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Ring Constructions and Generation of the Unbounded Derived Module Category

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Abstract
We consider the smallest triangulated subcategory of the unbounded derived module category of a ring that contains the injective modules and is closed under set indexed coproducts. If this subcategory is the entire derived category, then we say that injectives generate for the ring. In particular, we ask whether, if injectives generate for a collection of rings, do injectives generate for related ring constructions, and vice versa. We provide sufficient conditions for this statement to hold for various constructions including recollements, ring extensions and module category equivalences.

Keywords Derived categories · Homological conjectures · Recollements

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1 Introduction

The finitistic dimension conjecture is a longstanding open problem in the representation theory of finite dimensional algebras. Recently, Rickard proved that if the derived category of a finite dimensional algebra is 'generated' by its injective modules, then the algebra satisfies the finitistic dimension conjecture [32, Theorem 4.3]. In this paper we provide techniques to detect if this generation property holds for rings obtained from various ring constructions.

The finitistic dimension conjecture is prolific in the area of representation theory, not least because if it was true, then many other conjectures would follow, including the Nunke condition and the generalised Nakayama conjecture. Happel provides a summary of these conjectures and the relationship between them in [15]. In a talk in 2001, Keller [22] noted that a finite dimensional algebra satisfies both the Nunke condition and the generalised Nakayama conjecture if its derived module category is generated, as a triangulated category

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with coproducts, by the injective modules. If this is the case, then we say that *injectives generate* for the algebra. Keller asked whether, 'injectives generate' also implied the finitistic dimension conjecture, to which Rickard [32, Theorem 4.3] provided an affirmative answer.

Since a finite dimensional algebra satisfies the finitistic dimension conjecture if injectives generate for that algebra, it is natural to ask, for which classes of algebras do injectives generate? There are many classes of finite dimensional algebras for which injectives generate, including commutative algebras, Gorenstein algebras and monomial algebras [32, Theorem 8.1]. In fact, there is currently no known example of a finite dimensional algebra over a field for which injectives do not generate. However, it is not the case that injectives generate for all rings; one counterexample is the polynomial ring in infinitely many variables [32, Theorem 3.5].

There is extensive work into finding methods that can be used to identify rings that satisfy particular homological properties. One such approach is to consider a collection of related rings and ask whether, if some of the rings have a property, do the others as well. For example, this method has been applied to the finitistic dimension conjecture with recollements of derived categories [10, 16], ring homomorphisms [35, 37–39], and operations defined on the quiver of a quiver algebra [8, 13, 14]. We apply the same philosophy to the property 'injectives generate'. This leads us to employ reduction techniques, originally used in calculating the finitistic dimension of a ring, to check if injectives generate for a ring, including the arrow removal for quiver algebras defined by Green, Psaroudakis and Solberg [14, Section 4].

One of the most general ring constructions is given by two rings A and B and a ring homomorphism $f\colon B \to A$ between them. In Section 5, we provide sufficient conditions on the ring homomorphism f such that, if injectives generate for A, then injectives generate for B, and vice versa. The conditions we supply are satisfied by many familiar ring constructions, including those shown in Theorem 1.1.

Theorem 1.1 *Let A and B be rings. Suppose that injectives generate for B. If one of the following holds, then injectives generate for A.*

(i) *A is the trivial extension $B \ltimes M$ for a (B, B)-bimodule M such that the ideal $(0, M) \leq A$ has finite flat dimension as a left A-module. [Lemma 5.9, Example 5.10]*
(ii) *A is a Frobenius extension of B. [Lemma 5.14, Example 5.15]*
(iii) *A is an almost excellent extension of B. [Lemma 5.14, Example 5.15]*

In particular, consider the triangular matrix ring of two rings B and C and a (C, B)-bimodule M, denoted by

$$A := \begin{pmatrix} C & {}_CM_B \\ 0 & B \end{pmatrix}.$$

The ring A is isomorphic to the trivial extension $(C \times B) \ltimes M$. Hence, if the nilpotent ideal

$$\begin{pmatrix} 0 & {}_CM_B \\ 0 & 0 \end{pmatrix} \leq A,$$

has finite flat dimension as a left A-module, and injectives generate for $C \times B$, then Theorem 1.1 applies and injectives generate for A.

The triangular matrix ring A also induces a recollement of the derived module category of A with respect to the derived module categories of B and C. A *recollement* is a diagram of six functors between three derived module categories emulating a short exact sequence of rings introduced by Beĭlinson, Bernstein and Deligne [6]. In Section 6, we focus on the interaction between rings that appear in a recollement with respect to injective generation. In particular, we consider recollements of unbounded derived categories that restrict to recollements of bounded (above or below) derived categories. Angeleri Hügel, Koenig, Liu and Yang provide necessary and sufficient conditions for a recollement to restrict to a bounded (above) recollement [2]. In Proposition 6.14 we prove an analogous result to characterise when a recollement restricts to a bounded below recollement. These characterisations can be used to prove Theorem 1.2.

Theorem 1.2 *Let* $(R) = (\mathcal{D}(B), \mathcal{D}(A), \mathcal{D}(C))$ *be a recollement of unbounded derived module categories with* A *a finite dimensional algebra over a field. Suppose that injectives generate for both* B *and* C*. If one of the following conditions holds, then injectives generate for* A*.*

(i) *The recollement* (R) *restricts to a recollement of bounded above derived categories. [Proposition 6.13]*
(ii) *The recollement* (R) *restricts to a recollement of bounded below derived categories. [Proposition 6.15]*

The recollement induced by a triangular matrix algebra restricts to a recollement of bounded above derived categories, so Theorem 1.2 can be applied to triangular matrix algebras.

The techniques used in the proof of Theorem 1.2 can be used to prove similar results about recollements of more general triangulated categories. In Section 7, we focus on the recollement of triangulated categories induced from the recollement of module categories,

$$(\text{Mod-}(A/AeA), \text{Mod-}A, \text{Mod-}eAe),$$

where A is a ring and $e \in A$ is an idempotent. This recollement of module categories has been studied extensively by Green, Psaroudakis and Solberg [14, Section 3, Section 5] as a tool to compare the finitistic dimensions of A/AeA, A and eAe. Note that the little finitistic dimension of a finite dimensional algebra Λ is defined as

$$\text{findim}(\Lambda) = \sup\{\text{proj.dim}_\Lambda (M_\Lambda) : M_\Lambda \in \text{mod-}\Lambda, \text{proj.dim}_\Lambda (M_\Lambda) < \infty\},$$

and the big finitistic dimension is defined as

$$\text{FinDim}(\Lambda) = \sup\{\text{proj.dim}_\Lambda (M_\Lambda) : M_\Lambda \in \text{Mod-}\Lambda, \text{proj.dim}_\Lambda (M_\Lambda) < \infty\}.$$

In particular, in [14, Theorem B] it is shown that, when A is a finite dimensional algebra over an algebraically closed field and the semi-simple right A-module

$$S = (1-e)A/\text{rad}\,((1-e)A),$$

has finite injective dimension, if $\text{findim}(eAe) < \infty$, then $\text{findim}(A) < \infty$. This extends the work of Fuller and Saorín [13], where they prove that if $\text{findim}(eAe) < \infty$, then $\text{findim}(A) < \infty$ when S_A has projective dimension at most one. In Section 7, we provide similar results for injective generation. Moreover, we consider the big finitistic dimensions of A/AeA, A and eAe when A/AeA has finite flat dimension as a left A-module.

Theorem 1.3 (Theorem 7.7) *Let* A *be a finite dimensional algebra over a field and* $e \in A$ *be an idempotent. Suppose that* A/AeA *has finite flat dimension as a left* A*-module. If* *FinDim*$(A/AeA) < \infty$ *and* *FinDim*$(eAe) < \infty$*, then* *FinDim*$(A) < \infty$*.*

Note that there is a 'dual' property to 'injectives generate'. In particular, if the derived module category of a ring is generated, as a triangulated category with products, by the projective modules, then we say that *projectives cogenerate* for the ring. Rickard proves that, if projectives cogenerate for a finite dimensional algebra A, then the finitistic dimension conjecture holds for A^{op} [32, Proposition 5.2]. In this paper, alongside the results about injective generation, we provide analogous results for projective cogeneration.

1.1 Layout of the Paper

In Section 2, we recall the definitions and some useful properties of localising and colocalising subcategories. Section 3 showcases the techniques used in this paper to prove injective generation statements through a straightforward example, namely, for a tensor product algebra over a field. In Section 4, we show that separable equivalence preserves the property 'injectives generate'. Section 5 considers general ring homomorphisms and includes the proof of Theorem 1.1. The final two sections of the paper concern the interaction between injective generation statements and rings that appear in recollements. In particular, Section 6 focuses on recollements of derived categories of rings and includes Theorem 1.2. Section 7 concerns recollements of module categories and their induced recollements of triangulated categories, including the proof of Theorem 1.3.

2 Preliminaries

In this section, we provide the required definitions and preliminary results that will be used throughout the paper. Section 2.2 recalls the definition of localising and colocalising subcategories of the derived module category which we require for the definition of injective generation. In Section 2.4 we focus on showing when triangle functors of the derived category preserve specific properties of complexes.

2.1 Notation

Firstly, we fix some notation. All rings and ring homomorphisms are unital, and modules are right modules unless otherwise stated. For a ring A and a left A-module M, we denote M^* to be the right A-module $\mathrm{Hom}_{\mathbb{Z}}({}_AM, \mathbb{Q}/\mathbb{Z})$.

We use the following notation for various categories.

- Mod-A is the category of all A-modules, and mod-A is the category of all finitely generated A-modules.
- Inj-A is the category of all injective A-modules.
- Proj-A is the category of all projective A-modules.
- $\mathcal{K}(A)$ is the unbounded homotopy category of cochain complexes of A-modules.
- $\mathcal{D}(A)$ is the unbounded derived category of cochain complexes of A-modules. $\mathcal{D}^*(A)$ with $* \in \{-, +, b\}$ is the bounded above, bounded below and bounded derived category respectively.

We shall be considering functors between derived categories and properties that they preserve. For brevity, we shall often abuse terminology by writing, for example, that a functor preserves bounded complexes of projectives when we mean that it preserves the property of being quasi-isomorphic to a bounded complex of projectives.

Let A and B be rings and $f: B \to A$ be a ring homomorphism. The functors induced by f will be denoted as follows.

- Induction, $\mathrm{Ind}_B^A := -\otimes_B A_A : \text{Mod-}B \to \text{Mod-}A$,
- Restriction, $\mathrm{Res}_B^A := \mathrm{Hom}_A\left({}_BA, -\right) : \text{Mod-}A \to \text{Mod-}B$,
- Coinduction, $\mathrm{Coind}_B^A := \mathrm{Hom}_B\left({}_AA, -\right) : \text{Mod-}B \to \text{Mod-}A$.

2.2 (Co)Localising Subcategories

There are many ways to generate the unbounded derived category of a ring, here we focus on generation via localising and colocalising subcategories. First we recall their definitions.

Definition 2.1 ((Co)Localising Subcategory) Let A be a ring and $\mathcal{S}$ be a class of complexes in $\mathcal{D}(A)$.

- A localising subcategory of $\mathcal{D}(A)$ is a triangulated subcategory of $\mathcal{D}(A)$ closed under set indexed coproducts. The smallest localising subcategory of $\mathcal{D}(A)$ containing $\mathcal{S}$ is denoted by $\mathrm{Loc}_A(\mathcal{S})$.
- A colocalising subcategory of $\mathcal{D}(A)$ is a triangulated subcategory of $\mathcal{D}(A)$ closed under set indexed products. The smallest colocalising subcategory of $\mathcal{D}(A)$ containing $\mathcal{S}$ is denoted by $\mathrm{Coloc}_A(\mathcal{S})$.

There are many well-known properties of localising and colocalising subcategories, some of which we recall now.

Lemma 2.2 [32, Proposition 2.1] *Let A be a ring and $\mathcal{C}$ be a triangulated subcategory of $\mathcal{D}(A)$.*

(i) *If $\mathcal{C}$ is either a localising subcategory or a colocalising subcategory of $\mathcal{D}(A)$, then $\mathcal{C}$ is closed under direct summands.*
(ii) *Let $X = \left(X^i, d^i\right)_{i\in\mathbb{Z}} \in \mathcal{D}(A)$ be bounded. If the module X^i is in $\mathcal{C}$ for all $i \in \mathbb{Z}$, then X is in $\mathcal{C}$.*

Remark 2.3 If A is a ring, then every bounded complex of injective A-modules is in $\mathrm{Loc}_A(\text{Inj-}A)$ by Lemma 2.2. Similarly, every bounded complex of projective A-modules is in $\mathrm{Coloc}_A(\text{Proj-}A)$.

Throughout this paper we investigate when a localising subcategory or colocalising subcategory of $\mathcal{D}(A)$ generated by some class of complexes $\mathcal{S}$ is the entire unbounded derived module category.

Definition 2.4 Let A be a ring and $\mathcal{S}$ be a class of complexes in $\mathcal{D}(A)$.

