)$ such that $\phi \circ \lambda = f$ and $\phi \circ g = h$. Given that $F$ is torsionfree, $\phi$ factors through the canonical map $\pi : N' \to N'/\Delta(N')$; denote the induced map $N'/\Delta(N') \to F$ by $\overline{\phi}$, and note that the composition $\overline{\lambda} := \pi \circ \lambda$ is still an embedding, since $\mathrm{Im}(\lambda) \cap \Delta(N') = 0$; it is thus harmless to view $\overline{\lambda}$ as a set inclusion. We will ascertain that the pair $\big(N'/\Delta(N'), \overline{\phi}\big)$ gives rise to a class in $\mathcal{E}_f$: Indeed, the cokernel of $\overline{\lambda}$ belongs to $\mathcal{T}$, since it is an epimorphic image of $N/\nabla(N)$. Torsionfreeness of $N'/\Delta(N')$ moreover guarantees that $\overline{\lambda}$ is an essential extension. That $\overline{\phi}$ extends $f$, is immediate from our construction. The inequality $[\big(N'/\Delta(N'), \overline{\phi}\big)] \preceq [(M, p)]$ now yields a monomorphism $\upsilon : N'/\Delta(N') \to M$ such that $p \circ \upsilon = \overline{\phi}$. It is straightforward to deduce that $h$ factors through $p$ as required.

To see that $p$ is a right minimal morphism, consider a decomposition $p = \big(p_1 \ 0\big) : M = U \oplus V \longrightarrow F$, where $p_1$ is right minimal. Due to the fact that $[(M, p)] \in \mathcal{E}_f$, the domain $M$ of $p$ is an essential extension of $X^\dagger/\Delta(X^\dagger)$ such that the quotient of $M$ modulo $X^\dagger/\Delta(X^\dagger)$ belongs to $\mathcal{T}$. This means that $eM = e\big(X^\dagger/\Delta(X^\dagger)\big)$, whence the induced map $p' = p|_{eM} : eM \to eF$ coincides with the restriction $f'$ of $f$ to $e\big(X^\dagger/\Delta(X^\dagger)\big)$. In light of the natural equivalence $\mathbf{e} \circ (\Lambda e \otimes_{e\Lambda e} -) \cong 1_{e\Lambda e\text{-mod}}$, this identifies $f'$ with the minimal approximation $q : X \to eF$. Due to the matrix decomposition $p' = \big(p'_1 \ 0\big) : X \cong eU \oplus eV \to eF$, where $p'_1 = p_{1|eU}$, right minimality of $q$ thus yields $eV = 0$, so that $V \in \mathcal{T}$. On the other hand, $V \in \mathcal{F}$; this

follows from the fact that $M \in \mathcal{F}$, because the torsionfree module $X^{\dagger}/\Delta(X^{\dagger})$ is an essential submodule of $M$. Consequently, $V = 0$, which proves minimality of $p$ as claimed.

$\square$

We apply Theorem 4.6 to the situation where the $\mathcal{P}^{<\infty}$-categories of $e\Lambda e$-mod and (equivalently) $\Lambda$-mod are contravariantly finite in the corresponding ambient module categories. The proposition below picks up the theme of Lemma 4.4 and Remark 4.7. It reinforces our understanding of the links among the minimal approximations of objects $M \in \Lambda$-mod and those of the corresponding $\Lambda$-modules $M/\Delta(M)$, $M_{\sigma}$, and $\mathfrak{core}(M)$. These connections underlie the upcoming exploration of the basic algebra $\widetilde{\Lambda}$ that results from strongly tilting $\Lambda$-mod.

**Proposition 4.8** *Continue to adopt conditions 4.1(i) and (ii), and suppose that $\mathcal{P}^{<\infty}(e\Lambda e$-mod) is contravariantly finite in $e\Lambda e$-mod. $\big($By Theorem 4.6, this implies contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda$-mod) in $\Lambda$-mod.$\big)$ For $M \in \Lambda$-mod, denote by $\mathcal{A}(M)$ the minimal right $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation of $M$.*

(a) *If $M \in \mathcal{G}$, then $\mathcal{A}(M) \in \mathcal{G}$.*
(b) *If $M$ is torsionfree, then $\mathcal{A}(M)$ naturally embeds into $\mathcal{A}(M_{\sigma})$, and the core of $\mathcal{A}(M)$ coincides with that of $\mathcal{A}(M_{\sigma})$.*
(c) *For arbitrary $M \in \Lambda$-mod, $\mathcal{A}\big(M/\Delta(M)\big) \cong \mathcal{A}(M)/\Delta(\mathcal{A}(M))$, and the torsion submodule $\Delta(\mathcal{A}(M))$ is isomorphic to $\Delta(M)$. In particular, there is a natural homomorphism $\rho : \mathcal{A}(M) \to \mathcal{A}(M_{\sigma})$ such that $\mathrm{Ker}(\rho)$ and $\mathrm{Coker}(\rho)$ belong to $\mathcal{T}$.*

*Proof* Part (a) is a consequence of Lemma 4.4. We re-encounter this fact in a notational setup that is convenient towards parts (b) and (c).

Suppose that $M \in \Lambda$-mod is torsionfree, and fix an injective envelope $E$ of $\mathcal{A}(M)$. Since $\mathcal{A}(M)$ is again torsionfree (Lemma 4.5), so is $\mathrm{soc}\big(\mathcal{A}(M)\big)$, which implies that $\mathrm{soc}\big(\mathcal{A}(M)\big)$ is contained in the submodule $\mathfrak{core}\big(\mathcal{A}(M)\big)$ of $\mathcal{A}(M)$. Therefore $E$ is also an injective envelope of $\mathfrak{core}\big(\mathcal{A}(M)\big)$. Moreover, let $p : \mathcal{A}(M) \to M$ be a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation, and set $X = e\mathcal{A}(M)$. By Theorem 4.6 and Remark 4.7, the restriction $\mathbf{e}(p) : e\mathcal{A}(M) \to eM$ is then a minimal $\mathcal{P}^{<\infty}(e\Lambda e$-mod)-approximation of $eM$, and $p$ is maximal in the set $\mathcal{E}_f$ of essential $\Delta$-extensions of the composition

$$f : \frac{\Lambda e \otimes_{e\Lambda e} X}{\Delta(\Lambda e \otimes_{e\Lambda e} X)} \cong \mathfrak{core}(\mathcal{A}(M)) \longrightarrow \mathfrak{core}(M) \hookrightarrow M. \qquad (\bullet)$$

(a) Suppose $M = M_{\sigma}$. In view of the fact that $\mathbf{e}(p) : X = e\mathcal{A}(M) \to eM$ is a minimal $\mathcal{P}^{<\infty}(e\Lambda e$-mod)-approximation of $eM$, Lemma 4.4 guarantees that $(\Lambda e \otimes_{e\Lambda e} X)_{\sigma}$ is a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation of $M_{\sigma} = M$, i.e., $\mathcal{A}(M) \cong \big(\mathcal{A}(M)\big)_{\sigma}$ as claimed.
(b) In light of (a), $(\Lambda e \otimes_{e\Lambda e} X)_{\sigma} \cong \big(\frac{\Lambda e \otimes_{e\Lambda e} X}{\Delta(\Lambda e \otimes_{e\Lambda e} X)}\big)_{\sigma}$ is a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation of $M_{\sigma}$. Hence we infer from $(\bullet)$ that

$$\mathcal{A}(M_{\sigma}) \cong \big(\mathfrak{core}\,\mathcal{A}(M)\big)_{\sigma} \cong \mathcal{A}(M)_{\sigma}.$$

We may thus identify $\mathcal{A}(M_{\sigma})$ with a submodule of $E$ containing $\mathcal{A}(M)$ such that $\mathcal{A}(M_{\sigma})/\mathcal{A}(M) \in \mathcal{T}$. The latter condition implies that $\mathfrak{core}\big(\mathcal{A}(M_{\sigma})\big) = \mathfrak{core}\big(\mathcal{A}(M)\big)$.
(c) Now let $M$ be arbitrary, and let $\pi : M \twoheadrightarrow \overline{M} = M/\Delta(M)$ be the canonical map. Moreover, suppose that $q : \mathcal{A}(\overline{M}) \to \overline{M}$ is a minimal $\mathcal{P}^{<\infty}(\Lambda$-mod)-approximation

of $\overline{M}$. On the model of previous arguments, it is straightforward to check that the map $q' : N \to M$ in the following pullback is a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $M$.

$$
\begin{array}{ccc}
N & \xrightarrow{\ q'\ } & M \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathcal{A}(\overline{M}) & \xrightarrow{\ q\ } & \overline{M}
\end{array}
$$

The fact that $\mathrm{Ker}(\pi') \cong \mathrm{Ker}(\pi)$ shows that $\mathrm{Ker}(\pi')$ is a torsion module, necessarily equal to $\Delta(N)$ because $\mathcal{A}(\overline{M})$ is torsionfree; hence $q'$ induces an isomorphism $\Delta(q') : \Delta(N) \cong \Delta(M)$. To check right minimality of $q'$, suppose $u \in \mathrm{End}_\Lambda(N)$ satisfies $q' \circ u = q'$. On applying the functors $\Delta$ and $(1 : \Delta)$, the latter acting as $M \to (1 : \Delta)(M) := \overline{M}$, we find that $\Delta(u) \in \mathrm{End}_\Lambda\big(\Delta(N)\big)$ is an isomorphism, since $\Delta(q')$ is an isomorphism; further $(1 : \Delta)(u) \in \mathrm{End}_\Lambda\big(N/\Delta(N)\big) \cong \mathrm{End}_\Lambda\big(\mathcal{A}(\overline{M})\big)$ is an isomorphism, since $(1 : \Delta)(q') \cong q : \mathcal{A}(\overline{M}) \to \overline{M}$ was chosen right minimal. Consequently, also $u$ is an isomorphism, which proves $q$ to be right minimal as claimed.

