Affinoid Dixmier Modules and the Deformed Dixmier-Moeglin Equivalence

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Abstract

The affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$ of a free, finitely generated $\mathbb{Z}_p$ -Lie algebra $\mathcal{L}$ has proven to be useful within the representation theory of compact p -adic Lie groups, and we aim to further understand its algebraic structure. To this end, we define the notion of a Dixmier module over $\widehat{U(\mathcal{L})}_K$, a generalisation of the Verma module, and we prove that when $\mathcal{L}$ is nilpotent, all primitive ideals of $\widehat{U(\mathcal{L})}_K$ can be described in terms of annihilator ideals of Dixmier modules. Using this, we take steps towards proving that this algebra satisfies a version of the classical Dixmier-Moeglin equivalence.

Keywords Non-commutative algebra · Dixmier-Moeglin equivalence · Lie lattices · Affinoid envelopes

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1 Motivation

Throughout, fix p a prime, $K \setminus \mathbb{Q}_p$ a finite extension, $\mathcal{O}$ the valuation ring of K , $\pi \in \mathcal{O}$ a uniformiser.

1.1 Classical Motivation – The Dixmier-Moeglin Equivalence

This research is partially inspired by a problem in classical non-commutative algebra. Let k be any field of characteristic 0, and let R be a Noetherian k -algebra. We say that a prime ideal P of R is:

- *Primitive* if $P = \text{Ann}_R M = \{r \in R : rM = 0\}$ for some irreducible R -module M .

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- *Weakly rational* if $Z(R/P)$ is an algebraic field extension of k .
- *Rational* if $Z(Q(R/P))$ is an algebraic field extension of k , where $Q(R/P)$ is the Goldie ring of quotients of R/P , in the sense of [16, Theorem 2.3.6].
- *Locally closed* if $P \neq \cap \{Q \trianglelefteq R : Q \text{ prime, } P \subsetneq Q\}$, i.e. if $\{P\}$ is a locally closed subset of $\text{Spec } R$ with respect to the Zariski topology.

It is not difficult to see that if P is rational then it is weakly rational.

Definition 1.1 Let R be a Noetherian k -algebra. We say that R satisfied the *Dixmier-Moeglin equivalence* if for all prime ideals P of R , we have:

P is primitive $\iff P$ is rational $\iff P$ is locally closed.

Note that if R is commutative and k is algebraically closed, then this condition is essentially a strong version of Hilbert's Nullstellensatz, i.e. that all prime ideals P of R arise as an intersection of maximal ideals $\mathfrak{m}$ of R of codimension 1.

This condition is known to be satisfied for numerous examples of Noetherian k -algebras, most notably the enveloping algebra $U(\mathfrak{h})$ for a finite dimensional k -Lie algebra $\mathfrak{h}$ as proved independently by Dixmier and Moeglin in [7] and [15]. An ongoing project is to classify algebras satisfying the equivalence, and in [3], a detailed analysis of what is known to date on this subject is given. In this paper, however, we will not be concerned with the classical picture, but with a p -adic analogue.

Specifically, let K be our p -adic field, and let $\widehat{R}$ be a Noetherian, Banach K -algebra. Unfortunately, to prove the full Dixmier-Moeglin equivalence for such algebras may prove to be impractical, since the techniques classically used typically involve discrete generating sets, which are absent here. However, there is a weaker statement that might be more approachable in this setting.

For each $n \in \mathbb{N}$, in Definition 2.3 below we will define the *level n deformation* $\widehat{R}_n$ of $\widehat{R}$, which is a dense Banach subalgebra of $\widehat{R}$, and using this we make the following definition:

Definition 1.2 We say that the Noetherian, Banach K -algebra $\widehat{R}$ satisfies the *deformed Dixmier-Moeglin equivalence* if for all prime ideals P of R , there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

$P \cap \widehat{R}_n$ is primitive $\iff P \cap \widehat{R}_n$ is weakly rational $\iff P \cap \widehat{R}_n$ is locally closed.

Key example If we take $R = U(\mathfrak{g})$ and let $\mathcal{R} := U(\mathcal{L})$ for some $\mathcal{O}$ -Lie lattice $\mathcal{L}$ in $\mathfrak{g}$, then the π -adic completion $\widehat{R} = \widehat{U(\mathcal{L})}_K$ of R with respect to $\mathcal{R}$ is called the *affinoid enveloping algebra* (or *affinoid envelope*) of $\mathcal{L}$, and $\widehat{R}_n$ is just $\widehat{U(\pi^n \mathcal{L})}_K$.

We believe that the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$ should satisfy the deformed Dixmier-Moeglin equivalence, but to date, this is only known in the case where $\mathfrak{g}$ is nilpotent and contains an abelian ideal of codimension 1, as shown in [13, Corollary 1.6.2].

1.2 Second Motivation – Iwasawa Algebras

The other main motivation for this research lies within representation theory of compact p -adic Lie groups.

Fix G a uniform pro- p group in the sense of [8, Definition 4.1]. When studying the p -adic representation theory of G , there are several avenues we can consider. If we were interested in general, abstract K -linear representations of G , naturally we would study modules over

the standard group algebra $K[G]$, but since G is not discrete, this object is of little practical use. Instead, define the *rational Iwasawa algebra*:

$$KG := \varprojlim_{U \trianglelefteq_o G} \mathcal{O}[G/U] \otimes_{\mathcal{O}} K,$$

this is a Noetherian, topological K -algebra whose module structure completely describes the *continuous* representations of G , a class which includes all finite dimensional, smooth and locally analytic representations.

Let $\mathcal{L} := \frac{1}{p} \log(G)$ be the $\mathbb{Z}_p$ -Lie algebra of G , as defined in [2, Section 10], and let $\widehat{U(\mathcal{L})}_K$ be the *affinoid envelope* of $\mathcal{L}$ with coefficients in K , as defined above.

Using [2, Theorem 10.4], we see that there exists a dense embedding of topological K -algebras $KG \rightarrow \widehat{U(\mathcal{L})}_K$, i.e. $\widehat{U(\mathcal{L})}_K$ is a Banach completion of KG , and thus we see that representations of the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$ naturally have the structure of KG -modules, which allows us to explore the representation theory of G via the representation theory of $\mathcal{L}$.

In this paper, we will study the algebraic structure of the affinoid envelope, and in a forthcoming work [12], we will show how we can use our results to deduce information about primitive ideals in KG .

1.3 Main Results

The key aim of this paper is to describe the primitive ideal structure of the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$, focusing on the case where $\mathcal{L}$ is nilpotent. Specifically, we aim to prove that under suitable conditions this algebra satisfies the deformed Dixmier-Moeglin equivalence.

To this end, we will follow the representation theoretic approach outlined by Jaques Dixmier in [7] when studying the classical enveloping algebra $U(\mathfrak{g})$. This approach was to define a class of irreducible induced representations of $U(\mathfrak{g})$ whose annihilator ideals completely describe the primitive ideals in $U(\mathfrak{g})$. In Section 2, we will adapt this approach to the affinoid envelope.

Specifically, to each linear form $\lambda \in \text{Hom}_{\mathbb{Z}_p}(\mathcal{L}, \mathcal{O})$, we will associate an induced module $\widehat{D(\lambda)}$ over $\widehat{U(\mathcal{L})}_K$, which we call a *Dixmier module*.

We will prove in Section 3 that Dixmier modules are generally irreducible, and thus the ideal $I(\lambda) := \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}$ is primitive in $\widehat{U(\mathcal{L})}_K$. We call ideals of this form *Dixmier annihilators*. Our main result, which we will prove in Section 6, describes all weakly rational ideals of $\widehat{U(\mathcal{L})}_K$ in terms of Dixmier annihilators:

Theorem A *Let $\mathcal{L}$ be an $\mathcal{O}$ -Lie lattice in a nilpotent Lie algebra $\mathfrak{g}$, and let P be a weakly rational ideal in $\widehat{U(\mathcal{L})}_K$. Then there exists $N \in \mathbb{N}$, $\lambda \in \text{Hom}_{\mathcal{O}}(\pi^N \mathcal{L}, \mathcal{O}_{\overline{K}})$ such that for all $n \geq N$:*

$$P \cap \widehat{U(\pi^n \mathcal{L})}_K = I(\lambda) = \text{Ann}_{\widehat{U(\pi^n \mathcal{L})}_K} \widehat{D(\lambda)}$$

Note This makes sense because if λ takes values in the algebraic closure $\overline{K}$ of K , then it must take values in F for some finite extension F of K , and we can consider the action of $\widehat{U(\mathcal{L})}_K$ on $\widehat{D(\lambda)}$ via the embedding of $\widehat{U(\mathcal{L})}_K$ into $\widehat{U(\mathcal{L})}_F$.

This result aims towards a correspondence between orbits of linear forms in $\text{Hom}_{\mathbb{Z}_p}(\mathcal{L}, \mathcal{O}_{\overline{K}})$ and primitive ideals in $\widehat{U(\mathcal{L})}_K$, in line with the classical result of Dixmier [7, Theorem 6.2.4]. In a subsequent paper [12], Theorem A will become a key step in classifying primitive ideals in KG for G nilpotent.

Finally, in Section 7, we will explore the properties of Dixmier modules and Dixmier annihilators further, and prove the deformed Dixmier-Moeglin equivalence in a specific case:

Theorem B *Let $\mathfrak{g}$ be a nilpotent K -Lie algebra such that $[\mathfrak{g}, \mathfrak{g}]$ is abelian, and let $\mathcal{L}$ be an $\mathcal{O}$ -Lie lattice in $\mathfrak{g}$. Then $\widehat{U(\mathcal{L})}_K$ satisfies the deformed Dixmier-Moeglin equivalence. In fact, for all weakly rational ideals P of $\widehat{U(\mathcal{L})}_K$, $n \in \mathbb{N}$ sufficiently high, $\frac{\widehat{U(\pi^n \mathcal{L})}_K}{P \cap \widehat{U(\pi^n \mathcal{L})}_K}$ is a simple domain.*

This result provides an insight into the deeper algebraic structure of the affinoid envelope, and we hope it will be of independent interest within the field of non-commutative algebra, and that it will advance the Dixmier-Moeglin project into the p -adic setting.

2 Preliminaries

In this section, we sometimes want to consider the classical picture. So throughout, take k to be any field of characteristic 0.

2.1 Affinoid Algebras and Rigid Geometry

First, we will recap some notions from rigid geometry. For a more detailed discussion, see [5].

Recall from [5, Definition 2.2.2] that if R is a ring carrying a complete, separated filtration w , the *Tate algebra* in d variables $t_1, \dots, t_d$ over R is the algebra:

$$R\langle t_1, \dots, t_d \rangle := \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_{\alpha} t_1^{\alpha_1} \cdots t_d^{\alpha_d} : w(\lambda_{\alpha}) \rightarrow \infty \text{ as } \alpha \rightarrow \infty \right\}.$$

In other words, the Tate algebra is the ring of power series with coefficients in R that converge on the unit disc R_+^d , we call these *Tate power series*. This ring carries a separated

filtration given by $w_{\inf} \left(\sum_{\alpha \in \mathbb{N}^d} \lambda_{\alpha} t_1^{\alpha_1} \cdots t_d^{\alpha_d} \right) := \inf \{ w(\lambda_{\alpha}) : \alpha \in \mathbb{N}^d \}$.

Normally, R is assumed to be commutative, and in our case, we will usually take $R = K$, in which case the Tate algebra is Noetherian, and the filtration $w_{\inf}$ is *Zariskian* in the sense of [14, Ch. II Definition 2.1.1]. Recall from [5, Definition 3.1.1] that we define an affinoid algebra to be any quotient of a Tate algebra over a complete, discretely valued field. Clearly any affinoid algebra A will carry a complete, Zariskian filtration w_A given by the quotient filtration with respect to $w_{\inf}$.

Affinoid algebras play a similar role in rigid geometry as commutative algebras play in standard algebraic geometry. Specifically, if A is an affinoid algebra, we define $\text{Sp } A$ to be the space of maximal ideals of A . We call this an *affinoid variety*, and we can realise A as a ring of $\overline{K}$ -valued functions on $\text{Sp } A$, where $a(\mathfrak{p}) := a + \mathfrak{p}$. This takes values in $\overline{K}$ since any maximal ideal in the Tate algebra has finite codimension by [5, Corollary 2.2.12].

We say that A is the *ring of analytic functions* on the affinoid variety $\mathrm{Sp} A$. Note that for any ring homomorphism $\phi : A \rightarrow B$ between affinoid algebras induces a map $\phi^\# : \mathrm{Sp} B \rightarrow \mathrm{Sp} A$, $q \mapsto \phi^{-1}(q)$, continuous with respect to the Zariski topology, and we call this a *morphism of affinoid varieties*. Therefore, we can realise affinoid varieties as a category, equivalent to the category of affinoid algebras, via an equivalence where each variety is sent to its ring of analytic functions.

Affinoid varieties are useful in p -adic analysis, since they can indeed be realised as non-archimedean spaces. Recall that for each $\epsilon \in \mathbb{R}$, we define the d -dimensional polydisc of radius ϵ to be the space

$$\mathbb{D}_\epsilon^d := \{\underline{\alpha} \in \overline{K}^d : v_\pi(\alpha_i) \geq \epsilon \text{ for each } i\}.$$

When $\epsilon = 0$ we call this the *unit disc*. We can consider this disc an affinoid space, isomorphic to $\mathrm{Sp} K\langle u_1, \dots, u_d \rangle$, and thus all discs are isomorphic, regardless of the radius.

Note that the Tate algebra $K\langle u_1, \dots, u_d \rangle$ is precisely the set of power series converging on $\mathbb{D}_0^d$, so we can indeed realise the Tate algebra as the ring of analytic functions on the unit disc. Moreover, for each $n \in \mathbb{N}$, the subalgebra $K\langle \pi^n u_1, \dots, \pi^n u_d \rangle$ is precisely those functions which converge on $\mathbb{D}_{-n}^d$.

Now, recall the following definition ([5, Definition 3.3.9]):

Definition 2.1 Let X be an affinoid variety. A subset U of X is called an *affinoid subdomain* if there exists an affinoid variety Y with a morphism $\alpha : X \rightarrow Y$ such that $\alpha(X) \subseteq U$, and the following universal property is satisfied: If $\beta : Z \rightarrow X$ is any morphism of affinoid varieties such that $\beta(Z) \subseteq U$ then there exists a unique morphism $\gamma : Z \rightarrow Y$ such that $\beta = \alpha \circ \gamma$.

Note 1. In the above definition, it follows from [5, Lemma 3.3.10] that the map α is an embedding of affinoid spaces, and its image is equal to U , so we may sometimes identify Y with U .

2. If X is an affinoid variety, then X carries a Grothendieck topology, in the sense of [5, Definition 5.1.1], where the admissible open subsets are the affinoid subdomains U of X .

Roughly speaking, we define a *rigid space* to be a space carrying a Grothendieck topology, that is locally equivalent to an affinoid variety. For example, we can consider the affine plane $\mathbb{A}_d(\overline{K}) = \overline{K}^d$ to be a rigid variety, where its admissible open subsets consist of the unit discs $\mathbb{D}_\epsilon^d$ and their affinoid subdomains. Indeed, for any affine variety X , a similar procedure known as *analytification* allows us to realise X as a rigid variety.

2.2 Lattices and Completions

Recall the following definition [2, Definition 2.7]:

Definition 2.2 Let V be a K -vector space.

- An $\mathcal{O}$ -lattice in V is an $\mathcal{O}$ -submodule N of V such that $\bigcap_{n \in \mathbb{N}} \pi^n N = 0$ and $N \otimes_{\mathcal{O}} K = V$.
- The *completion* of N is the $\mathcal{O}$ -module $\widehat{N} := \varprojlim_{n \in \mathbb{N}} \frac{N}{\pi^n N}$.
- The *completion* of V with respect to N is the K -vector space $\widehat{V} = \widehat{V}_N := \widehat{N} \otimes_{\mathcal{O}} K$.

Note that if V is finite dimensional then $\widehat{V} = V$ for any choice of lattice. More generally, every $\mathcal{O}$ -lattice N in V induces an exhaustive, separated filtration w_N on V given by

$$w_N(u) := \sup\{n \in \mathbb{Z} : u \in \pi^n N\}.$$

This is the π -adic filtration associated to N , and the topological completion of V with respect to w_N is precisely the completion $\widehat{V}_N$.

Now, let R be a Noetherian K -algebra, and let $\mathcal{R}$ be a Noetherian $\mathcal{O}$ -lattice subalgebra of R . Then the completion $\widehat{R}$ of R with respect to $\mathcal{R}$ is a Noetherian, Banach K -algebra.

Definition 2.3 Let $V \subseteq R$ be a K -vector subspace, and suppose that the lattice $M := V \cap \mathcal{R}$ generates $\mathcal{R}$ as an $\mathcal{O}$ -algebra. Then $\mathcal{R}_n := \mathcal{O}\langle \pi^n M \rangle \subseteq \mathcal{R}$ is an $\mathcal{O}$ -subalgebra lattice in R , and we define the *level n deformation* of $\widehat{R}$ to be $\widehat{R}_n$, the completion of R with respect to $\mathcal{R}_n$.

Example If R is the polynomial ring $K[t_1, \dots, t_r]$ and $\mathcal{R} = \mathcal{O}[t_1, \dots, t_r]$, then the completion $\widehat{R}$ of R with respect to $\mathcal{R}$ is the Tate algebra $K\langle t_1, \dots, t_r \rangle$. Moreover, we can take the deformed Tate algebra $K\langle \pi^n t_1, \dots, \pi^n t_r \rangle$ to be the level n deformation of $\widehat{R}$.

Now, let $\mathfrak{g}$ be a finite dimensional K -Lie algebra, and let $\mathcal{L}$ be an $\mathcal{O}$ -Lie lattice in $\mathfrak{g}$, i.e. $\mathcal{L}$ is an $\mathcal{O}$ -lattice in $\mathfrak{g}$ and it is closed under the Lie bracket. Note that the enveloping algebra $U(\mathcal{L})$ is an $\mathcal{O}$ -lattice in $U(\mathfrak{g})$.

Definition 2.4 Define the *affinoid enveloping algebra* of $\mathcal{L}$ with coefficients in $\mathcal{O}$ to be $\widehat{U(\mathcal{L})}$, the completion of $U(\mathcal{L})$ with respect to its π -adic filtration.

Also, define the *affinoid enveloping algebra* of $\mathcal{L}$ with coefficients in K to be

$$\widehat{U(\mathcal{L})}_K := \widehat{U(\mathcal{L})} \otimes_{\mathcal{O}} K.$$

This is the completion of $U(\mathfrak{g})$ with respect to the π -adic filtration associated to $U(\mathcal{L})$.

Note We can take the level n deformation of $\widehat{U(\mathcal{L})}_K$ to be $\widehat{U(\pi^n \mathcal{L})}_K$.

The following lemma is a straightforward consequence of the Poincaré-Birkhoff Witt theorem, see e.g. [13, Proposition 2.5.1] for the proof.

Lemma 2.1 If we let $\{x_1, \dots, x_d\}$ be a K -basis for $\mathfrak{g}$ which forms an $\mathcal{O}$ -basis for $\mathcal{L}$, then $\widehat{U(\mathcal{L})}$ is isomorphic as an $\mathcal{O}$ -module to the Tate algebra $\mathcal{O}\langle x_1, \dots, x_d \rangle$, and hence $\widehat{U(\mathcal{L})}_K$ is isomorphic to $K\langle x_1, \dots, x_d \rangle$ as a K -vector space.

Now, let M be a $\mathfrak{g}$ -representation, i.e. a $U(\mathfrak{g})$ -module, and let N be an $\mathcal{O}$ -lattice in M such that $\mathcal{L} \cdot N \subseteq N$, and suppose that N is π -adically complete. Then it follows that M has the structure of a $\widehat{U(\mathcal{L})}_K$ -module. Unless otherwise stated, we will assume that all modules are left modules.

Proposition 2.2 Let M be a finitely generated $\widehat{U(\mathcal{L})}_K$ -module. Then M contains an $\mathcal{O}$ -lattice N such that M is complete with respect to N and $\mathcal{L} \cdot N \subseteq N$.

Proof $M = \widehat{U(\mathcal{L})}_K m_1 + \dots + \widehat{U(\mathcal{L})}_K m_s$, so let $N := \widehat{U(\mathcal{L})} m_1 + \dots + \widehat{U(\mathcal{L})} m_s$, clearly N is an $\mathcal{O}$ -lattice in M .

Note that since $\widehat{U(\mathcal{L})}$ is π -adically complete and $\text{gr } \widehat{U(\mathcal{L})} \cong \frac{\mathcal{O}}{\pi\mathcal{O}}[t, u_1, \dots, u_d]$ is Noetherian, it follows from [14, Ch.II, Theorem 2.1.2] that the π -adic filtration on $\widehat{U(\mathcal{L})}$ is Zariskian, and hence any left submodule of $\widehat{U(\mathcal{L})}^s$ is closed in $\widehat{U(\mathcal{L})}^s$.

Since $N \cong \frac{\widehat{U(\mathcal{L})}^s}{J}$ for some left submodule J of $\widehat{U(\mathcal{L})}$, it follows that N is π -adically complete. $\square$

2.3 Polarisation and Dixmier Modules

Let $\mathfrak{h}$ be a finite dimensional Lie algebra over k . First recall the following definition [7, 1.12.8]:

Definition 2.5 Given $\lambda \in \mathfrak{h}^*$, define a *polarisation* of $\mathfrak{h}$ at λ to be a solvable subalgebra $\mathfrak{b}$ of $\mathfrak{h}$ such that for all $u \in \mathfrak{h}$, $\lambda([u, \mathfrak{b}]) = 0$ if and only if $u \in \mathfrak{b}$.

In particular, if $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ , then $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$, i.e. λ restricts to a character of $\mathfrak{b}$. Note that polarisations need not always exist.