- If $\mathrm{Loc}_A(\mathcal{S}) = \mathcal{D}(A)$, then we say that $\mathcal{S}$ generates $\mathcal{D}(A)$.
- If $\mathrm{Coloc}_A(\mathcal{S}) = \mathcal{D}(A)$, then we say that $\mathcal{S}$ cogenerates $\mathcal{D}(A)$.

It is well-known that for a ring A, its unbounded derived category $\mathcal{D}(A)$ is generated by the projective A-modules and cogenerated by the injective A-modules, see, for example, [32, Proposition 2.2]. Since a localising subcategory is closed under set indexed coproducts and direct summands, it follows immediately that the regular module A_A also generates $\mathcal{D}(A)$.

Remark 2.5 Let A be a ring and M_A be a generator of Mod-A (in the sense that every A-module is a quotient of a set indexed coproduct of copies of M). Recall that, since M_A is a generator, every projective A-module is a direct summand of a set indexed coproduct of copies of M_A. Thus M_A generates $\mathcal{D}(A)$.

Similarly, suppose that M_A is a cogenerator of Mod-A (in the sense that every A-module is a submodule of a set indexed product of copies of M_A). Then every injective A-module is a direct summand of a set indexed product of copies of M_A, and M_A cogenerates $\mathcal{D}(A)$. In particular, the injective cogenerator $A_A^* = \mathrm{Hom}_{\mathbb{Z}}({}_A A, \mathbb{Q}/\mathbb{Z})$ cogenerates $\mathcal{D}(A)$.

In this paper we investigate when the derived category is generated as a localising subcategory by the injective modules and as a colocalising subcategory by the projective modules.

Definition 2.6 Let A be a ring.

- If $\mathrm{Loc}_A(\text{Inj-}A) = \mathcal{D}(A)$, then we say that injectives generate for A.
- If $\mathrm{Coloc}_A(\text{Proj-}A) = \mathcal{D}(A)$, then we say that projectives cogenerate for A.

2.3 Functors

Many of the results in this paper rely on using functors that preserve the properties that define localising and colocalising subcategories. Since these ideas are mentioned often, we collate them here.

Definition 2.7 ((Pre)image) Let A and B be rings and $F\colon \mathcal{D}(A) \to \mathcal{D}(B)$ be a triangle functor.

- Let $\mathcal{C}_B$ be a full triangulated subcategory of $\mathcal{D}(B)$. The preimage of $\mathcal{C}_B$ under F is the smallest full subcategory of $\mathcal{D}(A)$ consisting of the complexes X such that $F(X)$ is in $\mathcal{C}_B$. (Note that this subcategory is a triangulated subcategory of $\mathcal{D}(A)$.)
- Let $\mathcal{C}_A$ be a full triangulated subcategory of $\mathcal{D}(A)$. The image of F applied to $\mathcal{C}_A$ is the smallest full triangulated subcategory of $\mathcal{D}(B)$ that contains $F(X)$ for all X in $\mathcal{C}_A$. Denote the image of F by $\mathrm{im}(F)$.

Lemma 2.8 *Let A and B be rings and $F\colon \mathcal{D}(A) \to \mathcal{D}(B)$ be a triangle functor.*

(i) *If F preserves set indexed coproducts, then the preimage of a localising subcategory of $\mathcal{D}(B)$ is a localising subcategory of $\mathcal{D}(A)$.*
(ii) *If F preserves set indexed products, then the preimage of a colocalising subcategory of $\mathcal{D}(B)$ is a colocalising subcategory of $\mathcal{D}(A)$.*

Proof The result follows immediately from the definition of localising and colocalising subcategories. □

Proposition 2.9 *Let A and B be rings and $F\colon \mathcal{D}(A) \to \mathcal{D}(B)$ be a triangle functor. Let $\mathcal{S}$ and $\mathcal{T}$ be classes of complexes in $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively.*

(i) *Suppose that $\mathcal{S}$ generates $\mathcal{D}(A)$. If F preserves set indexed coproducts, and $F(S)$ is in $\mathrm{Loc}_B(\mathcal{T})$ for all S in $\mathcal{S}$, then $\mathrm{im}(F)$ is a subcategory of $\mathrm{Loc}_B(\mathcal{T})$.*

(ii) *Suppose that $\mathcal{S}$ cogenerates $\mathcal{D}(A)$. If F preserves set indexed products, and $F(S)$ is in $Coloc_B(\mathcal{T})$ for all S in $\mathcal{S}$, then $im(F)$ is a subcategory of $Coloc_B(\mathcal{T})$.*

Proof (i) Suppose that F preserves set indexed coproducts and that $F(S)$ is in $\mathrm{Loc}_B(\mathcal{T})$ for all S in $\mathcal{S}$. By Lemma 2.8, the preimage of $\mathrm{Loc}_B(\mathcal{T})$ under F is a localising subcategory. Furthermore, the preimage contains $\mathcal{S}$, so it also contains $\mathrm{Loc}_A(\mathcal{S}) = \mathcal{D}(A)$. Thus $F(X)$ is in $\mathrm{Loc}_B(\mathcal{T})$ for all $X \in \mathcal{D}(A)$.

(ii) This follows similarly to (i).

□

2.4 Adjoint Functors

Homomorphism groups in the derived category can be used to characterise certain properties of complexes. In this subsection we consider some of these properties and their interaction with adjoint pairs of functors. Some of these well-known results can be found in [31, Proof of Proposition 8.1] and [23, Proof of Theorem 1].

Remark 2.10 Let A be a ring. Let $X \in \mathcal{D}(A)$ and $n \in \mathbb{Z}$. Recall that, since A is a projective generator of Mod-A, $\mathrm{Hom}_{\mathcal{D}(A)}(A, X[n]) = 0$ if and only if $H^n(X) = 0$. Similarly, since A^* is an injective cogenerator of Mod-A, $\mathrm{Hom}_{\mathcal{D}(A)}(X, A^*[n]) = 0$ if and only if $H^{-n}(X) = 0$.

Definition 2.11 (Compact objects) Let $\mathcal{T}$ be a triangulated category and $C \in \mathcal{T}$. Then C is compact if for all sets I and objects $\{X_i\}_{i \in I} \subset \mathcal{T}$ the canonical morphism

$$\bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{T}}(C, X_i) \to \mathrm{Hom}_{\mathcal{T}}\left(C, \bigoplus_{i \in I} X_i\right),$$

is an isomorphism.

Recall that the compact objects of $\mathcal{D}(A)$ are the perfect complexes of $\mathcal{D}(A)$, i.e. the complexes that are quasi-isomorphic to bounded complexes of finitely generated projectives. See for example [7, Proposition 6.4].

Lemma 2.12 *Let A be a ring.*

(i) *A complex $X \in \mathcal{D}(A)$ is bounded in cohomology if and only if, for each compact object $C \in \mathcal{D}(A)$, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(C, X[n]) = 0$ for all $|n| > N$.*

(ii) *A complex $I \in \mathcal{D}(A)$ is quasi-isomorphic to a bounded complex of injectives if and only if, for each complex $X \in \mathcal{D}(A)$ that is bounded in cohomology, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(X, I[n]) = 0$ for all $|n| > N$.*

(iii) *A complex $P \in \mathcal{D}(A)$ is quasi-isomorphic to a bounded complex of projectives if and only if, for each complex $X \in \mathcal{D}(A)$ that is bounded in cohomology, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(P, X[n]) = 0$ for all $|n| > N$.*

Proof (i) Let $X \in \mathcal{D}(A)$. Suppose that, for each compact object $C \in \mathcal{D}(A)$, there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}(A)}(C, X[n]) = 0$ for all $|n| > N$. Then, as A is compact, $H^n(X) \cong \mathrm{Hom}_{\mathcal{D}(A)}(A, X[n]) = 0$ for all but finitely many $n \in \mathbb{Z}$.

Now suppose that $X \in \mathcal{D}(A)$ is bounded in cohomology. Then there exists $Y \in \mathcal{D}(A)$ such that Y is bounded, and X is quasi-isomorphic to Y. Let $C \in \mathcal{D}(A)$

be a compact object. Then C is quasi-isomorphic to a bounded complex of finitely generated projectives $P \in \mathcal{D}(A)$. Hence, for all $n \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{D}(A)}(C, X[n]) \cong \mathrm{Hom}_{\mathcal{K}(A)}(P, Y[n]).$$

Since both P and Y are bounded, $\mathrm{Hom}_{\mathcal{K}(A)}(P, Y[n]) = 0$ for all but finitely many $n \in \mathbb{Z}$. Hence, there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}(A)}(C, X[n]) = 0$ for all $|n| > N$.

(ii) Suppose that $I \in \mathcal{D}(A)$ is quasi-isomorphic to a bounded complex of injectives $J \in \mathcal{D}(A)$. Let $X \in \mathcal{D}(A)$ be bounded in cohomology. Then there exists $Y \in \mathcal{D}(A)$ such that Y is bounded, and X is quasi-isomorphic to Y. Thus

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, I[n]) \cong \mathrm{Hom}_{\mathcal{K}(A)}(Y, J[n]),$$

for all $n \in \mathbb{Z}$. Moreover, both Y and J are bounded, so $\mathrm{Hom}_{\mathcal{K}(A)}(Y, J[n]) = 0$ for all but finitely many $n \in \mathbb{Z}$. Hence, there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}(A)}(X, I[n]) = 0$ for all $|n| > N$.

Now let us consider the converse. Let $Z \in \mathcal{D}(A)$. Suppose that for each $X \in \mathcal{D}(A)$, such that X is bounded in cohomology, there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}(A)}(X, Z[n]) = 0$ for all $|n| > N$. Then, taking $X = A$, we have that $H^n(Z) \cong \mathrm{Hom}_{\mathcal{D}(A)}(A, Z[n]) = 0$ for all but finitely many $n \in \mathbb{Z}$. Consequently, Z has a bounded below $\mathcal{K}$-injective resolution $I = \left(I^i, d^i\right)_{i\in\mathbb{Z}} \in \mathcal{D}(A)$ that has nonzero cohomology in only finitely many degrees. Hence, there exists $N' \in \mathbb{Z}$ such that, for all $n > N'$, $H^n(I) = 0$, and,

$$0 = \mathrm{Hom}_{\mathcal{D}(A)}\left(\bigoplus_{i\in\mathbb{Z}} \ker(d^i), Z[n]\right) \cong \mathrm{Hom}_{\mathcal{K}(A)}\left(\bigoplus_{i\in\mathbb{Z}} \ker(d^i), I[n]\right).$$

In particular,

$$\mathrm{Hom}_{\mathcal{K}(A)}\left(\ker(d^n), I[n]\right) = 0,$$

for all $n > N'$. Hence, the epimorphism $d^{n-1}\colon I^{n-1} \to \mathrm{im}(d^{n-1}) \cong \ker(d^n)$ splits. Consequently, $\ker(d^n)$ is an injective A-module, and the good truncation

$$\tau_{\le n}(I) = \cdots \to 0 \to I^0 \to I^1 \to \cdots \to I^{n-1} \to \ker(d^n) \to 0 \to \dots,$$

is a bounded complex of injectives that is quasi-isomorphic to I.

(iii) This follows similarly to (ii).

□

Lemma 2.13 *Let A be a ring and $X \in \mathcal{D}(A)$.*

(i) *The following are equivalent:*
   (a) *X is bounded above in cohomology,*
   (b) *For each compact object $C \in \mathcal{D}(A)$, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(C, X[n]) = 0$ for all $n > N$,*
   (c) *For each $Y \in \mathcal{D}(A)$ that is bounded in cohomology, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(X, Y[n]) = 0$ for all $n < N$.*

(ii) *The following are equivalent:*
   (a) *X is bounded below in cohomology,*
   (b) *For each compact object $C \in \mathcal{D}(A)$, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(C, X[n]) = 0$ for all $n < N$,*

(c) *For each $Y \in \mathcal{D}(A)$ that is bounded in cohomology, there exists $N \in \mathbb{Z}$ such that $Hom_{\mathcal{D}(A)}(Y, X[n]) = 0$ for all $n < N$.*

Proof (i) The equivalence of (a) and (b) is similar to the proof of Lemma 2.12 (i).

Now we show that (c) implies (a). Suppose for each $Y \in \mathcal{D}(A)$ that is bounded in cohomology, there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{D}(A)}(X, Y[n]) = 0$ for all $n < N$. Then, taking $Y = A^*$, we have that $\mathrm{Hom}_{\mathcal{D}(A)}(X, A^*[n]) = 0$ for all $n < N$. Hence, $H^{-n}(X) = 0$ for all $n < N$.

Finally, we show that (a) implies (c). Suppose that $X \in \mathcal{D}(A)$ is bounded above in cohomology. Then X has a bounded above $\mathcal{K}$-projective resolution $P \in \mathcal{D}(A)$. Let $Y \in \mathcal{D}(A)$ be bounded in cohomology. Then there exists $Z \in \mathcal{D}(A)$ such that Z is bounded, and Y is quasi-isomorphic to Z. Thus,

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, Y[n]) \cong \mathrm{Hom}_{\mathcal{K}(A)}(P, Z[n]),$$

for all $n \in \mathbb{Z}$. Since P is bounded above, and Z is bounded, there exists $N \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{K}(A)}(P, Z[n]) = 0$ for all $n < N$.

(iii) This follows similarly to (i).