We conclude that $q'$ coincides with $p$ up to isomorphism, whence $\mathcal{A}(M)/\Delta(\mathcal{A}(M)) \cong \mathcal{A}(\overline{M})$ and $\Delta\big(\mathcal{A}(M)\big) \cong \Delta(M)$. In particular, the composition

$$
\rho : \mathcal{A}(M) \twoheadrightarrow \mathcal{A}(M)/\Delta(M) \cong \mathcal{A}(\overline{M}) \hookrightarrow \mathcal{A}(\overline{M}_\sigma) = \mathcal{A}(M_\sigma)
$$

has the postulated properties. □

*Remark 4.9* Analyzing the proof of the preceding proposition, one observes the following: If $\pi_X : X \longrightarrow \overline{X} = X/\Delta(X)$ and $p_X : \mathcal{A}(X) \longrightarrow X$ denote the canonical projection and the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of a module $X$, then one has a composition of pullbacks

$$
\begin{array}{ccc}
\mathcal{A}(M) & \xrightarrow{\ p_M\ } & M \\
{\scriptstyle \pi_{\mathcal{A}(M)}}\downarrow & & \downarrow{\scriptstyle \pi_M} \\
\mathcal{A}(\overline{M}) & \xrightarrow{\ p_{\overline{M}}\ } & \overline{M} \\
\downarrow & & \downarrow \\
\mathcal{A}(M_\sigma) & \xrightarrow{\ p_{M_\sigma}\ } & M_\sigma
\end{array}
$$

such that the compositions of the vertical arrows are isomorphic to the canonical maps $\mu_{\mathcal{A}(M)} : \mathcal{A}(M) \longrightarrow \mathcal{A}(M)_\sigma$ and $\mu_M : M \longrightarrow M_\sigma$, respectively.

In applying Theorem 4.6, we typically decompose $e$ and $1-e$ into primitive idempotents: $e = e_1 + \cdots + e_m$ and $1 - e = e_{m+1} + \cdots + e_n$. Since part (2) of Theorem 4.6 aims at reducing the contravariant finiteness test for $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to that for $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$, we are interested in making $e\Lambda e$ as "small" as possible. Hence the situation where *all* simple left modules $S_i$ of finite projective dimension correspond to idempotents $e_i$ for $i \geq m + 1$ is of particular interest. A modification of an example by Fuller-Saorín (see [7, Example 4.2]) shows that, even for monomial algebras $\Lambda$, neither of the conditions Section 4.1 (i) nor Section 4.1 (ii) in the hypothesis of Theorem 4.6 is dispensable.

*Example 4.10* Let $\Lambda$ be the monomial algebra based on the quiver
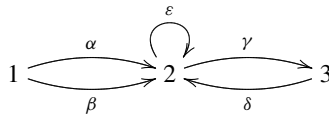


which is defined by the graphs of its indecomposable projective left $\Lambda$-modules, namely
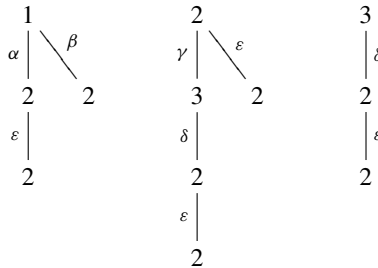


Clearly, the simple left $\Lambda$-modules $S_1$, $S_2$ have infinite projective dimension, whereas $\mathrm{pdim}_\Lambda S_3 = 2$. Let $e = e_1 + e_2$. Then condition Section 4.1 (i) is satisfied. On the other hand, the indecomposable projective left $e\Lambda e$-modules $e\Lambda e_i$ for $i = 1, 2$ and the $e\Lambda e$-module $e\Lambda(1-e) = \Lambda e_3$ have the following graphs:



the indecomposable
projective left
$e\Lambda e$-modules
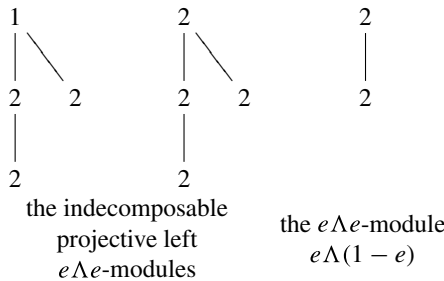
the $e\Lambda e$-module
$e\Lambda(1-e)$

This shows that the left $e\Lambda e$-module $e\Lambda(1-e)$ has infinite projective dimension. To see that the equivalence of Theorem 4.6 fails, observe that l.findim $e\Lambda e = 0$, whence $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$ is contravariantly finite in $e\Lambda e$-mod. Yet $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is not contravariantly finite in $\Lambda$-mod, since the simple module $S_1$ does not have a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation. Indeed, consider the family $(M_n)_{n\in\mathbb{N}}$ of objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ shown below:
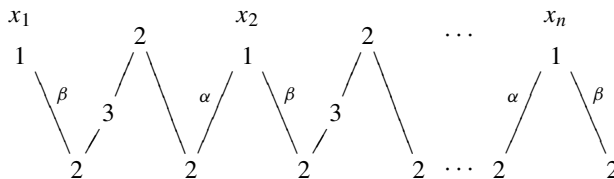


With the aid of Criterion 10 of [10], one readily checks that no homomorphism $\phi \in \mathrm{Hom}_\Lambda(\mathcal{P}^{<\infty}(\Lambda\text{-mod}), S_1)$ allows for factorization of *all* of the following maps $f_n \in \mathrm{Hom}_\Lambda(M_n, S_1)$ in the form $f_n = \phi \circ g_n$; here $f_n(x_1) = \overline{e_1} \in \Lambda e_1/Je_1$ and $f_n(x_i) = 0$ for $i > 1$.

We mention that, in the presence of condition Section 4.1 (ii), condition Section 4.1 (i) in the hypothesis of Theorem 4.6 is not superfluous either. Instances attesting to this are ubiquitous. In the above example, take $e = e_1$, and note that this choice makes $e\Lambda e$ semisimple.

# 5 The Basic Strong Tilting Object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and Its Endomorphism Algebra

In this section, we focus on the situation where $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. Let $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$, where $_\Lambda T$ is the basic strong tilting object in $\Lambda$-mod. The guiding question is this: When is $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$ in turn contravariantly finite in mod-$\widetilde{\Lambda}$? In light of Theorem 3.1, this amounts to the question of when the process of strongly tilting $\Lambda$-mod can be repeated arbitrarily. As witnessed by Example 3.3, the possibility of iteration is not automatic in case $\Lambda$-mod can be strongly tilted once.

*Setting 5.1* (Upgraded blanket hypothesis) Throughout this section, let $\Lambda$ be a basic Artin algebra and $e \in \Lambda$ an idempotent satisfying conditions (i) and (ii) of Setting 4.1, namely $\mathrm{pdim}_\Lambda \Lambda(1 - e)/J(1 - e) < \infty$ and $\mathrm{pdim}_{e\Lambda e} e\Lambda(1 - e) < \infty$, next to the additional condition that

(iii) $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$ is contravariantly finite in $e\Lambda e$-mod.

By Theorem 4.6, $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod in this setting. We will introduce an idempotent $\widetilde{e}$ in $\widetilde{\Lambda}$ which naturally corresponds to $e$. Our objective is to show that the blanket hypotheses Setting 4.1 carry over to mod-$\widetilde{\Lambda}$, meaning that the right $\widetilde{\Lambda}$-module $(1 - \widetilde{e})\widetilde{\Lambda}/(1 - \widetilde{e})\widetilde{J}$ in turn has finite projective dimension, and the corner $(1 - \widetilde{e})\widetilde{\Lambda}\widetilde{e}$ has finite projective dimension as a right $\widetilde{e}\widetilde{\Lambda}\widetilde{e}$-module. In light of Theorem 4.6, this will then allow us to deduce contravariant finiteness of $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$ from that of $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{e}\widetilde{\Lambda}\widetilde{e})$.

