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Are Special Biserial Algebras Homologically Tame?

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Abstract

Birge Huisgen-Zimmermann calls a finite dimensional algebra homologically tame provided the little and the big finitistic dimension are equal and finite. The question formulated in the title has been discussed by her in the paper "Representation-tame algebras need not be homologically tame", by looking for any $r \geq 1$ at a sequence of algebras Λ_m with big finitistic dimension $r+m$. As we will show, also the little finitistic dimension of Λ_m is $r+m$. It follows that contrary to her assertion, all the algebras Λ_m are homologically tame.

Keywords Finitistic dimension · Special biserial algebras

Mathematics Subject Classification (2010) Primary 16E10 · Secondary 16G20, 16G60

Birge Huisgen-Zimmermann calls a finite dimensional algebra *homologically tame* provided the little and the big finitistic dimension are equal and finite. The question formulated in the title has been discussed by her in the paper [1], by looking for any $r \geq 1$ at a sequence of algebras Λ_m with big finitistic dimension $r+m$. She presented a quite surprising infinite-dimensional Λ_m-module with projective dimension $r+m$, stressing that related finite-dimensional modules have infinite projective dimension. Nonetheless, as we will show, there do exist finite-dimensional Λ_m-modules with projective dimension $r+m$. Thus, also the little finitistic dimension of Λ_m is $r+m$. It follows that contrary to her assertion, all the algebras Λ_m are homologically tame.

Notation Let k be a field and $r \geq 1$ a fixed natural number. We will deal with a sequence of finite-dimensional k-algebras Λ_m with $m \geq 0$, with Λ_m being a factor algebra of Λ_{m+1} for all m (thus, Λ_m-modules can be considered as Λ_{m+1}-modules) such that the projective Λ_m-modules are also projective as Λ_{m+1}-modules. The modules to be considered are (not necessarily finitely generated, left) Λ_m-modules for some m. Given any module M, we

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denote by PM a projective cover, by ΩM the first syzygy module and by pd M the projective dimension of M. If x is a vertex of a quiver Q, the corresponding simple representation will also be denoted by x.

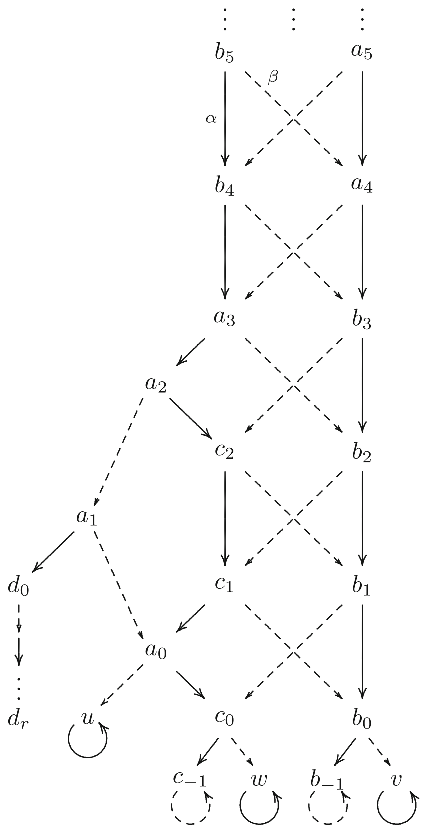
Outline In Section 1, we recall the definition of the special biserial algebras Λ_m considered in [1]. In Section 2, we exhibit for any m a finite-dimensional Λ_m-module Z_m of projective dimension $r+m$. Thus the little finitistic dimension fin. dim. Λ_m of Λ_m is at least $r+m$. Section 3 presents a proof of the assertion in [1] that the big finitistic dimension Fin. dim. Λ_m of Λ_m is at most $r+m$. Combining these results, we get

$$r+m \leq \text{fin. dim. } \Lambda_m \leq \text{Fin. dim. } \Lambda_m \leq r+m.$$

It follows that both the little and the big finitistic dimension are equal to $r+m$.