□

The properties that are considered in Remark 2.10, and Lemmas 2.12 and 2.13 are defined using homomorphism groups, so they interact well with adjoint functors. Recall that, for brevity, we shall often abuse terminology by writing, for example, that a functor preserves bounded complexes of projectives when we mean that it preserves the property of being quasi-isomorphic to a bounded complex of projectives.

Lemma 2.14 *Let A and B be rings. Let $F: \mathcal{D}(A) \to \mathcal{D}(B)$ and $G: \mathcal{D}(B) \to \mathcal{D}(A)$ be triangle functors such that (F, G) is an adjoint pair.*

(i) *If G preserves set indexed coproducts, then F preserves compact objects.*
(ii) *If F preserves compact objects, then G preserves complexes bounded (above or below) in cohomology.*
(iii) *If F preserves complexes bounded in cohomology, then G preserves bounded complexes of injectives and complexes bounded below in cohomology.*
(iv) *If G preserves bounded complexes of injectives, then F preserves complexes bounded (above or below) in cohomology.*
(v) *If G preserves complexes bounded in cohomology, then F preserves bounded complexes of projectives and complexes bounded above in cohomology.*
(vi) *If F preserves bounded complexes of projectives, then G preserves complexes bounded (above or below) in cohomology.*

Proof (i) Let $C \in \mathcal{D}(A)$ be a compact object, and let $\{X_i\}_{i\in I} \subset \mathcal{D}(B)$ for an index set I. Since C is compact and G preserves set indexed coproducts, there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(B)}\left(F(C), \bigoplus_{i\in I} X_i\right) &\cong \mathrm{Hom}_{\mathcal{D}(A)}\left(C, \bigoplus_{i\in I} G(X_i)\right), \\ &\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{D}(A)}(C, G(X_i)), \\ &\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{D}(B)}(F(C), X_i).\end{aligned}$$

Thus $F(C)$ is compact.

The remaining claims follow from Remark 2.10, Lemmas 2.12 and 2.13 and the adjunction

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, G(Y)[n]) \cong \mathrm{Hom}_{\mathcal{D}(B)}(F(X), Y[n]),$$

for appropriate choices of X or Y.

(ii) Let X run over all compact objects, and apply Lemmas 2.12 (i), 2.13 (i) and (ii).
(iii) Let X run over all complexes bounded in cohomology, and apply Lemmas 2.12 (ii) and 2.13 (ii).
(iv) Let $Y = B^*$, and apply Remark 2.10.
(v) Let Y run over all complexes bounded in cohomology, and apply Lemmas 2.12 (iii) and 2.13 (i).
(vi) Let $X = A$, and apply Remark 2.10.

□

3 Tensor Product Algebra

The first ring construction we consider is the tensor product of two finite dimensional algebras A and B over a field k. In particular, we prove that if injectives generate for the two algebras, then injectives generate for their tensor product. Firstly, we recall a description of the injective and projective modules of a tensor product algebra.

Lemma 3.1 [9, Chapter IX, Proposition 2.3], [36, Lemma 3.1] *Let A and B be finite dimensional algebras over a field k. Let M be an A-module and N be a B-module.*

(i) *If M is a projective A-module and N is a projective B-module, then $M \otimes_k N$ is a projective $(A \otimes_k B)$-module.*
(ii) *If M is an injective A-module and N is an injective B-module, then $M \otimes_k N$ is an injective $(A \otimes_k B)$-module.*

Notice that the structure of these modules is functorial in either argument. Let M be an A-module and N be a B-module. Define

$$F_N := - \otimes_k N : \text{Mod-}A \to \text{Mod-}(A \otimes_k B),$$
$$G_M := M \otimes_k - : \text{Mod-}B \to \text{Mod-}(A \otimes_k B).$$

Since k is a field, the functors F_N and G_M are exact. Hence, these functors extend to triangle functors $F_N : \mathcal{D}(A) \to \mathcal{D}(A \otimes_k B)$ and $G_M : \mathcal{D}(B) \to \mathcal{D}(A \otimes_k B)$.

Proposition 3.2 *Let A and B be finite dimensional algebras over a field k.*

(i) *If injectives generate for both A and B, then injectives generate for $A \otimes_k B$.*
(ii) *If projectives cogenerate for both A and B, then projectives cogenerate for $A \otimes_k B$.*

Proof (i) Since A is a finite dimensional algebra over the field k, every injective A-module is a direct summand of a set indexed coproduct of copies of $DA_A = \mathrm{Hom}_k({}_A A, k)$. Thus $\mathrm{Loc}_A(\text{Inj-}A) = \mathrm{Loc}_A(DA)$. Similarly, $\mathrm{Loc}_B(\text{Inj-}B) = \mathrm{Loc}_B(DB)$.

Define $C := A \otimes_k B$. By Lemma 3.1, $F_{DB}(DA) = DA \otimes_k DB$ is an injective C-module. Moreover, F_{DB} preserves set indexed coproducts. Suppose that injectives generate for A. Then, by Proposition 2.9, $\mathrm{im}(F_{DB})$ is a subcategory of Loc_C (Inj-C). In particular, $F_{DB}(A) = A \otimes_k DB$ is in Loc_C (Inj-C).

Now consider the functor $G_A := A \otimes_k -$. By the previous argument

$$G_A(DB) = A \otimes_k DB = F_{DB}(A) \in \mathrm{Loc}_C \text{ (Inj-}C\text{)} .$$

Moreover, G_A preserves set indexed coproducts. Suppose that injectives generate for B. Then, by Proposition 2.9, $\mathrm{im}(G_A)$ is a subcategory of Loc_C (Inj-C). In particular,

$$G_A(B) = A \otimes_k B = C \in \mathrm{Loc}_C \text{ (Inj-}C\text{)} .$$

Consequently, $\mathrm{Loc}_C(C) = \mathcal{D}(C)$ is a subcategory of Loc_C (Inj-C), and injectives generate for $C = A \otimes_k B$.

(ii) Both ${}_kB$ and DA_k are finite dimensional k-modules, so $F_B := - \otimes_k B$ and $G_{DA} := DA \otimes_k -$ preserve set indexed products. Hence the projectives cogenerate statement follows similarly to (i) by considering the functors F_B and G_{DA}.

□

The converse to Proposition 3.2 is shown as an application of the results about ring homomorphisms in Section 5. In particular, the converse statement follows immediately from Lemma 5.2.

4 Separable Equivalence

Rickard proved that if two algebras are derived equivalent, then injectives generate for one if and only if injectives generate for the other [32, Theorem 3.4]. Here we show the statement also holds for separable equivalence of rings, defined by Linckelmann [24, Section 3]. In particular, we prove the result using separably dividing rings [5, Section 2].

Definition 4.1 (Separably dividing rings.) Let A and B be rings. Then B separably divides A if there exist bimodules ${}_AM_B$ and ${}_BN_A$ such that

(i) The modules ${}_AM$, M_B, ${}_BN$ and N_A are all finitely generated projectives,
(ii) There exists a (B, B)-bimodule Y such that ${}_BN \otimes_A M_B$ and $B \oplus Y$ are isomorphic as (B, B)-bimodules.

Proposition 4.2 *Let A and B be rings such that B separably divides A.*

(i) *If injectives generate for A, then injectives generate for B.*
(ii) *If projectives cogenerate for A, then projectives cogenerate for B.*

Proof (i) Let ${}_AM_B$, ${}_BN_A$ and ${}_BY_B$ be as in Definition 4.1. Consider the adjoint functors

$$- \otimes_B N \colon \text{Mod-}B \to \text{Mod-}A,$$
$$\mathrm{Hom}_A(N, -) : \text{Mod-}A \to \text{Mod-}B.$$

Since both ${}_BN$ and N_A are projective, $- \otimes_B N$ and $\mathrm{Hom}_A(N, -)$ are exact and extend to triangle functors. As $\mathrm{Hom}_A(N, -)$ has an exact left adjoint it preserves injective modules. Furthermore, the module N_A is a finitely generated projective, so $\mathrm{Hom}_A(N, -)$ also preserves set indexed coproducts.

Suppose that injectives generate for A. Since $\mathrm{Hom}_A(N, -)$ preserves injective modules and set indexed coproducts, its image is a subcategory of $\mathrm{Loc}_B(\text{Inj-}B)$ by Proposition 2.9. By adjunction

$$\mathrm{Hom}_A({}_BN, \mathrm{Hom}_B(M_B, B_B)) \cong \mathrm{Hom}_B({}_BN \otimes_A M_B, B_B),$$

as right B-modules, so $\mathrm{Hom}_B({}_BN \otimes_A M_B, B_B)$ is in $\mathrm{Loc}_B(\text{Inj-}B)$. As (B, B)-bimodules $N \otimes_A M$ is isomorphic to $B \oplus Y$, so

$$\mathrm{Hom}_B({}_BN \otimes_A M_B, B_B) \cong B_B \oplus \mathrm{Hom}_B({}_BY_B, B_B),$$

as right B-modules. Moreover, localising subcategories are closed under direct summands, so B is in $\mathrm{Loc}_B(\text{Inj-}B)$, and injectives generate for B.

(ii) Both ${}_AM$ and M_B are finitely generated projective modules, so $-\otimes_A M_B$ preserves set indexed products and projective modules. Suppose that projectives cogenerate for A. Then $\mathrm{im}(-\otimes_A M_B)$ is a subcategory of $\mathrm{Coloc}_B(\text{Proj-}B)$ by Proposition 2.9. Moreover, B^* is a direct summand of $(B^* \otimes_B N) \otimes_A M$. Hence, B^* is in $\mathrm{Coloc}_B(\text{Proj-}B)$, and projectives cogenerate for B.

□

Definition 4.3 (Separable Equivalence) [24, Definition 3.1] Let A and B be rings. Then A and B are separably equivalent if A separably divides B via bimodules ${}_AM_B$ and ${}_BN_A$, and B separably divides A via the same bimodules ${}_AM_B$ and ${}_BN_A$.

Example 4.4 Let G be a finite group. Let k be a field of positive characteristic p, and H be a Sylow p-subgroup of G. Then the group algebras kG and kH are separably equivalent [24, Section 3].

Corollary 4.4.1 *Let A and B be separably equivalent rings.*

(i) *Injectives generate for A if and only if injectives generate for B.*
(ii) *Projectives cogenerate for A if and only if projectives cogenerate for B.*

Proof Since A and B are separably equivalent, A separably divides B and B separably divides A. □

5 Ring Homomorphisms

In this section we provide sufficient conditions on a ring extension $f: B \to A$ such that if injectives generate for A, then injectives generate for B, and vice versa. In particular, we focus on Frobenius extensions [21, 25], almost excellent extensions [40] and trivial extensions. In Section 5.1 we apply the results in this section to the arrow removal operation defined in [14, Section 4].

Recall that for a ring homomorphism $f: B \to A$ there exist three functors between the module categories of A and B, denoted as follows,

- Induction, $\mathrm{Ind}_B^A := -\otimes_B A_A : \text{Mod-}B \to \text{Mod-}A$,
- Restriction, $\mathrm{Res}_B^A := \mathrm{Hom}_A({}_BA, -) : \text{Mod-}A \to \text{Mod-}B$,
- Coinduction, $\mathrm{Coind}_B^A := \mathrm{Hom}_B({}_AA, -) : \text{Mod-}B \to \text{Mod-}A$.

Note that both $(\mathrm{Ind}_B^A, \mathrm{Res}_B^A)$ and $(\mathrm{Res}_B^A, \mathrm{Coind}_B^A)$ are adjoint pairs of functors.

Remark 5.1 If ${}_BA$ has finite flat dimension and $M \in \text{Mod-}B$, then $\text{Tor}_i^B(M, A) = 0$ for all but finitely many $i \in \mathbb{Z}$. Hence, $\mathbf{L}\text{Ind}_B^A(M)$ is bounded in cohomology. Moreover, any complex $X \in \mathcal{D}(B)$ that is bounded in cohomology is in the smallest triangulated subcategory generated by Mod-B. Consequently, $\mathbf{L}\text{Ind}_B^A(X)$ is bounded in cohomology. Thus, $\mathbf{L}\text{Ind}_B^A$ preserves complexes bounded in cohomology, and, by Lemma 2.14, Res_B^A preserves bounded complexes of injectives.

Similarly, if A_B has finite projective dimension, then $\text{Ext}_B^i(A, M) = 0$ for all but finitely many $i \in \mathbb{Z}$. Thus, $\mathbf{R}\text{Coind}_B^A$ preserves complexes bounded in cohomology, and, by Lemma 2.14, Res_B^A preserves bounded complexes of projectives.

Lemma 5.2 *Let A and B be rings with a ring homomorphism $f: B \to A$.*

(i) *Suppose that A has finite flat dimension as a left B-module and that $\text{Res}_B^A(\text{Mod-}A)$ generates $\mathcal{D}(B)$. If injectives generate for A, then injectives generate for B.*
(ii) *Suppose that A has finite projective dimension as a right B-module and that $\text{Res}_B^A(\text{Mod-}A)$ cogenerates $\mathcal{D}(B)$. If projectives cogenerate for A, then projectives cogenerate for B.*

Proof (i) If ${}_BA$ has finite flat dimension as a left B-module, then Res_B^A preserves bounded complexes of injectives by Remark 5.1. Furthermore, Res_B^A preserves set indexed coproducts. Hence, if injectives generate for A, then $\text{im}(\text{Res}_B^A)$ is a subcategory of $\text{Loc}_B(\text{Inj-}B)$ by Proposition 2.9. Thus, as $\text{Res}_B^A(\text{Mod-}A)$ generates $\mathcal{D}(B)$, injectives generate for B.