To this end, we will first assemble some information about the $\Lambda$-module structure of the basic strong tilting module $_\Lambda T$.

From Section 2.1 we know that add $T = $ add $\mathcal{A}$, where $\mathcal{A}$ is a minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of an injective cogenerator of $\Lambda$-mod. In particular, any such minimal approximation $\mathcal{A}$ is a strong tilting module. To pin down a candidate for $\mathcal{A}$, we decompose both $e$ and $1 - e$ into sums of primitive idempotents of $\Lambda$, say $e = e_1 + \cdots + e_m$ and $1 - e = e_{m+1} + \cdots + e_n$. Moreover, we let $S_i = \Lambda e_i/Je_i$ be the corresponding simple modules, and choose $\mathcal{A}_i$ to be the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of their injective envelopes $E(S_i)$. Clearly, $\mathcal{A} := \bigoplus_{1 \le i \le n} \mathcal{A}_i$ is then as required. Since the injective objects of the Giraud subcategory $\mathcal{G}$ of $\Lambda$-mod relative to the torsion pair $(\mathcal{T}, \mathcal{F})$ (see 2.4) are precisely the torsionfree injective $\Lambda$-modules, the subsum $\bigoplus_{1 \le i \le m} E(S_i)$ is an injective cogenerator in $\mathcal{G}$. Moreover, according to Lemma 2.2, left multiplication by $e$ induces an equivalence of categories $\mathcal{G} \cap \Lambda\text{-mod} \cong (e\Lambda e)\text{-mod}$, and $\bigoplus_{1 \le i \le m} eE(S_i)$ is an injective cogenerator in $e\Lambda e$-mod. We separately explore the direct sums $\bigoplus_{1 \le i \le m} \mathcal{A}_i$ and $\bigoplus_{m+1 \le i \le n} \mathcal{A}_i$.

**Proposition 5.2** *Assume the hypotheses 5.1, and let $T \in \Lambda$-mod be the basic strong tilting module. Then add($\bigoplus_{1 \le i \le m} \mathcal{A}_i$) contains precisely m isomorphism classes of indecomposable*

modules, represented by $T_1, \ldots, T_m$ say. If $T' = \bigoplus_{1 \le i \le m} T_i$, then the direct summand $T'$ of $T$ is an object of $\mathcal{G}$, and $eT'$ is the basic strong tilting object in $e\Lambda e$-mod, up to isomorphism.

Supplementary direct summands $T_{m+1}, \ldots, T_n$ of $T$ such that $T = T' \oplus \bigoplus_{m+1 \le i \le n} T_i$ are as follows: For $m + 1 \le i \le n$, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\mathcal{A}_i$ of $E(S_i)$ decomposes in the form $\mathcal{A}_i = T_i \oplus U_i$ with $\Delta(soc\ T_i) = S_i$ and $U_i \in add(T')$.

In particular, $\Delta(soc\ T) = S_{m+1} \oplus \cdots \oplus S_n$.

*Proof* By Theorem 4.6(1), $\bigoplus_{1 \le i \le m} e\mathcal{A}_i$ is a $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$-approximation of the injective cogenerator $\bigoplus_{1 \le i \le m} eE(S_i)$ in $e\Lambda e$-mod, which shows that $\bigoplus_{1 \le i \le m} e\mathcal{A}_i$ is a strong tilting object in $e\Lambda e$-mod. Since the rank of $K_0(e\Lambda e)$ equals $m$, this implies that $add(\bigoplus_{1 \le i \le m} e\mathcal{A}_i)$ contains precisely $m$ isomorphism classes of indecomposable $e\Lambda e$-modules (see Section 2.1). According to Proposition 4.8, the sum $\bigoplus_{1 \le i \le m} \mathcal{A}_i$ in turn belongs to $\mathcal{G}$, and therefore the category equivalence $\mathcal{G} \cap \Lambda\text{-mod} \cong e\Lambda e$-mod guarantees that $add(\bigoplus_{1 \le i \le m} \mathcal{A}_i)$ contains the same number of isomorphism classes of indecomposable objects. If these are represented by $T_1, \ldots, T_m$, then $e(\bigoplus_{1 \le i \le m} T_i)$ is the basic strong tilting $e\Lambda e$-module by construction. This proves the first claim.

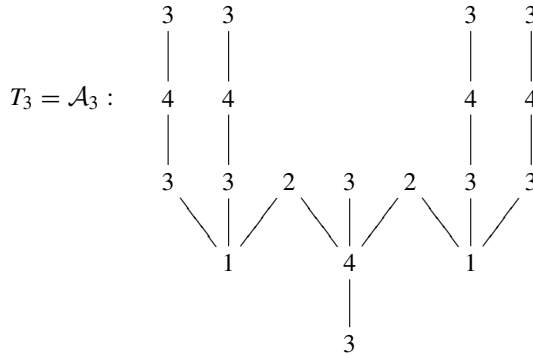For the final claims, let $i \ge m + 1$. We observe that $S_i = soc\ E(S_i)$ is the only simple torsion module which embeds into $\mathcal{A}_i$; indeed, this is immediate from the fact that $\Delta(\mathcal{A}_i) \cong \Delta(E(S_i))$ by Proposition 4.8(c). Hence precisely one of the indecomposable direct summands of $\mathcal{A}_i$ contains a copy of $S_i$ in its socle; say $\mathcal{A}_i = T_i \oplus U_i$ with $T_i$ indecomposable and $S_i = \Delta(soc\ T_i)$, while $\Delta(U_i) = 0$. Then $\bigoplus_{m+1 \le i \le n} T_i$ consists of $n - m$ pairwise nonisomorphic direct summands in $add(\mathcal{A})$, none of which occurs among $T_1, \ldots, T_m$, and we conclude that $T = \bigoplus_{1 \le i \le n} T_i$ up to isomorphism. Since the $U_i$ are torsionfree and hence do not belong to $add(\bigoplus_{m+1 \le i \le n} T_i)$, they belong to $add(\bigoplus_{1 \le i \le m} T_i)$.

In view of the fact that $T'$ is torsionfree, the final claim follows. $\qquad\square$

**Definition** We refer to Proposition 5.2. Let $\widetilde{e}_i \in \widetilde{\Lambda}$ be the projection $T \to T_i \subseteq T$ with respect to the decomposition $T = \bigoplus_{1 \le i \le n} T_i$. Viewing $\widetilde{e}_i$ as an endomorphism of $T$, we thus obtain a primitive idempotent in the tilted algebra $\widetilde{\Lambda}$. We define $\widetilde{e} := \sum_{1 \le i \le m} \widetilde{e}_i \in \widetilde{\Lambda}$. Thus $\widetilde{e}$ is the projection $T \to T' \subseteq T$ along $T'' := \bigoplus_{m+1 \le i \le n} T_i$.

*Example 5.3* (Return to Example 2.2.) Let $\Lambda$ be the algebra of Example 2.2. It is readily seen that $e = e_1 + e_2$ satisfies the conditions of Setting 5.1, that $e\Lambda e$ is a self-injective algebra and that the indecomposable injective modules $I_1$ and $I_2$ have finite projective dimension. Thus $T' = T_1 \oplus T_2 = I_1 \oplus I_2$. Recall from Proposition 5.2 that $T_3$ and $T_4$ are the indecomposable summands of $\mathcal{A}_3 \oplus \mathcal{A}_4$ which are not in $\mathcal{F}$. Let us fix $j \in \{3, 4\}$ in the sequel. By Proposition 4.8 and Remark 4.9, in order to calculate $\mathcal{A}_j$, we need to identify the minimal $\mathcal{P}^{<\infty}(\Lambda \text{ mod})$-approximation of $(I_j)_\sigma = S_2 \oplus S_2$ (see Example 2.6). By Lemma 4.4, such an approximation is of the form $(1 \otimes q)_\sigma \oplus (1 \otimes q)_\sigma$, where $q : X \longrightarrow eS_2$ is the minimal $\mathcal{P}^{<\infty}(e\Lambda e \text{ mod})$-approximation. The latter is the projective cover $e\Lambda e_2 \longrightarrow eS_2$ of $eS_2$ since $e\Lambda e$ is self-injective. It easily follows that $(1 \otimes q)_\sigma$ gets identified with the canonical projection $\rho : (\Lambda e_2)_\sigma = I_1 \longrightarrow S_2$. By Proposition 4.8 and Remark 4.9 again, the $\Lambda$-module $\mathcal{A}_j$ is the upper left corner of the pullback of $\rho \oplus \rho : I_1 \oplus I_1 \longrightarrow S_2 \oplus S_2$ and the projection $\mu_{I_j} : I_j \longrightarrow (I_j)_\sigma = S_2 \oplus S_2$. It then

follows that $T_3 = \mathcal{A}_3$ is given by the following diagram, and $T_4 = \mathcal{A}_4$ is obtained from $T_3$ by factoring out the copy of $S_3$ in the socle of $T_3$.

