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On the Extension-Closed Property for the Subcategory $\mathrm{Tr}(\Omega^2(\mathrm{mod}-A))$

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Abstract

We present a monomial quiver algebra A having the property that the subcategory $\mathrm{Tr}(\Omega^2(\mathrm{mod}-A))$ is not extension-closed. This answers a question raised by Idun Reiten.

Keywords Extension-Closed Subcategory · Auslander—Reiten Translate · Syzygies

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Introduction

For a noetherian ring Λ, let $\mathrm{mod}-\Lambda$ denote the category of finitely generated right Λ-modules. Moreover, let $\Omega^i(\mathrm{mod}-\Lambda)$ denote the full subcategory of i-th syzygy modules of finitely generated Λ-modules. Here a module N is an i-th syzygy module if it is a direct summand of a module of the form $P \oplus \Omega^i(M)$ with P being projective, M a module with projective resolution

$$\cdots \to P_i \xrightarrow{f_i} P_{i-1} \to \cdots \to P_0 \xrightarrow{f_0} M \to 0,$$

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Dedicated to the memory of Izzy and Leni

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and $\Omega^i(M)$ being the kernel of f_{i-1}. The study of syzygies is of major importance in homologcal algebra and representation theory, we mention for example Hilbert's famous syzygy theorem [6] and the work of Auslander and Reiten on the subcategories of syzygy modules for noetherian rings in [2]. Throughout, we assume that all subcategories are full. Let $\mathcal{C}$ denote a subcategory of $\text{mod}-\Lambda$ for a noetherian ring Λ and let Tr denote the Auslander−Bridger transpose that gives an equivalence between the stable categories $\underline{\text{mod}}-\Lambda$ and $\underline{\text{mod}}-\Lambda^{op}$, see for example [1]. Following [10], we define $\text{Tr}(\mathcal{C})$ to be the smallest additive subcategory of $\text{mod}-\Lambda^{op}$ containing the modules $\text{Tr}(M)$ for $M \in \mathcal{C}$ and all the projective Λ^{op}-modules. Recall that a subcategory $\mathcal{C}$ is said to be closed under extensions if for any $M, N \in \mathcal{C}$, also every module X is in $\mathcal{C}$ if there is an exact sequence of the form $0 \to M \to X \to N \to 0$. The extension-closedness of certain subcategories can give important information on the algebra. We mention for example Theorem 0.1 in [2] which shows that the property that the subcategories $\Omega^i(\text{mod}-\Lambda)$ are extension-closed for $1 \le i \le n$ for an algebra Λ is equivalent to Λ being quasi n-Gorenstein in the sense of [8]. In [10], it is noted that even if a subcategory $\mathcal{C}$ is not closed under extensions, the subcategory $\text{Tr}(\mathcal{C})$ can still be closed under extensions. This is for example true for $\mathcal{C} = \Omega^1(\text{mod}-\Lambda)$ as was remarked in [10]. Idun Reiten states in the paragraph before Proposition 1.1 in [10] that the answer to the following question is not known:

Question 0.1 *Let Λ be a noetherian ring and $i > 1$. Is $\text{Tr}(\Omega^i(\text{mod}-\Lambda))$ closed under extensions?*

In this article we give a negative answer to this question. An Appendix containing the performed computer calculations is also given.

Notation Throughout, let $A = KQ/I$ be the finite-dimensional quiver algebra over a field K where Q is given by $2 \underset{y}{\overset{z}{\rightleftarrows}} 1 \circlearrowright x$ and the relations are given by $I = \langle xy, yz, zx, x^3 \rangle$.