Lemma 2.3 Given $\lambda \in \mathfrak{h}^*$, let $\mathfrak{h}^\lambda$ be the subalgebra $\{u \in \mathfrak{h} : \lambda([u, \mathfrak{h}]) = 0\}$ of $\mathfrak{g}$. If $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ , then:

- $\mathfrak{h}^\lambda \subseteq \mathfrak{b}$.
- $\dim_F \mathfrak{b} = \frac{1}{2}(\dim_F \mathfrak{h} + \dim_F \mathfrak{h}^\lambda)$.
- If $\mathfrak{b}'$ is a subalgebra of $\mathfrak{h}$ such that $\lambda([\mathfrak{b}', \mathfrak{b}']) = 0$ and $\dim_F \mathfrak{b}' = \frac{1}{2}(\dim_F \mathfrak{h} + \dim_F \mathfrak{h}^\lambda)$, then $\mathfrak{b}'$ is a polarisation of $\mathfrak{h}$ at λ .
- $\mathfrak{b}$ contains every ideal $\mathfrak{a}$ of $\mathfrak{h}$ such that $\lambda([\mathfrak{h}, \mathfrak{a}]) = 0$. In particular, $\mathfrak{b}$ contains $Z(\mathfrak{h})$ and every ideal $\mathfrak{a}$ such that $\lambda(\mathfrak{a}) = 0$.

Proof The first statement follows from the definition of a polarisation, since $\lambda([\mathfrak{b}, \mathfrak{h}^\lambda]) = 0$, while the second and third follow from [7, 1.12.1]. The final statement follows from the first since if $\lambda([\mathfrak{a}, \mathfrak{h}]) = 0$ then $\mathfrak{a} \subseteq \mathfrak{h}^\lambda \subseteq \mathfrak{b}$. $\square$

Examples 1. If λ is a character of $\mathfrak{h}$, i.e. $\lambda([\mathfrak{h}, \mathfrak{h}]) = 0$, then $\mathfrak{h}$ is a polarisation of $\mathfrak{h}$ at λ . In particular, if $\mathfrak{h}$ is abelian, then for any $\lambda \in \mathfrak{h}^*$, $\mathfrak{h}$ is a polarisation of $\mathfrak{h}$ at λ .

2. If $\mathfrak{h} = \mathfrak{a} \rtimes kx$ for some abelian subalgebra $\mathfrak{a}$ of $\mathfrak{h}$, $x \in \mathfrak{h}$, then for any $\lambda \in \mathfrak{h}^*$, if $\lambda([\mathfrak{h}, \mathfrak{h}]) \neq 0$ then $\mathfrak{a}$ is a polarisation of $\mathfrak{h}$ at λ .

3. If k is algebraically closed, and $\mathfrak{h}$ is semisimple, then $\lambda \in \mathfrak{h}^*$ has a polarisation $\mathfrak{b}$ if and only if λ is *regular* in the sense of [7, 1.11.6]. In this case $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{h}$ by [7, Proposition 1.12.18], and hence λ is a character of a Cartan subalgebra.

In our case, we will be interested in the case where $\mathfrak{h}$ is solvable or nilpotent. Recall that $\mathfrak{h}$ is *completely solvable* if there exists a chain of ideals $0 = \mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \dots \subseteq \mathfrak{h}_d = \mathfrak{h}$ such that $\dim_k \mathfrak{h}_i = i$ for each i , e.g. if $\mathfrak{h}$ is nilpotent. Note that for any solvable Lie algebra $\mathfrak{h}$, we can choose a finite extension F/k such that $\mathfrak{h} \otimes_k F$ is completely solvable.

Proposition 2.4 Suppose that $\mathfrak{h}$ is completely solvable. Then given $\lambda \in \mathfrak{h}^*$, and an ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$, there exists a polarisation $\mathfrak{b}$ of $\mathfrak{h}$ at λ such that $\mathfrak{a} \subseteq \mathfrak{b}$.

Proof Choose a chain of ideals $0 = \mathfrak{h}_0 \subseteq \mathfrak{h}_1 \subseteq \cdots \subseteq \mathfrak{h}_d = \mathfrak{h}$ such that $\dim_k \mathfrak{h}_i = i$ for each i , and we may choose this chain such that $\mathfrak{h}_j = \mathfrak{a}$ for some j .

Setting $\lambda_i := \lambda|_{\mathfrak{h}_i}$ for each i , the subalgebra $\mathfrak{b} := \mathfrak{h}_1^{\lambda_1} + \cdots + \mathfrak{h}_d^{\lambda_d}$ is a polarisation of $\mathfrak{h}$ at λ by [7, Proposition 1.12.18]. Moreover, since $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$, it follows that $\mathfrak{h}_j^{\lambda_j} = \mathfrak{a}^{\lambda_j} = \mathfrak{a}$, and hence $\mathfrak{a} \subseteq \mathfrak{b}$ as required. $\square$

In particular, taking $\mathfrak{a} = 0$, this result ensures that polarisations always exist if $\mathfrak{h}$ is completely solvable. In this paper, we mainly focus on the case where $\mathfrak{h}$ is nilpotent.

Now, for any k -Lie algebra $\mathfrak{h}$, $\lambda \in \mathfrak{h}^*$, and any polarisation $\mathfrak{b}$ of $\mathfrak{h}$ at λ , since λ restricts to a character of $\mathfrak{b}$, it follows that $k_\lambda := k\nu$ is a $U(\mathfrak{b})$ -module via the $\mathfrak{b}$ -action $x \cdot v := \lambda(x)v$. This gives us the following definition:

Definition 2.6 Let $\lambda \in \mathfrak{h}^*$, and let $\mathfrak{b}$ be a polarisation of $\mathfrak{h}$ at λ . We define the $\mathfrak{b}$ -Dixmier module of $\mathfrak{h}$ at λ to be the $U(\mathfrak{h})$ -module:

$$D(\lambda) = D(\lambda)_{\mathfrak{b}} := U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} k_\lambda \quad (1)$$

Note that $D(\lambda)$ is a cyclic $U(\mathfrak{h})$ module, generated by a vector v_λ on which $U(\mathfrak{b})$ acts by scalars.

This definition is useful, because in the case where k is algebraically closed and $\mathfrak{h}$ is semisimple, these Dixmier modules are precisely the well-known Verma modules, which are fundamental within the representation theory of semisimple Lie algebras. So we may think of Dixmier modules as a generalisation of Verma modules.

Examples 1. If $\mathfrak{h}$ is semisimple, k is algebraically closed, the Verma module $D(\lambda)$ has a unique simple quotient $L(\lambda)$, known as a *simple highest weight module* with weight λ .

2. If $\mathfrak{h}$ is abelian, or more generally if λ is a character of $\mathfrak{h}$, then $D(\lambda) = k$ always.

3. If $\mathfrak{h} = \mathfrak{a} \rtimes kx$, for $\mathfrak{a}$ abelian, then if λ is not a character of $\mathfrak{h}$, $D(f) \cong k[t]$, where x acts by t , and each $u \in \mathfrak{a}$ acts by a polynomial in $k[\frac{d}{dt}]$.

Theorem 2.5 ([7, Theorem 6.1.1]) *Let $\mathfrak{h}$ be solvable, and let $\lambda \in \mathfrak{h}^*$. Then there exists a polarisation $\mathfrak{b}$ of $\mathfrak{h}$ at λ such that $D(\lambda)_{\mathfrak{b}}$ is an irreducible $U(\mathfrak{h})$ -module.*

The following result yields a complete description of primitive ideals in $U(\mathfrak{h})$:

Theorem 2.6 ([7, Theorem 6.1.7]) *Let $\mathfrak{h}$ be solvable, and let P be a primitive ideal of $\mathfrak{h}$. Then there exists a finite extension F/k , $\lambda \in \mathfrak{h}_F^*$ and a polarisation $\mathfrak{b}$ of $\mathfrak{h}_F$ at λ such that $P = \text{Ann}_{U(\mathfrak{h})} D(\lambda)_{\mathfrak{b}}$.*

Now we will return to the p -adic case, i.e. $K/\mathbb{Q}_p$ is a finite extension, $\mathfrak{g}$ is a K -Lie algebra and $\mathcal{L}$ is an $\mathcal{O}$ -lattice in $\mathfrak{g}$. We want to extend the conclusion of Theorems 2.5 and 2.6 to the affinoid setting.

In this case, when we choose $\lambda \in \mathfrak{g}^*$, we will always assume further that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$, or in other words $\lambda \in \mathcal{L}^* = \text{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$.

Firstly, note that for any polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ , if we set $\mathcal{B} := \mathfrak{b} \cap \mathcal{L}$, then $\mathcal{B}$ is an $\mathcal{O}$ -Lie lattice in $\mathfrak{b}$. Furthermore, if we let $K_\lambda := K\nu$ be the one dimensional $U(\mathfrak{b})$ -module induced by λ , then since $\pi^n U(\mathcal{B})v \subseteq p^n \mathcal{O}v$, it follows that K_λ carries the structure of a $\widehat{U(\mathcal{B})}_K$ -module.

Definition 2.7 Let $\lambda \in \mathfrak{g}^*$ such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$, and let $\mathfrak{b}$ be a polarisation of $\mathfrak{g}$ at λ . Define the $\mathfrak{b}$ -affinoid Dixmier module of $\mathcal{L}$ at λ to be the $\widehat{U(\mathcal{L})}_K$ -module defined by:

$$\widehat{D(\lambda)} = \widehat{D(\lambda)}_{\mathfrak{b}} := \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_{\lambda} \quad (2)$$

Notation If it is unclear what the ground field K is, we may sometimes write $\widehat{D(\lambda)}_K$ for $\widehat{D(\lambda)}$. Also, if it is unclear which lattice $\mathcal{L}$ we are considering, we may sometimes write $\widehat{D(\lambda)}_{\mathcal{B}}$ instead of $\widehat{D(\lambda)}_{\mathfrak{b}}$.

Note that as in the classical case, $\widehat{D(\lambda)}$ is a cyclic $\widehat{U(\mathcal{L})}_K$ -module, so $\widehat{D(\lambda)}_{\mathfrak{b}} = \widehat{U(\mathcal{L})}_K v_{\lambda}$, and $\widehat{U(\mathcal{B})}_K$ acts by scalars on v_{λ} .

In particular, using Proposition 2.2, we see that $\widehat{D(\lambda)}$ is π -adically complete with respect to some lattice. In fact it is a π -adic completion of the classical Dixmier module $D(\lambda)$.

Examples 1. If $\mathfrak{g}$ is split semisimple with Borel subalgebra $\mathfrak{b}$, the *affinoid Verma module* $\widehat{V(\lambda)}$ arises as a Dixmier module. This still has a unique simple quotient $\widehat{L(\lambda)}$.

2. If $\mathfrak{g}$ is abelian, or more generally if λ is a character of $\mathfrak{g}$, then $\widehat{D(\lambda)} = K$ always.

3. If $\mathfrak{g} = \mathfrak{a} \rtimes Kx$, for $\mathfrak{a}$ abelian, then if λ is not a character of $\mathfrak{g}$, $\widehat{D(f)} \cong K\langle t \rangle$, where x acts by t , and each $u \in \mathfrak{a}$ acts by a polynomial in $K[\frac{d}{dt}]$.

2.4 Reducing Quadruples

The following definition ([7, 4.7.7]) will be very useful to us throughout.

Definition 2.8 Let $\mathfrak{h}$ be a k -lie algebra. A *reducing quadruple* of $\mathfrak{h}$ is a 4-tuple $(x, y, z, \mathfrak{h}')$ where:

- $0 \neq x, y, z \in \mathfrak{h}$ and $\mathfrak{h}'$ is an ideal of $\mathfrak{h}$ of codimension 1,
- $y, z \in \mathfrak{h}'$ and $x \notin \mathfrak{h}'$,
- z is central in $\mathfrak{h}$ and y is central in $\mathfrak{h}'$,
- $[x, y] = \alpha z$ for some $0 \neq \alpha \in k$.

The following results link reducing quadruples to polarisations, and they can be found in the proof of [7, Theorem 6.1.1] and [7, Theorem 6.1.4]:

Proposition 2.7 Suppose that $\mathfrak{h}$ is nilpotent, $\dim(\mathfrak{h}) > 1$, and let $\lambda \in \mathfrak{h}^*$. If we assume that $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{h}$, then:

- There exist $x, y, z \in \mathfrak{h}$ and $\mathfrak{h}' \leq \mathfrak{h}$ such that $(x, y, z, \mathfrak{h}')$ forms a reducing quadruple for $\mathfrak{h}$.
- If $\mu := \lambda|_{\mathfrak{h}'}$ and $\mathfrak{b} \subseteq \mathfrak{h}'$ is a polarisation of $\mathfrak{h}'$ at μ , then $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ .

Proof (i) Firstly, for any $0 \neq z$ in the centre of $\mathfrak{h}$, Kz is an ideal of $\mathfrak{h}$, and hence $\lambda(z) \neq 0$.

Therefore, $\lambda : Z(\mathfrak{h}) \rightarrow K$ is injective, meaning that $Z(\mathfrak{h})$ must have dimension 0 or 1.

But since $\mathfrak{h}$ is nilpotent, the centre cannot be 0, hence $Z(\mathfrak{h}) = Kz$ for some $z \in Z(\mathfrak{h})$. Thus note that $\mathfrak{h}$ is non-abelian since $\dim(\mathfrak{h}) > 1 = \dim Z(\mathfrak{h})$.

Also using nilpotency of $\mathfrak{h}$ we may choose $y \in \mathfrak{h}$ such that $y \notin Z(\mathfrak{h})$ and $[y, \mathfrak{h}] \subseteq Z(\mathfrak{h}) = Kz$. Thus the linear map $\text{ad}(y) : \mathfrak{h} \rightarrow \mathfrak{h}$ has rank 1, and hence it must have kernel of dimension $\dim(\mathfrak{h}) - 1$. Let $\mathfrak{h}' := \ker(\text{ad}(y))$, and it follows that $\mathfrak{h}'$ is an abelian ideal in $\mathfrak{h}$ of codimension 1.

So, $\mathfrak{h} = \mathfrak{h}' \oplus Kx$ for some $x \in \mathfrak{h}$ such that $[x, y] \neq 0$, and hence $[x, y] = \alpha z$ for some $\alpha \in K$ with $\alpha \neq 0$. Hence $(x, y, z, \mathfrak{h}')$ is a reducing quadruple as required.

(ii) Since $\mathfrak{b}$ is a polarisation of $\mathfrak{h}'$ at μ , it must contain $Z(\mathfrak{h}')$ by Lemma 2.3, hence $y, z \in \mathfrak{b}$.

Clearly $\mathfrak{b} \subseteq \mathfrak{h}'$ is a solvable subalgebra of $\mathfrak{h}$, so suppose that $\mathfrak{b} \subseteq V \subseteq \mathfrak{h}$ with $\lambda([V, V]) = 0$. We will prove that $V \subseteq \mathfrak{h}'$, and it will follow that $V = \mathfrak{b}$ by the definition of a polarisation, thus proving that $\mathfrak{b}$ is a polarisation of $\mathfrak{h}$ at λ .

Suppose that $\beta x + u \in V$ for some $u \in \mathfrak{h}'$, $\beta \in K$. Then since $y \in \mathfrak{b} \subseteq V$, it follows that $\lambda([\beta x + u, y]) = 0$, and hence $\beta \alpha \lambda(z) = 0$. But since $\alpha, \lambda(z) \neq 0$, it follows that $\beta = 0$ and hence $V \subseteq \mathfrak{h}'$ as required. $\square$

Proposition 2.8 *Suppose that $\mathfrak{h}$ is nilpotent with $n := \dim_F \mathfrak{h} > 3$, and suppose that $\mathfrak{h}$ has a reducing quadruple $(x, y, z, \mathfrak{h}')$. We assume further that $\lambda \in \mathfrak{h}^*$ and $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{h}$. Then for any polarisation $\mathfrak{b}$ of $\mathfrak{h}$ at λ , there exists a polarisation $\mathfrak{b}'$ at λ , contained in $\mathfrak{h}'$, and a proper subalgebra $\mathfrak{t} \subsetneq \mathfrak{h}$ such that $\mathfrak{b}, \mathfrak{b}' \subseteq \mathfrak{t}$.*

Proof Firstly, note that $\mathfrak{h}^\lambda = \{u \in \mathfrak{h} : \lambda([u, \mathfrak{h}]) = 0\}$ is contained in $\mathfrak{b}$, otherwise $\mathfrak{b} \subsetneq \mathfrak{b} + \mathfrak{h}^\lambda$ and $\lambda([\mathfrak{b} + \mathfrak{h}^\lambda, \mathfrak{b} + \mathfrak{h}^\lambda]) = 0$, contradicting the definition of a polarisation.

If $\mathfrak{b} \subseteq \mathfrak{h}'$, then taking $\mathfrak{b}' = \mathfrak{b}$, $\mathfrak{t} = \mathfrak{h}'$, the statement is trivially true, so we may assume that $\mathfrak{b} \not\subseteq \mathfrak{h}'$.

Since y is central in $\mathfrak{h}'$ but not in $\mathfrak{h}$, it is clear that $\mathfrak{h}' = \ker(\text{ad}(y))$ and $Fz = \text{im}(\text{ad}(y))$. So since there exists $u \in \mathfrak{b} \setminus \mathfrak{h}'$, we have that $[u, y] \neq 0$, i.e. $[u, y] = \beta z$ for some $0 \neq \beta \in F$. But since Fz is a non-zero ideal of $\mathfrak{h}$, $\lambda(z) \neq 0$, and thus $\lambda([u, y]) \neq 0$. So since $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$ and $u \in \mathfrak{b}$, this means that $y \notin \mathfrak{b}$.

Let $\mathfrak{b}' := (\mathfrak{b} \cap \mathfrak{h}') \oplus Fy$. This is a subalgebra of $\mathfrak{h}$, and clearly it is contained in $\mathfrak{h}'$. Also, $\mathfrak{b} \cap \mathfrak{h}'$ has codimension 1 in $\mathfrak{b}$, therefore $\dim_F \mathfrak{b}' = \dim_F \mathfrak{b} \cap \mathfrak{h}' + 1 = \dim_F \mathfrak{b}$. But it is clear that $\lambda([\mathfrak{b}', \mathfrak{b}']) = 0$, so by Lemma 2.3, this means that $\mathfrak{b}'$ is a polarisation of $\mathfrak{h}$ at λ .

Now, let $\mathfrak{t} := \mathfrak{b} \oplus Fy$. Since $Fz = [y, \mathfrak{h}] \subseteq \mathfrak{b}$, this is a subalgebra of $\mathfrak{h}$, and clearly it contains $\mathfrak{b}$ and $\mathfrak{b}'$, so we only need to prove that $\mathfrak{t} \neq \mathfrak{h}$, so assume for contradiction that $\mathfrak{t} = \mathfrak{h}$. This means that $\mathfrak{b}$ has codimension 1 in $\mathfrak{h}$, so $\dim_F \mathfrak{b} = n - 1$. But $\dim_F \mathfrak{b} = \frac{1}{2}(n + \dim_F \mathfrak{h}^\lambda)$ by Lemma 2.3, and thus $\dim_F \mathfrak{h}^\lambda = n - 2$.

Furthermore, if $\beta x + \gamma y \in \mathfrak{h}^\lambda$, then $\beta \lambda([x, y]) = \gamma \lambda([y, x]) = 0$, which is only possible if $\beta = \gamma = 0$ since $\lambda(z) = \lambda(\alpha^{-1}[x, y]) \neq 0$. So $\text{Span}_F \{x, y\} \cap \mathfrak{h}^\lambda = 0$, and therefore, $\mathfrak{h} = Fx \oplus Fy \oplus \mathfrak{h}^\lambda$.

Let $\mathfrak{a} := \ker(\lambda) \cap \mathfrak{h}^\lambda$. Then since $z \in \mathfrak{h}^\lambda$ and $\lambda(z) \neq 0$, it follows that $\mathfrak{a}$ has codimension 1 in $\mathfrak{h}^\lambda$, which means that $\dim_F \mathfrak{a} = n - 3$.

It is clear that $\lambda(\mathfrak{a}) = 0$, so we will finish by proving that $\mathfrak{a}$ is an ideal of $\mathfrak{h}$, and this will imply that $\mathfrak{a} = 0$, and hence $n - 3 = 0$ and $n = \dim_F \mathfrak{h} = 3$ – contradicting our assumption.

By the definition of $\mathfrak{h}^\lambda$, it is clear that $\lambda([\mathfrak{h}^\lambda, \mathfrak{h}]) = 0$ and so $[\mathfrak{h}^\lambda, \mathfrak{a}] \subseteq \mathfrak{h}^\lambda \cap \ker(\lambda) = \mathfrak{a}$. So since $\mathfrak{h} = Fx \oplus Fy \oplus \mathfrak{h}^\lambda$, it remains to prove that $[y, \mathfrak{a}] \subseteq \mathfrak{a}$ and $[x, \mathfrak{a}] \subseteq \mathfrak{a}$.

Since $\mathfrak{a} \subseteq \mathfrak{h}'$, we have that $[y, \mathfrak{a}] = 0 \subseteq \mathfrak{a}$, and if we choose $u \in \mathfrak{b}$ such that $u \notin \mathfrak{h}'$, then $\mathfrak{b} = \mathfrak{h}^\lambda \oplus Fu$, so since $\mathfrak{h}$ is nilpotent and $\mathfrak{b}$ is a subalgebra, it follows that $[u, \mathfrak{h}^\lambda] \subseteq \mathfrak{h}^\lambda$, and hence $[u, \mathfrak{a}] \subseteq \mathfrak{h}^\lambda$. Also, since $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$, it follows that $[u, \mathfrak{a}] \subseteq \ker(\lambda)$, hence $[u, \mathfrak{a}] \subseteq \mathfrak{a}$.