(ii) This statement follows similarly to (i). In particular, if A_B has finite projective dimension as right B-module, then Res_B^A preserves bounded complexes of projectives by Remark 5.1. Then we apply Proposition 2.9 to Res_B^A.

□

There are many ways that $\text{Res}_B^A(\text{Mod-}A)$ could generate $\mathcal{D}(B)$. In particular, if A_B is a generator of Mod-B (in the sense that every B-module is a quotient of a set indexed coproduct of copies of A_B), then, by Remark 2.5, A_B generates $\mathcal{D}(B)$. Note that, if B_B is a direct summand of A_B, then A_B is a generator of Mod-B. Similarly, if ${}_BB$ is a direct summand of ${}_BA$, then B_B^* is a direct summand of $(A^*)_B$, and $(A^*)_B$ is a cogenerator of Mod-B. There are lots of examples of familiar ring extensions which satisfy both this property and the conditions of Lemma 5.2, some of which we list here.

- Tensor product algebra.

  Let A and B be finite dimensional algebras over a field k. Then the tensor product algebra $A \otimes_k B$ is an extension of both A and B. In particular, consider the ring homomorphism $f: A \to A \otimes_k B$ defined by $f(a) := a \otimes_k 1_B$ for $a \in A$. Note that $A \otimes_k B$ is free as an (A, A)-bimodule. Thus, $A \otimes_k B$ is projective as both a left and right A-module. Moreover, A is a direct summand of $A \otimes_k B$ as both a left and right A-module. Consequently, $A \otimes_k B$ is a generator of Mod-A, and $(A \otimes_k B)^*$ is a cogenerator of Mod-A.
- Frobenius extensions.

  Kasch [21] defined a generalisation of a Frobenius algebra called a free Frobenius extension. Nakayama and Tsuzuku [25] generalised this definition further to a Frobenius extension.

Definition 5.3 ((Free) Frobenius extension) Let A and B be rings with $f: B \to A$ a ring homomorphism. Then A is a (free) Frobenius extension of B if the following are satisfied.

– A_B is a finitely generated projective (respectively free) B-module.
– $\mathrm{Hom}_B({}_AA, {}_BB)$ is isomorphic as a (B, A)-bimodule to ${}_BA_A$.

The definition of a Frobenius extension implies that the two functors, Ind_B^A and Coind_B^A are isomorphic. Thus, both Ind_B^A and Coind_B^A are exact.

Example 5.4 There are well-known examples of Frobenius extensions which have the property that A_B is a generator of Mod-B and that $(A^*)_B$ is a cogenerator of Mod-B.

- Free Frobenius extensions
  Since A_B is a finitely generated free B-module, A_B is a generator of Mod-B. Moreover, ${}_BA_A$ and $\mathrm{Hom}_B({}_AA, {}_BB)$ are isomorphic as (B, A)-bimodules, so ${}_BA$ is free as a left B-module. Thus, B_B^* is a direct summand of $(A^*)_B$, and $(A^*)_B$ is a cogenerator of Mod-B.
- Strongly G-graded rings for a finite group G, [4, Example B].
  Let G be a finite group and A be a ring graded by G. Then A is strongly graded by G if $A_gA_h = A_{gh}$ for all $g, h \in G$. Let 1 be the identity element of G. Then A is a Frobenius extension of A_1. Moreover, A_1 is a direct summand of A as an (A_1, A_1)-bimodule. Thus, A_{A_1} is a generator of Mod-A_1, and $(A^*)_{A_1}$ is a cogenerator of Mod-A_1. This collection of graded rings includes skew group algebras, smash products and crossed products for finite groups.

- Almost excellent extensions.
  Almost excellent extensions were defined by Xue [40] as a generalisation of excellent extensions, which were first introduced by Passman [26]. The interaction of excellent extensions with various properties of rings has been studied in [17].

Definition 5.5 (Almost excellent extension) Let A and B be rings with $f: B \to A$ a ring homomorphism. Then A is an almost excellent extension of B if the following hold.

- There exist $a_1, a_2, ..., a_n \in A$ such that $A = \sum_{i=1}^{n} a_iB$ and $a_iB = Ba_i$ for all $1 \leq i \leq n$.
- A is right B-projective.
- ${}_BA$ is flat, and A_B is projective.

Recall the definition of right B-projective rings.

Definition 5.6 (Right B-projective) Let A and B be rings and $f: B \to A$ be a ring homomorphism. A short exact sequence of right A-modules

$$0 \to L_A \xrightarrow{f} K_A \xrightarrow{g} N_A \to 0,$$

is an (A, B)-exact sequence if it splits as a short exact sequence of right B-modules.

If every (A, B)-exact sequence also splits as a short exact sequence of right A-modules, then A is right B-projective.

By definition, ${}_BA$ is flat and A_B is projective, thus, to apply Lemma 5.2, all that is left to show is that $\mathrm{Res}^A_B(\text{Mod-}A)$ generates and cogenerates $\mathcal{D}(B)$. By [34, Corollary 4], $\mathrm{Coind}^A_B(M) = 0$ if and only if $M = 0$ for all $M \in \text{Mod-}B$. Consequently, $\mathrm{Hom}_B\left(\mathrm{Res}^A_B(A), M\right) = 0$ if and only if $M = 0$ by adjunction. Since $\mathrm{Res}^A_B(A) = A_B$ is projective, this is equivalent to A_B being a generator of Mod-B. Similarly, $(A^*)_B$ is a cogenerator for Mod-B since $\mathrm{Ind}^A_B(M) = 0$ if and only if $M = 0$ [33, Proposition 2.1].

- Trivial extension ring.

Definition 5.7 (Trivial extension ring) Let B be a ring and M be a (B, B)-bimodule. The trivial extension of B by M, denoted by $B \ltimes M$, is the ring with elements $(b, m) \in B \oplus M$, addition defined by,

$$(b, m) + (b', m') := (b + b', m + m'),$$

and multiplication defined by,

$$(b, m)(b', m') := (bb', bm' + mb').$$

Let $A := B \ltimes M$. Then there exists a ring homomorphism $f : B \to A$, defined by $f(b) = (b, 0)$ for all $b \in B$. Note that A is isomorphic to $B \oplus M$ as a (B, B)-bimodule. Thus A_B is a generator of Mod-B, and $(A^*)_B$ is a cogenerator of Mod-B. Hence, Lemma 5.2 (i) applies if ${}_BM$ has finite flat dimension, and Lemma 5.2 (ii) applies if M_B has finite projective dimension.

Example 5.8 – Let A be a ring, then $A \ltimes A$ is isomorphic to $A[x]/\langle x^2\rangle$.
– Let A and B be rings with ${}_AM_B$ an (A, B)-bimodule. Then the triangular matrix ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is isomorphic to $(A \times B) \ltimes M$.
– Green, Psaroudakis and Solberg [14, Section 4] use trivial extension rings to define an operation on quiver algebras called arrow removal. This operation is considered in Section 5.1.

The examples provided so far satisfy Lemma 5.2 because A_B generates Mod-B. One example of a ring construction which satisfies Lemma 5.2 without this assumption is a quotient ring $A := B/I$, where I is a nilpotent ideal of B. Since A_B is annihilated by I, A_B does not generate Mod-B. However, $\mathrm{Res}^A_B(\text{Mod-}A)$ does generate $\mathcal{D}(B)$.

Lemma 5.9 *Let B be a ring and I be a nilpotent ideal of B.*

(i) *Suppose that I has finite flat dimension as a left B-module. If injectives generate for B/I, then injectives generate for B.*
(ii) *Suppose that I has finite projective dimension as a right B-module. If projectives cogenerate for B/I, then projectives cogenerate for B.*

Proof The ring homomorphism we shall use is the quotient map $f : B \to B/I$. Let $A := B/I$. Then $\mathrm{Res}^A_B(\text{Mod-}A)$ consists of the B-modules annihilated by I. Note that every B-module M is an iterated extension of MI^m/MI^{m+1} for $m \in \mathbb{Z}_+$ via the short exact sequences

$$0 \to MI^{m+1} \to MI^m \to MI^m/MI^{m+1} \to 0.$$

Hence, as I is a nilpotent ideal, every B-module is in the triangulated subcategory of $\mathcal{D}(B)$ generated by $\mathrm{Res}^A_B(\text{Mod-}A)$. Thus, $\mathrm{Res}^A_B(\text{Mod-}A)$ both generates and cogenerates $\mathcal{D}(B)$.

(i) By the short exact sequence of left B-modules

$$0 \to I \to B \to A \to 0,$$

${}_BA$ has finite flat dimension if and only if ${}_BI$ has finite flat dimension. Hence, (i) follows from Lemma 5.2 (i).

(ii) Similarly, by the short exact sequence of right B-modules

$$0 \to I \to B \to A \to 0,$$

A_B has finite projective dimension if and only if I_B has finite projective dimension. Hence, (ii) follows from Lemma 5.2 (ii).

□

Example 5.10 Lemma 5.9 can be applied to trivial extension rings. In particular, let S be a ring and M be an (S, S)-bimodule. Let $R := S \ltimes M$. Then $(0, M)$ is a nilpotent ideal of R and $S \cong R/(0, M)$. Thus, if injectives generate for S, and ${}_R(0, M)$ has finite flat dimension, then Lemma 5.9 (i) applies and injectives generate for R. Similarly, Lemma 5.9 (ii) applies if $(0, M)_R$ has finite projective dimension.

5.1 Arrow Removal

Let $\Lambda := kQ/I$ be a path algebra with admissible ideal I. Let $a\colon v_e \to v_f$ be an arrow of Q which is not in a minimal generating set of I. Then Green, Psaroudakis and Solberg [14, Section 4] define the algebra obtained from Λ by removing the arrow a as $\Gamma := \Lambda/\Lambda\bar{a}\Lambda$, where $\bar{a} = a + I$. Moreover, they prove that this operation can be reformulated in terms of trivial extension rings.

Proposition 5.11 [14, Proposition 4.4 (iii)] *Let $\Lambda := kQ/I$ be an admissible quotient of the path algebra kQ over a field k. Suppose that there are arrows $a_i\colon v_{e_i} \to v_{f_i}$ in Q for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of I in kQ and $\mathrm{Hom}_\Lambda(e_i\Lambda, f_j\Lambda) = 0$ for all i and j in $\{1, 2, \ldots, t\}$. Let $\bar{a_i} = a_i + I$ in Λ. Let $\Gamma = \Lambda/\Lambda\{\bar{a_i}\}_{i=1}^t\Lambda$. Then Λ is isomorphic to the trivial extension $\Gamma \ltimes P$, where $P = \oplus_{i=1}^t \Gamma e_i \otimes_k f_i \Gamma$.*

Since the arrow removal operation can be thought of as a trivial extension, we can apply Lemmas 5.2 and 5.9 to prove that this operation preserves injective generation.

Proposition 5.12 *Let $\Lambda := kQ/I$ be an admissible quotient of the path algebra kQ over a field k. Suppose that there are arrows $a_i\colon v_{e_i} \to v_{f_i}$ in Q for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of I in kQ and $\mathrm{Hom}_\Lambda(e_i\Lambda, f_j\Lambda) = 0$ for all i and j in $\{1, 2, \ldots, t\}$. Let $\bar{a_i} = a_i + I$ in Λ. Let $\Gamma = \Lambda/\Lambda\{\bar{a_i}\}_{i=1}^t\Lambda$.*

(i) *Injectives generate for Λ if and only if injectives generate for Γ.*
(ii) *Projectives cogenerate for Λ if and only if projectives cogenerate for Γ.*

Proof Firstly, note that $P = \oplus_{i=1}^t \Gamma e_i \otimes_k f_i \Gamma$ is projective as both a left and right Γ-module. Moreover, ${}_\Gamma\Lambda_\Gamma$ is isomorphic to $\Gamma \oplus P$ as a (Γ, Γ)-bimodule. Hence, Λ is projective as both a left and right Γ-module. Consequently, Lemma 5.2 applies with $A = \Lambda$ and $B = \Gamma$. So, if injectives generate for Λ, then injectives generate for Γ, and the equivalent statement for projective cogeneration also holds.