Our main result is:

Theorem 0.2 *The subcategory $\text{Tr}(\Omega^2(\text{mod}-A))$ is not closed under extensions.*

The Proof

In this section we prove Theorem 0.2. We assume that the reader is familiar with the basics of representation theory and homological algebra of finite-dimensional algebras and refer for example to [3] and [11]. Let A be defined as above. Let S_i denote the simple A-modules, J the Jacobson radical of A, $D = \text{Hom}_K(-, K)$ the natural duality and $\tau = D\,\text{Tr}$ the Auslander-Reiten translate. The algebra A has vector space dimension 7 over the field K. Let e_i denote the primitive idempotents of the quiver algebra A corresponding to the vertices i for $i = 1, 2$. The indecomposable projective A-module $P_1 = e_1 A = \langle e_1, x, y, x^2 \rangle$ has dimension vector $[3, 1]$ and the indecomposable projective A-modules $P_2 = e_2 A = \langle e_2, z, zy \rangle$ has dimension vector $[1, 2]$. We remark that P_2 is isomorphic to the indecomposable injective module $I_2 = D(Ae_2)$. The indecomposable injective module $I_1 = D(Ae_1)$

has dimension vector [3, 1]. We give the Auslander−Reiten quiver of A here:

$$\begin{tikzcd}
& {[1,2]=P_2} \arrow[dr] & \\
{[1,1]=e_2J} \arrow[ur] \arrow[rr, dashed, shift left] \arrow[rr, dashed, shift right, <-] \arrow[drr] & & {[1,1]=P_2/S_2} \arrow[dll] \\
{[2,2]=A/(x+z)A=M_1} \arrow[u] \arrow[rr, dashed, shift left, <-] \arrow[rr, dashed, shift right] \arrow[drr] & & {[1,0]=S_1} \arrow[u] \arrow[dll] \\
{[2,1]=A/(x+y+z)A=U} \arrow[u] \arrow[rr, dashed, shift left] \arrow[rr, dashed, shift right, <-] \arrow[drr] & & {[2,1]=P_1/S_1} \arrow[u] \arrow[dll] \\
{[3,2]=A/e_2J} \arrow[u] \arrow[rr, dashed, shift left] \arrow[rr, dashed, shift right, <-] \arrow[drr] & & {[2,0]=P_1/soc(P_1)} \arrow[u] \arrow[dll] \\
{[3,1]=P_1} \arrow[u] \arrow[drr] & & {[3,1]=I_1} \arrow[u] \arrow[ll, dashed] \arrow[dll] \\
{[0,1]=S_2} \arrow[u] \arrow[rr, dashed, shift left] \arrow[rr, dashed, shift right, <-] & & {[3,0]=P_1/S_2} \arrow[u]
\end{tikzcd}$$

We remark that the algebra A is a representation-finite string algebra and thus the determination of the Auslander−Reiten quiver is well known, see for example [5]. We leave the verficiation of the Auslander−Reiten quiver of A to the reader. The Auslander−Reiten quiver can also be verified using [9], since by the theory in [5], the Auslander−Reiten quiver does not depend on the field because A is a representation-finite string algebra.

The dashed arrows in the Auslander−Reiten quiver indicate the Auslander−Reiten translates and the other arrows the irreducible maps. There are 13 indecomposable A-modules and we label them with their dimension vectors and an easy description via indecomposable projectives or injectives, their radicals, simple modules, or ideals in the algebra A. For finite-dimensional algebras we have $D\,\mathrm{Tr} = \tau$, the Auslander−Reiten translate. Since D is a duality, the subcategory $\mathrm{Tr}(\Omega^2(\mathrm{mod}-A))$ is extension-closed if and only if $\tau(\Omega^2(\mathrm{mod}-A))$ is extension-closed. We will consider the subcategory $\tau(\Omega^2(\mathrm{mod}-A))$ in the following. We sometimes use the notation $\tau_i := \tau\Omega^{i-1}$ for the higher Auslander−Reiten translates that play an important role for example in the theory of cluster-tilting subcategories, see [7]. Let $\tilde{I} := (x+z)A$ denote the right ideal generated by $x+z$, which has vector space basis $\{x+z, x^2, zy\}$. The module $M_1 := A/\tilde{I}$ is an indecomposable A-module with dimension vector [2, 2]. Moreover, we mention that M_1 is depicted as a quiver representation in the proof of Lemma 0.4. Let $M_2 := P_2/S_2 = e_2A/(zy)A$. This is an indecomposable A-module with dimension vector [1, 1]. See the Auslander−Reiten quiver of A.