$$
T_3 = \mathcal{A}_3 :
\begin{array}{ccccccccc}
3 & 3 & & & & & 3 & 3 \\
| & | & & & & & | & | \\
4 & 4 & & & & & 4 & 4 \\
| & | & & & & & | & | \\
3 & 3 & 2 & 3 & 2 & 3 & 3 & 3 \\
& 1 & & 4 & & 1 & & \\
& & & | & & & & \\
& & & 3 & & & &
\end{array}
$$

**Theorem 5.4** *Let $\Lambda$ be a basic Artin algebra. Assume the hypotheses 5.1, and let $T$ be the basic strong tilting module in $\Lambda$-mod. If $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$, the corner algebra $\widetilde{e}\widetilde{\Lambda}\widetilde{e}$ is isomorphic to $\mathrm{End}_{e\Lambda e}(eT')^{\mathrm{op}}$, where $eT'$ is the basic strong tilting module in $e\Lambda e$-mod specified in Proposition 5.2.*

*The pair $(\widetilde{\Lambda}, \widetilde{e})$ satisfies the right-hand versions of conditions 4.1: Namely, (i) the right $\widetilde{\Lambda}$-module $(1-\widetilde{e})\widetilde{\Lambda}/(1-\widetilde{e})\widetilde{J}$ has finite projective dimension, and (ii) the right $\widetilde{e}\widetilde{\Lambda}\widetilde{e}$-module $(1-\widetilde{e})\widetilde{\Lambda}\widetilde{e}$ has finite projective dimension.*

*Proof* The isomorphism $\widetilde{e}\widetilde{\Lambda}\widetilde{e} \cong \mathrm{End}_\Lambda(T')^{\mathrm{op}} \cong \big(\mathrm{End}_{e\Lambda e}(eT')\big)^{\mathrm{op}}$ is immediate from Proposition 5.2 and the comments that precede it. As above, suppose that $e = e_1 + \cdots + e_m$ and $1 - e = e_{m+1} + \cdots + e_n$, where the $e_i$ are primitive.

Regarding condition (i) of the claim: By strongness of $_\Lambda T$, the functor $\mathrm{Hom}_\Lambda(-, T) : \Lambda\text{-mod} \rightarrow \mathrm{mod}\text{-}\widetilde{\Lambda}$ induces a contravariant equivalence between $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and a certain subcategory of $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$, whence $\mathrm{pdim}_{\widetilde{\Lambda}} \mathrm{Hom}_\Lambda(S_i, T) < \infty$ for all $i \geq m + 1$ (see [12, Reference Theorem III and Theorem 1] or [13, Theorems 7,8]). So we only need to show that the right $\widetilde{\Lambda}$-module $\mathrm{Hom}_\Lambda(S_i, T)$ is isomorphic to $\widetilde{e}_i \widetilde{\Lambda}/\widetilde{e}_i \widetilde{J}$ for $m+1 \leq i \leq n$. Let $i \geq m + 1$. In light of Proposition 5.2, $S_i = \Delta(\mathrm{soc}\, T_i)$ is the only occurrence of $S_i$ in the socle of $T$, and therefore any homomorphism $S_i \rightarrow T$ maps $S_i$ onto $\Delta(\mathrm{soc}\, T_i)$. If $\mathrm{in}_i : S_i \hookrightarrow T_i \subseteq T$ is an embedding, we thus find that $\mathrm{Hom}_\Lambda(S_i, T)$ is isomorphic to $\mathrm{Hom}_\Lambda(S_i, \mathrm{in}_i S_i)$ as a $\widetilde{\Lambda}$-module. One checks that the latter module is annihilated by the radical $\widetilde{J}$ of $\widetilde{\Lambda}$, but not annihilated by $\widetilde{e}_i$, and concludes that $\mathrm{Hom}_\Lambda(S_i, T) \cong \widetilde{e}_i \widetilde{\Lambda}/\widetilde{e}_i \widetilde{J}$ as postulated.

To verify condition (ii), namely finiteness of $\mathrm{pdim}_{\widetilde{e}\widetilde{\Lambda}\widetilde{e}}(1 - \widetilde{e})\widetilde{\Lambda}\widetilde{e}$, we again use Proposition 5.2. The isomorphism $\widetilde{e}\widetilde{\Lambda}\widetilde{e} \cong \mathrm{End}_\Lambda(T')^{\mathrm{op}}$ shows that our claim amounts to finite projective dimension of $\mathrm{Hom}_\Lambda(T'', T')$ over $\Gamma = \mathrm{End}_\Lambda(T')^{\mathrm{op}}$. Since $\mathrm{add}(T'') \subseteq \mathrm{add}\big(\bigoplus_{m+1 \leq i \leq n} \mathcal{A}_i\big)$ by the proposition, it suffices to show that $\mathrm{pdim}_\Gamma \mathrm{Hom}_\Lambda(\mathcal{A}_i, T') < \infty$ for $i \geq m + 1$. Fix $i \geq m + 1$ in the following. Towards another reduction step, we apply Proposition 4.8 to a minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $\widehat{\mathcal{A}}_i$ of $E(S_i)_\sigma$. Part (c) of 4.8 provides us with a map $\rho : \mathcal{A}_i \rightarrow \widehat{\mathcal{A}}_i$ such that both $\mathrm{Ker}(\rho)$ and $\mathrm{Coker}(\rho)$ belong to $\mathcal{T}$. Since the restriction of the functor $\mathrm{Hom}_\Lambda(-, T)$ to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is exact, so is $\mathrm{Hom}_\Lambda(-, T')_{|\mathcal{P}^{<\infty}(\Lambda-\mathrm{mod})}$, and consequently torsionfreeness of $T'$ shows $\mathrm{Hom}_\Lambda(\rho, T') : \mathrm{Hom}_\Lambda(\widehat{\mathcal{A}}_i, T') \rightarrow \mathrm{Hom}_\Lambda(\mathcal{A}_i, T')$ to be an isomorphism of right $\Gamma$-modules. Lemma 4.3 moreover ensures that $e\widehat{\mathcal{A}}_i$ has finite projective dimension in $e\Lambda e$-mod. Returning to the category equivalence $\mathcal{G} \cap (\Lambda\text{-mod}) \approx e\Lambda e$-mod which sends $M$ to $eM$ (cf. Lemma 2.4), we thus find finiteness of $\mathrm{pdim}_\Gamma \mathrm{Hom}_\Lambda(\widehat{\mathcal{A}}_i, T')$ to be equivalent to finiteness of

the projective dimension of $\mathrm{Hom}_{e\Lambda e}(e\widehat{\mathcal{A}}_i, eT')$ over $\mathrm{End}_{e\Lambda e}(eT')^{\mathrm{op}}$ alias $\widetilde{e}\widetilde{\Lambda}\widetilde{e}$. Given that $\mathrm{Hom}_{e\Lambda e}(-, eT')$ takes objects in $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$ to objects in $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{e}\widetilde{\Lambda}\widetilde{e})$ [loc.cit., applied to $e\Lambda e$ in place of $\Lambda$], condition (ii) follows.                                                        □

Theorem 5.4 ensures that, with a "duplicate" in $\widetilde{\Lambda}$ of the original idempotent $e \in \Lambda$, the test provided by Theorem 4.6 is again available towards deciding whether $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$ is contravariantly finite in $\mathrm{mod}\text{-}\widetilde{\Lambda}$.

**Corollary 5.5** *Adopt the hypotheses and notation of Theorem 5.4. Then contravariant finiteness of $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$ in $\mathrm{mod}\text{-}\widetilde{\Lambda}$ is equivalent to contravariant finiteness of $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{e}\widetilde{\Lambda}\widetilde{e})$ in $\mathrm{mod}\text{-}\widetilde{e}\widetilde{\Lambda}\widetilde{e}$.*

In light of the fact that the basic strong tilt $\widetilde{e\Lambda e}$ of $e\Lambda e$ coincides with $\widetilde{e}\widetilde{\Lambda}\widetilde{e}$, we thus obtain: Not only is existence of a strong tilting object in $\Lambda$-mod equivalent to existence of a strong tilting object in $e\Lambda e$-mod under conditions 4.1, but the same hypothesis implies that $\Lambda$-mod allows for unlimited iteration of strong tilting precisely when this is true for $e\Lambda e$-mod. This conclusion compiles information from Theorems 3.1, 4.6 and Corollary 5.5.