Secondly, there exists a short exact sequence of (Λ, Λ)-bimodules

$$0 \to P \to \Lambda \to \Gamma \to 0.$$

By [14, Proposition 4.6 (vii)], $P \otimes_\Gamma P = 0$, so

$${}_\Lambda\Lambda \otimes_\Gamma P \cong {}_\Lambda\Gamma \otimes_\Gamma P \cong {}_\Lambda P.$$

Consequently, since ${}_\Gamma P$ is projective, so is ${}_\Lambda\Lambda \otimes_\Gamma P \cong {}_\Lambda P$. Similarly,

$$P \otimes_\Gamma \Lambda_\Lambda \cong P \otimes_\Gamma \Gamma_\Lambda \cong P_\Lambda,$$

and P_Γ is projective, so P_Λ is a projective right Λ-module. Consequently, Lemma 5.9 applies. In particular, Example 5.10 applies with $R = \Lambda$, $(0, {}_S M_S) = (0, {}_\Gamma P_\Gamma) = {}_\Lambda P_\Lambda$, and $S = R/(0, M) = \Gamma$. So, if injectives generate for Γ, then injectives generate for Λ, and the similar statement holds for projective cogeneration. $\square$

5.2 Frobenius Extensions And Almost Excellent Extensions

To prove the converse statement to Lemma 5.2 for Frobenius extensions and almost excellent extensions we prove a more general result that connects relatively B-injective A-modules to injective generation. Recall the definition of (A, B)-exact sequences (see Definition 5.6).

Definition 5.13 (Relatively projective/injective) Let A and B be rings with a ring homomorphism $f : B \to A$. Let M_A be an A-module.

- M_A is relatively B-projective if $\mathrm{Hom}_A(M, -)$ is exact on (A, B)-exact sequences.
- M_A is relatively B-injective if $\mathrm{Hom}_A(-, M)$ is exact on (A, B)-exact sequences.

Let A and B be rings with a ring homomorphism $f : B \to A$. Then any injective A-module I is relatively B-injective since $\mathrm{Hom}_A(-, I)$ is exact on all short exact sequences of A-modules. Similarly, any projective A-module is relatively B-projective. However, for both Frobenius extensions and almost excellent extensions all projective A-modules are relatively B-injective, see Example 5.15. This property can be used to prove the converse statement to Lemma 5.2 for both of these ring extensions.

Lemma 5.14 *Let A and B be rings with a ring homomorphism $f : B \to A$.*

(i) *Suppose that A_B is a finitely generated projective B-module, and that all projective A-modules are relatively B-injective. If injectives generate for B, then injectives generate for A.*
(ii) *Suppose that ${}_B A$ is a finitely generated projective B-module, and that all injective A-modules are relatively B-projective. If projectives cogenerate for B, then projectives cogenerate for A.*

Proof (i) Since A_B is a finitely generated projective, Coind_B^A is exact and preserves both set indexed coproducts and injective modules. Hence, if injectives generate for B, then $\mathrm{im}(\mathrm{Coind}_B^A)$ is a subcategory of $\mathrm{Loc}_A(\mathrm{Inj}\text{-}A)$ by Proposition 2.9. Let P be a projective A-module. Since P is relatively B-injective, P is isomorphic to a direct summand of

$\mathrm{Coind}_B^A \circ \mathrm{Res}_B^A(P)$, [19, Section 4.1]. Thus Proj-A is a subcategory of Loc_A (Inj-A), and injectives generate for A.

(ii) This follows similarly to (i). In particular, since ${}_BA$ is a finitely generated projective, Ind_B^A is exact and preserves set indexed products. Moreover, if an injective A-module I is relatively B-projective, then I is isomorphic to a direct summand of $\mathrm{Ind}_B^A \circ \mathrm{Res}_B^A(I)$, [19, Section 4.1].

□

Example 5.15 Lemma 5.14 applies to both Frobenius extensions and almost excellent extensions.

- Frobenius extensions.
  Let A and B be rings such that A is a Frobenius extension of B. Let M be an A-module. Then M is relatively B-injective if and only if M is relatively B-projective, [19, Proposition 4.1].
- Almost excellent extensions
  Recall that, if A is an almost excellent extension of B, then A is right B-projective. Hence, every A-module is both relatively B-injective and relatively B-projective, [40, Lemma 1.1]. Thus Lemma 5.14 applies if ${}_BA$ is a finitely generated projective B-module.

6 Recollements of Derived Module Categories

In this section we consider the interaction between rings that appear in a recollement of derived categories with respect to injective generation statements. Firstly, recall the definition of a recollement of derived categories, introduced by Beĭlinson, Bernstein and Deligne [6].

Definition 6.1 (Recollement) Let A, B and C be rings. A recollement is a diagram of six triangle functors as in Fig. 1 such that the following hold:

(i) The composition $j^* \circ i_* = 0$.
(ii) Each of the pairs (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) is an adjoint pair of functors.
(iii) The functors i_*, $j_!$ and j_* are fully faithful.
(iv) For all $X \in \mathcal{D}(A)$, there exist triangles:

$$j_!j^*X \to X \to i_*i^*X \to j_!j^*X[1]$$
$$i_*i^!X \to X \to j_*j^*X \to i_*i^!X[1]$$

$$\begin{tikzcd}
\mathcal{D}(B) \arrow[r, "i_*"] & \mathcal{D}(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{D}(C) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Fig. 1 Recollement of derived categories (R)

We denote a recollement of the form in Fig. 1 as $(R) = (\mathcal{D}(B), \mathcal{D}(A), \mathcal{D}(C))$. If a recollement (R) exists, then the properties of A, B and C are often related. This allows one to prove properties about A using B and C, and vice versa. Such a method has been exploited by Happel [16, Theorem 2] and Chen and Xi [10, Theorem 1.1] to prove various statements about the finitistic dimension of rings that appear in recollements. These results can be applied to recollements (R) which restrict to recollements on derived categories with various boundedness conditions. In this section, we say a recollement (R) restricts to a recollement (R^*) for $* \in \{-, +, b\}$ if the six functors of (R) restrict to functors on the essential image of $\mathcal{D}^*(\mathrm{Mod})$ in $\mathcal{D}(\mathrm{Mod})$. Note that such a restriction is not always possible, however in [2, Section 4] there are necessary and sufficient conditions for (R) to restrict to a recollement (R^-) or (R^b). In Proposition 6.14 we provide analogous conditions for (R) to restrict to a recollement (R^+).

Example 6.2 One example of a recollement of unbounded derived module categories can be defined using triangular matrix rings. Let B and C be rings and ${}_C M_B$ be a (C, B)-bimodule. Then the triangular matrix ring is defined as

$$A := \begin{pmatrix} C & {}_C M_B \\ 0 & B \end{pmatrix}.$$

In this situation A, B and C define a recollement (R). The functors of (R) are defined using idempotents of A. Let

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the recollement is defined by $j_! = - \otimes_C^L e_1 A$ and $i_* = - \otimes_B^L e_2 A$.

This section includes results about the dependence of A, B and C on each other with regards to 'injectives generate' and 'projectives cogenerate' statements. Theorem 6.3 is a summary of the results in this section that use the properties of B and C to prove generation statements about A.

Theorem 6.3 *Let (R) be a recollement.*

(i) *Suppose that injectives generate for both B and C. If one of the following conditions holds, then injectives generate for A.*
  (a) *The recollement (R) is in a ladder of recollements with height greater than or equal to 2. [Proposition 6.10]*
  (b) *The recollement (R) restricts to a bounded below recollement (R^+). [Proposition 6.15]*
  (c) *The recollement (R) restricts to a bounded above recollement (R^-), and A is a finite dimensional algebra over a field. [Proposition 6.13]*

(ii) *Suppose that projectives cogenerate for both B and C. If one of the following conditions holds, then projectives cogenerate for A.*
  (a) *The recollement (R) is in a ladder of recollements with height greater than or equal to 2. [Proposition 6.10]*
  (b) *The recollement (R) restricts to a bounded above recollement (R^-). [Proposition 6.13]*

(c) *The recollement* (R) *restricts to a bounded below recollement* (R^+)*, and* A *is a finite dimensional algebra over a field. [Proposition 6.15]*

To prove Theorem 6.3 we require some technical results which we state and prove now. We prove these results by using the fact that there are four pairs of adjoint functors in a recollement that preserve many properties of complexes, these properties are summarised in Table 1. The results in Table 1 follow from the definition of a recollement, standard properties of adjoint functors, and Lemma 2.14.

Lemma 6.4 *Let* (R) *be a recollement.*

(i) *Suppose that* j^* *preserves bounded complexes of injectives. If injectives generate for* A*, then injectives generate for* C*.*
(ii) *Suppose that* j^* *preserves bounded complexes of projectives. If projectives cogenerate for* A*, then projectives cogenerate for* C*.*

Proof (i) Suppose that injectives generate for A. Since j^* preserves bounded complexes of injectives and set indexed coproducts, $\mathrm{im}(j^*)$ is a subcategory of Loc_C (Inj-C) by Proposition 2.9. Furthermore, j^* is essentially surjective, as it is right adjoint to $j_!$ which is fully faithful. Hence, injectives generate for C.
(ii) This follows similarly to (i).

□

Proposition 6.5 *Let* (R) *be a recollement.*

(i) *Suppose that* $\mathrm{im}(i_*)$ *is a subcategory of* Loc_A *(Inj-*A*). If injectives generate for* C*, then injectives generate for* A*.*
(ii) *Suppose that* $\mathrm{im}(i_*)$ *is a subcategory of* Coloc_A *(Proj-*A*). If projectives cogenerate for* C*, then projectives cogenerate for* A*.*

Proof (i) Suppose that $\mathrm{im}(i_*)$ is a subcategory of Loc_A (Inj-A). Let $I \in \mathcal{D}(C)$ be a bounded complex of injectives, and consider the triangle,

$$j_!j^*(j_*(I)) \to j_*(I) \to i_*i^*(j_*(I)) \to j_!j^*(j_*(I))[1].$$

Since j_* preserves bounded complexes of injectives, $j_*(I)$ is in Loc_A (Inj-A). Hence, because Loc_A (Inj-A) is a triangulated subcategory, $j_!j^*(j_*(I))$ is in Loc_A (Inj-A).

Table 1 Properties of the triangle functors in a recollement

Property preserved	Functors with this property
Products	$i_*, i^!, j^*, j_*$.
Coproducts	$i^*, i_*, j_!, j^*$.
Compact objects	$i^*, j_!$.
Complexes bounded in cohomology	i_*, j^*.
Complexes bounded above in cohomology	$i^*, i_*, j_!, j^*$.
Complexes bounded below in cohomology	$i_*, i^!, j^*, j_*$.
Bounded complexes of projectives	$i^*, j_!$.
Bounded complexes of injectives	$i^!, j_*$.

Recall that j_* is fully faithful so $j_! j^* j_*(I) \cong j_!(I)$. Thus, $j_!$ maps bounded complexes of injectives to Loc_A (Inj-A).

Suppose that injectives generate for C. Then, by Proposition 2.9, $\mathrm{im}(j_!)$ is a subcategory of Loc_A (Inj-A). Since $\mathrm{im}(i_*)$ and $\mathrm{im}(j_!)$ are both subcategories of Loc_A (Inj-A), for all $X \in \mathcal{D}(A)$, both $i_* i^*(X)$ and $j_! j^*(X)$ are in Loc_A (Inj-A). Hence, X is in Loc_A (Inj-A) by the triangle

$$j_! j^*(X) \to X \to i_* i^*(X) \to j_! j^*(X)[1],$$

and injectives generate for A.

(ii) This follows similarly to (i).

□

Proposition 6.6 *Let* (R) *be a recollement.*

(i) *Suppose that* i_* *preserves bounded complexes of injectives.*

  (a) *If injectives generate for both* B *and* C*, then injectives generate for* A.
  (b) *If injectives generate for* A*, then injectives generate for* C.

(ii) *Suppose that* i_* *preserves bounded complexes of projectives.*

  (a) *If projectives cogenerate for both* B *and* C*, then projectives cogenerate for* A.
  (b) *If projectives cogenerate for* A*, then projectives cogenerate for* C.

Proof (i) We prove the first two statements as the other two follow similarly.

  (a) Suppose that injectives generate for B. Since i_* preserves bounded complexes of injectives and set indexed coproducts, by Proposition 2.9, $\mathrm{im}(i_*)$ is a subcategory of Loc_A (Inj-A). Thus, if injectives generate for C, then injectives generate for A by Proposition 6.5.
  (b) We claim that if i_* preserves bounded complexes of injectives, then j^* also preserves bounded complexes of injectives. Recall that $j_!$ preserves complexes bounded above in cohomology, see Table 1. Furthermore, since i_* preserves bounded complexes of injectives, by Lemma 2.14, i^* preserves complexes bounded below in cohomology. Let $X \in \mathcal{D}(C)$ be bounded below in cohomology and consider the triangle

$$\begin{tikzcd}
j_! j^*(j_*(X)) \arrow[r] \arrow[d, "\cong"] & j_*(X) \arrow[r] \arrow[d, "="] & i_* i^*(j_*(X)) \arrow[r] \arrow[d, "="] & j_! j^*(j_*(X))[1] \arrow[d, "\cong"] \\
j_!(X) \arrow[r] & j_*(X) \arrow[r] & i_* i^* j_*(X) \arrow[r] & j_!(X)[1].
\end{tikzcd}$$

Since i_*, i^* and j_* all preserve complexes bounded below in cohomology, by the triangle, $j_!$ also preserves complexes bounded below in cohomology. Hence, $j_!$ preserves both complexes bounded above and bounded below in cohomology. Consequently, $j_!$ preserves complexes bounded in cohomology, and j^* preserves bounded complexes of injectives by Lemma 2.14. Hence, the statement follows from Lemma 6.14.

(ii) This follows similarly to (i).