1 The Algebras Λ_m

As we mentioned, we consider the algebras Λ_m exhibited in [1]. This means, we deal with the following quiver with relations. The labels of the vertices are those used in [1], but we denote all arrows just by α or β, so that $\alpha\beta = 0 = \beta\alpha$ (and so that at any vertex, at most one α-arrow and at most one β-arrow start, and similarly, at most one α-arrow and at most one β-arrow end; finally, the arrow $a_1 \to d_0$ is an α-arrow). The α-arrows are drawn as solid arrows, the β-arrows are the dashed ones.



The path starting at d_0 is an alternating β-α-path of length r with vertices $d_0, d_1, \ldots, d_r$. The α-path of length 2 ending at a_m with $m \geq 3$ starts in b_{m+2}. Similarly, the α-path of length 2 ending at b_m with $m \geq 2$ starts in a_{m+2}.

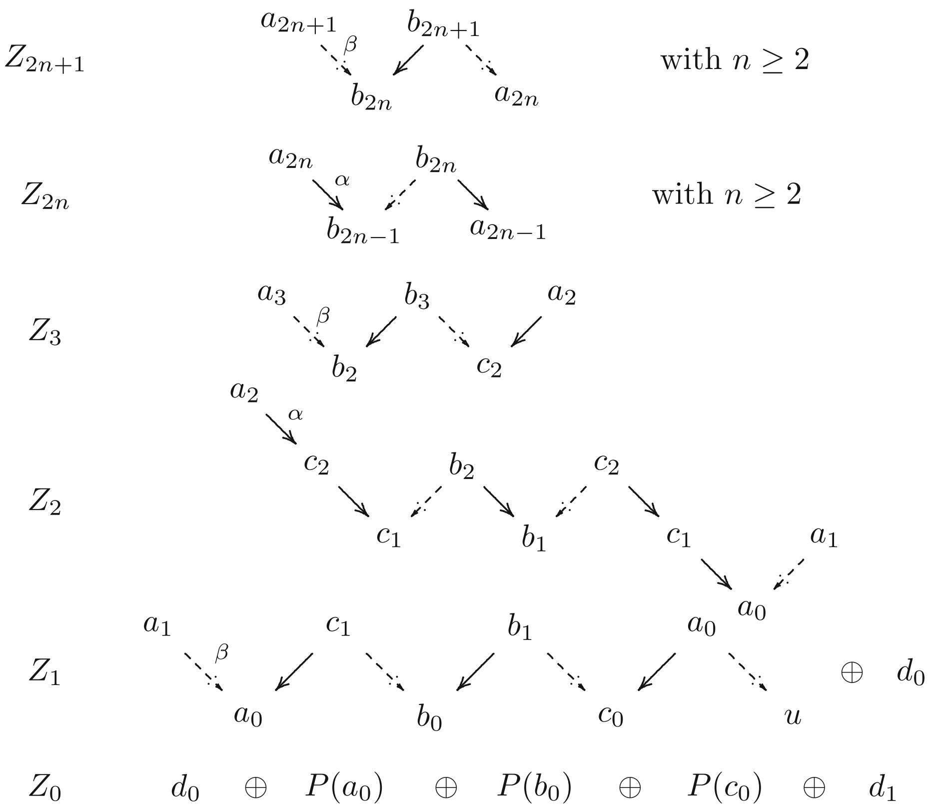
There are the following additional relations: the square of any loop is zero, and we have $\alpha^n = \beta^m$, whenever this makes sense.

It is easily seen that $\operatorname{pd} d_i = r - i$ (in particular $\operatorname{pd} d_0 = r$) and that $\operatorname{pd} u = \operatorname{pd} v = \operatorname{pd} w = \operatorname{pd} c_{-1} = \operatorname{pd} b_{-1} = \infty$.