**Corollary 5.6** *Let $\Lambda$ be a basic Artin algebra and $e \in \Lambda$ an idempotent satisfying the hypotheses 4.1 (i.e., all simple left $\Lambda$-modules of infinite projective dimension belong to $\mathrm{add}(\Lambda e/Je)$, and $\mathrm{pdim}_{e\Lambda e} e\Lambda(1-e) < \infty$). Then the category $\Lambda$-mod allows for unlimited iteration of strong tilting if and only if the same is true for the category $e\Lambda e$-mod.*

A first straightforward application of our techniques shows that, in testing for contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ or iterability of strong tilting, we may automatically discard the idempotents corresponding to the simple left $\Lambda$-modules of projective dimension at most 1. Namely:

**Proposition 5.7** *Let $\Lambda$ be a basic Artin algebra in which we fix the complete set of primitive idempotents $\{e_1, ..., e_r, e_{r+1}, ..., e_n\}$, ordered in such a way that $\mathrm{pdim}(\Lambda e_i/Je_i) \leq 1$ for $i > r$. For $e = \sum_{i=1}^{r} e_i$, the following assertions are equivalent:*

(1)  $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite in $\Lambda$-mod (resp., $\Lambda$-mod allows for unlimited iteration of strong tilting);*

(2)  $\mathcal{P}^{<\infty}(e\Lambda e\text{-mod})$ *is contravariantly finite in $e\Lambda e$-mod (resp., $e\Lambda e$-mod allows for unlimited iteration of strong tilting).*

*Proof* From the proof of [7, Proposition 2.1], we know that $e\Lambda(1-e)$ is projective as a left $e\Lambda e$-module. Thus the pair $(\Lambda\text{-mod}, e)$ satisfies the blanket hypotheses of Setting 4.1. Assertions 1 and 2 are thus consequences of Theorem 4.6 and Corollary 5.6, respectively.                                                        □

# 6 Applications and Examples

By way of the equivalences established in the previous sections, we can now easily secure contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\mathrm{mod}\text{-}\widetilde{\Lambda})$ in cases in which this originally required a considerable effort. Moreover, the unifying reasons behind these results become more transparent through the reduction and permit us to expand the settings to which they apply.

## 6.1 Precyclic/Postcyclic Vertices and Normed Loewy Lengths

An instance in which the simplification gained by reduction to corner algebras stands out is that of truncated path algebras and their strong tilts (see [12] and [13]); without a reduction technique, it is challenging to confirm that $\Lambda$-mod allows for unlimited iteration of strong tilting in this case.

In light of Theorem 4.6, the first step, namely to confirm contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda$-mod) for truncated $\Lambda$, has now been trivialized. To generalize it, recall that, given any path algebra modulo relations, $\Lambda = KQ/I$, a vertex $e_i$ of $Q$ (systematically identified with a primitive idempotent of $\Lambda$) is called *precyclic* if there exists a path of length $\geq 0$ which starts in $e_i$ and ends on an oriented cycle; the attribute *postcyclic* is dual, and $e_i$ is called *critical* if it is both pre- and postcyclic. Clearly, all vertices which give rise to simple modules of infinite projective dimension are among the precyclic ones; moreover $e_i \Lambda e_j = 0$ whenever $e_i$ is precyclic, but $e_j$ is not. Theorem 4.6 thus yields

**Proposition 6.1** *Let $\Lambda = KQ/I$ be an arbitrary path algebra modulo relations, and let $e$ be the sum of the primitive idempotents corresponding to the precyclic vertices of $Q$. If $\mathcal{P}^{<\infty}(e\Lambda e$-mod) is contravariantly finite in $e\Lambda e$-mod, e.g., if l.findim $e\Lambda e = 0$, then $\mathcal{P}^{<\infty}(\Lambda$-mod) is contravariantly finite in $\Lambda$-mod.*

For truncated $\Lambda$, the final condition concerning the left finitistic dimension of $e\Lambda e$ is clearly satisfied since all indecomposable projective left $e\Lambda e$-modules $e\Lambda e_i$ for precyclic $e_i$ have the same Loewy length; this is not necessarily true for the indecomposable injective left $e\Lambda e$-modules, but it is for those whose socles correspond to critical vertices; namely, if $e'$ is the sum of the critical vertices of $Q$, then the indecomposable injective left $e'\Lambda e'$-modules also have coinciding Loewy lengths in the truncated case. The combination of these two conditions which norm the Loewy lengths of certain projective or injective modules is, in fact, all that is needed to guarantee that $\Lambda$-mod allows for iterated strong tilting.

**Proposition 6.2** *Again, let $\Lambda = KQ/I$ be a path algebra modulo relations and $e$ the sum of the primitive idempotents corresponding to the precyclic vertices of $Q$. Moreover, let $e'$ be the sum of those idempotents which correspond to the critical vertices.*

*Suppose that all indecomposable projective left $e\Lambda e$-modules have the same Loewy length, and that the analogous equality holds for the Loewy lengths of the indecomposable injective left $e'\Lambda e'$-modules. Then $\Lambda$-mod allows for unlimited iteration of strong tilting, thus giving rise to a sequence of related module categories $\Lambda$-mod $\rightsquigarrow$ mod-$\widetilde{\Lambda}$ $\rightsquigarrow$ $\widetilde{\widetilde{\Lambda}}$-mod $\rightsquigarrow \cdots$. Starting with the first strong tilt, mod-$\widetilde{\Lambda}$, the Morita equivalence classes of these categories repeat periodically with period 2.*

*Proof* Suppose that $e = e_1 + \cdots + e_m$ and $e' = e_1 + \cdots + e_r$ for some $r \leq m$. Since l.findim $e\Lambda e = 0$ due to the first condition on Loewy lengths, $\mathcal{P}^{<\infty}(e\Lambda e$-mod) is contravariantly finite in $e\Lambda e$-mod. Proposition 6.1 thus guarantees that $\mathcal{P}^{<\infty}(\Lambda$-mod) is contravariantly finite in $\Lambda$-mod.

As we already pointed out above, the conditions (i) and (ii) of Setting 4.1 are satisfied for the pair ($\Lambda$-mod, $e$), whence, by Theorem 5.4, we only need to show that $e\Lambda e$-mod allows for unlimited iteration of strong tilting. Set $\Lambda' = e\Lambda e$, and let $Q'$, resp. $J'$, be the quiver and Jacobson radical of $\Lambda'$, respectively. We already know that the category $\mathcal{P}^{<\infty}(\Lambda'$-mod) is contravariantly finite in $\Lambda'$-mod, due to the vanishing of the left finitistic dimension of $\Lambda'$. The latter in fact entails that the basic strong left $\Lambda'$-module $T'$ is a copy

of the left regular module ${}_{\Lambda'}\Lambda'$. Consequently, the strongly tilted algebra $\widetilde{\Lambda}' = \widetilde{e\Lambda e}$, i.e., the opposite of $\mathrm{End}_{\Lambda'}(T')$, coincides with $\Lambda'$. By Theorem 4.6, it therefore suffices to check that $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ is contravariantly finite in mod-$\Lambda'$. To confirm this, we observe that the pair (mod-$\Lambda'$, $e'$) in turn satisfies the hypotheses 4.1: Indeed, the precyclic vertices of $(Q')^{\mathrm{op}}$ are precisely $e_1, \ldots, e_r$, whence the simple right $\Lambda'$-modules of infinite projective dimension are among the quotients $e_i\Lambda'/e_i J'$ for $i \leq r$, and $(e - e')\Lambda'e' = 0$. The right finitistic dimension of $e'\Lambda'e' = e'\Lambda e'$ is in turn zero, because all indecomposable projective right $e'\Lambda e'$-modules have the same Loewy length; indeed, this follows by duality from the second of our two hypotheses. Consequently, another application of Proposition 6.1 yields contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ as required.

The concluding statements are part of Theorem 3.1.                                                □

We deduce Theorem D of [12] as a special case.

**Corollary 6.3** *Suppose that* $\Lambda = KQ/I$ *is a truncated path algebra. Then* $\Lambda$-mod *allows for unlimited iteration of strong tilting.*

The proof of the following generalization of Proposition 6.2 is immediate from that of the latter.