□

$$\begin{tikzcd}[column sep=huge]
 & \vdots & & \vdots & \\
\mathcal{D}(B) \arrow[rr, bend left=50, "i_{n-2}"] \arrow[rr, "i_n"] \arrow[rr, bend right=50, "i_{n+2}"'] & & \mathcal{D}(A) \arrow[ll, bend right=15, "j_{n-1}"'] \arrow[ll, bend left=15, "j_{n+1}"] \arrow[rr, bend left=50, "j_{n-2}"] \arrow[rr, "j_n"] \arrow[rr, bend right=50, "j_{n+2}"'] & & \mathcal{D}(C) \arrow[ll, bend right=15, "i_{n-1}"'] \arrow[ll, bend left=15, "i_{n+1}"] \\
 & \vdots & & \vdots &
\end{tikzcd}$$

Fig. 2 Ladder of recollements

Lemma 6.7 *Let* (R) *be a recollement.*

(i) *Suppose that injectives generate for* A*. Then injectives generate for* B *if one of the following two conditions holds:*

(a) $i^!$ *preserves set indexed coproducts,*
(b) $i^*(\textit{Inj-}A)$ *is a subcategory of* $\mathrm{Loc}_B\,(\textit{Inj-}B)$.

(ii) *Suppose that projectives cogenerate for* A*. Then projectives cogenerate for* B *if one of the following two conditions holds:*

(a) i^* *preserves set indexed products,*
(b) $i^!(\textit{Proj-}A)$ *is a subcategory of* $\mathrm{Coloc}_B\,(\textit{Proj-}B)$.

Proof (i) Since i_* is fully faithful, both i^* and $i^!$ are essentially surjective. Hence, if either $\mathrm{im}(i^*)$ or $\mathrm{im}(i^!)$ is a subcategory of $\mathrm{Loc}_B\,(\text{Inj-}B)$, then injectives generate for B.

(a) By Proposition 2.9, this is a sufficient condition for $\mathrm{im}(i^!)$ to be a subcategory of $\mathrm{Loc}_B\,(\text{Inj-}B)$.
(b) By Proposition 2.9, this is a sufficient condition for $\mathrm{im}(i^*)$ to be a subcategory of $\mathrm{Loc}_B\,(\text{Inj-}B)$.

(ii) This follows similarly to (i).

□

6.1 Ladders of Recollements

The previous results about recollements can be applied to ladders of recollements. A ladder of recollements, [2, Section 3], is a collection of finitely or infinitely many rows of triangle functors between $\mathcal{D}(A)$, $\mathcal{D}(B)$ and $\mathcal{D}(C)$, of the form given in Fig. 2, such that any three consecutive rows form a recollement. The height of a ladder is the number of distinct recollements it contains.

Proposition 6.8 [2, Proposition 3.2] *Let* (R) *be a recollement.*

(i) *The recollement* (R) *can be extended one step downwards if and only if* j_* *(equivalently* $i^!$*) has a right adjoint. This occurs precisely when* j^* *(equivalently* i_**) preserves compact objects.*

(ii) *The recollement* (R) *can be extended one step upwards if and only if* $j_!$ *(equivalently* i^**) has a left adjoint. If* A *is a finite dimensional algebra over a field, this occurs precisely when* $j_!$ *(equivalently* i^**) preserves bounded complexes of finitely generated modules.*

If the recollement (R) can be extended one step downwards, then we have a recollement $(R_\downarrow)$ as in Fig. 3.

Example 6.9 As seen in Example 6.2, a triangular matrix ring defines a recollement (R). Moreover, this recollement extends one step downwards, [2, Example 3.4]. Recall that $i_* := -\otimes^L_B e_2 A$ where e_2 is an idempotent of A. In particular, note that $e_2 A_A$ is a finitely generated projective A-module, so i_* preserves compact objects. Thus, by Proposition 6.8, (R) extends one step downwards.

Proposition 6.10 *Let* (R) *be the top recollement in a ladder of height 2.*

(i) *If injectives generate for both* B *and* C*, then injectives generate for* A*.*
(ii) *If injectives generate for* A*, then injectives generate for* B*.*
(iii) *If projectives cogenerate for both* B *and* C*, then projectives cogenerate for* A*.*
(iv) *If projectives cogenerate for* A*, then projectives cogenerate for* C*.*

Proof The bottom recollement of the ladder $(R_\downarrow)$ is a recollement as in (R) but with the positions of B and C swapped. Hence, in this bottom recollement j_* acts as i_* does in the recollement (R). Moreover, j_* preserves bounded complexes of injectives by Table 1. Thus, (i) and (ii) follow from Proposition 6.6.

Moreover, by Proposition 6.8, i_* preserves compact objects. Hence, i_* preserves bounded complexes of projectives by Lemma 2.14. Consequently, Proposition 6.6 proves (iii) and (iv). $\square$

Example 6.11 By Proposition 6.10 it follows that for any triangular matrix ring

$$A = \begin{pmatrix} C & {}_C M_B \\ 0 & B \end{pmatrix},$$

if injectives generate for B and C, then injectives generate for A. Moreover, if injectives generate for A, then injectives generate for B.

Note that this is the 'injectives generate version' of the well-known result proved by Fossum, Griffith and Reiten [12, Corollary 4.21] that compares the finitistic dimensions of B, C and the triangular matrix ring A.

$$\begin{tikzcd}
\mathcal{D}(B) \arrow[r, shift left, "i_*"] \arrow[r, bend right=60, "i_{\downarrow}"'] & \mathcal{D}(A) \arrow[l, bend right=60, "i^*"'] \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] \arrow[r, bend right=60, "j^{\downarrow}"'] & \mathcal{D}(C) \arrow[l, bend right=60, "j_!"'] \arrow[l, shift left, "j_*"]
\end{tikzcd}$$

Fig. 3 Recollement of derived categories extended one step downwards $(R_\downarrow)$

Corollary 6.11.1 *Let (R) be a recollement in a ladder of height ≥ 3.*

(i) *Injectives generate for A if and only if injectives generate for both B and C.*
(ii) *Projectives cogenerate for A if and only if projectives cogenerate for both B and C.*

6.2 Bounded Above Recollements

In this section, we consider the case of a recollement that restricts to a bounded above recollement. In particular, we use a characterisation by [2].

Proposition 6.12 [2, Proposition 4.11] *Let (R) be a recollement. Then the following are equivalent:*

(i) *The recollement (R) restricts to a bounded above recollement (R^-),*
(ii) *The functor i_* preserves bounded complexes of projectives.*

If A is a finite dimensional algebra over a field, then both conditions are equivalent to:

(iii) *The functor i_* preserves compact objects.*

Note that if $i_*(B)$ is compact, then the recollement (R) also extends one step downwards by Proposition 6.8 [2, Proposition 3.2].

Proposition 6.13 *Let (R) be a recollement that restricts to a bounded above recollement (R^-).*

(i) *If projectives cogenerate for B and C, then projectives cogenerate for A.*
(ii) *If projectives cogenerate for A, then projectives cogenerate for C.*

Moreover, suppose that A is a finite dimensional algebra over a field.

(iii) *If injectives generate for B and C, then injectives generate for A.*
(iv) *If injectives generate for A, then injectives generate for B.*

Proof Since (R) restricts to a recollement of bounded above derived categories, i_* preserves bounded complexes of projectives by Proposition 6.12. Hence, (i) and (ii) follow from Proposition 6.6. Furthermore, if A is a finite dimensional algebra over a field, then i_* preserves compact objects by Proposition 6.12. Then the recollement also extends one step downwards by Proposition 6.8 so (iii) and (iv) follow from Proposition 6.10. □

6.3 Bounded Below Recollements

Now we consider recollements (R) that restrict to bounded below recollements (R^+). First we prove an analogous statement to Proposition 6.12 about the conditions under which a recollement (R) restricts to a recollement (R^+).

Proposition 6.14 *Let (R) be a recollement. Then the following are equivalent:*

(i) *The recollement (R) restricts to a bounded below recollement (R^+),*
(ii) *The functor i_* preserves bounded complexes of injectives.*

If A is a finite dimensional algebra over a field, then both conditions are equivalent to:

(iii) *The functor $j_!$ preserves bounded complexes of finitely generated modules.*

Proof First we prove that (ii) implies (i). Suppose that i_* preserves bounded complexes of injectives. Then, by the proof of Proposition 6.6, all six functors preserve complexes bounded below in cohomology. Hence, the recollement (R) restricts to a bounded below recollement (R^+).

For the converse statement, suppose that (R) restricts to a bounded below recollement (R^+), that is, all six functors preserve complexes bounded below in cohomology. Moreover, recall that i^* preserves complexes bounded above in cohomology, see Table 1. Hence, i^* preserves both complexes bounded above and bounded below in cohomology. Thus, i^* preserves complexes bounded in cohomology, and, by Lemma 2.14, i_* preserves bounded complexes of injectives.

Now, we show that (iii) implies (ii). By Proposition 6.8, i^* has a left adjoint $i_\uparrow$. The application of Lemma 2.14 to the triple of adjoint functors $(i_\uparrow, i^*, i_*)$ shows that i_* preserves bounded complexes of injectives.

Finally, we show that (i) implies (iii). Suppose that A is a finite dimensional algebra over a field. Let $X \in \mathcal{D}(C)$ be a bounded complex of finitely generated modules. Since A is a finite dimensional algebra over a field, $j_!(X)$ is a bounded above complex of finitely generated modules [2, Lemma 2.10 (b)]. Suppose that (R) restricts to a bounded below recollement (R^+). Then $j_!(X)$ is also bounded below in cohomology, so we can truncate $j_!(X)$ from below, and $j_!(X)$ is quasi-isomorphic to a bounded complex of finitely generated modules. □

We can use these results to get an analogous statement to Proposition 6.13 about bounded below recollements.

Proposition 6.15 *Let (R) be a recollement that restricts to a bounded below recollement (R^+).*

(i) *If injectives generate for both B and C, then injectives generate for A.*
(ii) *If injectives generate for A, then injectives generate for C.*

Moreover, suppose that A is a finite dimensional algebra over a field.

(iii) *If projectives cogenerate for both B and C, then projectives cogenerate for A.*
(iv) *If projectives cogenerate for A, then projectives cogenerate for B.*

Proof This is an application of Propositions 6.6 and 6.10 using 6.14, in a similar way to the proof of Proposition 6.13. □

6.4 Bounded Recollements

Finally, we consider the case of a recollement (R) that restricts to a bounded recollement (R^b).

Proposition 6.16 *Let (R) be a recollement that restricts to a bounded recollement (R^b).*

(i) *If injectives generate for both B and C, then injectives generate for A.*
(ii) *If injectives generate for A, then injectives generate for C.*
(iii) *If projectives cogenerate for both B and C, then projectives cogenerate for A.*
(iv) *If projectives cogenerate for A, then projectives cogenerate for C.*

Moreover, suppose that A is a finite dimensional algebra over a field.

(v) *Injectives generate for A if and only if injectives generate for both B and C.*

(vi) *Projectives cogenerate for A if and only if projectives cogenerate for both B and C.*

Proof Since (R^b) is a recollement of bounded derived categories, both i^* and $i^!$ preserve complexes bounded in cohomology. Hence, by Lemma 2.14, i_* preserves both bounded complexes of injectives and bounded complexes of projectives. So, (R) restricts to a bounded above recollement by Proposition 6.12, and a bounded below recollement by Proposition 6.14. Consequently, the results follow from Propositions 6.13 and 6.15. $\square$

7 Recollements of Module Categories

In this section we consider the interaction of rings that appear in a recollement of module categories with respect to injective generation. First, we recall the definition of a recollement of module categories, see for example [14, 28, 29].

Definition 7.1 (Recollement of module categories) Let A, B and C be rings. A recollement of module categories is a diagram of additive functors as in Fig. 4 such that the following hold:

1. Each of the pairs (q, i), (i, p), (l, e) and (e, r) is an adjoint pair of functors,
2. The functors i, l and r are fully faithful,
3. The composition $e \circ i$ is zero.

We denote a recollement of module categories as in Fig. 4 by (Mod-B, Mod-A, Mod-C).

Example 7.2 Let A be a ring and $e \in A$ be an idempotent. Then there exists a recollement of module categories

$$(\text{Mod-}(A/AeA), \text{Mod-}A, \text{Mod-}eAe),$$

with functors given by,

$$\begin{aligned} q &:= \operatorname{Ind}_A^{A/AeA}, & l &:= -\otimes_{eAe} eA, \\ i &:= \operatorname{Res}_A^{A/AeA}, & e &:= \operatorname{Hom}_A(eA, -) \cong -\otimes_A Ae, \\ p &:= \operatorname{Coind}_A^{A/AeA}, & r &:= \operatorname{Hom}_{eAe}(Ae, -). \end{aligned}$$

Note that q, i, and p correspond to the functors induced by the ring epimorphism $\pi : A \to A/AeA$, see Section 2.1 for the explicit descriptions.