But since $u \notin \mathfrak{h}'$, we have that $u = \beta x + \gamma y$, where $\beta \neq 0$, so it follows immediately that $[x, \mathfrak{a}] \subseteq \mathfrak{a}$ as required. $\square$

Reducing quadruples will play an important role in many of the proofs in this paper, since they allow us to use an inductive strategy commonly employed by Dixmier in [7], which we outline below, very roughly:

Dixmier's Induction Strategy We have a statement $\mathcal{P}$ involving a nilpotent k -Lie algebra $\mathfrak{h}$ and a linear form $\lambda : \mathfrak{h} \rightarrow k$.

- Step 1: The base case is where $\dim_k \mathfrak{h} = 1$, this case should be straightforward. So we can assume that $\dim_k \mathfrak{h} > 1$ and $\mathcal{P}$ is true for all nilpotent Lie algebras of dimension less than $\dim_k \mathfrak{h}$.
- Step 2: If there exists a non-zero ideal $\mathfrak{a}$ of $\mathfrak{h}$ such that $\lambda(\mathfrak{a}) = 0$, then we can replace $\mathfrak{h}$ by $\frac{\mathfrak{h}}{\mathfrak{a}}$ and apply induction. So we may assume that $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$.
- Step 3: Applying Proposition 2.7, we can choose a reducing quadruple $(x, y, z, \mathfrak{h}')$ of $\mathfrak{h}$. Since we know that $\mathcal{P}$ holds for $\mathfrak{h}'$ and $\lambda|_{\mathfrak{h}'}$ by induction, we can hopefully induce up to $\mathfrak{h}$.

2.5 Tate-Weyl Algebras

Let us first recall the definition of the Weyl algebra, given in [16, Section 1.3]:

Definition 2.9 Let $d \in \mathbb{N}$. We define the d 'th Weyl algebra $A_d(k)$ to be the k -algebra in $2d$ -variables $x_1, \dots, x_d, y_1, \dots, y_d$ satisfying the following relations for $1 \leq i, j \leq d$, $i \neq j$:

$$x_i x_j - x_j x_i = 0, y_i y_j - y_j y_i = 0, x_i y_j - y_j x_i = 0, x_i y_i - y_i x_i = 1.$$

Lemma 2.9 $A_d(k)$ is isomorphic as a k -vector space to the polynomial ring $k[x_1, \dots, x_d, \partial_1, \dots, \partial_d]$, subject to the relations that $x_1, \dots, x_d$ commute, $\partial_1, \dots, \partial_d$ commute, $x_i \partial_j = \partial_j x_i$ if $i \neq j$ and $x_i \partial_i = \partial_i x_i + 1$.

Note It is well known and quite straightforward to prove that $A_n(k)$ is a simple k -algebra domain.

The Weyl algebra is of great importance within algebraic and differential geometry, since it can be realised as the ring of differential operators on the affine plane $\mathbb{A}_k^d$. The following result highlights the usefulness of this object within representation theory, since it can be explicitly realised as a ring of endomorphisms:

Lemma 2.10 Let $k[t_1, \dots, t_d]$ be a polynomial ring in d -variables, and for each $i = 1, \dots, d$, let $x_i \in \text{End}_k k[t_1, \dots, t_d]$ be left multiplication by t_i , and let $\partial_i := \frac{d}{dt_i} \in \text{End}_k k[t_1, \dots, t_d]$. Then the natural map $A_d(k) \rightarrow \text{End}_k k[t_1, \dots, t_d]$ sending x_i to x_i and ∂_i to ∂_i is an injective ring homomorphism.

Now, if we fix $\mathfrak{h}$ a nilpotent k -Lie algebra, the following result of Dixmier ([7, Theorem 4.7.9]) shows us that all weakly rational ideals in $U(\mathfrak{h})$ are maximal, which completes the Dixmier-Moeglin equivalence in the nilpotent case:

Theorem 2.11 *Let I be a two-sided ideal of $U(\mathfrak{h})$ such that $Z\left(\frac{U(\mathfrak{h})}{I}\right) = k$, then there exists $d \in \mathbb{N}$ such that $\frac{U(\mathfrak{h})}{I} \cong A_d(k)$. The integer d is called the weight of I .*

The proof of this Theorem is inductive, not constructive, so we cannot usually explicitly write down this isomorphism. However, using Theorem 2.6, we know that all weakly rational ideals of $U(\mathfrak{h})$ arise as annihilators of Dixmier modules, and using this notion there are cases where we can construct an isomorphism.

Specifically, if $I = \text{Ann}_{U(\mathfrak{h})} D(\lambda)$ for some linear form λ of $\mathfrak{h}$, with polarisation $\mathfrak{b}$ of codimension d , then we know that $\frac{U(\mathfrak{h})}{I}$ is a ring of k -linear endomorphisms of $D(\lambda) \cong k[u_1, \dots, u_d]$. Since we know from Lemma 2.10 that $A_d(k)$ can be realised as a ring of endomorphisms of $k[u_1, \dots, u_d]$, we want to show that these two rings can be identified.

Roughly speaking, we want to show that elements in the polarisation $\mathfrak{b}$ act by polynomials in $\partial_1, \dots, \partial_d$, and that each basis vector u_i of $\mathfrak{g}/\mathfrak{b}$ acts by x_i . In the next section, we will see how to do this explicitly.

Now, we want to try and generalise Theorem 2.11 to the affinoid case, so we need to define an affinoid version of the Weyl algebra:

Definition 2.10 Let $A_d(\mathcal{O})$ be the $\mathcal{O}$ -lattice subalgebra of $A_d(K)$ consisting of polynomials in $x_1, \dots, x_d, \partial_1, \dots, \partial_d$ with coefficients in $\mathcal{O}$. We define the d 'th Tate-Weyl algebra, denoted $\widehat{A_d(K)}$ to be the completion of $A_d(K)$ with respect to $A_d(\mathcal{O})$

Similarly to $A_d(K)$, the Tate-Weyl algebra $\widehat{A_d(K)}$ is also a simple domain.

Lemma 2.12 *The d 'th Tate-Weyl algebra $\widehat{A_d(K)}$ is isomorphic as a K -vector space to $K\langle x_1, \dots, x_d, \partial_1, \dots, \partial_d \rangle$, where $x_1, \dots, x_d$ commute, $\partial_1, \dots, \partial_d$ commute, $x_i \partial_j = \partial_j x_i$ if $i \neq j$ and $x_i \partial_i = \partial_i x_i + 1$.*

Moreover, the level n deformation $\widehat{A_d(K)}_n$ of $\widehat{A_d(K)}$ is the subalgebra consisting the elements of $K\langle \pi^n x_1, \dots, \pi^n x_d, \pi^n \partial_1, \dots, \pi^n \partial_d \rangle$.

Using this definition, we have the following affinoid version of Lemma 2.10:

Lemma 2.13 *Let $K\langle t_1, \dots, t_d \rangle$ be a Tate algebra in d -variables, and for each $i = 1, \dots, d$, let $x_i \in \text{End}_K K\langle t_1, \dots, t_d \rangle$ be left multiplication by t_i , and let $\partial_i := \frac{d}{dt_i} \in \text{End}_K K\langle t_1, \dots, t_d \rangle$. Note that if $\lambda_\alpha \in K$ for each $\alpha \in \mathbb{N}^{2d}$, and $\lambda_\alpha \rightarrow 0$ as $|\alpha| \rightarrow \infty$, then the series $\sum_{\alpha \in \mathbb{N}^{2d}} \lambda_\alpha x_1^{\alpha_1} \dots x_d^{\alpha_d} \partial_1^{\alpha_{d+1}} \dots \partial_d^{\alpha_{2d}}$ is a well defined K -linear endomorphism of $K\langle t_1, \dots, t_d \rangle$.*

Then the natural map $A_d(k) \rightarrow \text{End}_K K\langle t_1, \dots, t_d \rangle$ sending x_i to x_i and ∂_i to ∂_i is an injective ring homomorphism.

In a similar vein to Theorem 2.11, we would like to prove that all weakly rational quotients of the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$ are isomorphic to Tate-Weyl algebras. However, it was shown in [13] that this need not always be the case. In Section 7 we will explore how to prove a related, weaker statement.

2.6 The Coadjoint Action

Now we will recall some Lie theory. Assume $\mathfrak{h}$ is nilpotent Lie algebra, and note that for every $u \in \mathfrak{h}$, the map $\text{ad}(u)$ is a nilpotent derivation of $\mathfrak{h}$. So we can define:

$$\exp(\text{ad}(u)) := \sum_{n \geq 0} \frac{1}{n!} \text{ad}(u)^n : \mathfrak{h} \rightarrow \mathfrak{h} \quad (3)$$

Since $\text{ad}(u)$ is a derivation, it follows that $\exp(\text{ad}(u))$ is a Lie-automorphism of $\mathfrak{h}$.

Definition 2.11 Define the *adjoint algebraic group* of $\mathfrak{h}$ to be $\mathbb{H}(\mathfrak{h}) := \{\exp(\text{ad}(u)) \in \text{Aut}(\mathfrak{h}) : u \in \mathfrak{h}\}$

Then $\mathbb{H}(\mathfrak{h})$ is a subgroup of $\text{Aut}(\mathfrak{h})$, and if we define the functor $\mathbb{H} : k\text{-Alg} \rightarrow \text{Grp}, R \mapsto \mathbb{H}(\mathfrak{h} \otimes_k R)$, then $\mathbb{H}$ is an affine algebraic group in the sense of [10, Definition I.2.1], and it is unipotent.

Note If we view the space $\text{ad}(\mathfrak{h}) \subseteq \text{End}_k(\mathfrak{h})$ as an affine variety over k , then the map $\exp : \text{ad}(\mathfrak{h}) \rightarrow \mathbb{H}$ is an isomorphism of varieties, with inverse $\log$.

Now, let $\mathfrak{h}^* : k\text{-Alg} \rightarrow \text{Set}$ be the linear dual of $\mathfrak{h}$, i.e. $\mathfrak{h}^*(R) = \text{Hom}_R(\mathfrak{h} \otimes_k R, R) \cong \mathfrak{h}^*(k) \otimes_k R$. Then $\mathfrak{h}^*$ is an affine scheme in the sense of [10, Definition I.1.3].

Definition 2.12 Define an action of $\mathbb{H}$ on $\mathfrak{g}^*$, i.e. a morphism of varieties $\mathbb{H} \times \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, by $(g \cdot f)(u) = f(g^{-1}u)$. This is the *coadjoint action*, and the orbits of this action in $\mathfrak{h}^*$ are called *coadjoint orbits*.

Given $\lambda \in \mathfrak{h}^*(k)$, let X be the coadjoint orbit of λ in $\mathfrak{h}^*$, and let S be the stabiliser of λ in $\mathbb{H}$, i.e. $S(R) := \{g \in \mathbb{H}(R) : g \cdot \lambda = \lambda\}$, an affine algebraic subgroup of $\mathbb{H}$.

Lemma 2.14 *There exists an isomorphism of varieties $\mathbb{H} \cong S \times X$ such that the map $\mathbb{H} \rightarrow X, g \mapsto g \cdot \lambda$ is just the natural projection $S \times X \rightarrow X$.*

Proof Since $\mathbb{H}$ is affine and S is closed in $\mathbb{H}$, we see using [4, Theorem II.6.8] that the quotient variety $\mathbb{H}/S$ exists. Since the orbit map $\mathbb{H} \rightarrow X$ is surjective and k has characteristic 0, it follows that this map is *separable* in the sense of [4, I.8.2], and hence using [4, Theorem I.17.3] and [4, Proposition II.6.7] it follows that $X = \mathbb{H}/S$ and that the map $\mathbb{H} \rightarrow X$ is the quotient map. Since X is closed in $\mathfrak{h}^*$, it follows that $\mathbb{H}/S$ is affine.

Now, using [10, I.5.6(1)] we see that $\mathbb{H} \times S \cong \mathbb{H} \times_{\mathbb{H}/S} \mathbb{H}$ as varieties, and using this isomorphism, it follows that the natural map $\mathbb{H} \rightarrow \mathbb{H}/S$ is an S -torsor in the sense of [6, Ch.III Definition 4.1.3].

Therefore, since S is unipotent and $\mathbb{H}/S$ is affine, it follows from [6, Ch.IV Proposition 3.7(b)] that the torsor $\mathbb{H} \rightarrow \mathbb{H}/S$ is trivial, i.e. $\mathbb{H} \cong S \times \mathbb{H}/S = S \times X$ as varieties and the map $\mathbb{H} \rightarrow X$ is just the projection to the second factor. $\square$

3 The Action of $\widehat{U(\mathcal{L})}_K$ on $\widehat{D(\lambda)}$

In this section, we will prove our affinoid version of Theorem 2.5, at least in the case where $\mathfrak{g}$ is nilpotent. Throughout, assume that $\mathfrak{g}$ is a finite dimensional K -Lie algebra, with $\mathcal{O}$ -Lie lattice $\mathcal{L}$.

3.1 Induced Modules

Since the affinoid Dixmier module $\widehat{D(\lambda)}$ is an induced $\widehat{U(\mathcal{L})}_K$ -module, we will first explore some general properties of induced modules.

Lemma 3.1 *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ such that $\mathfrak{a} \subseteq \mathfrak{h}$. Let $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{a}$, $\mathfrak{h}_1 := \mathfrak{h}/\mathfrak{a}$. Also, set $\mathcal{H} := \mathcal{L} \cap \mathfrak{h}$, $\mathcal{A} := \mathcal{L} \cap \mathfrak{a}$, $\mathcal{L}_1 := \mathcal{L}/\mathcal{A}$, $\mathcal{H}_1 := \mathcal{H}/\mathcal{A}$, which are Lie lattices in $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{g}_1$ and $\mathfrak{h}_1$ respectively. Then:*

- (i) *There is a continuous surjection of K -algebras $\widehat{U(\mathcal{L})}_K \twoheadrightarrow \widehat{U(\mathcal{L}_1)}_K$ induced by the surjection $\mathcal{L} \twoheadrightarrow \mathcal{L}_1$. The kernel of this surjection is $\mathfrak{a}\widehat{U(\mathcal{L})}_K$*
- (ii) *If M is a finitely generated $\widehat{U(\mathcal{H}_1)}_K$ -module, then M has the structure of a $\widehat{U(\mathcal{H})}_K$ -module via the surjection in (i), and $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \cong \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{H}_1)}_K} M$ as $\widehat{U(\mathcal{L})}_K$ -modules.*

Proof (i) It is clear that the surjection $\mathcal{L} \twoheadrightarrow \mathcal{L}_1$ induces a surjection $U(\mathcal{L}) \twoheadrightarrow U(\mathcal{L}_1)$ sending $\pi^n U(\mathcal{L})$ to $\pi^n U(\mathcal{L}_1)$, so this yields a continuous map $\widehat{U(\mathcal{L})} \rightarrow \widehat{U(\mathcal{L}_1)}$.

If we fix a basis $\{x_1, \dots, x_d\}$ for $\mathcal{L}$ such that $\{x_{r+1}, \dots, x_d\}$ is a basis for $\mathcal{A}$, then using Lemma 2.1, we see that every element of $\widehat{U(\mathcal{L}_1)}_K$ has the form $\sum_{\alpha \in \mathbb{N}^r} \lambda_\alpha (x_1 + \mathcal{A})^{\alpha_1} \cdots (x_r + \mathcal{A})^{\alpha_r}$, where $\lambda_\alpha \in \mathcal{O} \rightarrow 0$ as $\alpha \rightarrow \infty$. Clearly under the map $\widehat{U(\mathcal{L})} \rightarrow \widehat{U(\mathcal{L}_1)}$, x_i maps to $x_i + \mathcal{A}$ for each i , and hence the map is surjective.

Moreover, we can write any element of $\widehat{U(\mathcal{L})}$ as $\sum_{\alpha \in \mathbb{N}^r} x_1^{\alpha_1} \cdots x_r^{\alpha_r} c_\alpha$ for some $c_\alpha \in \widehat{U(\mathcal{A})}$ converging to zero, and this maps to 0 if and only if c_α maps to 0 for each α . But each c_α has the form $\sum_{\beta \in \mathbb{N}^{d-r}} \mu_\beta x_{r+1}^{\beta_{r+1}} \cdots x_d^{\beta_d}$, and this maps to zero if and only if $\mu_0 = 0$, i.e. $c_\alpha \in \widehat{U(\mathcal{A})}\mathcal{A}$. Hence the kernel of the surjection is $\widehat{U(\mathcal{L})}\mathcal{A}$ and part (i) follows.

(ii) Let $\phi : \widehat{U(\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L}_1)}_K$ be the surjection from part (i), and define a map:

$$\Theta : \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \rightarrow \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{H}_1)}_K} M, r \otimes m \mapsto \phi(r) \otimes m.$$

It is clear that this is a well defined map of $\widehat{U(\mathcal{L})}_K$ -modules, we want to prove that it is an isomorphism.

Every element $s \in \widehat{U(\mathcal{L}_1)}_K$ can be written uniquely in the form

$$s = \sum_{\alpha \in \mathbb{N}^r} \lambda_\alpha (x_1 + \mathcal{A})^{\alpha_1} \cdots (x_r + \mathcal{A})^{\alpha_r} = \phi \left(\sum_{\alpha \in \mathbb{N}^r} \lambda_\alpha x_1^{\alpha_1} \cdots x_r^{\alpha_r} \right),$$

so there is a unique element in $K\langle x_1, \dots, x_r \rangle$ that maps to s under ϕ . We call this element $\phi^{-1}(s)$, and it is clear that this defines a K -linear map $\phi^{-1} : \widehat{U(\mathcal{L}_1)} \rightarrow K\langle x_1, \dots, x_r \rangle$.

Therefore, we can define a K -linear map $\Psi : \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{H}_1)}_K} M \rightarrow \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ sending $s \otimes m$ to $\phi^{-1}(s) \otimes m$. We can show that this is well defined by choosing an appropriate basis for $\mathcal{H}_1$ that extends to a basis for $\mathcal{L}_1$, and clearly it is a right inverse to Θ .

Using the fact that $\widehat{U(\mathcal{L})}_K$ is isomorphic as a K -vector space to $K\langle x_1, \dots, x_r \rangle \langle x_{r+1}, \dots, x_d \rangle$, and $x_{r+1}, \dots, x_d \in \mathcal{A} \subseteq \mathcal{H}$, we see that every simple tensor $s \otimes m \in \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ can be written as an infinite sum of simple tensors $s_n \otimes m_n$ converging to zero as $n \rightarrow \infty$, with $s_n \in K\langle x_1, \dots, x_r \rangle$. We know this sum converges by Proposition 2.2.

Therefore, for any $s \in \widehat{U(\mathcal{L})}_K$, $m \in M$, $\Psi\Theta(s \otimes m) = \sum_{n \in \mathbb{N}} \Psi\Theta(s_n \otimes m_n) = \sum_{n \in \mathbb{N}} \Psi(\phi(s_n) \otimes m_n)$, and since $s_n \in K\langle x_1, \dots, x_s \rangle$ for each n , $\phi^{-1}(\phi(s_n)) = s_n$, and hence $\Psi\Theta(s \otimes m) = s \otimes m$. Thus Ψ and Θ are mutually inverse bijections. $\square$

Now, recall from [1] that if A is a Banach K -algebra, and M is a left A -module, π -adically complete with respect to some lattice $N \subseteq M$, then we may define the *Tate module*:

$$M\langle t_1, \dots, t_d \rangle := \left\{ \sum_{\alpha \in \mathbb{N}^d} t_1^{\alpha_1} \cdots t_d^{\alpha_d} s_\alpha : s_\alpha \in M, s_\alpha \rightarrow 0 \text{ as } |\alpha| \rightarrow \infty \right\}.$$

Note that we don't necessarily give $M\langle t_1, \dots, t_d \rangle$ the structure of an A -module, a priori it is just a K -vector space.

Proposition 3.2 *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, and let $\mathcal{H} := \mathfrak{h} \cap \mathcal{L}$, so $\mathcal{H}$ is a Lie lattice in $\mathfrak{h}$. Suppose that M is a finitely generated $\widehat{U(\mathcal{H})}_K$ -module. Then if $r = \dim_K \mathfrak{g}/\mathfrak{h}$, there is an isomorphism of K -vector spaces $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \cong M\langle t_1, \dots, t_r \rangle$, where $t_i v$ corresponds to $x_i \otimes v$ for some $\mathcal{O}$ -basis $\{x_1, \dots, x_r\}$ for $\mathcal{L}/\mathcal{H}$. Thus $M\langle t_1, \dots, t_r \rangle$ carries the structure of a $\widehat{U(\mathcal{L})}_K$ module.*

Moreover, if $r = 1$, so $\mathcal{L} = \mathcal{H} \oplus \mathcal{O}x$ for some $x \in \mathcal{L}$, then we can choose this isomorphism $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \cong M\langle t \rangle$ such that:

- (i) x acts by t on $M\langle t \rangle$.
- (ii) If $y, z \in \mathcal{H}$ act on M by scalars $\beta_y, \beta_z \in \mathcal{O}$, $[x, z] = 0$ and $[y, x] = \alpha z$ for some $\alpha \in K$, then y acts on $M\langle t \rangle$ by $\alpha\beta_z \frac{d}{dt} + \beta_y$.
- (iii) If $\alpha, \beta_z \neq 0$ and M is irreducible over $\widehat{U(\mathcal{H})}_K$, then $M\langle t \rangle$ is irreducible over $\widehat{U(\mathcal{L})}_K$.

Proof Let $\{x_1, \dots, x_d\}$ be an $\mathcal{O}$ -basis for $\mathcal{L}$ such that $\{x_{r+1}, \dots, x_d\}$ is a basis for $\mathcal{H}$. Then by Lemma 2.1, writing $\underline{x}^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, we have: $\widehat{U(\mathcal{L})}_K = \{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \underline{x}^\alpha : \lambda_\alpha \in \mathcal{O}, \lambda_\alpha \rightarrow 0 \text{ as } |\alpha| \rightarrow \infty \}$.