**Corollary 6.4** *Let* $\Lambda = KQ/I$, *and let* $e$ *and* $e'$ *be as in the statement of 6.2. If* l.findim $e\Lambda e = 0$ *and* $\mathcal{P}^{<\infty}(\text{mod-}e'\Lambda e')$ *is contravariantly finite in mod-$e'\Lambda e'$, then* $\Lambda$-mod *allows for unlimited iteration of strong tilting.*

Recall that an Artin algebra $\Lambda$ is said to be *left serial* if all indecomposable projective left $\Lambda$-modules are uniserial. The algebras which are left and right serial are also called *Nakayama algebras*. In [5] it was shown that, for any split left serial algebra $\Lambda$, the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod. The conclusion actually carries over to the category mod-$\Lambda$ of right $\Lambda$-modules. Namely:

**Proposition 6.5** *Suppose that* $\Lambda$ *is a path algebra modulo relations. If* $\Lambda$ *is left serial, then* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite in* $\Lambda$-mod *and* $\mathcal{P}^{<\infty}(\text{mod-}\Lambda)$ *is contravariantly finite in* mod-$\Lambda$.

*Proof* The assertion for left modules was proved in [5]. We only address right $\Lambda$-modules.

By hypothesis, $\Lambda \cong KQ/I$ is left serial. In particular, this means that $Q$ is free of double arrows. Without loss of generality, we assume that $Q$ is a connected quiver; we may further assume that it is not a tree, since otherwise $\Lambda$ has finite global dimension, which renders the contravariant finiteness claim trivial. By left seriality, $Q$ then contains a single oriented cycle such that all off-cycle vertices are pre- but not postcyclic. Consequently, any vertex of $Q^{\mathrm{op}}$ either belongs to said cycle or else is post- but not precyclic. On letting $e$ be the sum of the precyclic vertices of $Q^{\mathrm{op}}$, i.e., the vertices located on the oriented cycle in the present situation, we thus obtain a Nakayama algebra $e\Lambda e$. Hence the $\mathcal{P}^{<\infty}$-categories in both $e\Lambda e$-mod and mod-$e\Lambda e$ are contravariantly finite in the corresponding ambient module categories by [5, Theorem 5.2]. (The latter may alternatively be deduced from the fact that Nakayama algebras have finite representation type.) On combining this with Proposition 6.1, we obtain the claim.                                                □

## 6.2 Applications to Morita Contexts

In this subsection we shall see that Morita contexts provide a tool to construct examples of $\mathcal{P}^{<\infty}$-contravariant finiteness and iteration of strong tilting. Recall that a *Morita context* (over a ground commutative ring $K$) consists of a sextuple $(A, B, M, N, \varphi, \psi)$, where $A$ and $B$ are $K$-algebras, $M$ and $N$ are an $A$-$B$- and a $B$-$A$-bimodule, which we always assume with the same action of $K$ on the left and on the right, and $\varphi : M \otimes_B N \longrightarrow A$ and $\psi : N \otimes_A M \longrightarrow B$ are morphisms of $A$-$A$- and $B$-$B$-bimodules, respectively, satisfying certain compatibiliity conditions (see [18]) which are exactly the ones that make $\Lambda = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ into a $K$-algebra with the obvious multiplication. Recall that $\tau_A = \mathrm{Im}(\varphi)$ and $\tau_B = \mathrm{Im}(\psi)$ are two-sided ideals of $A$ and $B$, respectively, called the *trace ideals* of the Morita context.

The Morita contexts in which we are interested have some additional properties. We assume that $K$ is artinian and $A$, $B$, $M$ and $N$ are finitely generated as $K$-modules. Such a Morita context will be called a *basic Morita context of Artin algebras* if $\Lambda$ is basic or, equivalently, if $A$ and $B$ are basic, $\tau_A \subseteq J(A)$ and $\tau_B \subseteq J(B)$, where $J(-)$ denotes the Jacobson radical.

The following is the main result of the subsection.

**Theorem 6.6** *Let $(A, B, M, N, \varphi, \psi)$ be a basic Morita context of Artin algebras, where $M$ is projective as a left $A$-module. Suppose also that $\psi : N \otimes_A M \longrightarrow B$ is a monomorphism and the algebra $B/\tau_B$ has finite global dimension.*

*Set $\Lambda = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$. Then $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod (resp., $\Lambda$-mod allows for unlimited iteration of strong tilting) if and only if this is true for $\mathcal{P}^{<\infty}(A\text{-mod})$ (resp., $A$-mod).*

*In particular, $\Lambda$-mod allows for unlimited iteration of strong tilting when $A$ is a Gorenstein algebra or a truncated path algebra or when l.findim $A = 0 = $ r.findim $A$.*

*Proof* Take $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The goal is to check that the pair $(\Lambda\text{-mod}, e)$ satisfies the blanket hypotheses of Setting 4.1. Bearing in mind that the algebra $e\Lambda e$ is isomorphic to $A$, the result will then be a direct consequence of Theorem 4.6 and Corollary 5.6. Note that $B$ and $(1-e)\Lambda(1-e)$ are also isomorphic algebras and hence the $e\Lambda e$-$(1-e)\Lambda(1-e)$-bimodule $e\Lambda(1-e)$ is isomorphic to the $A$-$B$-bimodule $M$, with the obvious meaning. Hence $\mathrm{pdim}_{e\Lambda e}(e\Lambda(1-e)) = 0$ and we need to check that $\mathrm{pdim}_\Lambda(\frac{\Lambda(1-e)}{J(\Lambda)(1-e)}) < \infty$.

We first prove that $\Lambda e\Lambda(1-e) = \begin{pmatrix} 0 & M \\ 0 & \tau_B \end{pmatrix} \cong \begin{pmatrix} M \\ \tau_B \end{pmatrix}$ is a projective left $\Lambda$-module. Indeed the adjunction map $\epsilon_{\Lambda(1-e)} : \Lambda e \otimes_{e\Lambda e} e\Lambda(1-e) \longrightarrow \Lambda(1-e)$ gets identified, in the obvious way, with the map $\begin{pmatrix} A \\ N \end{pmatrix} \otimes_A M \longrightarrow \begin{pmatrix} M \\ B \end{pmatrix}$, that can be expressed matricially as $\begin{pmatrix} \mu & 0 \\ 0 & \psi \end{pmatrix} : \begin{pmatrix} A \otimes_A M \\ N \otimes_A M \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ B \end{pmatrix}$. Here $\mu : A \otimes_A M \longrightarrow M$ is the canonical isomorphism given by multiplication and $\psi : N \otimes_A M \longrightarrow B$ is the map in the Morita context. It follows that $\epsilon_{\Lambda(1-e)}$ is a monomorphism, and so

$$\Lambda e \otimes_{e\Lambda e} e\Lambda(1-e) \cong \Lambda e\Lambda(1-e) = \mathrm{Im}(\epsilon_{\Lambda(1-e)}).$$

But $\Lambda e \otimes_{e\Lambda e} e\Lambda(1-e) \in \mathrm{add}(\Lambda e \otimes_{e\Lambda e} e\Lambda e) = \mathrm{add}(\Lambda e)$ since $e\Lambda(1-e) \in e\Lambda e$-proj $= \mathrm{add}(_{e\Lambda e}e\Lambda e)$. Therefore $\Lambda e\Lambda(1-e)$ is a projective left $\Lambda$-module.

Next we should notice that there is an algebra isomorphism $\frac{\Lambda}{\Lambda e \Lambda} \cong \frac{B}{\tau_B}$, so that these isomorphic algebras have finite global dimension. Moreover, we have an isomorphism $\frac{\Lambda}{\Lambda e \Lambda} \cong \frac{\Lambda(1-e)}{\Lambda e \Lambda(1-e)}$ in $\Lambda$-mod, which, by the previous paragraph, implies that $\frac{\Lambda}{\Lambda e \Lambda}$ has projective dimension $\leq 1$ as a left $\Lambda$-module. Note also that $\frac{\Lambda(1-e)}{J(\Lambda)(1-e)}$ is canonically a left $\frac{\Lambda}{\Lambda e \Lambda}$-module. Fix a minimal projective resolution $0 \to Q_t \to ... \to Q_1 \to Q_0 \to \frac{\Lambda(1-e)}{J(\Lambda)(1-e)} \to 0$ in $\frac{\Lambda}{\Lambda e \Lambda}$-mod. This is also an exact sequence in $\Lambda$-mod, and we have that $\mathrm{pdim}_\Lambda(Q_k) \leq 1$ since $Q_k \in \mathrm{add}(\frac{\Lambda}{\Lambda e \Lambda})$, for all $k = 0, 1, ..., t$. It then follows that $\mathrm{pdim}_\Lambda(\frac{\Lambda(1-e)}{J(\Lambda)(1-e)}) < \infty$, as desired.

The final statement clearly follows from assertion 3 since in those examples $A$-mod allows for unlimited iteration of strong tilting. □

The last theorem and its proof have the following consequence for triangular matrix algebras.