The recollement of module categories (Mod-(A/AeA), Mod-A, Mod-eAe) induces a recollement of triangulated categories as in Fig. 5, [11], [28, Proof of Theorem 8.3].

$$\begin{tikzcd}
\text{Mod-}B \arrow[r, "i"] & \text{Mod-}A \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & \text{Mod-}C \arrow[l, bend right, "l"'] \arrow[l, bend left, "r"]
\end{tikzcd}$$

Fig. 4 Recollement of module categories

$$\begin{tikzcd}
\ker(-\otimes_A Ae) \arrow[r, "i_*"] & \mathcal{D}(A) \arrow[l, bend right=50, "i^*"'] \arrow[l, bend left=50, "i^!"] \arrow[r, "-\otimes_A Ae"] & \mathcal{D}(eAe) \arrow[l, bend right=50, "-\otimes^L_{eAe} eA"'] \arrow[l, bend left=50, "\mathrm{RHom}_{eAe}(Ae,-)"]
\end{tikzcd}$$

Fig. 5 Recollement of triangulated categories induced from a recollement of module categories

When $\pi: A \to A/AeA$ is a homological epimorphism, the triangulated category $\ker(-\otimes_A Ae)$ is equivalent to $\mathcal{D}(A/AeA)$, and we get a recollement $(\mathcal{D}(A/AeA), \mathcal{D}(A), \mathcal{D}(eAe))$ [11], [1, Subsection 1.7], [3, Proposition 2.1]. Note that, in general, $\ker(-\otimes_A Ae)$ is not necessarily equivalent to the derived category of a ring. However, if, for example, A is a finite dimensional algebra over a field, then $\ker(-\otimes_A Ae)$ is equivalent to the derived category of a differentially graded algebra [20, Proposition 2.10].

Throughout the rest of this section we focus on proving generation statements relating the three rings A, eAe and A/AeA. Although there is not necessarily a recollement of the derived categories of these rings, the proofs in Section 6 can be applied to the induced recollement in Fig. 5. To do this we focus on the relationship between the subcategories $\mathrm{im}(i_*) = \ker(j^*)$ and $\mathrm{Loc}_A\left(\mathrm{Res}(A/AeA)\right)$. First we require a well-known result about when a complex is in the (co)localising subcategory generated by its cohomology modules.

Lemma 7.3 *Let A be a ring with $X \in \mathcal{D}(A)$. Let $\mathcal{T}$ be a triangulated subcategory of $\mathcal{D}(A)$. Suppose that all of the cohomology modules of X are in $\mathcal{T}$.*

- (i) *If X is bounded in cohomology, then X is in $\mathcal{T}$.*
- (ii) *If X is bounded below in cohomology, and $\mathcal{T}$ is a localising subcategory of $\mathcal{D}(A)$, then X is in $\mathcal{T}$.*
- (iii) *If X is bounded above in cohomology, and $\mathcal{T}$ is a colocalising subcategory of $\mathcal{D}(A)$, then X is in $\mathcal{T}$.*

Proof (i) A complex bounded in cohomology is in the triangulated subcategory of the derived category generated by its cohomology modules.

(ii) Recall that a complex $X = \left(X^i, d^i\right)_{i\in\mathbb{Z}} \in \mathcal{D}(A)$ that is bounded below in cohomology is the colimit of its bounded above good truncations

$$\tau_{\leq n}(X) := \cdots \to X^{n-2} \to X^{n-1} \to \ker d^n \to 0 \to \ldots.$$

Moreover, the bounded above good truncations of a complex bounded below in cohomology are bounded in cohomology. Thus, by (i), a complex bounded below in cohomology is in the localising subcategory of the derived category generated by its cohomology modules.

(iii) This follows similarly to (ii) since a complex that is bounded above in cohomology is the limit of its bounded below good truncations.

□

Corollary 7.3.1 *Let A be a ring and $e \in A$ an idempotent. Let $\mathcal{S}$ be a class of complexes in $\mathcal{D}(A/AeA)$. Consider the functor*

$$j^* = -\otimes_A Ae\colon \mathcal{D}(A) \to \mathcal{D}(eAe).$$

(i) *If $\mathcal{S}$ generates $\mathcal{D}(A/AeA)$, then ker$(-\otimes_A Ae) \cap \mathcal{D}^+(A)$ is a subcategory of* $Loc_A\left(Res_A^{A/AeA}(\mathcal{S})\right)$.

(ii) *If $\mathcal{S}$ cogenerates $\mathcal{D}(A/AeA)$, then ker$(-\otimes_A Ae) \cap \mathcal{D}^-(A)$ is a subcategory of* $Coloc_A\left(Res_A^{A/AeA}(\mathcal{S})\right)$.

Proof (i) Denote $\mathrm{Res}_A^{A/AeA}$ by Res. Let $X \in \ker(-\otimes_A Ae) \cap \mathcal{D}^+(A)$. Since $-\otimes_A Ae$ is exact, the cohomology modules of X are annihilated by e, and so $H^n(X)$ is in Res(Mod-(A/AeA)) for all $n \in \mathbb{Z}$. Moreover, Res preserves set indexed coproducts, and $\mathcal{S}$ generates $\mathcal{D}(A/AeA)$ so im(Res) is a subcategory of $\mathrm{Loc}_A\left(\mathrm{Res}(\mathcal{S})\right)$ by Proposition 2.9.

(ii) Follows similarly to (i).

□

7.1 The Big Finitistic Dimension

Recollements of module categories interact well with the finitistic dimension as seen in [14, Section 3, Section 5]. In this section, we use a characterisation of the finitistic dimension zconjecture by Rickard [32, Theorem 4.4] to show dependencies between the finitistic dimensions of A, eAe and A/AeA. Recall the definition of the big finitistic dimension of a ring.

Definition 7.4 (Big finitistic dimension) Let Λ be a ring. The big finitistic dimension of Λ is

$$\mathrm{FinDim}(\Lambda) = \sup\{\mathrm{proj.dim}_\Lambda(M_\Lambda) : M_\Lambda \in \text{Mod-}\Lambda, \mathrm{proj.dim}_\Lambda(M_\Lambda) < \infty\}.$$

The finitistic dimension conjecture states that, if Λ is a finite dimensional algebra over a field, then $\mathrm{FinDim}(\Lambda) < \infty$. An overview of the history of the finitistic dimension conjecture is given by Huisgen-Zimmermann in [18]. Rickard proved that the finitistic dimension conjecture has an equivalent form in terms of perpendicular subcategories of the derived category using Bousfield localisation [32, Theorem 4.4]. Recall the definition of a perpendicular subcategory of $\mathcal{D}(\Lambda)$ for a ring Λ.

Definition 7.5 (Perpendicular categories) Let Λ be a ring and $X \in \mathcal{D}(\Lambda)$. Then the right perpendicular category of X is

$$X^\perp := \{Y \in \mathcal{D}(\Lambda) : \mathrm{Hom}_{\mathcal{D}(\Lambda)}(X, Y[m]) = 0 \text{ for all } m \in \mathbb{Z}\}.$$

The left perpendicular category of X is

$${}^\perp X := \{Y \in \mathcal{D}(\Lambda) : \mathrm{Hom}_{\mathcal{D}(\Lambda)}(Y, X[m]) = 0 \text{ for all } m \in \mathbb{Z}\}.$$

Note that $X^\perp$ is a colocalising subcategory of $\mathcal{D}(\Lambda)$, and ${}^\perp X$ is a localising subcategory of $\mathcal{D}(\Lambda)$. In particular, if there exists $Y \in \mathcal{D}(A)$ such that $Y \in {}^\perp X$, then $\mathrm{Loc}_A(Y)$ is a subcategory of ${}^\perp X$.

Theorem 7.6 [32, Theorem 4.4] *Let Λ be a finite dimensional algebra over a field. Then FinDim$(\Lambda) < \infty$ if and only if $D\Lambda^\perp \cap \mathcal{D}^+(\Lambda) = \{0\}$.*

Theorem 7.7 *Let A be a finite dimensional algebra over a field and $e \in A$ be an idempotent. Suppose that A/AeA has finite flat dimension as a left A-module. If $\mathrm{FinDim}(A/AeA) < \infty$ and $\mathrm{FinDim}(eAe) < \infty$, then $\mathrm{FinDim}(A) < \infty$.*

Proof Let Λ be a finite dimensional algebra over a field k. Then, by Theorem 7.6, $\mathrm{FinDim}(\Lambda) < \infty$ if and only if $D\Lambda^{\perp} \cap \mathcal{D}^{+}(\Lambda) = \{0\}$, where $D\Lambda_{\Lambda} = \mathrm{Hom}_k ({}_{\Lambda}\Lambda, k)$. Hence, we prove that, if $D(A/AeA)^{\perp} \cap \mathcal{D}^{+}(A/AeA) = \{0\}$, and $D(eAe)^{\perp} \cap \mathcal{D}^{+}(eAe) = \{0\}$, then $DA^{\perp} \cap \mathcal{D}^{+}(A) = \{0\}$.

Let $\mathbf{L}$Ind, Res and $\mathbf{R}$Coind denote the functors induced from the ring epimorphism $\pi : A \to A/AeA$. Suppose that $X \in DA^{\perp} \cap \mathcal{D}^{+}(A)$. Then DA is in ${}^{\perp}X$. Also, because ${}^{\perp}X$ is a localising subcategory, $\mathrm{Loc}_A\,(DA)$ is a subcategory of ${}^{\perp}X$. Since A is a finite dimensional algebra over a field, every injective A-module is a direct summand of a set indexed coproduct of copies of DA, and $\mathrm{Loc}_A\,(\text{Inj-}A) = \mathrm{Loc}_A\,(DA)$. Consequently, $\mathrm{Loc}_A\,(\text{Inj-}A)$ is a subcategory of ${}^{\perp}X$.

Since $\mathrm{Res}(A/AeA)$ has finite flat dimension as a left A-module, Res preserves bounded complexes of injectives by Remark 5.1. Hence, $\mathrm{Res}(D(A/AeA))$ is in $\mathrm{Loc}_A\,(\text{Inj-}A)$. Consequently, for all $m \in \mathbb{Z}$,

$$0 = \mathrm{Hom}_{\mathcal{D}(A)}\left(\mathrm{Res}(D(A/AeA)), X[m]\right) \cong \mathrm{Hom}_{\mathcal{D}(A/AeA)}\left(D(A/AeA), \mathbf{R}\mathrm{Coind}(X)[m]\right),$$

and $\mathbf{R}\mathrm{Coind}(X)$ is in $D(A/AeA)^{\perp}$. Moreover, $\mathbf{R}$Coind is a right derived functor, so $\mathbf{R}\mathrm{Coind}(X)$ is bounded below in cohomology, and $\mathbf{R}\mathrm{Coind}(X)$ is in $D(A/AeA)^{\perp} \cap \mathcal{D}^{+}(A/AeA) = \{0\}$.

Since $\mathbf{R}\mathrm{Coind}(X) = 0$, by adjunction,

$$\mathrm{Hom}_{\mathcal{D}(A)}\left(\mathrm{Res}(A/AeA), X[m]\right) \cong \mathrm{Hom}_{\mathcal{D}(A/AeA)}\left(A/AeA, \mathbf{R}\mathrm{Coind}(X[m])\right) = 0,$$

for all $m \in \mathbb{Z}$, so X is in $\mathrm{Res}(A/AeA)^{\perp}$. Recall that the idempotent $e \in A$ gives rise to a recollement of triangulated categories as in Fig. 5, with $j^{*} = - \otimes_A Ae$. Moreover, $j^{*}(\mathrm{Res}(A/AeA)) = 0$, so $\mathrm{Res}(A/AeA)$ is in $\ker(j^{*}) = \mathrm{im}(i_{*})$, and $j_{*}j^{*}(X)$ is in $\mathrm{Res}(A/AeA)^{\perp}$. Hence, by the triangle

$$i_{*}i^{!}(X) \to X \to j_{*}j^{*}(X) \to i_{*}i^{!}(X)[1],$$

we have that $i_{*}i^{!}(X)$ is also in $\mathrm{Res}(A/AeA)^{\perp}$. Note that j_{*} is a right derived functor, so $j_{*}j^{*}(X)$ is bounded below in cohomology, and therefore, $i_{*}i^{!}(X)$ is also bounded below in cohomology.

Now consider ${}^{\perp}i_{*}i^{!}(X)$. Since ${}^{\perp}i_{*}i^{!}(X)$ is a localising subcategory that contains $\mathrm{Res}(A/AeA)$, $\mathrm{Loc}_A\left(\mathrm{Res}(A/AeA)\right)$ is a subcategory of ${}^{\perp}i_{*}i^{!}(X)$. Moreover, $i_{*}i^{!}(X)$ is bounded below in cohomology, so $i_{*}i^{!}(X)$ is in $\mathrm{Loc}_A\left(\mathrm{Res}(A/AeA)\right)$ by Corollary 7.3.1. Hence, $i_{*}i^{!}(X)$ is in ${}^{\perp}i_{*}i^{!}(X)$, so $i_{*}i^{!}(X) = 0$, and $X \cong j_{*}j^{*}(X)$.