The algebra Λ_m with $m \geq 0$ is given by the full subquiver with vertices a_i, b_i, c_i where $i \leq m$ and all the vertices $u, v, w, d_0, \ldots, d_r$.

2 A Finite-Dimensional Λ_m-Module Z_m of Projective Dimension $r + m$

We are going to exhibit a sequence of Λ_m-modules Z_m. All are direct sums of string modules.



For Z_m with m even, the southeast arrows are α-arrows; for m odd, the southeast arrows are β-arrows.

Proposition *For $m \geq 0$, we have $\Omega Z_{m+1} = Z_m$, and* $\operatorname{pd} Z_m = r + m$.

Proof The first assertion is easily verified. Since $\operatorname{pd} d_0 = r$ and $\operatorname{pd} d_1 = r - 1$, the second assertion is an immediate consequence, using induction. □

Remark The modules Z_m with $m \geq 1$ are finite dimensional Λ_m-modules, but not Λ_{m-1}-modules. Since the projective dimension of any Z_m is finite, the modules Z_m with $m \geq 2$ are counter-examples to Claim 2 of [1].

3 The Big Finitistic Dimension of Λ_m

Let Λ'_1 be obtained from Λ_2 by deleting the vertices a_2 and b_2. We note the following: *Let M be a Λ_m-module. If $m = 1$ or $m \geq 3$, then ΩM is a Λ_{m-1}-module. If $m = 2$, then ΩM is a Λ'_1-module.*

Let $\mathcal{X}$ be the set of the following 10 isomorphism classes of Λ'_1-modules; these are string modules X with $X_{c_2} \neq 0$.

$$\begin{tikzcd}[column sep=small, row sep=small]
& & & & c_2 \arrow[dl] \arrow[dr, dotted] \\
& a_1 \arrow[dl] \arrow[dr, dotted] & & c_1 \arrow[dl] & & b_1 \\
d_0 & & a_0
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & & c_2 \arrow[dl] \arrow[dr, dotted] \\
a_1 \arrow[dr, dotted] & & c_1 \arrow[dl] & & b_1 \\
& a_0
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & c_2 \arrow[dl] \arrow[dr, dotted] \\
& c_1 \arrow[dl] & & b_1 \\
a_0
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& c_2 \arrow[dl] \arrow[dr, dotted] \\
c_1 & & b_1
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
c_2 \arrow[dr, dotted] \\
& b_1
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
& & & & c_2 \arrow[dl] \\
& a_1 \arrow[dl] \arrow[dr, dotted] & & c_1 \arrow[dl] \\
d_0 & & a_0
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & & c_2 \arrow[dl] \\
a_1 \arrow[dr, dotted] & & c_1 \arrow[dl] \\
& a_0
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& & c_2 \arrow[dl] \\
& c_1 \arrow[dl] \\
a_0
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small, row sep=small]
& c_2 \arrow[dl] \\
c_1
\end{tikzcd}
\quad
c_2$$

Lemma 1 *The modules in $\mathcal{X}$ have infinite projective dimension.*

Proof For the modules X in the first row, c_{-1} is a direct summand of $\Omega^2 X$. For the modules X in the second row, v is a direct summand of $\Omega^3 X$. □

Lemma 2 *Any Λ'_1-module is the direct sum of a Λ_1-module, of copies of $P(c_2)$, and of copies of modules in $\mathcal{X}$.*

Proof Let M be a Λ'_1-module without a direct summand of the form $P(c_2)$. Let U be the subquiver of the quiver of Λ'_1 with vertices $d_0, a_1, a_0, c_1, c_2, b_1$.

Since U is a Dynkin quiver, any representation of U is a direct sum of finite-dimensional indecomposable representations. We decompose the restriction $M|U$ of M to U as follows: $M|U = X \oplus Y$, where X is a direct sum of copies of modules in $\mathcal{X}$ and $Y_{c_2} = 0$.