**Corollary 6.7** *Let $K$ be a commutative Artinian ring, let $A$ and $B$ be basic Artin $K$-algebras and let $M$ and $N$ be a finitely generated $A$-$B$-bimodule and $B$-$A$-bimodule, respectively. Suppose that $B$ has finite global dimension and $\mathcal{P}^{<\infty}(A$-mod$)$ is contravariantly finite in $A$-mod. The following assertions hold:*

(1)  *If $\Lambda = \begin{pmatrix} A & 0 \\ N & B \end{pmatrix}$, then $\mathcal{P}^{<\infty}(\Lambda$-mod$)$ is contravariantly finite in $\Lambda$-mod. Moreover if $A$-mod allows for unlimited iteration of strong tilting, so does $\Lambda$-mod.*

(2)  *If $\mathrm{pdim}(_A M) < \infty$, the conclusions of assertion 1 remain true on replacement of $\Lambda$ by $\Gamma = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.*

*Proof* Assertion 1 is a direct consequence of Theorem 6.6. As for assertion 2, note that if $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $\mathrm{pdim}_{e\Gamma e}(e\Gamma(1-e)) < \infty$ since we have $\mathrm{pdim}(_A M) < \infty$. We will prove that $\mathrm{pdim}_\Gamma(\Gamma e \Gamma(1-e)) < \infty$ and, arguing as in the last paragraph of the proof of Theorem 6.6, we will conclude that the pair $(\Gamma$-mod, $e)$ satisfies the blanket hypotheses of Setting 4.1, and the result will follow from Theorem 4.6 and Corollary 5.6.

Note that $0 = (1-e)\Gamma e$ and so $\Gamma e = e\Gamma e$ is projective as right $e\Gamma e$-module. Moreover we have a commutative diagram of $K$-modules

$$
\begin{array}{ccc}
e\Gamma e \otimes_{e\Gamma e} e\Gamma(1-e) & \xrightarrow{\;\cong\;} & e\Gamma(1-e) \\
\| & & \| \\
\Gamma e \otimes_{e\Gamma e} e\Gamma(1-e) & \xrightarrow{\hspace{2cm}} & \Gamma e \Gamma(1-e)
\end{array}
\quad,
$$

where the upper horizontal arrow is the canonical isomorphism and the lower horizontal one is the multiplication map. It follows that this latter arrow is an isomorphism, which implies that $\mathrm{pdim}_\Gamma(\Gamma e \Gamma(1-e)) < \infty$ since $_{e\Gamma e} e\Gamma(1-e)$ has finite projective dimension and the functor $\Gamma e \otimes_{e\Gamma e} - : e\Gamma e$-mod $\longrightarrow \Gamma$-mod is exact and takes projectives to projectives. □

We end the paper by giving non-triangular examples to which Theorem 6.6 applies. We start with the following elementary observation.

*Remark 6.8* Let $A$ and $B$ be basic Artin $K$-algebras and $M$ and $N$ finitely generated $A$-$B$- and $B$-$A$-bimodules, respectively. If $N \otimes_B M = 0$ then any morphism $\varphi : M \otimes_B N \longrightarrow A$ of $A$-$A$-bimodules such that $\mathrm{Im}(\varphi) \subseteq J(A)$ gives rise to a basic Morita context of Artin algebras $(A, B, M, N, \varphi, 0)$ since the required compatibility conditions hold (see, e.g., [19, Exercise IV.35]).

*Examples 6.9* Let $A$, $B$ be basic finite dimensional algebras over an algebraically closed field $K$ and let us fix complete sets of primitive idempotents $\{e_1, ...., e_m\}$ and $\{e_{m+1}, ...., e_n\}$ in $A$ and $B$, respectively. Consider either one of the following two situations, where the unadorned $\otimes$ means $\otimes_K$:

(a) $\mathrm{gldim}(B) < \infty$, $M$ is any finitely generated $A$-$B$-bimodule that is projective as a left $A$-module, $N$ is any finitely generated $B$-$A$-bimodule such that $N e_i = 0$ whenever the simple left $A$-module $Ae_i / J(A)e_i$ embeds in $\mathrm{top}(_A M)$, and $\varphi : M \otimes_B N \longrightarrow A$ is any morphism of $A$-$A$-bimodules such that $\mathrm{Im}(\varphi) \subseteq J(A)$. Then $(A, B, M, N, \varphi, 0)$ is a basic Morita context.

(b) Suppose that the quiver $Q_B$ of $B$ has no oriented cycles, let $\mathrm{Soc}(B)$ be the socle of $B$ as a $B$-$B$-bimodule and fix any index $k \in \{m+1, ..., n\}$ such that $W := \mathrm{Soc}(B)e_k \neq 0$. Note that $W$ is a subbimodule of $_B B_B$ isomorphic to $Y \otimes \frac{e_k B}{e_k J(B)}$, for some semisimple left $B$-module $Y$ that, due to the absence of oriented cycles in $Q_B$, satisfies $e_k Y = 0$. Take $M = Ae_i \otimes \frac{e_k B}{e_k J(B)}$ and $N = Y \otimes \frac{e_i A}{e_i J(A)}$, for some $i = 1, ..., m$. We then have $M \otimes_B N = 0$ and an isomorphism of $B$-$B$-bimodules

$$N \otimes_A M \cong Y \otimes \frac{e_i A e_i}{e_i J(A) e_i} \otimes \frac{e_k B}{e_k J(B)} \cong Y \otimes \frac{e_k B}{e_k J(B)} \cong W.$$

Taking as $\psi : N \otimes_A M \longrightarrow B$ the composition of this latter isomorphism followed by the inclusion $W \hookrightarrow B$, we get a basic Morita context $(A, B, M, N, 0, \psi)$ (cf. the left-right symmetric version of Remark 6.8).

In the situations (a) and (b), the associated algebra $\Lambda = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$ satisfies that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod (resp., $\Lambda$-mod allows for unlimited iteration of strong tilting) if, and only if, so does the algebra $A$.

*Proof* In situation (a), we have an isomorphism of left $A$-modules $M \cong \oplus_{i \in I} Ae_i^{t_i}$, where $I$ is the set of $i \in \{1, ..., m\}$ such that the simple left $A$-module $Ae_i / J(A)e_i$ embeds in $\mathrm{top}(_A M)$ and $t_i > 0$ for all $i \in I$. It follows that $N \otimes_A M = 0$ and the existence of the mentioned Morita context follows by Remark 6.8. Hence in both situations $M$ is projective as a left $A$-module and the map $\psi : N \otimes_A M \longrightarrow B$ is a monomorphism. In (a) we have that $\tau_B = 0$ and in (b) the quiver of $B/\tau_B$ has no oriented cycles. Therefore $\mathrm{gl.dim}(B/\tau_B) < \infty$ in both cases and assertions 1 and 2 are direct consequences of Theorem 6.6. $\square$

The following example is a combinatorial version of Example 6.9(a).

*Example 6.10* Let $A := K Q_e / \langle R_e \rangle$ and $B := K Q_{1-e} / \langle R_{1-e} \rangle$ be finite dimensional algebras given as quotients of path algebras modulo relations, on which we fix bases $B_e$ and $B_{1-e}$ consisting of paths in $Q_e$ and $Q_{1-e}$, respectively. Consider the algebra $\Lambda = K Q / \langle R \rangle$, also given by quiver and relations, where:

(1)   The quiver $Q$ is obtained from the disjoint union quiver $Q_e \bigsqcup Q_{1-e}$ by adding two finite sets of arrows (some possibly empty) $\{\alpha_1, ..., \alpha_s\}$ and $\{\beta_1, ..., \beta_t\}$, where the $\alpha_i$ go from vertices in $Q_{1-e}$ to vertices in $Q_e$ and the $\beta_j$ from vertices in $Q_e$ to vertices in $Q_{1-e}$, with the only restriction that $t(\alpha_k) \neq o(\beta_l)$ for all $k = 1, ..., s$ and $l = 1, ..., t$ (here $o(\gamma)$ and $t(\gamma)$ denote, respectively, the origin and terminus of any arrow $\gamma$);

(2)   The set of relations is $R = R_e \cup R_{1-e} \cup R_{1-e,e} \cup R_{e,e}$, where
$R_{1-e,e} = \{\beta_l q : l \in \{1, ..., t\}$ and $q \in \bigcup_{k=1}^{s} e_{o(\beta_l)} B_e e_{t(\alpha_k)}\}$, and
$R_{e,e} = \{\alpha_k p \beta_l : l \in \{1, ..., t\}, k \in \{1, ..., s\}$ and $p \in e_{o(\alpha_k)} B_{1-e} e_{t(\beta_l)}\}$.