Finally, note that j_{*} preserves injective modules as $j^{*} = - \otimes_A Ae$ is exact. Thus, $j_{*}(D(eAe))$ is in $\mathrm{Loc}_A\,(\text{Inj-}A)$, and $\mathrm{Hom}_{\mathcal{D}(A)}\,(j_{*}(D(eAe)), X) = 0$. Since $X \cong j_{*}j^{*}(X)$, and j_{*} is fully faithful,

$$\mathrm{Hom}_{\mathcal{D}(eAe)}\left(D(eAe), j^{*}(X)\right) \cong \mathrm{Hom}_{\mathcal{D}(A)}\,(j_{*}(D(eAe)), X) = 0.$$

Consequently, $j^{*}(X) \in D(eAe)^{\perp} \cap \mathcal{D}^{+}(eAe) = \{0\}$, and $X \cong j_{*}j^{*}(X) = 0$. $\square$

7.2 Recollements where $Ae \otimes^{L}_{eAe} eA$ is Bounded in Cohomology

Now we restrict to the case when $j_! j^*(A) = Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology. This property is satisfied when, for example, Ae has finite flat dimension as a right eAe-module or eA has finite flat dimension as a left eAe-module. First, we prove a similar result to Corollary 7.3.1 concerning the interaction of $\mathrm{Loc}_A\left(\mathrm{Res}(A/AeA)\right)$ and the subcategory $\mathrm{im}(i_*)$.

Lemma 7.8 *Let A be a ring and $e \in A$ be an idempotent. Let $\mathcal{S}$ be a class of complexes in $\mathcal{D}(A/AeA)$. Consider the functor*

$$j^* = - \otimes_A Ae \colon \mathcal{D}(A) \to \mathcal{D}(eAe)\,.$$

Suppose that $Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology.

(i) *If $\mathcal{S}$ generates $\mathcal{D}(A/AeA)$, then $ker(-\otimes_A Ae) = Loc_A\left(Res_A^{A/AeA}(\mathcal{S})\right)$.*

(ii) *If $\mathcal{S}$ cogenerates $\mathcal{D}(A/AeA)$, then $ker(-\otimes_A Ae) = Coloc_A\left(Res_A^{A/AeA}(\mathcal{S})\right)$.*

Proof Denote the restriction functor $\mathrm{Res}_A^{A/AeA}$ by Res. Recall that the idempotent $e \in A$ gives rise to a recollement of triangulated categories as in Fig. 5, with $j^* = -\otimes_A Ae$. Moreover, by the definition of a recollement $\ker(j^*) = \mathrm{im}(i_*)$.

(i) Since i_* is fully faithful and preserves set indexed coproducts, $\mathrm{im}(i_*)$ is a localising subcategory. Moreover, $j^*(\mathrm{Res}(\mathcal{S})) = \{0\}$, so $\mathrm{Loc}_A\left(\mathrm{Res}(\mathcal{S})\right)$ is a subcategory of $\ker(j^*) = \mathrm{im}(i_*)$.

Since $j_! j^*(A) = Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology, by the triangle

$$j_! j^*(A) \to A \to i_* i^*(A) \to j_! j^*(A)[1],$$

we have that $i_* i^*(A)$ is also bounded in cohomology. In particular, $i_* i^*(A)$ is in $\ker(j^*) \cap \mathcal{D}^+(A)$, so, by Corollary 7.3.1, $i_* i^*(A)$ is in $\mathrm{Loc}_A\left(\mathrm{Res}(\mathcal{S})\right)$. Moreover, $i_* i^* \colon \mathcal{D}(A) \to \mathcal{D}(A)$ preserves set indexed coproducts, and i^* is essentially surjective, so $\mathrm{im}(i_*) = \mathrm{im}(i_* i^*) = \mathrm{Loc}_A(i_* i^*(A))$. Thus, $\mathrm{im}(i_*)$ is a subcategory of $\mathrm{Loc}_A\left(\mathrm{Res}(\mathcal{S})\right)$.

(ii) Similarly to (i) $\mathrm{Coloc}_A\left(\mathrm{Res}(\mathcal{S})\right)$ is a subcategory of $\mathrm{im}(i_*)$.

For the converse, note that for all $m \in \mathbb{Z}$,

$$H^m\left(j_* j^*(A^*)\right) \cong \mathrm{Hom}_{\mathcal{D}(A)}\left(A, j_* j^*(A^*)[m]\right) \cong \mathrm{Hom}_{\mathcal{D}(A)}\left(j_! j^*(A), A^*[m]\right).$$

Consequently, since $j_! j^*(A)$ is bounded in cohomology, and A^* is an injective cogenerator of Mod-A, $j_* j^*(A^*)$ is bounded in cohomology. Hence, $i_* i^!(A^*)$ is also bounded in cohomology by the triangle

$$i_* i^!(A^*) \to A^* \to j_* j^*(A^*) \to i_* i^!(A^*)[1].$$

Thus, similarly to (i), $\mathrm{im}(i_*)$ is a subcategory of $\mathrm{Coloc}_A\left(\mathrm{Res}(\mathcal{S})\right)$ by Corollary 7.3.1.

□

Now we prove an analogous result to Theorem 7.7 for injective generation.

Proposition 7.9 *Let A be a ring and $e \in A$ be an idempotent such that $Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology.*

(i) *Suppose that A/AeA has finite flat dimension as a left A-module. If injectives generate for both A/AeA and eAe, then injectives generate for A.*
(ii) *Suppose that A/AeA has finite projective dimension as a right A-module. If projectives cogenerate for both A/AeA and eAe, then projectives cogenerate for A.*

Proof Recall that there exists a recollement of triangulated categories as in Fig. 5 with $j^* = -\otimes_A Ae$, and $\ker(j^*) = \operatorname{im}(i_*)$.

(i) Denote $\operatorname{Res}_A^{A/AeA}$ by Res and $\mathbf{L}\operatorname{Ind}_A^{A/AeA}$ by **L**Ind. If injectives generate for A/AeA, then, by Lemma 7.8,

$$\operatorname{im}(i_*) = \ker(j^*) = \operatorname{Loc}_A\left(\operatorname{Res}(\text{Inj-}(A/AeA))\right).$$

If A/AeA has finite flat dimension as a left A-module, then Res preserves bounded complexes of injectives by Remark 5.1. Hence, $\operatorname{Res}(\text{Inj-}(A/AeA))$ is a subcategory of $\operatorname{Loc}_A(\text{Inj-}A)$. Thus, $\operatorname{im}(i_*)$ is a subcategory of $\operatorname{Loc}_A(\text{Inj-}A)$, and the result follows from the proof of Proposition 6.5.
(ii) Follows similarly to (i).

□

Following the ideas of Fuller and Saorín [13, Section 1] and Green, Psaroudakis and Solberg [14, Section 5] we now suppose that the semi-simple A-module $(1-e)A/\operatorname{rad}((1-e)A)$ has finite projective dimension or finite injective dimension, where A is a finite dimensional algebra. Note that we can characterise the projective (injective) dimension of an A/AeA-module in terms of the projective (injective) dimension of $(1-e)A/\operatorname{rad}((1-e)A)$, as we see in Lemma 7.10.

Lemma 7.10 *Let A be a finite dimensional algebra over a field and $e \in A$ be an idempotent. Let S be the semi-simple A-module $(1-e)A/rad((1-e)A)$. Let N be an A-module that is annihilated by e.*

(i) *If S_A has finite injective dimension, then N_A has finite injective dimension.*
(ii) *If S_A has finite projective dimension, then N_A has finite projective dimension.*

Proof Since Ne is zero, the radical series of N contains only direct summands of set indexed coproducts of S. Hence, if S has finite injective (projective) dimension, then N has finite injective (respectively projective) dimension. □

The proof of Lemma 7.10 requires that every A/AeA-module has a finite radical series which is not necessarily true when A is not a finite dimensional algebra. In this situation we consider a homological dimension used by Fuller and Saorín [13, Section 1], Psaroudakis [27, Section 4], and Green, Psaroudakis and Solberg [14, Section 3].

Definition 7.11 (Relative projective (injective) global dimension) Let A be a ring and $e \in A$ be an idempotent. Then the A/AeA-relative projective global dimension of A is

$$\operatorname{pgl}_A(A/AeA) := \sup\{\operatorname{proj.dim}_A\left(\operatorname{Res}(M)_A\right) : M \in \text{Mod-}(A/AeA)\}.$$

The A/AeA-relative injective global dimension of A is

$$\operatorname{igl}_A(A/AeA) := \sup\{\operatorname{inj.dim}_A\left(\operatorname{Res}(M)_A\right) : M \in \text{Mod-}(A/AeA)\}.$$

Note that, for a finite dimensional algebra over a field, if the semi-simple A-module $S = (1-e)A/\mathrm{rad}\,((1-e)A)$ has finite projective dimension, then $\mathrm{pgl}_A\,(A/AeA) < \infty$. Similarly, if S has finite injective dimension, then $\mathrm{igl}_A\,(A/AeA) < \infty$.

Proposition 7.12 *Let A be a ring and $e \in A$ be an idempotent such that $Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology.*

(i) *Suppose that $\mathrm{igl}_A\,(A/AeA) < \infty$. If injectives generate for eAe, then injectives generate for A.*
(ii) *Suppose that $\mathrm{pgl}_A\,(A/AeA) < \infty$. If projectives cogenerate for eAe, then projectives cogenerate for A.*

Proof (i) Since $Ae \otimes^{L}_{eAe} eA$ is bounded in cohomology,

$$\mathrm{im}(i_*) = \mathrm{Loc}_A\left(\mathrm{Res}(A/AeA)\right),$$

by Lemma 7.8. Furthermore, as $\mathrm{igl}_A\,(A/AeA) < \infty$, $\mathrm{Res}(A/AeA)$ has finite injective dimension as an A-module. Hence, $\mathrm{Res}(A/AeA)$ is in $\mathrm{Loc}_A\,(\text{Inj-}A)$, and $\mathrm{im}(i_*)$ is a subcategory of $\mathrm{Loc}_A\,(\text{Inj-}A)$. Thus, the proof of Proposition 6.5 applies.
(ii) Follows similarly to (i).

□

When the induced recollement of triangulated categories in Fig. 5 is equivalent to a recollement of rings, then the results in Section 6 apply directly. This occurs, for example, when $\mathrm{pgl}_A\,(A/AeA) \leq 1$ [14, Proposition 3.5 (iv)].

Lemma 7.13 *Let A be a ring and $e \in A$ be an idempotent.*

(i) *Suppose that $\mathrm{igl}_A\,(A/AeA) \leq 1$.*

  (a) *Injectives generate for A if and only if injectives generate for eAe.*

  Moreover, suppose that A is a finite dimensional algebra over a field.

  (a) *If projectives cogenerate for eAe, then projectives cogenerate for A.*

(ii) *Suppose that $\mathrm{pgl}_A\,(A/AeA) \leq 1$.*

  (a) *Projectives cogenerate for A if and only if projectives cogenerate for eAe.*

  Moreover, suppose that A is a finite dimensional algebra over a field.

  (a) *If injectives generate for eAe, then injectives generate for A.*

Proof Denote the functors induced from the ring homomorphism $\pi : A \to A/AeA$ by **L**Ind, Res **R**Coind. If either $\mathrm{igl}_A\,(A/AeA) \leq 1$ or $\mathrm{pgl}_A\,(A/AeA) \leq 1$, then $\pi : A \to A/AeA$ is a homological ring epimorphism [14, Proposition 3.5 (iv), Remark 5.9]. Moreover, π is a homological ring epimorphism if and only if $\mathrm{Res}_A^{A/AeA}$ is a homological embedding [27, Corollary 3.13]. In this situation the recollement of module categories (Mod-(A/AeA), Mod-A, Mod-eAe) lifts to a recollement of derived categories of the same rings, $(R) = (\mathcal{D}(A/AeA), \mathcal{D}(A), \mathcal{D}(eAe))$ with $i_* = \mathrm{Res}$ and $j^* = -\otimes_A Ae$, [11], [28, Theorem 8.3].

(i) Suppose that $\mathrm{igl}_A\,(A/AeA) \leq 1$. We claim that A/AeA has finite global dimension. Let N be an A/AeA-module. Then $\mathrm{Res}(N)$ has finite injective dimension as an A-module, so $\mathbf{R}\mathrm{Coind} \circ \mathrm{Res}(N)$ is quasi-isomorphic to a bounded complex of injectives.

Since (R) is a recollement of derived module categories, Res is fully faithful as a functor of derived categories. Thus $\mathbf{R}\mathrm{Coind} \circ \mathrm{Res}(N)$ is quasi-isomorphic to N, and N has finite injective dimension as an A/AeA-module. Consequently, A/AeA has finite global dimension. Therefore, injectives generate for A/AeA, and projectives cogenerate for A/AeA.

Since $i_* = \mathrm{Res}$ preserves bounded complexes of injectives, Propositions 6.14 and 6.15 apply.

(ii) Similarly, if $\mathrm{pgl}_A(A/AeA) \leq 1$, then A/AeA has finite global dimension by considering $\mathbf{L}\mathrm{Ind} \circ \mathrm{Res}(N)$ for each A/AeA-module N. Moreover, $i_* = \mathrm{Res}$ preserves bounded complexes of projectives. Thus the statements follow from Propositions 6.12 and 6.13. □

Remark 7.14 If A is a finite dimensional algebra over a field k, and $A/\mathrm{rad}(A)$ is separable over k, then Qin [30] shows that the converse of Proposition 7.12 holds and, also that the converse of Lemma 7.13 (i)(b) and (ii)(b) hold.

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