Define a map:

$$\Theta : \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M \rightarrow M\langle t_1, \dots, t_r \rangle, \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \underline{x}^\alpha \otimes v \mapsto \sum_{\beta \in \mathbb{N}^r} t_1^{\beta_1} \cdots t_r^{\beta_r} \left(\sum_{\gamma \in \mathbb{N}^{d-r}} \lambda_{(\beta, \gamma)} \underline{x}^\gamma v \right).$$

Note that here (β, γ) refers to the d -tuple whose first r terms are the terms of β , and the last $d - r$ terms are the terms of γ . It is straightforward but technical to show that this is a well defined K -linear map, so we need to prove that it is an isomorphism.

Firstly, $M = \widehat{U(\mathcal{H})}_K v_1 + \cdots + \widehat{U(\mathcal{H})}_K v_t$, so any element of $M\langle t_1, \dots, t_r \rangle$ will have the form $\sum_{\beta \in \mathbb{N}^r} t_1^{\beta_1} \cdots t_r^{\beta_r} (a_{1,\beta} v_1 + \cdots + a_{t,\beta} v_t)$ for some $a_{i,\beta} \in \widehat{U(\mathcal{H})}_K$. This is the image of $\sum_{\beta \in \mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} a_{1,\beta} \otimes v_1 + \cdots + \sum_{\beta \in \mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} a_{t,\beta} \otimes v_t$, so Θ is surjective.

Furthermore, if $\sum_{\beta \in \mathbb{N}^r} t_1^{\beta_1} \cdots t_r^{\beta_r} (a_{1,\beta} v_1 + \cdots + a_{t,\beta} v_t) = 0$ then $a_{1,\beta} v_1 + \cdots + a_{t,\beta} v_t = 0$ for all β . Since $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ is finitely generated, it is complete by Proposition 2.2, thus

$$\begin{aligned} & \sum_{\beta \in \mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} a_{1,\beta} \otimes v_1 + \cdots + \sum_{\beta \in \mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} a_{t,\beta} \otimes v_t \\ &= \sum_{\beta \in \mathbb{N}^r} x_1^{\beta_1} \cdots x_r^{\beta_r} \otimes (a_{1,\beta} v_1 + \cdots + a_{t,\beta} v_t) = 0, \end{aligned} \quad (4)$$

hence Θ is injective.

Hence Θ is an isomorphism of K -vector spaces, and $\Theta(x_i \otimes v) = t_i v$ for all $i \leq r$, $v \in M$. So clearly we can define an action of $\widehat{U(\mathcal{L})}_K$ on $M\langle t_1, \dots, t_r \rangle$ making Θ into an isomorphism of $\widehat{U(\mathcal{L})}_K$ -modules.

- (i) Since $\Theta(x^n v) = t^n v$ for all $v \in M$, $n \in \mathbb{N}$, it is clear that the action of x on $M\langle t \rangle$ is given by multiplication by t .
- (ii) Since y and z act by scalars on M , their action on $M\langle t \rangle$ is determined entirely by their action on the powers of t .

Since $[x, z] = 0$, it follows that z commutes with all powers of x , and hence $z \cdot t^n v = z\Theta(x^n \otimes v) = \Theta(x^n \otimes z \cdot v) = \Theta(\beta_z(x^n \otimes v)) = \beta_z t^n v$, so z acts on $M\langle t \rangle$ via β_z .

Clearly $y \cdot v = \beta_y v$, so we will assume that for some $n \geq 0$, $y \cdot t^n v = n\alpha\beta_z t^{n-1} v + \beta_y t^n v$ and show that $y \cdot t^{n+1} v = (n+1)\alpha\beta_z t^n v + \beta_y t^{n+1} v$, and it will follow using induction that y acts by $\alpha\beta_z \frac{d}{dt} + \beta_y$:

$$\begin{aligned} y \cdot t^{n+1} v &= y\Theta(x^{n+1} \otimes v) = \Theta(yx^{n+1} \otimes v) = \Theta([y, x]x^n + xyx^n \otimes v) \\ &= \alpha z\Theta(x^n \otimes v) + xy\Theta(x^n \otimes v) = \alpha z t^n v + xy t^n v \\ &= \alpha\beta_z t^n v + x n\alpha\beta_z t^{n-1} v + x\beta_y t^n v = (n+1)\alpha\beta_z t^n v + \beta_y t^{n+1} v \end{aligned} \quad (5)$$

- (iii) Let $\partial := \frac{d}{dt}$, and let $\rho : \widehat{U(\mathcal{L})}_K \rightarrow \text{End}_K(M\langle t \rangle)$ be the action, then by part (ii), $\rho(y) = \alpha\beta_z \partial + \beta_y$, so $\partial = (\alpha\beta_z)^{-1}(\rho(y) - \beta_y) = \rho((\alpha\beta_z)^{-1}(y - \beta_y)) \in \text{im}(\rho)$.

Hence for each $n \in \mathbb{N}$, $\partial^{[n]} = \frac{1}{n!} \partial^n \in \text{im}(\rho)$.

So, suppose that $0 \neq T \leq M\langle t \rangle$ is a submodule, i.e there exists $\sum_{m \geq 0} t^m s_m \in T$, $s_m \in M$, s_m not all zero, $s_m \rightarrow 0$ as $m \rightarrow \infty$.

Then since $\partial^{[n]}(T) \subseteq T$ for all n , it follows that $\sum_{m \geq n} \binom{m}{n} t^{m-n} s_m \in T$, hence we may assume that $s_0 \neq 0$.

Set $s := s_0 \in M \setminus \{0\}$, and define a sequence of elements in T by $r_0 := \sum_{m \geq 0} t^m s_m$, and for $i > 0$, $r_i := r_{i-1} - t^i \partial^{[i]}(r_{i-1})$.

Now, if $r_i = s + \sum_{m>i} t^m s_{i,m}$, then $t^i \partial^{[i]}(r_i) = \sum_{m>i} t^m \binom{m}{i} s_{i,m}$, so

$$r_{i+1} = r_i - t^i \partial^{[i]}(r_i) = s + \sum_{m>i+1} t^m (s_{i,m} - \binom{m}{i} s_{i,m}).$$

So inductively, we get that for each $i \in \mathbb{N}$, $r_i = s + \sum_{m>i} s_{i,m} t^m$ for some $s_{i,m} \in M$ with $v(s_{i,m}) \geq v(s_{i-1,m})$. It follows easily that $r_i \rightarrow s$ in $M\langle t \rangle$ as $i \rightarrow \infty$.

But since $M\langle t \rangle$ is finitely generated over $\widehat{U(\mathcal{L})}_K$, which is Noetherian, it follows that T is finitely generated over $\widehat{U(\mathcal{L})}_K$, and hence T is π -adically complete by Proposition 2.2. Therefore, since each $r_i \in T$, this means that $s \in T$. So $0 \neq s \in M \cap T$, and thus $M \cap T \neq 0$. But since M is irreducible, $M \cap T = 0$ or M , hence $M \cap T = M$.

It follows that $M\langle t \rangle = \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{L})}_K} M \subseteq T$, and hence $T = M\langle t \rangle$. Since our choice of T was arbitrary, this implies that $M\langle t \rangle$ is irreducible as required. $\square$

3.2 An Explicit Formula

From now on, we will assume that $\mathfrak{g}$ is nilpotent. We will now examine the action of $\widehat{U(\mathcal{L})}_K$ on $\widehat{D(\lambda)}$ more closely, and write down an explicit formula for the action of elements of $\mathfrak{g}$.

First, let $\lambda : \mathfrak{g} \rightarrow K$ be a linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$, and let $\mathfrak{b}$ be a polarisation of $\mathfrak{g}$ at λ with $\mathcal{B} = \mathfrak{b} \cap \mathcal{L}$. Fix a basis $\{u_1, \dots, u_r\}$ for $\mathcal{L}/\mathcal{B}$, and it follows from Proposition 3.2 that $\widehat{D(\lambda)}_{\mathfrak{b}} \cong K\langle u_1, \dots, u_r \rangle$ as a K -vector space.

So, for each $i = 1, \dots, r$, recall from Section 2.4 that we can define the endomorphisms $x_i, \partial_i \in \text{End}_K \widehat{D(\lambda)}$ by $x_i(f) := u_i f$ and $\partial_i(f) := \frac{df}{du_i}$. Note that for each $u \in \mathfrak{g}$, we can write $u = v_u + \alpha_{1,u} u_1 + \dots + \alpha_{r,u} u_r$ for some unique $v_u \in \mathfrak{b}, \alpha_{i,u} \in K$. Using this, we define:

$$\mu : \mathfrak{g} \rightarrow \text{End}_K \widehat{D(\lambda)}_{\mathfrak{b}}, u \mapsto \lambda(v_u) + \alpha_{1,u} x_1 + \dots + \alpha_{r,u} x_r. \quad (6)$$

Naively, one might think that this defines the action of $\mathfrak{g}$ on $\widehat{D(\lambda)}$, but this is not true. However, the following result gives the explicit formula which allows us to completely describe this action:

Proposition 3.3 *Let $\rho : \widehat{U(\mathcal{L})}_K \rightarrow \text{End}_K \widehat{D(\lambda)}_{\mathfrak{b}}$ be the natural action of $\widehat{U(\mathcal{L})}_K$ on $\widehat{D(\lambda)}_{\mathfrak{b}}$, then we may choose a basis $\{u_1, \dots, u_r\}$ for $\mathcal{L}/\mathcal{B}$ such that for every $u \in \mathfrak{g}$, the action of u is given by:*

$$\rho(u) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_r!} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r}$$

Note This is a finite sum, since $\mathfrak{g}$ is nilpotent. Also, we define $\text{ad}(x)(y) := [y, x]$, as opposed to the more conventional $\text{ad}(x)(y) = [x, y]$.

Moreover, if $u \in \mathfrak{b}$ and $\text{ad}(\mathfrak{g})^n(u) \subseteq \mathfrak{b}$ for all $n \in \mathbb{N}$, then this formula holds for any choice of basis.

Proof Since $\mathfrak{g}$ is nilpotent, we can choose a basis $\{u_1, \dots, u_r\}$ for $\mathcal{L}/\mathcal{B}$ such that $[u_i, \mathfrak{g}] \subseteq \mathfrak{b} \oplus \text{Span}_K\{u_i + 1, \dots, u_r\}$ for all i , and we will fix such basis throughout the proof.

So, define $\rho' : \mathfrak{g} \rightarrow \text{End}_K \widehat{D(\lambda)}, u \mapsto \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_r!} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r}$. We want to prove that $\rho(u) = \rho'(u)$ for all $u \in \mathfrak{g}$, i.e. that $\rho(u)(f) = \rho'(u)(f)$ for all $f \in \widehat{D(\lambda)}$.

Firstly, since by Lemma 3.2, every element of $\widehat{D(\lambda)}$ can be written as a Tate power series in the variables $u_1^{\beta_1} \cdots u_r^{\beta_r}$, it suffices to prove that $\rho(u)(u_1^{\beta_1} \cdots u_r^{\beta_r}) = \rho'(u)(u_1^{\beta_1} \cdots u_r^{\beta_r})$ for every $\beta_1, \dots, \beta_d \in \mathbb{N}$.

Note that $\rho'(u)(u_1^{\beta_1} \cdots u_r^{\beta_r}) = \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) u_1^{\beta_1 - \alpha_1} \cdots u_r^{\beta_r - \alpha_r}$, where $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for all i , and $\binom{\beta}{\alpha} := \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_r}{\alpha_r}$. So, we will prove by induction on $|\beta| := \beta_1 + \cdots + \beta_r$ that $\rho(u)(u_1^{\beta_1} \cdots u_r^{\beta_r}) = \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) u_1^{\beta_1 - \alpha_1} \cdots u_r^{\beta_r - \alpha_r}$.

First, suppose that $\beta = 0$, so $u_1^{\beta_1} \cdots u_r^{\beta_r} = 1$, and hence $\rho'(u)(u_1^{\beta_1} \cdots u_r^{\beta_r}) = \mu(u)(1)$.

Note that under the isomorphism $\widehat{D(\lambda)} \rightarrow K\langle u_1, \dots, u_r \rangle$ given by Lemma 3.2, the inverse image of 1 is $1 \otimes v$, where v generates the one dimensional subspace K_λ of $\widehat{D(\lambda)} = \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_\lambda$. Write $u = v_u + \alpha_{1,u}u_1 + \cdots + \alpha_{r,u}u_r$ for $v_u \in \mathfrak{b}$ and $\alpha_{i,u} \in K$, so $\mu(u) = \lambda(v_u) + \alpha_{1,u}x_1 + \cdots + \alpha_{r,u}x_r$ by definition. Then:

$$\begin{aligned} \rho(u)(1) &= \rho(u)(1 \otimes v) = u \otimes v = v_u \otimes v + \alpha_{1,u}u_1 \otimes v + \cdots + \alpha_{r,u}u_r \otimes v \\ &= \lambda(v_u) \otimes v + \alpha_{1,u}u_1 \otimes v + \cdots + \alpha_{r,u}u_r \otimes v, \end{aligned}$$

and the image of this under the isomorphism $D(\lambda) \rightarrow K\langle u_1, \dots, u_r \rangle$ is $\lambda(v_u) + \alpha_{1,u}u_1 + \cdots + \alpha_{r,u}u_r$ by Lemma 3.2, and clearly this is equal to $\mu(u)(1) = \rho'(u)(1)$, so $\rho(u)(1) = \rho'(u)(1)$ as required.

So, now assume that $n > 0$ and the result holds whenever $|\beta| < n$. Choose γ with $|\gamma| = n$, and choose $i \geq 1$ minimal such that $\gamma_i \neq 0$, so $u_1^{\gamma_1} \cdots u_r^{\gamma_r} = u_i^{\gamma_i} \cdots u_r^{\gamma_r}$. Let $\beta_i := \gamma_i - 1$, $\beta_j = \gamma_j$ for all $j > i$, so that $u_i^{\gamma_i} \cdots u_r^{\gamma_r} = u_i^{\beta_i+1} u_{i+1}^{\beta_{i+1}} \cdots u_r^{\beta_r}$.

Then $\rho(u)(u_i^{\gamma_i} \cdots u_r^{\gamma_r}) = \rho(u)(u_i^{\beta_i+1} \cdots u_r^{\beta_r} \otimes v) = uu_i^{\beta_i+1} \cdots u_r^{\beta_r} \otimes v = [u, u_i]u_i^{\beta_i} \cdots u_r^{\beta_r} \otimes v + u_iuu_i^{\beta_i} \cdots u_r^{\beta_r} \otimes v = \rho([u, u_i])u_i^{\beta_i} \cdots u_r^{\beta_r} + \rho(u_i)\rho(u)u_i^{\beta_i} \cdots u_r^{\beta_r}$. Since $|\beta| = |\gamma| - 1 < n$, applying induction gives that

$$\begin{aligned} \rho([u, u_i])u_i^{\beta_i} \cdots u_r^{\beta_r} &= \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}([u, u_i])) u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r} \\ &= \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i+1}(u)) u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r} \\ &= \sum_{\alpha_j \leq \beta_j} \binom{\beta_i}{\alpha_i} \cdots \binom{\beta_r}{\alpha_r} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i+1}(u)) u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r}. \end{aligned}$$

and

$$\rho(u_i)\rho(u)u_i^{\beta_i} \cdots u_r^{\beta_r} = \rho(u_i) \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r}.$$

By our choice of basis, $\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u) \in \mathfrak{b} \oplus \text{Span}_K \{u_{i+1}, \dots, u_r\}$ for all α , and thus $\mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) \in K \oplus \text{Span}_K \{x_r, \dots, x_{i+1}\}$.

This means that for all α , $\mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u))u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r}$ is linear combination of monomials of the form $u_i^{\delta_i} \cdots u_r^{\delta_r}$.

So since $\rho(u_i)(u_i^{\delta_i} \cdots u_r^{\delta_r}) = u_i^{\delta_i+1} u_{i+1}^{\delta_{i+1}} \cdots u_r^{\delta_r} = x_i(u_i^{\delta_i} \cdots u_r^{\delta_r})$ for all δ , and x_i commutes with $x_r, \dots, x_{i+1}$, it follows that:

$$\begin{aligned} & \rho(u_i) \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r} \\ &= \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\beta_i - \alpha_i + 1} \cdots u_r^{\beta_r - \alpha_r} \\ &= \sum_{\alpha_j \leq \beta_j} \binom{\beta_i}{\alpha_i} \cdots \binom{\beta_r}{\alpha_r} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\beta_i - \alpha_i + 1} \cdots u_r^{\beta_r - \alpha_r}. \end{aligned}$$

Note that this is the only point in the proof where we use our particular choice of basis. Whereas if we assume that $\text{ad}(\mathfrak{g})^n(u) \in \mathfrak{b}$ for all n then, $\mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) \in K$ for all α , and clearly this commutes with $\rho(u_i)$ regardless of the choice of basis.

Now, collecting terms gives us:

$$\begin{aligned} \rho(u)(u_i^{\beta_i+1} \cdots u_r^{\beta_r}) &= \sum_{\alpha_j \leq \beta_j} \binom{\beta_i}{\alpha_i} \cdots \binom{\beta_r}{\alpha_r} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i+1}(u)) u_i^{\beta_i - \alpha_i} \cdots u_r^{\beta_r - \alpha_r} \\ &+ \sum_{\alpha_j \leq \beta_j} \binom{\beta_i}{\alpha_i} \cdots \binom{\beta_r}{\alpha_r} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\beta_i - \alpha_i + 1} \cdots u_r^{\beta_r - \alpha_r} \\ &= u_i^{\beta_i+1} \cdots u_r^{\beta_r} + \sum_{1 \leq \alpha_i \leq \beta_i} \left(\binom{\beta_i}{\alpha_i} + \binom{\beta_i}{\alpha_i - 1} \right) \binom{\beta_{i+1}}{\alpha_{i+1}} \cdots \binom{\beta_r}{\alpha_r} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\beta_i+1 - \alpha_i} \cdots u_r^{\beta_r - \alpha_r} \\ &+ \sum_{\alpha_i = \beta_i} \binom{\beta_{i+1}}{\alpha_{i+1}} \cdots \binom{\beta_r}{\alpha_r} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\beta_i+1}(u)) u_{i+1}^{\beta_{i+1} - \alpha_{i+1}} \cdots u_r^{\beta_r - \alpha_r} \\ &= \sum_{\alpha \leq \gamma} \binom{\gamma}{\alpha} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_i)^{\alpha_i}(u)) u_i^{\gamma_i - \alpha_i} \cdots u_r^{\gamma_r - \alpha_r} \end{aligned}$$

The last equality follows since $\binom{\beta_i}{\alpha_i} + \binom{\beta_i}{\alpha_i - 1} = \binom{\beta_i + 1}{\alpha_i} = \binom{\gamma_i}{\alpha_i}$. $\square$

Note 1. This proof is purely classical, and defines a formula for the action of $\mathfrak{g}$ on the classical Dixmier module $D(\lambda)$. In fact, this result gives us a ring homomorphism $U(\mathfrak{g}) \rightarrow A_r(K)$ whose kernel is $\text{Ann}_{U(\mathfrak{g})} D(\lambda)$. We suspect this map is surjective in general, since we know from [7, Proposition 6.2.2] that $U(\mathfrak{g}) / \text{Ann } D(\lambda) \cong A_r(K)$.

2. If $u \in \mathfrak{b}$ and $\text{ad}(\mathfrak{g})^n(u) \subseteq \mathfrak{b}$ for all $n \in \mathbb{N}$, i.e. $u \in \mathfrak{a}$ for some ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) = \lambda(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u))$ for all $\alpha \in \mathbb{N}^r$, so it follows from this proposition that u acts on $\widehat{D(\lambda)}$ by a polynomial in $K[\partial_1, \dots, \partial_r]$.

Corollary 3.4 Assume that $\mathcal{L}$ is a powerful Lie lattice in $\mathfrak{g}$, i.e. $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$. Then for any linear form $\lambda : \mathfrak{g} \rightarrow K$ with $\lambda(\mathcal{L}) \subseteq \mathcal{O}$, if $\mathfrak{g}$ has a polarisation $\mathfrak{b}$ at λ of codimension r , and $I := \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}}$, then there exists an injective ring homomorphism $\widehat{U(\mathcal{L})}_K / I \rightarrow \widehat{A_r(K)}$, and thus $\widehat{U(\mathcal{L})}_K / I$ is a domain.

Proof The natural action $\rho : \widehat{U(\mathcal{L})}_K \rightarrow \text{End}_K \widehat{D(\lambda)}$ has kernel I , and since $\widehat{D(\lambda)} \cong K\langle u_1, \dots, u_r \rangle$ by Lemma 3.2, the Tate-Weyl algebra $\widehat{A_r(K)}$ embeds as a subalgebra into

$\widehat{\text{End}_K D(\lambda)}$ by Lemma 2.13. Therefore, it remains to prove that the image of ρ is contained in $\widehat{A_r(K)}$.

Using Proposition 3.3, we know that we can fix a basis $\{u_1, \dots, u_r\}$ for $\mathcal{L}/\mathcal{B}$ such that for each $u \in \mathfrak{g}$, $\rho(u) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_r!} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r}$.

Since $\mu(\mathfrak{g}) \subseteq K \oplus Kx_1 \oplus \cdots \oplus Kx_r \subseteq A_r(K)$ and $\partial_1, \dots, \partial_r \in A_r(K)$, it follows that $\rho(\mathfrak{g}) \subseteq A_r(K)$, and hence $\rho(U(\mathfrak{g})) \subseteq A_r(K)$.