Lemma 0.3 *The following assertions hold:*

(1) $\tau_3(M_1) \cong M_1$ *and hence* $M_1 \in \tau(\Omega^2(\mathrm{mod}-A))$.
(2) $\tau_3(M_2) \cong M_2$ *and hence* $M_2 \in \tau(\Omega^2(\mathrm{mod}-A))$.

Proof

(1) We have the exact sequences

$$0 \to (x+z)A \to A \to A/(x+z)A \to 0$$

and

$$0 \to S_1 \to e_1 A \to (x+z)A \to 0.$$

It follows that $\Omega^2(M_1) \cong S_1$. We can see from the Auslander–Reiten quiver that $\tau(S_1) \cong M_1$. Therefore $\tau(\Omega^2(M_1)) \cong \tau(S_1) \cong M_1$.

(2) We have the exact sequences

$$0 \to S_2 \to e_2 A \to e_2 A/S_2 \to 0$$

and

$$0 \to e_2 J^1 \to e_2 A \to S_2 \to 0.$$

It follows that $\Omega^2(M_2) \cong e_2 J$ and we can see from the Auslander–Reiten quiver that $\tau(e_2 J) \cong M_2$. Therefore $\tau(\Omega^2(M_2)) \cong \tau(e_2 J) \cong M_2$.

□

Now, we construct a module W such that W is an extension of M_2 by M_1, but W does not lie in $\tau(\Omega^2(\mathrm{mod}-A))$.

Lemma 0.4 *There is a short exact sequence*

$$0 \to M_1 \xrightarrow{u} W \to M_2 \to 0.$$

We have $W \cong P_2 \oplus U$, *where* $U = A/((x+y+z)A)$ *is indecomposable. The short exact sequence is therefore not split exact.*

Proof We define $u = (u_1, u_2)$ and W as follows using quiver representations:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
k^2 \arrow[loop left, "\begin{pmatrix}0&1\\0&0\end{pmatrix}=x"] \arrow[rr, bend left, "y=\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[d, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}=u_1"'] & & k^2 \arrow[ll, bend left, "z=\begin{pmatrix}0&0\\0&-1\end{pmatrix}"] \arrow[d, "u_2=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"] \\
k^3 \arrow[loop left, "\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}=x"] \arrow[rr, bend left, "y=\begin{pmatrix}1&0&0\\0&0&0\\1&0&0\end{pmatrix}"] & & k^3 \arrow[ll, bend left, "z=\begin{pmatrix}0&0&0\\0&-1&0\\0&0&1\end{pmatrix}"]
\end{tikzcd}
$$

It is then easy to see that u is a monomorphism whose domain is isomorphic to M_1, and that the cokernel of u is isomorphic to M_2. The quiver representation of P_2 looks as follows:

$$
\begin{tikzcd}[column sep=huge]
k \arrow[loop left, "(0)=x"] \arrow[rr, bend left, "y=\begin{pmatrix}0&1\end{pmatrix}"] & & k^2 \arrow[ll, bend left, "z=\begin{pmatrix}1\\0\end{pmatrix}"]
\end{tikzcd}
$$

The quiver representation of U looks as follows:

$$\begin{tikzcd} k^2 \arrow[loop left, "x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}"] \arrow[r, bend left, "y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}"] & k \arrow[l, bend left, "z = \begin{pmatrix} 0 & -1 \end{pmatrix}"] \end{tikzcd}$$

Note that the module U is isomorphic to the module $A/(x+y+z)A$ and is indecomposable. A direct verification that we leave to the reader shows that $W \cong P_2 \oplus U$. This proves that the short exact sequence is not split exact, since W is not isomorphic to the direct sum of the indecomposable modules M_1 and M_2. $\square$

Lemma 0.5 *The module $U = A/(x+y+z)A$ does not lie in $\tau(\Omega^2(\mathrm{mod} - A))$.*