We claim that X is a submodule of M. For the proof, we use that the maps $\alpha : X_{c_2} \to X_{c_1}, \alpha : X_{c_1} \to X_{a_0}, \alpha : X_{a_1} \to X_{d_0}$, and $\beta : X_{c_2} \to X_{b_1}$ are surjective. Since M has no direct summand of the form $P(c_2)$, we have $\alpha^3 M_{c_2} = \beta^2 M_{c_2} = 0$, thus $\alpha X_{a_0} = 0$ and $\beta X_{b_1} = 0$. The relations $\alpha\beta = 0 = \beta\alpha$ show that also the subspaces $\beta X_{d_0},\ \beta X_{a_0},\ \beta X_{c_1},\ \alpha X_{b_1}$ all are zero.

Let M' be defined by $M'|U = Y$ and $M'_x = M_x$ for those vertices x in the quiver of Λ'_1 which do not belong to U. Clearly, M' is a submodule of M and we have $M = X \oplus M'$. By construction, $M'_{c_2} = 0$, thus M' is a Λ_1-module. □

Corollary *If M is a Λ_2-module of finite projective dimension, then ΩM is a Λ_1-module.*

Proof The syzygy-module ΩM is a Λ_1'-module of finite projective dimension, thus according to Lemma 1 and Lemma 2 a Λ_1-module. □

Proposition *Any Λ_m-module of finite projective dimension has projective dimension at most $r + m$.*

Proof Let M be a Λ_m-module of finite projective dimension.

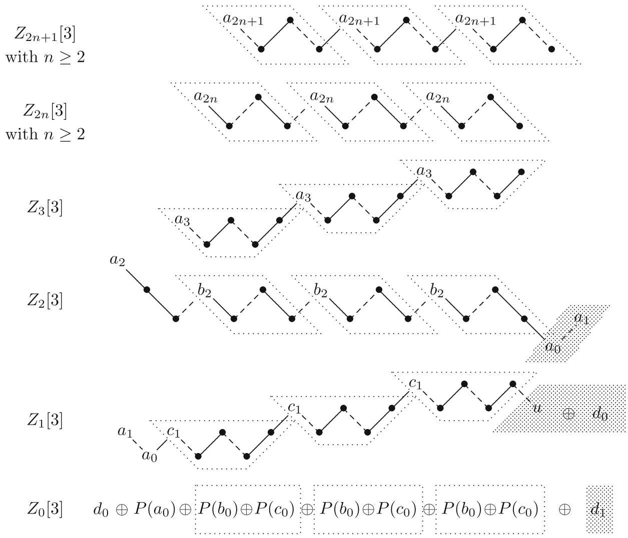
First, let $m = 0$. The algebra Λ_0 is the product of an algebra of global dimension r (with vertices $d_0, \dots, d_r$) and an algebra (with vertices $a_0, c_0, b_0, u, v, w, b_{-1}, c_{-1}$) whose non-projective modules have infinite projective dimension. Thus $\operatorname{pd} M \le r$.

Now, let $m \ge 1$. Then ΩM is a Λ_{m-1}-module of finite projective dimension. By induction $\operatorname{pd} \Omega M \le r + m - 1$, thus $\operatorname{pd} M \le r + m$. □

4 Direct Limits

The abstract of [1] claims that there exist infinite dimensional Λ-modules of finite projective dimension which are not direct limits of finitely generated representations of finite projective dimension. Apparently, the author refers to the Λ_m-modules labelled M_m which are presented in Claim 4 of [1] (these are the only infinite dimensional Λ-modules exhibited in the paper; they are used in order to show that $\operatorname{Fin.dim.} \Lambda_m \ge r + m$). Indeed, these modules M_m have finite projective dimension, namely $\operatorname{pd} M_m = r + m$. However, *the modules M_m* ***are*** *direct limits of finitely generated modules of finite projective dimension,* as we will show.