Therefore, it remains to prove that $\rho : U(\mathfrak{g}) \rightarrow A_r(K)$ is continuous with respect to the π -adic topologies on $U(\mathfrak{g})$ and $A_r(K)$ respectively. And since ρ is linear, this just means proving that the image of $U(\mathcal{L})$ under ρ is contained in $A_r(\mathcal{O}) = \mathcal{O}[x_1, \dots, x_r, \partial_1, \dots, \partial_r]$.

Since $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$, it follows that for all $u \in \mathcal{L}$, $\alpha \in \mathbb{N}^r$, $\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u) \in p^{|\alpha|}\mathcal{L}$. And since $\mu : \mathcal{L} \rightarrow \mathcal{O} \oplus \mathcal{O}x_1 \oplus \cdots \mathcal{O}x_r$ is linear, $\mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \in p^{|\alpha|}A_r(\mathcal{O})$.

Therefore, since $\frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_r!} p^{|\alpha|} = \frac{p^{\alpha_1}}{\alpha_1!} \cdots \frac{p^{\alpha_r}}{\alpha_r!} \in \mathcal{O}$, it follows that:

$$\rho(u) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_r!} \mu(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \in A_r(\mathcal{O}).$$

Therefore, the image of $\mathcal{L}$ under ρ is contained in $A_r(\mathcal{O})$, and therefore so is the image of $U(\mathcal{L})$ as required. $\square$

Now, fix an ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{a} \subseteq \mathfrak{b}$, and define

$$\mathfrak{a}^\perp := \{u \in \mathfrak{g} : \lambda([u, \mathfrak{a}]) = 0\},$$

then $\mathfrak{a}^\perp$ is a subalgebra of $\mathfrak{g}$ with $\mathfrak{b} \subseteq \mathfrak{a}^\perp$, so we can fix a basis $\{u_1, \dots, u_r\}$ for $\mathcal{L}/\mathcal{B}$ such that $\{u_{s+1}, \dots, u_r\}$ is a basis for $(\mathfrak{a}^\perp \cap \mathcal{L})/\mathcal{B}$ for some $s \leq r$.

Proposition 3.5 *Every element of $\mathfrak{a}$ acts on $\widehat{D(\lambda)}$ by a polynomial in $K[\partial_1, \dots, \partial_s]$, and the image of $U(\mathfrak{a})$ under the action $\rho : \widehat{U(\mathcal{L})}_K \rightarrow \widehat{\text{End}_K D(\lambda)}$ is precisely $K[\partial_1, \dots, \partial_s]$.*

Proof Firstly, using Proposition 3.3, we know that for every $u \in \mathfrak{a}$,

$$\rho(u) = \sum_{\alpha \in \mathbb{N}^r} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_r!} \lambda(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \partial_1^{\alpha_1} \cdots \partial_r^{\alpha_r} \in K[\partial_1, \dots, \partial_r].$$

Also, if $\alpha_i \neq 0$ for any $r \geq i > s$, then assuming i is maximal such that $\alpha_i \neq 0$, $\lambda(\text{ad}(u_r)^{\alpha_r} \cdots \text{ad}(u_1)^{\alpha_1}(u)) = \lambda(\text{ad}(u_i)^{\alpha_i} \cdots \text{ad}(u_1)^{\alpha_1}(u)) = 0$ since $u_i \in \mathfrak{a}^\perp$.

Therefore, $\rho(u) = \sum_{\alpha \in \mathbb{N}^s} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_s!} \lambda(\text{ad}(u_s)^{\alpha_s} \cdots \text{ad}(u_1)^{\alpha_1}(u)) \partial_1^{\alpha_1} \cdots \partial_s^{\alpha_s} \in$

$K[\partial_1, \dots, \partial_s]$ as required.

For the second statement, clearly $\rho(U(\mathfrak{a})) \subseteq K[\partial_1, \dots, \partial_s]$, so we just need to show that $\partial_1, \dots, \partial_s \in \rho(U(\mathfrak{a}))$.

We will first need to construct our basis appropriately. For convenience, set $\mathcal{A} = \mathfrak{a} \cap \mathcal{L}$ and $\mathcal{A}^\perp = \mathfrak{a}^\perp \cap \mathcal{L}$. First, write $\mathcal{L}/\mathcal{B} = (\mathcal{A}^\perp/\mathcal{B}) \oplus V$, for some complement V of $\mathcal{A}^\perp/\mathcal{B}$ in $\mathcal{L}/\mathcal{B}$.

We define the *upper central series* by $Z_1(\mathcal{L}) := Z(\mathcal{L})$ and $Z_i(\mathcal{L}) := \{u \in \mathcal{L} : [u, \mathcal{L}] \subseteq Z_{i-1}(\mathcal{L})\}$ for all $i > 1$. Since $\mathcal{L}$ is nilpotent, $Z_c(\mathcal{L}) = \mathcal{L}$ for some $c \geq 1$. Therefore, since $V \cap \mathfrak{a}^\perp = 0$, and hence $\lambda([\mathfrak{a}, V]) \neq 0$, we can choose $m_1 > 1$ minimal such that $\lambda([\mathcal{A} \cap Z_{m_1}(\mathcal{L}), V]) \neq 0$.

So, choose $y_1 \in \mathcal{A} \cap Z_{m_1}(\mathcal{L})$ such that $\lambda([y_1, V]) \neq 0$. Then $\lambda \circ \text{ad}(y_1) : V \otimes K \rightarrow K$ is a non-zero linear form, so we may write $V = \mathcal{O}u_1 \oplus V_1$ where $\lambda([y_1, V_1]) = 0$ and $\lambda([y_1, u_1]) \neq 0$.

Now, let us suppose for induction, that for some $i < s = \dim_K \mathfrak{g}/\mathfrak{a}^\perp$, we have the following data:

- Integers $1 < m_1 \leq m_2 \leq \dots \leq m_i \leq c$.
- Subspaces $V_i \subseteq V_{i-1} \subseteq \dots \subseteq V_1 \subseteq V_0 = V$ with $\lambda([V_{j-1}, \mathfrak{a} \cap Z_{m_{j-1}}(\mathcal{L})]) = 0$ for each $j \geq 1$.
- Elements $u_j \in V_{j-1}$ such that $V_{j-1} = \mathcal{O}u_j \oplus V_j$ for each $j \geq 1$.
- Elements $y_j \in \mathcal{A} \cap Z_{m_j}(\mathcal{L})$ such that $\lambda([y_j, V_j]) = 0$ and $\lambda([y_j, u_j]) \neq 0$.

Note that for each j , $\dim_K(V_j \otimes K) = s - j$. So since $i < s$, $V_i \cap \mathfrak{a}^\perp = 0$, so $\lambda([V_i, \mathfrak{a}]) \neq 0$.

But since $V_i \subseteq V_{i-1}$, we know that $[V_i, \mathfrak{a} \cap Z_{m_{i-1}}(\mathcal{L})] = 0$, so choose $m_{i+1} \geq m_i$ minimal such that $\lambda([V_i, \mathfrak{a} \cap Z_{m_{i+1}}(\mathcal{L})]) \neq 0$, and choose $y_{i+1} \in \mathcal{A} \cap Z_{m_{i+1}}(\mathcal{L})$ such that $\lambda([y_{i+1}, V_i]) \neq 0$.

Again, $\lambda \circ \text{ad}(y_{i+1}) : V_i \otimes K \rightarrow K$ is a non-zero linear form, so $V_i = \mathcal{O}u_{i+1} \oplus V_{i+1}$ for some $u_{i+1} \in V_i$, where $\lambda([y_{i+1}, V_{i+1}]) = 0$ and $\lambda([y_{i+1}, u_{i+1}]) \neq 0$.

Therefore, applying induction gives us a basis $\{u_1, \dots, u_s\}$ for V such that for each i , $\lambda([u_i, \mathcal{A} \cap Z_{m_{i-1}}(\mathcal{L})]) = 0$, and elements $y_1, \dots, y_s \in \mathcal{A}$ with $y_i \in Z_{m_i}(\mathcal{L})$ such that $\lambda([y_i, u_j]) = 0$ for all $j < i$ and $\lambda([y_i, u_i]) \neq 0$.

So, applying our explicit formula with the basis $\{u_1, \dots, u_s\}$, we get that $\rho(y_i) = \sum_{\alpha \in \mathbb{N}^s} \frac{1}{\alpha_1!} \dots \frac{1}{\alpha_s!} \lambda(\text{ad}(u_s)^{\alpha_s} \dots \text{ad}(u_1)^{\alpha_1}(y_i)) \partial_1^{\alpha_1} \dots \partial_s^{\alpha_s}$ for each i . Let us suppose that for all $0 \leq j < i$, $\partial_j \in \rho(U(\mathfrak{g}))$.

Then since $[\mathcal{L}, y_i] \subseteq Z_{m_{i-1}}(\mathcal{L})$ and $\lambda([u_j, Z_{m_{i-1}}(\mathcal{L})]) = 0$ for all $j \geq i$, it follows that if $\lambda(\text{ad}(u_s)^{\alpha_s} \dots \text{ad}(u_1)^{\alpha_1}(y_{i+1})) \neq 0$ then either $\alpha_j = 0$ for all $j \geq i$, or $\alpha_j = 0$ for all $j < i$ and $\alpha_i + \dots + \alpha_s = 1$. Therefore:

$$\rho(y_i) = f(\partial_1, \dots, \partial_{i-1}) + \lambda([y_i, u_i])\partial_i + \dots + \lambda([y_i, u_s])\partial_s \text{ for some polynomial } f.$$

But since $\partial_1, \dots, \partial_{i-1} \in \rho(U(\mathfrak{a}))$, $\lambda([y_i, u_j]) = 0$ for all $j > i$, and $\lambda([y_i, u_i]) \neq 0$, it follows that $\partial_i \in \rho(U(\mathfrak{a}))$. So applying induction, we see that $\partial_1, \dots, \partial_s \in \rho(U(\mathfrak{a}))$ as required.

Finally, if $\{u'_1, \dots, u'_s\}$ is any other basis for V , then since each u'_i is a non-zero linear combination in $u_1, \dots, u_s$, it follows from the chain rule that $\partial_{u'_i} = \beta_i \partial_{u_i}$ for some non-zero $\beta_i \in K$, and hence $\partial_{u'_1}, \dots, \partial_{u'_s} \in \rho(U(\mathfrak{a}))$. $\square$

3.3 Irreducibility

Now we will prove the main theorem of this section.

Theorem 3.6 *Suppose that $\mathfrak{g}$ is nilpotent, $\lambda \in \mathfrak{g}^*$ with $\lambda(\mathcal{L}) \subseteq \mathcal{O}$. Then there exists a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ such that the affinoid Dixmier module $\widehat{D(\lambda)}_{\mathfrak{b}}$ of $\mathcal{L}$ at λ with respect to $\mathfrak{b}$ is irreducible as a $\widehat{U(\mathcal{L})}_K$ -module.*

Proof We will use induction on $n = \dim_K \mathfrak{g}$, so first suppose that $n = 1$. Then $\mathfrak{g}$ is abelian, so λ is a character of $\mathfrak{g}$, and $\widehat{D(\lambda)} = K$, which is clearly irreducible.

For the inductive step, we will assume that the result holds for all $m < n$:

Suppose first that there exists a non-zero ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$ such that $\lambda(\mathfrak{a}) = 0$. Let $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{a}$, so since $\mathfrak{a} \neq 0$, $\dim_K \mathfrak{g}_1 < n$, so we may apply the inductive hypothesis to $\mathfrak{g}_1$.

Let $\mathcal{A} := \mathcal{L} \cap \mathfrak{a}$, and let $\mathcal{L}_1 := \frac{\mathcal{L}}{\mathcal{A}}$, then $\mathcal{A}, \mathcal{L}_1$ are lattices in $\mathfrak{a}, \mathfrak{g}_1$ respectively.

Now, let λ_1 be the linear form of $\mathfrak{g}_1$ induced by λ , and clearly $\lambda_1(\mathcal{L}_1) \subseteq \mathcal{O}$. So by the inductive hypothesis, there exists a polarisation $\mathfrak{b}_1$ of $\mathfrak{g}_1$ at λ_1 such that $\widehat{D(\lambda_1)_{\mathfrak{b}_1}}$ is irreducible over $\widehat{U(\mathcal{L}_1)_K}$.

Since $\mathfrak{b}_1 = \frac{\mathfrak{b}}{\mathfrak{a}}$ for some subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, it follows that $\mathfrak{b}$ is a polarisation of $\mathfrak{g}$ at λ , so let $\mathcal{B} := \mathfrak{b} \cap \mathcal{L}$, $\mathcal{B}_1 := \mathfrak{b}_1 \cap \mathcal{L}_1$, and let $M := \widehat{D(\lambda_1)_{\mathfrak{b}_1}} = \widehat{U(\mathcal{L}_1)_K} \otimes_{\widehat{U(\mathcal{B}_1)_K}} K_{\lambda_1}$.

Using the surjection $\widehat{U(\mathcal{L})_K} \twoheadrightarrow \widehat{U(\mathcal{L}_1)_K}$ given by Lemma 3.1(i), we see that M has the structure of an irreducible $\widehat{U(\mathcal{L})_K}$ -module, and using the fact that $\mathcal{B}_1 = \frac{\mathcal{B}}{\mathcal{A}}$ and Lemma 3.1(ii) we see that $M \cong \widehat{U(\mathcal{L})_K} \otimes_{\widehat{U(\mathcal{B})_K}} K_\lambda = \widehat{D(\lambda)_{\mathfrak{b}}}$, and hence $\widehat{D(\lambda)_{\mathfrak{b}}}$ is irreducible as required.

So from now on, we may assume that $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$.

Using Proposition 2.7, we can find a reducing quadruple $(x, y, z, \mathfrak{g}')$ of $\mathfrak{g}$ such that any polarisation $\mathfrak{b} \subseteq \mathfrak{g}'$ of $\mathfrak{g}'$ at $\mu := \lambda|_{\mathfrak{g}'}$ is in fact a polarisation of $\mathfrak{g}$ at λ . Note that since y, z are central in $\mathfrak{g}'$, $y, z \in \mathfrak{b}$ by Lemma 2.3.

Set $\mathcal{L}' := \mathcal{L} \cap \mathfrak{g}'$. Then since $\dim_K \mathfrak{g}' = n - 1 < n$, using the inductive hypothesis we can choose a polarisation $\mathfrak{b}$ of $\mathfrak{g}'$ at μ such that $\widehat{D(\mu)} = \widehat{U(\mathcal{L}')_K} \otimes_{\widehat{U(\mathcal{B})_K}} K_\mu$ is irreducible over $\widehat{U(\mathcal{L}')_K}$.

Now, since $\widehat{U(\mathcal{L})_K} = \widehat{U(\mathcal{L})_K} \otimes_{\widehat{U(\mathcal{L}')_K}} \widehat{U(\mathcal{L}')_K}$, it follows that $\widehat{D(\lambda)_{\mathfrak{b}}} = \widehat{U(\mathcal{L})_K} \otimes_{\widehat{U(\mathcal{L}')_K}} \widehat{D(\mu)_{\mathfrak{b}}}$.

Therefore, setting $M := \widehat{D(\mu)_{\mathfrak{b}}}$, we have that $\widehat{D(\lambda)_{\mathfrak{b}}} \cong M\langle t \rangle$ as a K -vector space by Proposition 3.2(i), where x acts on $M\langle t \rangle$ by t .

Also, y, z are central in $\mathfrak{g}'$ and $[y, x] = \alpha z$ for some $0 \neq \alpha \in K$. Therefore, since y and z act on M by scalars $\lambda(y), \lambda(z)$ respectively, we see using Proposition 3.2(ii) that y acts on $M\langle t \rangle$ by $\alpha\lambda(z)\frac{d}{dt} + \lambda(y)$.

So, finally, since Kz is an ideal of $\mathfrak{g}$, $\lambda(z) \neq 0$, so since $\alpha, \lambda(z) \neq 0$, it follows from Proposition 3.2(iii) that $\widehat{D(\lambda)_{\mathfrak{b}}} = M\langle t \rangle$ is irreducible over $\widehat{U(\mathcal{L})_K}$ as required. $\square$

4 Dixmier Annihilators

From now on, we will always assume that $\mathfrak{g}$ is nilpotent. We are interested in the annihilators inside $\widehat{U(\mathcal{L})_K}$ of affinoid Dixmier modules.

Since for any $\lambda \in \mathfrak{g}^*$ with $\lambda(\mathcal{L}) \subseteq \mathcal{O}$, there exists a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ such that $\widehat{D(\lambda)_{\mathfrak{b}}}$ is irreducible by Theorem 3.6, it follows by definition that $\text{Ann}_{\widehat{U(\mathcal{L})_K}} \widehat{D(\lambda)_{\mathfrak{b}}}$ is a primitive ideal of $\widehat{U(\mathcal{L})_K}$.

Ideally, we want to prove that all primitive ideals arise as annihilators of affinoid Dixmier modules. But in this section, we will first show that these Dixmier annihilators are always primitive, regardless of the choice of polarisation.

4.1 Reducing Ideals

First we need some preliminary results. The first is an affinoid version of [7, Proposition 5.1.7]:

Lemma 4.1 *Let $\mathfrak{h} \leq \mathfrak{g}$ be a subalgebra, let $\mathcal{H} := \mathfrak{h} \cap \mathcal{L}$, and let M be a finitely generated $\widehat{U(\mathcal{H})}_K$ -module, with $J := \text{Ann}_{\widehat{U(\mathcal{H})}_K} M$. Then $\text{Ann}_{\widehat{U(\mathcal{L})}_K} (\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M)$ is the largest two-sided ideal in $\widehat{U(\mathcal{L})}_K$ contained in $\widehat{U(\mathcal{L})}_K J$. It follows that if M, N are finitely generated $\widehat{U(\mathcal{H})}_K$ -modules such that $\text{Ann}_{\widehat{U(\mathcal{H})}_K} M = \text{Ann}_{\widehat{U(\mathcal{H})}_K} N$ then*

$$\text{Ann}_{\widehat{U(\mathcal{L})}_K} (\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M) = \text{Ann}_{\widehat{U(\mathcal{L})}_K} (\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} N).$$

Proof Fix an $\mathcal{O}$ -basis $\{x_1, \dots, x_d\}$ for $\mathcal{L}$ such that $\{x_1, \dots, x_r\}$ is a basis for $\mathcal{H}$. Then by Lemma 2.1, every element $r \in \widehat{U(\mathcal{L})}_K$ can be written as $r = \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, for some $\lambda_\alpha \in K$ converging to zero as $|\alpha| \rightarrow \infty$, i.e. $r = \sum_{\alpha \in \mathbb{N}^r} \underline{x}^\alpha s_\alpha$ for some $s_\alpha \in \widehat{U(\mathcal{H})}_K$ such that $s_\alpha \rightarrow 0$ as $|\alpha| \rightarrow \infty$.

Using Proposition 3.2, we see that $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ is isomorphic as a K -vector space to the Tate module $M\langle t_1, \dots, t_r \rangle = \{ \sum_{\alpha \in \mathbb{N}^r} t_1^{\alpha_1} \cdots t_r^{\alpha_r} s_\alpha : s_\alpha \in M, s_\alpha \rightarrow 0 \text{ as } |\alpha| \rightarrow \infty \}$ via an isomorphism Ψ sending $\underline{x}^\alpha \otimes m$ to $t_1^{\alpha_1} \cdots t_r^{\alpha_r} m$. It is clear that the set of all elements in $\widehat{U(\mathcal{L})}_K$ that annihilate the set M inside $M\langle t_1, \dots, t_r \rangle$ on the left contains the left ideal $\widehat{U(\mathcal{L})}_K J$.

Moreover, if $rM = 0$ for some $r = \sum_{\alpha \in \mathbb{N}^r} \underline{x}^\alpha s_\alpha \in \widehat{U(\mathcal{L})}_K$, then for all $m \in M$:

$$0 = rm = \sum_{\alpha \in \mathbb{N}^r} \underline{x}^\alpha s_\alpha m = \Psi^{-1} \left(\sum_{\alpha \in \mathbb{N}^r} t_1^{\alpha_1} \cdots t_r^{\alpha_r} s_\alpha m \right), \text{ and hence } s_\alpha m = 0 \text{ for all } \alpha \in \mathbb{N}^r.$$

Thus $s_\alpha \in J$ for all α , and hence $r \in \widehat{U(\mathcal{L})}_K J$. Therefore the right ideal $\widehat{U(\mathcal{L})}_K J$ is the set of all elements of $\widehat{U(\mathcal{L})}_K$ that annihilate the set M .

It follows that if $r \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M = 0$, then $r \widehat{U(\mathcal{L})}_K$ annihilates M , so the two-sided ideal generated by r is contained in $\widehat{U(\mathcal{L})}_K J$. Since our choice of r was arbitrary, it follows that the annihilator of $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$ is contained in $\widehat{U(\mathcal{L})}_K J$.

Furthermore, if $I \subseteq \widehat{U(\mathcal{L})}_K J$ is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, then I annihilates M , so since $I \widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K I$, it must also annihilate the submodule generated by M inside $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{H})}_K} M$, which is clearly the whole module, and the result follows. $\square$

The next result will be essential to several of our proofs, since it allows us to safely pass to and from a reducing quadruple when studying two-sided ideals in $\widehat{U(\mathcal{L})}_K$.

Theorem 4.2 Suppose that $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$ with $x, y, z \in \mathcal{L}$, and let $\mathcal{L}' := \mathcal{L} \cap \mathfrak{g}'$. Then if I is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$ such that $z + I$ is not a zero divisor in $\widehat{U(\mathcal{L})}_K/I$, then I is controlled by $\mathcal{L}'$, i.e.:

$$I = \left(I \cap \widehat{U(\mathcal{L}')}_K \right) \widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K \left(I \cap \widehat{U(\mathcal{L}')}_K \right).$$

Proof Using Lemma 2.1, we see that every element of $\widehat{U(\mathcal{L})}_K$ can be written as $g(x)$ for some Tate power series g with coefficients in $\widehat{U(\mathcal{L}')}_K$. We will prove that if $g(x) \in I$ then the coefficients of g all lie in I , and the result follows.