Proof We prove the equivalent statement that $\tau^{-1}(U)$ is not a second syzygy module. From the Auslander–Reiten quiver we can see that $\tau^{-1}(U) = P_1/S_1$. Now we use the result that a general module X over an Artin algebra is a direct summand of an n-th syzygy module if and only if X is a direct summand of $P' \oplus \Omega^n(\Omega^{-n}(X))$ for some projective module P', see for example Proposition 3.2 of [4]. The two short exact sequences

$$0 \to P_1/S_1 \to D(A) \to S_2 \oplus U \to 0$$

and

$$0 \to S_2 \oplus U \to D(A) \to S_1 \oplus M_2 \to 0$$

give rise to minimal injective coresolutions. Now $\Omega^2(S_1) \cong S_1 \oplus S_2 \oplus e_2 J^1$ and $\Omega^2(M_2) \cong e_2 J^1$.

But the indecomposable module P_1/S_1 is not a direct summand of a module of the form

$$P' \oplus \Omega^2(\Omega^{-2}(P_1/S_1)) = P' \oplus S_1 \oplus S_2 \oplus e_2 J^1 \oplus e_2 J^1$$

for any projective module P'. Hence, $\tau^{-1}(U)$ is not a 2^{nd} syzygy module. $\square$

The subcategory $\tau(\Omega^2(\mathrm{mod} - A))$ and equivalently the subcategory $\mathrm{Tr}(\Omega^2(\mathrm{mod} - A))$ is therefore not extension-closed.

Appendix: QPA Calculations

In this section we illustrate how to verify that our result is correct over the field with three elements with the help of QPA. We maintain the notation from the previous chapter. The reader can copy and paste the following code into GAP. A hashmark indicates the beginning of a comment.

```
LoadPackage("qpa");
Q:=Quiver(2,[[1,1,"x"],[1,2,"y"],[2,1,"z"]]);kQ:=PathAlgebra
(GF(3),Q);
AssignGeneratorVariables(kQ); # in order to recognize x,y,z as
elements of kQ
rel:=[x*y,y*z,z*x,x^3];A:=kQ/rel; # define the admissible ideal
```

```
and the algebra
projA:=IndecProjectiveModules(A); # define the PIMs of A
P1:=projA[1];P2:=projA[2];
M2:=CoKernel(SocleOfModuleInclusion(P2));
I_neu:=RightIdeal(A, [A.x+A.z]); # next, transform this ideal
into the module A/I_neu:
O:=RightAlgebraModuleToPathAlgebraMatModule(RightAlgebraModule
(A, \*, I_neu));
M1:=TransposeOfModule(NthSyzygy(TransposeOfModule(O),1));
IsomorphicModules(M1,DTr(NthSyzygy(M1,2)));
IsomorphicModules(M2,DTr(NthSyzygy(M2,2)));
IsIndecomposableModule(M1);
IsIndecomposableModule(M2);
ext:=ExtOverAlgebra(M2,M1);
maps:=ext[2];
Size(maps); # compute the vector space dimension of Ext(M2,M1)
monos:=List(maps,h->PushOut(ext[1],h)[1]);
u:=monos[1];
IsInjective(u);
IsomorphicModules(Source(u),M1);
IsomorphicModules(CoKernel(u),M2);
W:=Range(u);
WW:=DecomposeModule(W);
OO:=WW[1];IsomorphicModules(OO,P2);
U:=WW[2];
IsIndecomposableModule(U);
K:=TrD(U);
IsNthSyzygy(K,2);
```

Hereby, we have constructed an injective map $u : M_1 \to W$, where the modules M_1 and $M_2 := \mathrm{Coker}(u)$ are indecomposable and satisfy $M_1 \cong \tau_3(M_1)$ and $M_2 \cong \tau_3(M_2)$. It is verified that the module W is isomorphic to $P_2 \oplus U$, where P_2 is the indecomposable projective module corresponding to the second vertex e_2 and U is some indecomposable module that is not in $\tau(\Omega^2(\mathrm{mod}-A))$, since $\tau^{-1}(U)$ is not a second syzygy module. The subcategory $\tau(\Omega^2(\mathrm{mod}-A))$ is therefore not extension-closed.

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Declarations

Competing interests Non-financial interests: none.

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