When $\mathrm{gldim}(B) < \infty$, the subcategory $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod (resp. $\Lambda$-mod allows for arbitrary iteration of strong tilting) if, and only if, the corresponding property is true for the algebra $A$.

*Proof* Let $\{e_1, ..., e_m, e_{m+1}, ..., e_n\}$ be the set of primitive idempotents of $\Lambda$ corresponding to the vertices of $Q$, where we assume that $i$ is a vertex of $Q_e$ if and only if $i \leq m$. We have the canonical Morita context associated to $e := e_1 + ... + e_m$, so that $\Lambda \cong \begin{pmatrix} e\Lambda e & e\Lambda(1-e) \\ (1-e)\Lambda e & (1-e)\Lambda(1-e) \end{pmatrix}$. The relations $R_{1-e,e}$ and $R_{e,e}$ guarantee that the nonzero paths in $e\Lambda e$ and $(1-e)\Lambda(1-e)$ consist exclusively of arrows of $Q_e$ and $Q_{1-e}$, respectively. That is, we have $e\Lambda e \cong A$ and $(1-e)\Lambda(1-e) \cong B$.

On the other hand, one can check that the chosen set of relations implies that $e\Lambda(1-e) = \sum_{k=1}^{s} \sum_{p \in e_{o(\alpha_k)} B_{1-e}} e\Lambda\alpha_k p$, that this sum is direct and that the map $e\Lambda e_{t(\alpha_k)} \longrightarrow e\Lambda\alpha_k p$ ($x \to x\alpha_k p$) is an isomorphism, for all $k = 1, ..., s$. Therefore $e\Lambda(1-e)$ is a projective left $e\Lambda e$-module. Moreover the simple left $e\Lambda e$-module $e\Lambda e_i / eJe_i$ embeds in $\mathrm{top}(_{e\Lambda e}e\Lambda(1-e))$ if and only if $i = t(\alpha_k)$ for some $k = 1, ..., s$. But the relations in $R_{1-e,e}$ imply that $(1-e)\Lambda e_{t(\alpha_k)} = 0$, for all $k = 1, ..., s$.

We are thus in the situation of Example 6.9(a), and the conclusions follow from that example.     □

## 6.3 A Specific Path Algebra Modulo Relations

The final example is a non-monomial path algebra modulo relations whose category of left modules allows for unlimited iteration of strong tilting. Our reduction technique renders verification of this fact significantly less labor-intensive.

*Example 6.11* Let $\Lambda = KQ/I$ be a specimen of the following class of finite dimensional algebras over a field $K$, which depends on 4 parameters $c_1, ..., c_4 \in K^*$. The quiver $Q$ is
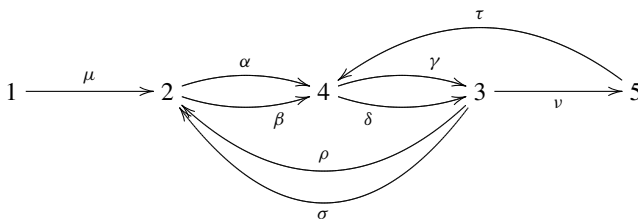
and $I \subseteq KQ$ is the ideal generated by $\gamma\alpha - c_1\delta\beta$, $\gamma\beta - c_2\delta\alpha$, $\alpha\rho - c_3\beta\sigma$, and $\alpha\rho - c_4\tau\nu$, next to monomial relations which are apparent from the graphs of the indecomposable projective left $\Lambda$-modules:
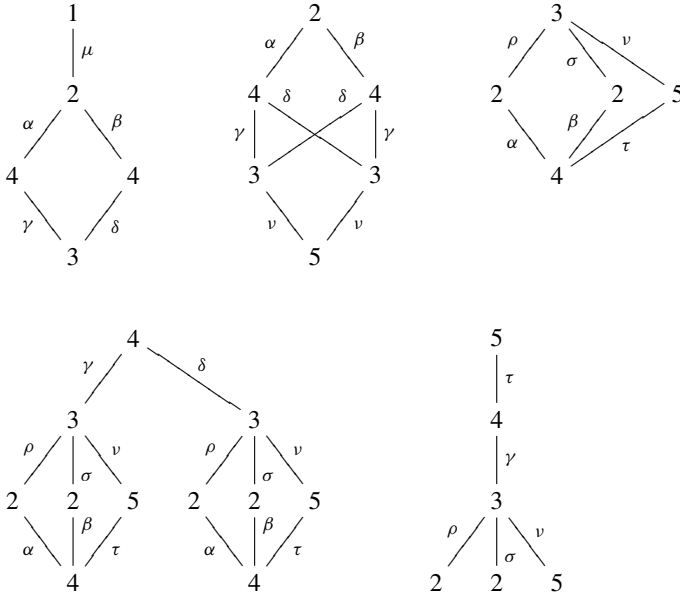
Then $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ is contravariantly finite in $\Lambda$-mod, and $\Lambda$-mod allows for unlimited iteration of strong tilting.

*Proof* First one checks that the simple left $\Lambda$-modules corresponding to the vertices $e_4$ and $e_5$ have finite projective dimension, namely $\mathrm{pdim}_\Lambda S_4 = 1$ and $\mathrm{pdim}_\Lambda S_5 = 3$. Choose $e :=$ $e_1 + e_2 + e_3$. To check that $e\Lambda(1 - e)$ is projective in $e\Lambda e$-mod, observe that $\Lambda' := e\Lambda e$ is the algebra determined by the quiver $Q'$ and the indecomposable projective left $\Lambda'$-modules shown below.
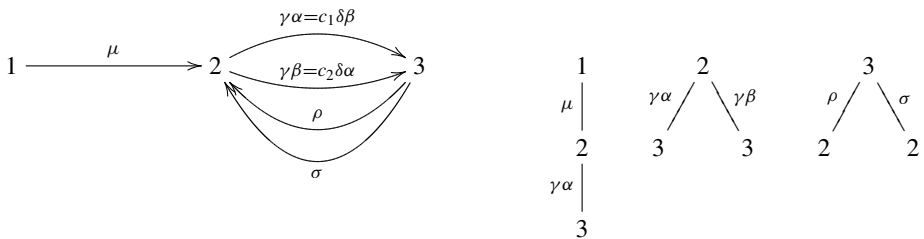
We now read off that $e\Lambda e_4 \cong (\Lambda' e_3)^2$ and $e\Lambda e_5 \cong \Lambda' e_3$ in $\Lambda'$-mod, to find that the left $\Lambda'$-module $e\Lambda(1 - e)$ is indeed projective. To see that $\mathcal{P}^{<\infty}(\Lambda'\text{-mod})$ is contravariantly finite in $\Lambda'$-mod, we check that $\mathrm{l.findim}\ \Lambda' = 0$ by applying Bass's criterion [4]: Indeed, the graphs of the indecomposable projective right $\Lambda'$-modules,
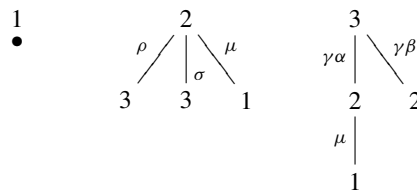
show that all simple right $\Lambda'$-modules embed into the right socle of $\Lambda'$. Consequently, Theorem 4.6 yields contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in $\Lambda$-mod.

To confirm that $\Lambda$-mod even allows for unlimited iteration of strong tilting, we show that this is true for $\Lambda'$-mod, whence Corollary 5.6 will yield our claim. Since l.findim $\Lambda' = 0$, the basic strong tilting object in $\Lambda'$-mod is $_{\Lambda'}T' = {}_{\Lambda'}\Lambda'$, which yields $\widetilde{\Lambda'} \cong \Lambda'$.

It is not difficult to directly ascertain that $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ is in turn contravariantly finite in mod-$\Lambda'$, but another application of Theorem 4.6 cuts this task short: Setting $e' = e'_2 + e'_3$, where the $e'_i$ are the primitive idempotents of $\Lambda'$ corresponding to the vertices of $Q'$, it is effortless to check that $e'$ satisfies the conditions of Setting 4.1 relative to mod-$\Lambda'$, and that r.findim $e'\Lambda'e' = 0$. Hence $\mathcal{P}^{<\infty}(\text{mod-}e'\Lambda'e')$ is contravariantly finite in mod-$e'\Lambda'e'$, and consequently so is $\mathcal{P}^{<\infty}(\text{mod-}\Lambda')$ in mod-$\Lambda'$. By Theorem 3.1, we thus conclude that $\Lambda'$-mod allows for unlimited iteration of strong tilting. This completes the argument. $\qquad\square$

**Data Availability** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

## Declarations

**Conflict of Interests** This paper is in compliance with the ethical requirements for this sort of publication. The authors have no financial or propriety interests in any material discussed in this article.

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