It will suffice to show that if $g(x) = c_0 + c_1x + c_2x^2 + \cdots \in I$ then the formal derivative $g'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots$ also lies in I . Then using an argument similar to the proof of Proposition 3.2(iii), we can construct a sequence of elements in I converging to c_0 . By closure of I in $\widehat{U(\mathcal{L})}_K$, it follows that $c_0 \in I$, so repeating the argument for $\frac{1}{n!}g^{(n)}(x)$ for each x , it follows that all coefficients of $g(x)$ lie in I as required.

To prove that $g'(x)$ lies in I , consider the action of y on $\widehat{U(\mathcal{L})}_K/I$:

Since y is central in $\mathcal{L}'$, y commutes with everything in $\widehat{U(\mathcal{L}')}_K$. Also, since $[x, y] = \alpha z$, clearly $y \cdot x = xy - \alpha z$, and an easy induction shows that $y \cdot x^n = x^n y - n\alpha x^{n-1}z$. So if l_y is the left action of y on $\widehat{U(\mathcal{L})}_K/I$, r_y is the right action, then $l_y - r_y = -\alpha z \frac{d}{dx}$. Therefore, since z is not a zero divisor modulo I , and $\alpha \neq 0$, it follows that if $g(x) \in I$ then $\frac{d}{dx}(g(x)) \in I$ as required. $\square$

4.2 Independence Theorem

We will now prove the main result of this section, namely that Dixmier annihilator ideals are independent of the choice of polarisation. First, we deal with a special case.

Lemma 4.3 Suppose that $\mathcal{L}$ has an $\mathcal{O}$ -basis $\{x, y, z\}$ such that z is central and $[x, y] = \alpha z$ for some $0 \neq \alpha \in \mathcal{O}$. Then for any $0 \neq \beta \in \mathcal{O}$, the ideal $(z - \beta)\widehat{U(\mathcal{L})}_K$ is a maximal two-sided ideal of $\widehat{U(\mathcal{L})}_K$.

Proof Let I be a proper ideal of $\widehat{U(\mathcal{L})}_K$ containing $z - \beta$. Then since $\beta \neq 0$, $z + I$ is not a zero divisor in $\widehat{U(\mathcal{L})}_K/I$. So setting $\mathfrak{g}' := \text{Span}_K\{y, z\}$, since $(x, y, z, \mathfrak{g}')$ is a reducing quadruple, it follows from Theorem 4.2 that I is controlled by $\mathcal{L}' = \mathfrak{g}' \cap \mathcal{L}$. Therefore, if we can prove that $I \cap \widehat{U(\mathcal{L}')}_K = (z - \beta)\widehat{U(\mathcal{L}')}_K$ then it follows that $I = (z - \beta)\widehat{U(\mathcal{L})}_K$.

Let $\bar{y}$ be the image of y in $\widehat{U(\mathcal{L}')}_K/(z - \beta)\widehat{U(\mathcal{L}')}_K$, then given $r \in \widehat{U(\mathcal{L}')}_K$, by Lemma 2.1, the image of r in $\widehat{U(\mathcal{L}')}_K/(z - \beta)\widehat{U(\mathcal{L}')}_K$ has the form $\bar{r} = \sum_{n \geq 0} \lambda_n \bar{y}^n$ for some $\lambda_n \in K$,

$\lambda_n \rightarrow 0$ as $n \rightarrow \infty$.

Since $[x, y] = \alpha z$, we have that $x \cdot \bar{y} = \bar{y}x + \alpha\beta$, and an easy induction shows that $x \cdot \bar{y}^n = \bar{y}^n x + \alpha\beta \bar{y}^{n-1}$ for all n , i.e. if l_x and r_x are the respective left and right actions of

x on $\widehat{U(\mathcal{L}')}_K/(z - \beta)$, then $l_x - r_x = \alpha\beta \frac{d}{d\bar{y}}$. Since $\alpha, \beta \neq 0$ and I is a two-sided ideal, $\frac{d}{d\bar{y}}$ preserves $I \cap \widehat{U(\mathcal{L}')}_K/(z - \beta)$.

So if $g(\bar{y}) = \lambda_0 + \lambda_1\bar{y} + \lambda_2\bar{y}^2 + \dots \in I \cap \widehat{U(\mathcal{L}')}_K/(z - \beta)\widehat{U(\mathcal{L}')}_K$, it follows that $\frac{1}{n!}g^{(n)}(\bar{y}) \in I$ for all $n \in \mathbb{N}$, and using an argument similar to the proof of Proposition 3.2(iii), we can construct a sequence of elements in $I \cap \widehat{U(\mathcal{L}')}_K$ converging to $\lambda_0 \in K$.

By closure of $I \cap \widehat{U(\mathcal{L}')}_K$, this implies that $\lambda_0 \in I$, and hence $\lambda_0 = 0$, and it follows after replacing $g(\bar{y})$ by $\frac{1}{n!}g^{(n)}(\bar{y})$ that $\lambda_n = 0$ for all n , i.e. $g(\bar{y}) = 0$. Therefore $I \cap \widehat{U(\mathcal{L}')}_K = (z - \beta)\widehat{U(\mathcal{L}')}_K$, and $I = (z - \beta)\widehat{U(\mathcal{L})}_K$. So since our choice of I was arbitrary, $(z - \beta)\widehat{U(\mathcal{L})}_K$ is maximal as required. $\square$

Theorem 4.4 *Suppose $\mathfrak{g}$ is nilpotent, and let $\lambda \in \mathcal{L}^*$. Then for any polarisations $\mathfrak{b}_1, \mathfrak{b}_2$ of $\mathfrak{g}$ at λ , $\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_1} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_2}$.*

Proof If $\mathfrak{g}$ is abelian then $\mathfrak{b}_1 = \mathfrak{b}_2 = \mathfrak{g}$ so the statement is obvious. Since all nilpotent Lie algebras of dimension 1 and 2 are abelian, we may assume that $\dim_K \mathfrak{g} \geq 3$.

If $\mathfrak{g}$ is non-abelian and $\dim_K \mathfrak{g} = 3$ then it is straightforward to show that $\mathcal{L}$ has basis $\{x, y, z\}$ with z central and $[x, y] = \alpha z$ for some $\alpha \in \mathcal{O} \setminus 0$. If $\lambda(z) = 0$ then λ is a character of $\mathfrak{g}$, so $\mathfrak{g}$ is the only polarisation and the statement is trivially true. If $\lambda(z) \neq 0$, then for any polarisation $\mathfrak{b}$, z acts on $\widehat{D(\lambda)}_{\mathfrak{b}}$ by $\lambda(z)$, and so the $\widehat{U(\mathcal{L})}_K$ -annihilator must contain $(z - \lambda(z))$, which is a maximal ideal by Lemma 4.3, hence this must be the annihilator in all cases as we require.

So from now on, we may assume that $n = \dim_K \mathfrak{g} \geq 4$ and we will proceed by Dixmier's induction strategy on n :

Suppose first that there exists a non-zero ideal $\mathfrak{a} \trianglelefteq \mathfrak{g}$ such that $\lambda(\mathfrak{a}) = 0$, so λ induces a linear form λ_0 of $\mathfrak{g}_0 := \mathfrak{g}/\mathfrak{a}$. Setting $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$, $\mathcal{L}_0 := \frac{\mathcal{L}}{\mathcal{A}}$, it is clear that $\mathcal{L}_0$ is a Lie lattice in $\mathfrak{g}_0$ and $\lambda_0(\mathcal{L}_0) \subseteq \mathcal{O}$.

Note that $\mathfrak{a} \subseteq \mathfrak{b}_1, \mathfrak{b}_2$ by Lemma 2.3, so set $\mathfrak{b}_{i,0} := \mathfrak{b}_i/\mathfrak{a}$ for $i = 1, 2$, and $\mathfrak{b}_{1,0}, \mathfrak{b}_{2,0}$ are polarisations of $\mathfrak{g}_0$ at λ_0 .

Since $\dim_K \mathfrak{g}_0 < n$, it follows from induction that $\text{Ann}_{\widehat{U(\mathcal{L}_0)_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{1,0}} = \text{Ann}_{\widehat{U(\mathcal{L}_0)_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{2,0}}$.

Using Lemma 3.1, we see that for $i = 1, 2$, $\widehat{D(\lambda_0)}_{\mathfrak{b}_{i,0}} = \widehat{U(\mathcal{L}_0)_K} \otimes_{\widehat{U(\mathcal{B}_{i,0})_K}} K_{\lambda}$ is naturally a $\widehat{U(\mathcal{L})}_K$ -module, and that it is isomorphic to $\widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B}_i)_K}} K_{\lambda} = \widehat{D(\lambda)}_{\mathfrak{b}_i}$.

If $\phi : \widehat{U(\mathcal{L})}_K \twoheadrightarrow \widehat{U(\mathcal{L}_0)_K}$ is the natural surjection, then it is clear that $\text{Ann}_{\widehat{U(\mathcal{L}_0)_K}} \widehat{D(\lambda_0)}_{\mathfrak{b}_{i,0}} = \frac{\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_0)}_{\mathfrak{b}_{i,0}}}{\ker(\phi)}$ for $i = 1, 2$. Thus $\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_0)}_{\mathfrak{b}_{1,0}} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_0)}_{\mathfrak{b}_{2,0}}$, and hence:

$\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_1} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_0)}_{\mathfrak{b}_{1,0}} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_0)}_{\mathfrak{b}_{2,0}} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_2}$ as required.

So from now on, we may assume that $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$. Using Proposition 2.7, we see that this means that there exists a reducing quadruple $(x, y, z, \mathfrak{g}')$ for $\mathfrak{g}$. Since we are assuming $\dim_K \mathfrak{g} > 3$, we may apply Proposition 2.8 to get that for each $i = 1, 2$ there exists a proper subalgebra $\mathfrak{h}_i$ of $\mathfrak{g}$ containing $\mathfrak{b}_i$, and a polarisation $\mathfrak{b}'_i$ of $\mathfrak{g}$ at λ contained in $\mathfrak{h}_i$ and $\mathfrak{g}'$.

By induction, since $\dim_K \mathfrak{g}' < n$, we get that $\text{Ann}_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\lambda|_{\mathfrak{g}'})}_{\mathfrak{b}'_1} = \text{Ann}_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\lambda|_{\mathfrak{g}'})}_{\mathfrak{b}'_2}$, so by Lemma 4.1, $\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}'_1} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}'_2}$.

Similarly, since $\mathfrak{h}_1, \mathfrak{h}_2$ are proper subalgebras of $\mathfrak{g}$, we also have that $\text{Ann}_{\widehat{U(\mathcal{H}_i)_K}} \widehat{D(\lambda|_{\mathfrak{h}_i})}_{\mathfrak{b}_i} = \text{Ann}_{\widehat{U(\mathcal{H}_i)_K}} \widehat{D(\lambda|_{\mathfrak{h}_i})}_{\mathfrak{b}'_i}$ for $i = 1, 2$ by induction, and applying Lemma 4.1 again gives that $\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_i} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}'_i}$, and it follows that $\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_1} = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{\mathfrak{b}_2}$ as required. $\square$

Now that we have established that the annihilator of a Dixmier module does not depend on the choice of polarisation, we can unambiguously make the following definition:

Definition 4.1 Let F/K be a finite extension, and let $\lambda \in \mathcal{L}_F^* := (\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F)^*$ be a linear form. Define the *Dixmier annihilator* in $\widehat{U(\mathcal{L})}_K$ associated to λ to be the two sided ideal $I(\lambda) := \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$ (or $I(\lambda)_F$ if it is unclear what the base field is).

Note This definition makes sense because there is a natural embedding $\widehat{U(\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L}_F)_F}$ for any finite extension F/K . Using Theorem 3.6, we see that $I(\lambda)$ is a primitive ideal of $\widehat{U(\mathcal{L})}_K$ whenever $F = K$.

5 Locally Closed Ideals

Now we will study some general ring theoretic properties of the affinoid enveloping algebra.

5.1 Prime Ideals in $\widehat{U(\mathcal{L})}_K$

Proposition 5.1 Let P be a prime ideal of $\widehat{U(\mathcal{L})}_K$, and let J be a two-sided ideal of $R := \widehat{U(\mathcal{L})}_K/P$. Then if $J \neq 0$ there exists an element $a \in J$ such that a^n does not converge to 0 as $n \rightarrow \infty$.

Proof Let w be the π -adic filtration on $\widehat{U(\mathcal{L})}_K$ corresponding to the lattice $\widehat{U(\mathcal{L})}$, and let $\bar{w}$ be the quotient filtration on $R := \widehat{U(\mathcal{L})}_K/P$. Then since R is complete with respect to $\bar{w}$ and $\text{gr}_{\bar{w}} R \cong \frac{\text{gr}_{\bar{w}} \widehat{U(\mathcal{L})}_K}{\text{gr}_{\bar{w}} P}$, it follows from [14, Ch II Theorem 2.2.1] that $\bar{w}$ is a Zariskian filtration on R .

Also, since $\text{gr}_w \widehat{U(\mathcal{L})}_K \cong U(\mathcal{L}/\pi\mathcal{L})[t, t^{-1}]$ by [2, Lemma 3.1], and it is well known that $U(\mathcal{L}/\pi\mathcal{L})$ is finitely generated over its centre, it follows that $\text{gr}_{\bar{w}} R$ is also finitely generated over its centre.

Furthermore, $t = \text{gr } \pi$ is central of positive degree in $\text{gr } R$, and it is non-nilpotent, so it follows that $\text{gr } R$ is finitely generated over a central Noetherian subring whose positive part is non-nilpotent. Hence after applying [11, Theorem 3.3], we can find a filtration v on the

ring of quotients $Q(R)$ such that v restricts to a valuation of the centre, and the inclusion $(R, \bar{w}) \rightarrow (Q(R), v)$ is continuous.

Suppose for contradiction that for every element $a \in J$, $a^n \rightarrow 0$ as $n \rightarrow \infty$. Choose an arbitrary $a \in J$, and following [11, Definition 3.2], define the growth rate function $\rho : Q(R) \rightarrow \mathbb{R} \cup \{\infty\}$, $q \mapsto \lim_{n \rightarrow \infty} \frac{v(q^n)}{n}$, and let $m := \lceil \rho(a) \rceil$.

If we assume that $m < \infty$, then set $b := \pi^{-(m+1)}a$, and since π is central in R , we see using [11, Lemma 3.7(v)] that $\rho(b) = \rho(a) - (m+1)v(\pi) \leq \rho(a) - (\rho(a) + 1)v(\pi) < 0$, and hence b^n does not converge to 0 as $n \rightarrow \infty$ – contradiction since $b \in J$.

Therefore $m = \rho(a) = \infty$, so since $Q(R)$ is simple and artinian, it follows from [11, Lemma 3.7(iv)] that a is nilpotent.

Since our choice of a was arbitrary, this means that every element of J is nilpotent, and using [7, Lemma 3.1.14] it follows that J is a nilpotent ideal of R . Since R is prime, this means that $J = 0$ as required. $\square$

The following result is the affinoid version of [7, Proposition 3.1.15]:

Theorem 5.2 *Let I be a two sided ideal of $\widehat{U(\mathcal{L})}_K$. Then I is semiprime if and only if I is an intersection of primitive ideals.*

Proof Clearly if I is an intersection of primitive ideals, then it is semiprime, so it remains only to prove the converse, i.e. that if I is semiprime then it is the intersection of primitive ideals.

Since semiprime ideals arise as an intersection of primes, we can assume that I is prime in $\widehat{U(\mathcal{L})}_K$, and we will show that $J(\widehat{U(\mathcal{L})}_K/I) = 0$, from which the result follows.

Assume for contradiction that $J := J(\widehat{U(\mathcal{L})}_K/I) \neq 0$, then since I is prime it follows from Proposition 5.1 that we can choose an element $a \in J$ such that a^n does not converge to zero as $n \rightarrow \infty$.

Let $R := \frac{\widehat{U(\mathcal{L})}_K}{I}$, and let $C := R\langle X \rangle$ be the Tate algebra in one variable over R . Then if we set $\mathfrak{g}_0 := \mathfrak{g} \times K$, $\mathcal{L}_0 := \mathcal{L} \times \mathcal{O}$, it is clear that $\mathcal{L}_0$ is a Lie lattice in $\mathfrak{g}_0$ and it follows from Lemma 2.1 that $C \cong \widehat{U(\mathcal{L}_0)_K}/I\widehat{U(\mathcal{L}_0)_K}$.

Consider the element $1 - aX \in C$. If this element is a unit, its inverse must have the form $a_0 + a_1X + a_2X^2 + \cdots$ for some $a_i \in \widehat{U(\mathcal{L})}_K/I$ with $a_n \rightarrow 0$ as $n \rightarrow \infty$. But since $1 = (1 - aX)(a_0 + a_1X + a_2X^2 + \cdots)$, it follows that $a_0 = 1$, $a_1 = a$, $a_2 = a^2$, $\cdots$, $a_n = a^n$, and hence $a^n \rightarrow 0$ as $n \rightarrow \infty$ – contradiction.

Therefore $1 - aX$ is not a unit in C , so there exists a maximal left ideal of C containing $1 - aX$, i.e. there exists an irreducible C -module M and an element $0 \neq m \in M$ such that $(1 - aX)m = 0$.

Now, X does not act by zero on M , otherwise $1 - aX$ would act by 1, and we would have $(1 - aX)m = m \neq 0$. So using Schur's Lemma, the action of X is invertible, and using [17, Theorem 6.4.6] we see that the action of X^{-1} is algebraic over K , i.e. there exists $f(t) = a_0 + a_1t + \cdots + a_nt^n$ for some $a_i \in K$ such that $f(X^{-1}) = 0$, and we may assume that $a_0 \neq 0$. So let $g(t) := a_0^{-1}f(t) = 1 + b_1t + \cdots + b_nt^n$.

Since $aXm = m$, we have that $am = X^{-1}aXm = X^{-1}m$, hence $a^r m = X^{-r}m$ for all $r \in \mathbb{N}$, and thus $g(a)m = g(X^{-1})m = 0$.

But $g(a) = 1 + (b_1 + b_2a + \cdots + b_na^{n-1})a$, so since $a \in J(\widehat{U(\mathcal{L})}_K/I)$, this means that $g(a)$ is a unit in $\widehat{U(\mathcal{L})}_K/I$. Therefore, since $m \neq 0$, $g(a)m \neq 0$ – contradiction.

Therefore $J(\widehat{U(\mathcal{L})}_K/I) = 0$ as we require. $\square$

5.2 Towards the Dixmier-Moeglin Equivalence

Now, recall from the introduction that a prime ideal P of $\widehat{U(\mathcal{L})}_K$ is *locally closed* if

$$P \neq \cap \{Q \trianglelefteq \widehat{U(\mathcal{L})}_K : Q \text{ prime}, P \subsetneq Q\}.$$

Proposition 5.3 *Let P be a prime ideal of $\widehat{U(\mathcal{L})}_K$. Then:*

P is locally closed $\implies P$ is primitive $\implies P$ is weakly rational.

Proof First, suppose that P is primitive, i.e. $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} M$ for some simple $\widehat{U(\mathcal{L})}_K$ -module M . Then it follows from [17, Theorem 6.4.6] that every element of $\text{End}_{\widehat{U(\mathcal{L})}_K} M$ is algebraic over K . So since $Z(\widehat{U(\mathcal{L})}_K/P)$ is a domain, and embeds into $\text{End}_{\widehat{U(\mathcal{L})}_K} M$, it follows that $Z(\widehat{U(\mathcal{L})}_K/P)$ is an algebraic field extension of K , and hence P is weakly rational.

Using Theorem 5.2, we see that if P is equal to an intersection of primitive ideals. So if P is locally closed, i.e. not equal to the intersection of all prime ideals properly containing it, then P must be primitive. $\square$

Note These implications also hold for primes P in $U(\mathfrak{g})$, and in the case where $\mathfrak{g}$ is nilpotent, P weakly rational $\implies P$ maximal [7, Proposition 4.7.4]. We suspect that this is also true in the affinoid case, but this is only known in the case where $\mathfrak{g}$ contains an abelian ideal of codimension 1 [13, Corollary 1.6.2].

Using this result, we see that to prove the Dixmier-Moeglin equivalence for $\widehat{U(\mathcal{L})}_K$, it remains to show that weakly rational ideals are locally closed. Taking steps in this direction, the aim of this section is to prove that all locally closed prime ideals P of $\widehat{U(\mathcal{L})}_K$ have the form of a Dixmier annihilator $I(\lambda)_F$ for some finite extension F of K .

Proposition 5.4 *Let $\mathfrak{g}$ be a nilpotent K -Lie algebra, and let $\mathcal{L}$ be an $\mathcal{O}$ -Lie lattice in $\mathfrak{g}$ such that every locally closed prime ideal in $\widehat{U(\mathcal{L})}_K$ has the form $I(\lambda)_F$ for some finite extension F/K and some K -linear map $\lambda : \mathfrak{g} \rightarrow F$ such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$.*

Then given any prime ideal P in $\widehat{U(\mathcal{L})}_K$, P arises as an intersection of Dixmier annihilators.

Proof Note that since $\text{gr } \widehat{U(\mathcal{L})}_K \cong U(\mathcal{L}/\pi\mathcal{L})[t, t^{-1}]$ has finite left and right Krull dimension, it follows from [14, Ch.I Theorem 7.1.3] that $\widehat{U(\mathcal{L})}_K$ has finite left and right Krull dimension. Therefore, using [16, Lemma 6.4.5], it follows that $\widehat{U(\mathcal{L})}_K$ has finite *classical Krull dimension*, i.e. there is a finite upper bound on the length of chains of prime ideals in $\widehat{U(\mathcal{L})}_K$.