For $m \ge 0$ and $t \ge 1$, let us introduce a Λ_m-module $Z_m[t]$ such that $Z_m[1] = Z_m$, with a submodule $U_{mt} \subset Z_m[t]$, as well as a map $\phi_{mt} : Z_m[t] \to Z_m[t+1]$ with kernel U_{mt}. Below, we display the modules $Z_m[t]$ with $t = 3$. The submodule U_{mt} is the zero module in case $m \ge 3$, and is the shaded part in case $m \le 2$. The module $X = Z_m[t]/U_{mt}$ has a filtration $0 \subseteq X_0 \subset X_1 \subset \cdots \subset X_t \subseteq Z_m[t]/U_{mt}$ with isomorphic subfactors X_s/X_{s-1} for $1 \le s \le t$. In our display, we enclose the subfactors X_s/X_{s-1} with $1 \le s \le 3$ by dotted lines.

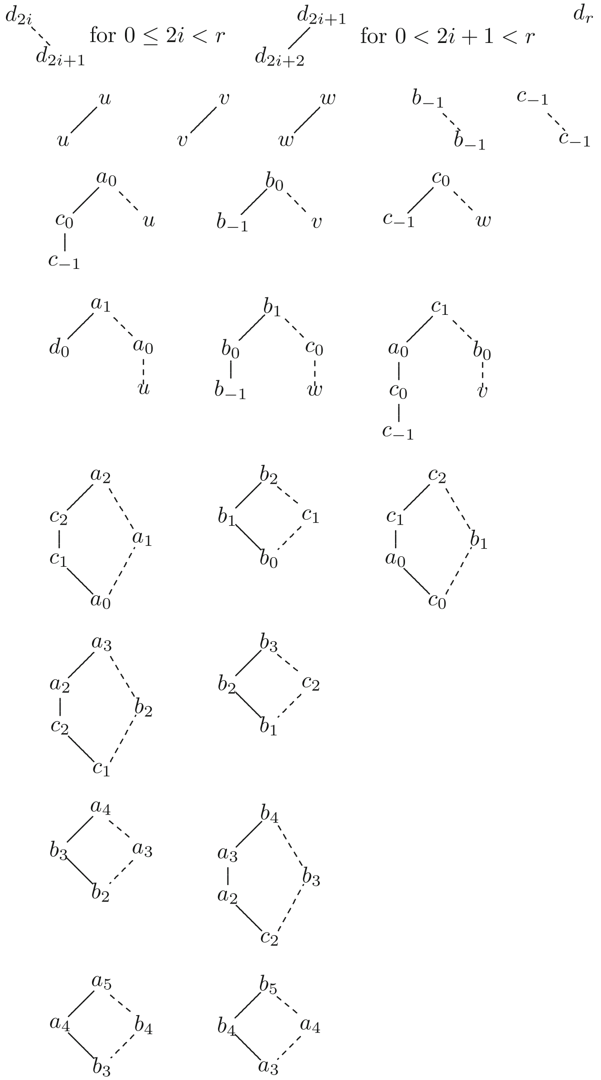


The map ϕ_{mt} is given by the obvious embedding of $Z_m[t]/U_{mt}$ into $Z_m[t{+}1]$ and we define $M_m = \lim_t(Z_m[t], \phi_{mt})$. For $m \geq 1$, the modules M_m are those presented in [1], Claim 4. As in Section 2, one easily checks that $\Omega(Z_{m+1}[t]) = Z_m[t]$ for any $m \geq 0$ and $t \geq 1$, so that pd $Z_m[t] = r + m$. Also, $\Omega M_{m+1} = M_m$, and therefore pd $M_m = r + m$.

Remark For $m \geq 3$, the module M_m is just a Prüfer module for its support algebra (which is hereditary).

Appendix: The shape of the Indecomposable Projective Λ_5-Modules

These graphical displays can be found in [1]. But the referee has suggested to provide the pictures also here.



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