So, given a prime ideal P of R , define the *dimension* $\dim(P)$ of P to be the largest integer $n \geq 0$ such that there exists a chain of prime ideals $P = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of $\widehat{U(\mathcal{L})}_K$. We will proceed by induction on $\dim(P)$.

If $\dim(P) = 0$, then P is maximal, and hence locally closed, so $P = I(\lambda)_F$ for some finite extension F , $\lambda : \mathfrak{g} \rightarrow F$ as required. So suppose the result holds whenever $\dim(P) < n$.

If $\dim(P) = n$ then for every prime ideal Q of $\widehat{U(\mathcal{L})}_K$ with $P \subsetneq Q$, Q arises as an intersection of Dixmier annihilators by the inductive hypothesis.

If P is locally closed, then P is a Dixmier annihilator by assumption, otherwise P is equal to the intersection of all prime ideals properly containing it, and hence it is an intersection of Dixmier annihilators as required. $\square$

5.3 Classification

Now we will complete the classification of locally closed ideals of $\widehat{U(\mathcal{L})}_K$ in terms of Dixmier annihilators. First, we need some technical results:

Lemma 5.5 *Suppose that $\dim_K \mathfrak{g} > 1$, and let I be an ideal of $\widehat{U(\mathcal{L})}_K$. Suppose further that $Z(\widehat{U(\mathcal{L})}_K/I) = K$, and $I \cap \mathfrak{g} = 0$. Then $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$.*

Proof Firstly, suppose $u, v \in \mathfrak{g}$ are central, then $u + I, v + I \in Z(\widehat{U(\mathcal{L})}_K/I) = K$, hence they are K -linearly dependent. So there exist non-zero $\alpha, \beta \in K$ such that $\alpha u + \beta v \in I \cap \mathfrak{g} = 0$, hence u, v are K -linearly dependent in $\mathfrak{g}$.

Since $\mathfrak{g}$ is nilpotent, $Z(\mathfrak{g}) \neq 0$, so it follows that $Z(\mathfrak{g})$ has dimension 1, i.e. $Z(\mathfrak{g}) = Kz$ for some $z \in Z(\mathfrak{g})$.

So, since $\dim_K \mathfrak{g} > 1$, $\mathfrak{g}$ is non-abelian, and again using nilpotence of $\mathfrak{g}$, we can find $y \in \mathfrak{g}$ such that y is not central, but $[y, \mathfrak{g}] \subseteq Z(\mathfrak{g}) = Kz$.

Let $\mathfrak{g}' := \ker(\text{ad}(y))$. Then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$, and since $\text{ad}(y) : \mathfrak{g} \rightarrow Kz$ is non-zero, $\mathfrak{g}'$ must have codimension 1 in $\mathfrak{g}$, so $\mathfrak{g} = \mathfrak{g}' \oplus Kx$, and it is clear that $(x, y, z, \mathfrak{g}')$ is a reducing quadruple for $\mathfrak{g}$. $\square$

Proposition 5.6 *Let I be a two sided ideal of $\widehat{U(\mathcal{L})}_K$ such that $F = Z(\widehat{U(\mathcal{L})}_K/I)$ is a finite field extension of K . Then $U(\widehat{\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F})_F \cong \widehat{U(\mathcal{L})}_K \otimes_K F$, and there exists a surjection $U(\widehat{\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F})_F \rightarrow \widehat{U(\mathcal{L})}_K/I$ of F -algebras with kernel containing $I \otimes_K F$.*

Proof To see that $U(\widehat{\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F})_F \cong \widehat{U(\mathcal{L})}_K \otimes_K F$, note that:

$$\begin{aligned} U(\mathcal{L}) \otimes_{\mathcal{O}} \mathcal{O}_F &\leftrightarrow U(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F) \\ a \otimes \alpha &\mapsto \alpha a \\ u \otimes \alpha &\leftarrow u \otimes \alpha \end{aligned}$$

Are isomorphisms of $\mathcal{O}_F$ algebras, preserving the π -adic filtration. Hence they induce an isomorphism $U(\widehat{\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F}) \cong \widehat{U(\mathcal{L})} \otimes_{\mathcal{O}} \mathcal{O}_F$, and the result follows.

Since $F = Z(\widehat{U(\mathcal{L})}_K/I) \subseteq \widehat{U(\mathcal{L})}_K/I$, it is clear that $\widehat{U(\mathcal{L})}_K/I$ is an F -algebra, and the map $\widehat{U(\mathcal{L})}_K \otimes_K F \rightarrow \widehat{U(\mathcal{L})}_K/I, r \otimes (\alpha + I) \mapsto \alpha r + I$ is clearly a surjection of F -algebras sending $I \otimes F$ to 0 as required. $\square$

Lemma 5.7 *Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ nilpotent, let $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$ and let $\mathcal{L}_0 := \mathcal{L}/\mathcal{A}$. Let P be a prime ideal of $\widehat{U(\mathcal{L})}_K$, containing $\mathfrak{a}$, such that the image $P_0 \trianglelefteq \widehat{U(\mathcal{L}_0)}_K$ of P under the surjection $\widehat{U(\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L}_0)}_K$ is a Dixmier annihilator. Then P is a Dixmier annihilator.*

Proof We know that $P_0 = \text{Ann}_{\widehat{U(\mathcal{L}_0)}_K} \widehat{D(\mu)}_F$ for some finite extension F/K , $\mu \in (\mathcal{L}/\mathcal{A})_F^*$. Clearly μ is induced from a linear form λ of $\mathfrak{g} \otimes_K F$ such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$ and $\lambda(\mathfrak{a}) = 0$. We will prove that $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$.

Choose a polarisation $\mathfrak{b}$ of $\mathfrak{g} \otimes_K F$ at λ , and since the annihilator is independent of the choice of polarisation by Theorem 4.4, we may assume that $\mathfrak{a} \subseteq \mathfrak{b}$, i.e. $\mathfrak{b}/\mathfrak{a}$ is a polarisation of $\mathfrak{g}/\mathfrak{a}$ at μ . Using Lemma 3.1(iii), we see that $\widehat{D(\lambda)}_F = \widehat{U(\mathcal{L})}_F \otimes_{\widehat{U(\mathcal{B})}_F} F \cong \widehat{U(\mathcal{L}/\mathcal{A})}_F \otimes_{\widehat{U(\mathcal{B}/\mathcal{A})}_F} F = \widehat{D(\mu)}_F$.

Using Lemma 3.1(i), we know that $\widehat{U(\mathcal{L})}/\mathfrak{a}\widehat{U(\mathcal{L})} \cong \widehat{U(\mathcal{L}/\mathcal{A})}_K$, and hence $P_0 = P/\mathfrak{a}\widehat{U(\mathcal{L})}_K$. Therefore, since $P_0 = \text{Ann}_{\widehat{U(\mathcal{L}_0)}_K} \widehat{D(\mu)}_F$, and hence $P\widehat{D(\mu)}_F = 0$, it follows that $P\widehat{D(\lambda)}_F = 0$, i.e. $P \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$.

Moreover, if $x\widehat{D(\lambda)}_F = 0$ then $x\widehat{D(\mu)}_F = 0$ so $x + \mathfrak{a}\widehat{U(\mathcal{L})}_K \in P_0$ and hence $x \in P$. Therefore $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$ as required. $\square$

Now we can prove the main theorem of this section, classifying locally closed ideals in terms of Dixmier annihilators.

Theorem 5.8 *Let $\mathfrak{g}$ be a nilpotent K -lie algebra, with $\mathcal{O}$ -Lie lattice $\mathcal{L}$, and let P be a locally closed prime ideal of $\widehat{U(\mathcal{L})}_K$. Then there exists a finite extension F/K and a K -linear map $\lambda : \mathfrak{g} \rightarrow F$ with $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$ such that $P = I(\lambda)_F$.*

Proof We will use induction on $n = \dim_K \mathfrak{g}$:

First suppose that $n = 1$, and hence $\widehat{U(\mathcal{L})}_K \cong K\langle u \rangle$ by Lemma 2.1. So if P is a locally closed ideal, then it is primitive, and hence maximal since $\widehat{U(\mathcal{L})}_K$ is commutative. So let $F := \widehat{U(\mathcal{L})}_K/P$, then F is a field.

Furthermore, using [5, Corollary 2.2.12], we see that F is a finite extension of K , so define $\lambda : \mathfrak{g} \rightarrow F$, $x \mapsto x + P$, and clearly this map is K -linear. Also, $\widehat{U(\mathcal{L})}/P \cap \widehat{U(\mathcal{L})} = \mathcal{O}\langle u \rangle/P \cap \mathcal{O}\langle u \rangle$ is a lattice in $F = K\langle u \rangle/P$. Thus $\widehat{U(\mathcal{L})}/P \cap \widehat{U(\mathcal{L})} \subseteq \mathcal{O}_F$ so clearly $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$.

So $\widehat{D(\lambda)}_F = F$, where $x \in \widehat{U(\mathcal{L})}_K$ acts by zero if and only if $\lambda(x) = 0$, i.e. if and only if $x \in P$, so $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F = I(\lambda)_F$ as required.

So now suppose that the result holds whenever $\dim_K \mathfrak{g} < n$.

Again, suppose that P is a locally closed ideal of $\widehat{U(\mathcal{L})}_K$, and let $\mathfrak{a} := P \cap \mathfrak{g}$, $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$. Clearly $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, contained in P , and $\mathcal{A}$ is a Lie lattice in $\mathfrak{a}$. We will suppose first that $\mathfrak{a} \neq 0$.

Let P_0 be the image of P under the surjection $\widehat{U(\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L}/\mathcal{A})}_K$, then P_0 is a locally closed ideal of $\widehat{U(\mathcal{L}/\mathcal{A})}_K$. Since $\dim_K \mathfrak{g}/\mathfrak{a} < n$, it follows from induction that P_0 is a Dixmier annihilator. Therefore, using Lemma 5.7, P is a Dixmier annihilator as required.

So from now on we may assume that $\mathfrak{a} = P \cap \mathfrak{g} = 0$.

Since we know by Proposition 5.3 that P is primitive, it follows from [17, Theorem 6.4.6] that $F = Z(\widehat{U(\mathcal{L})}_K/P)$ is an algebraic field extension of K . Since the centre of $\widehat{U(\mathcal{L})}_K/P$ is closed and $\widehat{U(\mathcal{L})}_K/P$ is complete, it follows that F is complete, so it must in fact be a finite extension of K .

We will assume for now that $F = K$, so applying Lemma 5.5, we see that $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$. So let $\mathcal{L}' := \mathfrak{g}' \cap \mathcal{L}$, then since $z \notin P$, it is clear that $z + P \in Z(\widehat{U(\mathcal{L})}_K/P) = K$ is not a zero divisor, so using Theorem 4.2, we see that P is controlled by $\mathcal{L}'$, i.e. $P = \widehat{U(\mathcal{L})}_K(P \cap \widehat{U(\mathcal{L}')}_K)$.

Let $Q := P \cap \widehat{U(\mathcal{L}')}_K$, then Q is a semiprime ideal of $\widehat{U(\mathcal{L}')}_K$, so since all locally closed prime ideals in $\widehat{U(\mathcal{L}')}_K$ are Dixmier annihilators by induction, it follows from Proposition 5.4 that all semiprime ideals arise as an intersection of Dixmier annihilators, i.e. there exist finite extensions F_j/K , $\mu_j \in (\mathcal{L}')_{F_j}^*$, as j ranges over some indexing set X , and

$$Q = \bigcap_{j \in X} I(\mu_j)_{F_j}.$$

Since $z \notin Q$ and $Z(\widehat{U(\mathcal{L})}_K/P) = K$, there exists $0 \neq \beta \in K$ such that $z - \beta \in Q$. Therefore $z - \beta \in I(\mu_j)_{F_j}$ for each j . Since $\beta \neq 0$, this means that $z \notin I(\mu_j)_{F_j}$, i.e. $\mu_j(z) \neq 0$.

Now, it is clear that $(x \otimes 1, y \otimes 1, z \otimes 1, \mathfrak{g}' \otimes_K F_j)$ is a reducing quadruple for $\mathfrak{g} \otimes_K F_j$, so applying Proposition 2.7(ii) gives that if $\mathfrak{b}$ is a polarisation of $\mathfrak{g}' \otimes_K F_j$ at μ_j and λ_j is an extension of μ_j to $\mathfrak{g} \otimes_K F_j$, then $\mathfrak{b}$ is a polarisation of $\mathfrak{g} \otimes_K F_j$ at λ_j .

Therefore, $\widehat{D(\lambda_j)}_{F_j} \cong \widehat{U(\mathcal{L})}_{F_j} \otimes_{\widehat{U(\mathcal{L}')}_{F_j}} \widehat{D(\mu_j)}_{F_j}$, so by Lemma 4.1, $I(\lambda_j)_{F_j} = \text{Ann}_{\widehat{U(\mathcal{L})}_{F_j}} \widehat{D(\lambda_j)}_{F_j}$ is the largest two-sided ideal of $\widehat{U(\mathcal{L})}_{F_j}$ contained in $\widehat{U(\mathcal{L})}_{F_j} \text{Ann}_{\widehat{U(\mathcal{L}')}_F} \widehat{D(\mu_j)}_{F_j}$.

But $P = \widehat{U(\mathcal{L})}_K Q \subseteq \widehat{U(\mathcal{L})}_K I(\mu_j)_{F_j}$, and by Proposition 5.6, $\widehat{U(\mathcal{L})}_{F_j} = \widehat{U(\mathcal{L})}_K \otimes_K F_j$, hence $P \otimes_K F_j \subseteq \widehat{U(\mathcal{L})}_{F_j} I(\mu_j)_{F_j} \subseteq \widehat{U(\mathcal{L})}_{F_j} \text{Ann}_{\widehat{U(\mathcal{L}')}_{F_j}} \widehat{D(\mu_j)}_{F_j}$.

Thus $P \otimes_K F_j \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_{F_j}} \widehat{D(\lambda_j)}_{F_j}$ and $P \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_j)}_{F_j} = I(\lambda_j)_{F_j}$.

Furthermore, given $r \in \bigcap_{j \in X} I(\lambda_j)_{F_j}$, we have that $r = \sum_{i \geq 0} x^i r_i$ for some $r_i \in \widehat{U(\mathcal{L}')}_K$ by Lemma 2.1, with $r_i \rightarrow 0$ as $i \rightarrow \infty$. Then since each $I(\lambda_j)_{F_j}$ is a prime ideal of $\widehat{U(\mathcal{L})}_K$, and $z \notin I(\lambda_j)_{F_j}$, it follows from Theorem 4.2 that each r_i lies in $I(\lambda_j)_{F_j}$ for every j .

This means that $r_i \widehat{D(\lambda_j)}_{F_j} = 0$ for all i, j , so $r_i \widehat{D(\mu_j)}_{F_j} = 0$ and thus $r_i \in \bigcap_{j \in X} I(\mu_j)_{F_j} = Q$ for every i . Therefore $r \in \widehat{U(\mathcal{L})}_K Q = P$. Since our choice of r was arbitrary, it follows that:

$$P = \bigcap_{j \in X} I(\lambda_j)_{F_j}.$$

Since P is locally closed and each $I(\lambda_j)_{F_j}$ is a prime ideal of $\widehat{U(\mathcal{L})}_K$ containing P , it follows that $P = I(\lambda_j)_{F_j}$ for some $j \in X$ as we require.

Finally, take P to be a general locally closed prime ideal. Then $F = Z(\widehat{U(\mathcal{L})}_K/P)$ is a finite extension of K , so let $\mathfrak{g}_0 := \mathfrak{g} \otimes_K F$, $\mathcal{L}_0 := \mathcal{L} \otimes_O \mathcal{O}_F$. Then $\dim_F \mathfrak{g}_0 = \dim_K \mathfrak{g} = n$, $\mathcal{L}_0$ is a Lie lattice in $\mathfrak{g}_0$, and by Proposition 5.6, there exists a surjection of F -algebras

$\widehat{U(\mathcal{L}_0)}_F = \widehat{U(\mathcal{L})}_K \otimes_K F \rightarrow \widehat{U(\mathcal{L})}_K/P$ whose kernel contains $P \otimes_K F$. Let J be this kernel.

Then J is a locally closed prime ideal of $\widehat{U(\mathcal{L}_0)}_F$ and $\widehat{U(\mathcal{L}_0)}_F/J \cong \widehat{U(\mathcal{L})}_K/P$. But $Z(\widehat{U(\mathcal{L})}_F/J) \cong Z(\widehat{U(\mathcal{L})}_K/P) = F$ so it follows from the above discussion that $J = \text{Ann}_{\widehat{U(\mathcal{L}_0)}_F} \widehat{D(\lambda)}_{F'}$ for some finite extension F'/F and some linear form λ of $\mathfrak{g}_0 \otimes_F F'$ such that $\lambda(\mathcal{L}_0) \subseteq \mathcal{O}_{F'}$.

It is clear that $J \cap \widehat{U(\mathcal{L})}_K = P$, and hence $P = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_{F'} = I(\lambda)_{F'}$ as required. $\square$

Corollary 5.9 *Let $\mathcal{L}$ be a Lie lattice in $\mathfrak{g}$ nilpotent. Then given a prime ideal P of $\widehat{U(\mathcal{L})}_K$, P arises as an intersection of Dixmier annihilators.*

Proof This is immediate from Theorem 5.8 and Proposition 5.4. $\square$

6 Weakly Rational Ideals

In this section, we will prove our main result, Theorem A, which allows us to describe all weakly rational ideals in $\widehat{U(\mathcal{L})}_K$ in terms of Dixmier annihilators.

6.1 Dixmier Annihilators under Isomorphisms

In Section 2, we saw how the adjoint algebraic group $\mathbb{G}$ of $\mathfrak{g}$ acts on $\mathfrak{g}^*$ via the coadjoint action. More generally, if σ is any Lie automorphism of $\mathfrak{g}$, then for any λ in $\mathfrak{g}^*$, we can similarly define $\sigma \cdot \lambda : \mathfrak{g} \rightarrow K, u \mapsto \lambda(\sigma^{-1}(u))$. Note that if $\mathcal{L}$ is a Lie lattice in $\mathfrak{g}$ then $\sigma(\mathcal{L})$ also is.

More generally, let us suppose that $\mathcal{L}_1, \mathcal{L}_2$ are Lie lattices in $\mathfrak{g}$ and $\sigma : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is an $\mathcal{O}$ -linear Lie isomorphism. Then σ extends to a K -linear isomorphism $\sigma : \widehat{U(\mathcal{L}_1)}_K \rightarrow \widehat{U(\mathcal{L}_2)}_K$ of affinoid enveloping algebras.

Given a linear form $\lambda \in \text{Hom}_{\mathcal{O}}(\mathcal{L}_1, \mathcal{O})$, $\sigma \cdot \lambda \in \text{Hom}_{\mathcal{O}}(\mathcal{L}_2, \mathcal{O})$, and if $\mathcal{B}_1$ is a polarisation at λ then $\mathcal{B}_2 = \sigma \mathcal{B}_1$ is a polarisation at $\sigma \cdot \lambda$.

Lemma 6.1 *Let $I(\lambda) := \text{Ann}_{\widehat{U(\mathcal{L}_1)}_K} \widehat{D(\lambda)}_{\mathcal{B}_1} \trianglelefteq \widehat{U(\mathcal{L}_1)}_K$, $I(\sigma \cdot \lambda) := \text{Ann}_{\widehat{U(\mathcal{L}_2)}_K} \widehat{D(\sigma \cdot \lambda)}_{\mathcal{B}_2} \trianglelefteq \widehat{U(\mathcal{L}_2)}_K$. Then $\sigma(I(\lambda)) = I(\sigma \cdot \lambda)$*

Proof By definition, $\widehat{D(\lambda)}_{\mathcal{B}_1} = \widehat{U(\mathcal{L}_1)}_K \otimes_{\widehat{U(\mathcal{B}_1)}_K} K_\lambda$ and $\widehat{D(\sigma \cdot \lambda)}_{\mathcal{B}_2} = \widehat{U(\mathcal{L}_2)}_K \otimes_{\widehat{U(\mathcal{B}_2)}_K} K_{\sigma \cdot \lambda} = \widehat{U(\sigma \mathcal{L}_1)}_K \otimes_{\widehat{U(\sigma \mathcal{B}_1)}_K} K_{\sigma \cdot \lambda}$. So consider the map $\Theta : \widehat{D(\lambda)}_{\mathcal{B}_1} \rightarrow \widehat{D(\sigma \cdot \lambda)}_{\mathcal{B}_2}, x \otimes v \mapsto \sigma(x) \otimes v$.

We will show that Θ is a K -linear isomorphism such that $\Theta(xm) = \sigma(x)\Theta(m)$ for all $x \in \widehat{U(\mathcal{L})}_K, m \in \widehat{D(\lambda)}_{\mathcal{B}_1}$. It will follow from this that $x\widehat{D(\lambda)}_{\mathcal{B}_1} = 0$ if and only if $\sigma(x)\widehat{D(\sigma \cdot \lambda)}_{\mathcal{B}_2} = 0$, and hence $\sigma(I(\lambda)) = I(\sigma \cdot \lambda)$ as required.

It is clear that Θ is K -linear, and that it has an inverse defined by $x \otimes v \mapsto \sigma^{-1}(x) \otimes v$, hence it is an isomorphism of vector spaces.

Finally, $\Theta(x(y \otimes v)) = \Theta(xy \otimes v) = \sigma(xy) \otimes v = \sigma(x)(\sigma(y) \otimes v) = \sigma(x)\Theta(y \otimes v)$. $\square$

This result becomes particularly useful when comparing Dixmier annihilators, particularly using the following lemma.

Lemma 6.2 *Suppose $\lambda, \mu \in \text{Hom}_{\mathcal{O}}(\mathcal{L}, \mathcal{O})$ such that $I(\lambda) \cap U(\mathfrak{g}) = I(\mu) \cap U(\mathfrak{g})$, e.g. if $I(\lambda) \subseteq I(\mu)$. Then there exists $g \in \mathbb{G}(K)$ such that $\mu = g \cdot \lambda$.*

Proof Since the classical Dixmier module $D(\lambda)$ is dense in $\widehat{D(\lambda)}$, it follows that $r\widehat{D(\lambda)} = 0$ if and only if $rD(\lambda) = 0$, and hence $I(\lambda) \cap U(\mathfrak{g}) = \text{Ann}_{U(\mathfrak{g})}D(\lambda)$. Therefore, if $I(\lambda) \cap U(\mathfrak{g}) = I(\mu) \cap U(\mathfrak{g})$ then $\text{Ann}_{U(\mathfrak{g})}D(\lambda) = \text{Ann}_{U(\mathfrak{g})}D(\mu)$, and using [7, Proposition 6.2.3], it follows that λ and μ lie in the same coadjoint orbit as required.

Also, if $I(\lambda) \subseteq I(\mu)$ then $\text{Ann}_{U(\mathfrak{g})}D(\lambda) \subseteq \text{Ann}_{U(\mathfrak{g})}D(\mu)$. But $\text{Ann}_{U(\mathfrak{g})}D(\lambda)$ is a weakly rational ideal of $U(\mathfrak{g})$, and hence it is maximal by [7, Proposition 4.7.4], therefore $I(\lambda) \cap U(\mathfrak{g}) = \text{Ann}_{U(\mathfrak{g})}D(\lambda) = \text{Ann}_{U(\mathfrak{g})}D(\mu) = I(\mu) \cap U(\mathfrak{g})$. $\square$

6.2 The Coadjoint Action

Now, let $\lambda, \mu : \mathfrak{g} \rightarrow F$ be linear forms such that $\lambda(\mathcal{L}), \mu(\mathcal{L}) \subseteq \mathcal{O}_F$ for some finite extension F/K . We want to compare the Dixmier annihilators $I(\lambda)$ and $I(\mu)$ in $\widehat{U(\mathcal{L})}_K$ in the case where λ and μ lie in the same coadjoint orbit, i.e. $\mu = g \cdot \lambda$ for some $g \in \mathbb{G}(F)$. Explicitly, $g = \exp(\text{ad}(u))$ for some $u \in \mathfrak{g} \otimes_K F$.

Our first results ensure that it is sufficient to consider the case where $F = K$.

Proposition 6.3 *Let F/K be a finite extension, and let $\lambda : \mathfrak{g} \rightarrow K$ be K -linear. Then there exists a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ such that $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g} \otimes_K F$ at the extension $\lambda_F : \mathfrak{g} \otimes_K F \rightarrow F$.*

Proof Using induction on $\dim(\mathfrak{g})$. If $\dim(\mathfrak{g}) = 1$ then it is obvious, because $\mathfrak{g}$ and $\mathfrak{g} \otimes F$ are the only polarisations. So suppose the result holds whenever $\dim(\mathfrak{g}) < n$.

If $\lambda(\mathfrak{a}) = 0$ for some non-zero ideal $\mathfrak{a}$ of $\mathfrak{g}$, then using induction we may choose a polarisation $\mathfrak{a} \subseteq \mathfrak{b}$ such that $\frac{\mathfrak{b}}{\mathfrak{a}} \otimes_K F = \frac{\mathfrak{b} \otimes_K F}{\mathfrak{a} \otimes_K F}$ is a polarisation for $\frac{\mathfrak{g} \otimes_K F}{\mathfrak{a} \otimes_K F}$ at λ_F . Hence $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g} \otimes_K F$ at λ_F .

So from now on, we may assume that $\lambda(\mathfrak{a}) = 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$. Then it follows from Proposition 2.7 that $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$, and $\lambda(z) \neq 0$. Clearly $(x \otimes 1, y \otimes 1, z \otimes 1, \mathfrak{g}' \otimes_K F)$ is a reducing quadruple for $\mathfrak{g} \otimes_K F$.

Let $\mathfrak{b}$ be a polarisation for $\mathfrak{g}'$ at $\lambda|_{\mathfrak{g}'}$ such that $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g}' \otimes_K F$. Then using Lemma 2.7 we see that $\mathfrak{b}$ is a polarisation for $\mathfrak{g}$ at λ , and $\mathfrak{b} \otimes_K F$ is a polarisation for $\mathfrak{g} \otimes_K F$ at λ_F as required. $\square$

Corollary 6.4 *Let F/K be a finite extension, and let $\lambda : \mathfrak{g} \rightarrow F$ be K -linear such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$. Then for any finite extension L/F , $I(\lambda)_F = I(\lambda)_L$.*

Proof This is immediate from Proposition 6.3 and Theorem 4.4. $\square$

So, from now on, we will assume that λ and μ take values in K , and $\mu = \exp(\text{ad}(u)) \cdot \lambda$ for some $u \in \mathfrak{g}$. Let $\sigma := \exp(\text{ad}(u)) \in \mathbb{G}(K)$, and fix a natural number $N \in \mathbb{N}$ such that $u \in p^{-N}\mathcal{L}$. Also let c be the nilpotency class of $\mathfrak{g}$, i.e. c is minimal such that $\text{ad}(\mathfrak{g})^c = 0$.

Since σ is a Lie automorphism of $\mathfrak{g}$, it follows that $\sigma\mathcal{L}$ is an $\mathcal{O}$ -Lie lattice in $\mathfrak{g}$, hence there exists a natural number $n \in \mathbb{N}$ such that $p^n\mathcal{L} \subseteq \sigma\mathcal{L}$ and $p^n\sigma\mathcal{L} \subseteq \mathcal{L}$.

Lemma 6.5 *For any $n \geq cN + v_p(c!)$, $p^n\mathcal{L} \subseteq \sigma\mathcal{L}$ and $p^n\sigma\mathcal{L} \subseteq \mathcal{L}$.*

Proof Since $\sigma = \exp(\text{ad}(u))$, where $u = p^{-N}v$ for some $v \in \mathcal{L}$, it follows that for all $w \in \mathcal{L}$:

$$\sigma(w) = w + p^{-N}[v, w] + \frac{1}{2}p^{-2N}[v, [v, w]] + \cdots + \frac{1}{c!}p^{-cN}(\text{ad}(v))^c(w) \quad (7)$$

But for each $0 \leq i \leq c$, $(\text{ad}(v))^i(w) \in \mathcal{L}$, $v_p(\frac{1}{i!}p^{-iN}) = -iN - v_p(i!) \geq -cN - v_p(c!) \geq -n$, so $\frac{1}{i!}p^{-iN}(\text{ad}(v))^i(w) \in p^{-n}\mathcal{L}$. Hence $\sigma\mathcal{L} \subseteq p^{-n}\mathcal{L}$, and $p^n\sigma\mathcal{L} \subseteq \mathcal{L}$.

Also, σ is an isomorphism, and $\sigma^{-1} = \exp(\text{ad}(-u))$, with $-u \in p^{-N}\mathcal{L}$. So since $\sigma^{-1} : \sigma\mathcal{L} \rightarrow \mathcal{L}$ is a Lie-isomorphism, it follows from the above discussion that $p^n\mathcal{L} \subseteq \sigma\mathcal{L}$. $\square$

It is clear that since $\sigma : \mathcal{L} \rightarrow \sigma\mathcal{L}$ is a continuous isomorphism of $\mathcal{O}$ -Lie lattices, it extends to a continuous isomorphism $\sigma : \widehat{U(\mathcal{L})}_K \rightarrow \widehat{U(\sigma\mathcal{L})}_K$ of K -algebras. Moreover, for any $n \in \mathbb{N}$, σ induces an isomorphism $\sigma : \widehat{U(p^n\mathcal{L})}_K \rightarrow \widehat{U(p^n\sigma\mathcal{L})}_K$, and thus using Lemma 6.5, for $n \geq cN + v_p(c!)$, there is an injective K -algebra homomorphism $\sigma : \widehat{U(p^n\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L})}_K$.

Proposition 6.6 *Given $n \in \mathbb{N}$ such that $n \geq cN + v_p(c!)$, if I is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, then $\sigma : \widehat{U(p^n\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L})}_K$ maps $I \cap \widehat{U(p^n\mathcal{L})}_K$ into I .*

Proof Consider the sequence of continuous $\mathcal{O}$ -linear maps $\sigma_i := \sum_{0 \leq j \leq i} \frac{1}{j!}(\text{ad}(u))^j : \widehat{U(p^n\mathcal{L})}_K \rightarrow \widehat{U(\mathcal{L})}_K$. Clearly each of these sends $I \cap \widehat{U(p^n\mathcal{L})}_K$ into I .

We will show that σ_i converges pointwise to σ as $i \rightarrow \infty$, and since all ideals in $\widehat{U(\mathcal{L})}_K$ are closed, it will follow that $\sigma(I \cap \widehat{U(p^n\mathcal{L})}_K) \subseteq I$.

Let $\delta := \text{ad}(u)$, and let v be the p -adic filtration on $\widehat{U(\mathcal{L})}_K$ induced from $\widehat{U(\mathcal{L})}$. Then for all $u \in \mathcal{L}$, $v(\delta(u)) \geq v(u) - N$.

Since δ is a derivation, a standard inductive argument shows that for all $a_1, \dots, a_r \in \widehat{U(\mathcal{L})}_K$:

$$\sum_{0 \leq j \leq i} \frac{1}{j!} \delta^j(a_1 a_2 \cdots a_r) = \sum_{0 \leq j \leq i} \left(\sum_{j_1 + \cdots + j_r = j} \left(\prod_{1 \leq m \leq r} \frac{1}{j_m!} \delta^{j_m}(a_j) \right) \right) \quad (8)$$

So, if $x \in \widehat{U(\mathcal{L})}_K$, then $x = \sum_{(r,u)} \lambda_u u_1 \cdots u_r$, where the sum is taken over all $r \geq 0$, $u = u_1 \cdots u_r$ for $u_i \in \mathcal{L}$, and $v_p(\lambda_u) - nr \rightarrow \infty$ as $r \rightarrow \infty$. Therefore, fixing $t \in \mathbb{N}$, we have:

$$(\sigma - \sigma_t)(x) = \sum_{(r,u)} \lambda_u (\sigma - \sigma_t)(u_1 \cdots u_r) = \sum_{(r,u)} \lambda_u \left(\sum_{j > t} \left(\sum_{j_1 + \cdots + j_r = j} \left(\prod_{1 \leq m \leq r} \frac{1}{j_m!} \delta^{j_m}(u_m) \right) \right) \right) \quad (9)$$

For each $r \geq 0$, let $A_r := \{\alpha \in [c]^r : |\alpha| > t\}$, where $[c] = \{0, \dots, c-1\}$. Note that $\delta^c(u) = 0$ for all $u \in \mathcal{L}$.

Then $(\sigma - \sigma_t)(x) = \sum_{(r,u)} \lambda_u \left(\sum_{\alpha \in A_r} \prod_{1 \leq m \leq r} \frac{1}{\alpha_m!} \delta^{\alpha_m}(u_m) \right)$, and since A_r is finite, $\sum_{\alpha \in A_r} \prod_{1 \leq m \leq r} \frac{1}{\alpha_m!} \delta^{\alpha_m}(u_m)$ is a finite sum.

Since $\alpha_m < c$ for all $\alpha \in A_r$, we have that $v_p(\alpha_m!) \leq v_p(c!)$. Also, since $v(\delta(u)) \geq v(u) - N$, it follows that $v(\delta^{\alpha_m}(u)) \geq v(u) - \alpha_m N$. Therefore $v(\frac{1}{\alpha_m!} \delta^{\alpha_m}(u_m)) \geq v(u_m) - \alpha_m N - v_p(c!)$ for all $m \leq r$.

Thus for each pair (r, u) , $v(\sum_{\alpha \in A_r} \prod_{1 \leq m \leq r} \frac{1}{\alpha_m!} \delta^{\alpha_m}(u_m)) \geq v(u_1) + \dots + v(u_r) - (|\alpha|N + v_p(c!)r) \geq -(|\alpha|N + rv_p(c!)) \geq -r(cN + v_p(c!))$, the last inequality follows since $|\alpha| \leq rc$.

Therefore, $v\left(\lambda_u \left(\sum_{\alpha \in A_r} \prod_{1 \leq m \leq r} \frac{1}{\alpha_m!} \delta^{\alpha_m}(u_m) \right)\right) \geq v_p(\lambda_u) - r(cN + v_p(c!)) \geq v_p(\lambda_u) - nr \rightarrow \infty$ as $r \rightarrow \infty$.

Moreover, for $r \leq \frac{t}{c}$, $A_r = \emptyset$, so we have:

$$(\sigma - \sigma_t)(x) = \sum_{(r,u), r > \frac{t}{c}} \lambda_u \left(\sum_{\alpha \in A_r} \prod_{1 \leq m \leq r} \frac{1}{\alpha_m!} \delta^{\alpha_m}(u_m) \right) \quad (10)$$

Therefore, $v((\sigma - \sigma_t)(x)) \geq \inf\{v_p(\lambda_u) - nr : u = u_1 \dots u_r \text{ with } r > \frac{t}{c}\}$, and this tends to infinity as $t \rightarrow \infty$. Hence $(\sigma - \sigma_t)(x) \rightarrow 0$ as $t \rightarrow \infty$.

So $\sigma(x) = \lim_{t \rightarrow \infty} \sigma_t(x)$, so if $x \in I$ then $\sigma(x) \in I$ as required. $\square$

Now, let $\mathfrak{b}$ be a polarisation for $\mathfrak{g}$ at λ , and let $\mathfrak{b}'$ be a polarisation for $\mathfrak{g}$ at μ . Since $\mu = \sigma \cdot \lambda$, it follows that $\sigma \mathfrak{b}$ is also a polarisation for $\mathfrak{g}$ at μ . Also, it is clear that $\sigma \mathfrak{b} \cap \sigma \mathcal{L} = \sigma(\mathfrak{b} \cap \mathcal{L})$, so let $\mathcal{B} := \mathfrak{b} \cap \mathcal{L}$ and $\mathcal{B}' := \mathfrak{b}' \cap \mathcal{L}$.

Consider the Dixmier modules $\widehat{D(\lambda)}_{\mathcal{B}} := \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B})}_K} K_{\lambda}$, $\widehat{D(\mu)}_{\sigma \mathcal{B}} := \widehat{U(\sigma \mathcal{L})}_K \otimes_{\widehat{U(\sigma \mathcal{B})}_K} K_{\mu}$, $\widehat{D(\mu)}_{\mathcal{B}'} := \widehat{U(\mathcal{L})}_K \otimes_{\widehat{U(\mathcal{B}')}_K} K_{\mu}$.

Then $\widehat{D(\lambda)}_{\mathcal{B}}$ and $\widehat{D(\mu)}_{\mathcal{B}'}$ are $\widehat{U(\mathcal{L})}_K$ -modules, topological completions of the $U(\mathfrak{g})$ -modules $D(\lambda)_{\mathfrak{b}}$ and $D(\mu)_{\mathfrak{b}'}$ respectively, while $\widehat{D(\mu)}_{\sigma \mathcal{B}}$ is a $\widehat{U(\sigma \mathcal{L})}_K$ -module, a topological completion of $D(\mu)_{\sigma \mathfrak{b}}$.

Let $I(\mu) := \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu)}_{\mathcal{B}'} \trianglelefteq \widehat{U(\mathcal{L})}_K$, and let $I'(\mu) := \text{Ann}_{\widehat{U(\sigma \mathcal{L})}_K} \widehat{D(\mu)}_{\sigma \mathcal{B}} \trianglelefteq \widehat{U(\sigma \mathcal{L})}_K$.

Lemma 6.7 $I'(\mu) = \sigma(I(\lambda))$, and given $n \in \mathbb{N}$ such that $p^n \mathcal{L} \subseteq \sigma \mathcal{L}$, $I(\mu) \cap \widehat{U(p^n \mathcal{L})}_K = I'(\mu) \cap \widehat{U(p^n \mathcal{L})}_K$.

Proof Since $\sigma : \mathcal{L} \rightarrow \sigma \mathcal{L}$ is a Lie isomorphism, it follows from Lemma 6.1 that $I'(\mu) = \sigma(I(\lambda))$.

Let $\mathcal{C} := \sigma \mathfrak{b} \cap p^n \mathcal{L}$, then the $\widehat{U(p^n \mathcal{L})}_K$ -affinoid Dixmier module $\widehat{D(\mu)}_{\mathcal{C}} := \widehat{U(p^n \mathcal{L})}_K \otimes_{\widehat{U(\mathcal{C})}_K} K_{\mu}$ embeds densely into $\widehat{D(\mu)}_{\sigma \mathcal{B}}$. So if $x \in \widehat{U(p^n \mathcal{L})}_K$ then $x \widehat{D(\mu)}_{\mathcal{C}} = 0$ if and only if $x \widehat{D(\mu)}_{\sigma \mathcal{B}} = 0$.

But since $\sigma\mathfrak{b}$ and $\mathfrak{b}'$ are polarisations of $\mathfrak{g}$ at μ with $\sigma\mathfrak{b} \cap p^n\mathcal{L} = C$ and $\mathfrak{b}' \cap p^n\mathcal{L} = p^n\mathcal{B}'$, we can apply Theorem 4.4 to get that $\text{Ann}_{\widehat{U(p^n\mathcal{L})}_K} \widehat{D(\mu)}_C = \text{Ann}_{\widehat{U(p^n\mathcal{L})}_K} \widehat{D(\mu)}_{p^n\mathcal{B}'}$. Therefore, given $x \in \widehat{U(p^n\mathcal{L})}_K$:

$$x\widehat{D(\mu)}_{\sigma\mathcal{B}} = 0 \iff x\widehat{D(\mu)}_C = 0 \iff \widehat{D(\mu)}_{p^n\mathcal{B}'} = 0 \iff x\widehat{D(\mu)}_{\mathcal{B}'} = 0$$

Therefore $I(\mu) \cap \widehat{U(p^n\mathcal{L})}_K = I'(\mu) \cap \widehat{U(p^n\mathcal{L})}_K$ as required. $\square$

Now we can prove the main result of this section, which allows us to compare Dixmier annihilators for $\lambda, \mu \in \mathfrak{g}^*$ in the same coadjoint orbit:

Theorem 6.8 *Let $\mathfrak{g}$ be a nilpotent K -Lie algebra, with nilpotency class c , and let $\mathcal{L}$ be an $\mathcal{O}$ -Lie lattice in $\mathfrak{g}$. Let $\lambda, \mu : \mathfrak{g} \rightarrow \overline{K}$ be K -linear maps such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$, and suppose that $\mu = \exp(\text{ad}(u)) \cdot \lambda$ for some $u \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$. Then given $n \in \mathbb{N}$ such that $n \geq Nc + v_p(c!)$, $I(\lambda) \cap \widehat{U(p^{2n}\mathcal{L})}_K = I(\mu) \cap \widehat{U(p^{2n}\mathcal{L})}_K$.*

Proof Firstly, note that u lies in $p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_F)$ for some finite extension F of K , and we may assume further that λ and μ take values in F , possibly after extending F .

Using Corollary 6.4, we see that $I(\lambda) = I(\lambda)_F$, i.e. $I(\lambda) = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda)}_F$, and similarly $I(\mu) = \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu)}_F$, so we may safely assume that $F = K$. So λ and μ are K -linear forms of $\mathfrak{g}$, $u \in p^{-N}\mathcal{L}$, and setting $\sigma := \exp(\text{ad}(u))$, since $n \geq Nc + v(c!)$ we see using Lemma 6.5 that $p^n\mathcal{L} \subseteq \sigma\mathcal{L}$ and $p^n\sigma\mathcal{L} \subseteq \mathcal{L}$.

We will prove that $I(\mu) \cap \widehat{U(p^{2n}\mathcal{L})}_K \subseteq I(\lambda)$, and after replacing σ by $\sigma^{-1} = \exp(\text{ad}(-u))$, it will follow that $I(\lambda) \cap \widehat{U(p^{2n}\mathcal{L})}_K \subseteq I(\mu)$ as required.

By Lemma 6.7, we see that $I(\mu) \cap \widehat{U(p^{2n}\mathcal{L})}_K = \sigma(I(\lambda)) \cap \widehat{U(p^{2n}\mathcal{L})}_K \subseteq \sigma(I(\lambda)) \cap \widehat{U(p^n\sigma\mathcal{L})}_K = \sigma(I(\lambda) \cap \widehat{U(p^n\mathcal{L})}_K)$.

But since $I(\lambda)$ is a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, it follows from Proposition 6.6 that $\sigma(I(\lambda) \cap \widehat{U(p^n\mathcal{L})}_K) \subseteq I(\lambda)$, and hence $I(\mu) \cap \widehat{U(p^{2n}\mathcal{L})}_K \subseteq I(\lambda)$ as required. $\square$

6.3 Proof of Theorem A

Now, let P be a weakly rational ideal of $\widehat{U(\mathcal{L})}_K$. Using Theorem 5.8 and Proposition 5.4, we see that P arises as an intersection of Dixmier annihilators:

$$P = \bigcap_{j \in X} I(\lambda_j)$$

for some $\lambda_j : \mathfrak{g} \rightarrow \overline{K}$ K -linear, such that $\lambda_j(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$ for each j .

Also, $P \cap U(\mathfrak{g})$ is a weakly rational ideal of $U(\mathfrak{g})$ and hence it is maximal by [7, Proposition 4.7.4]. So since $P \cap U(\mathfrak{g}) \subseteq I(\lambda_j) \cap U(\mathfrak{g})$ for all j , it follows that $I(\lambda_j) \cap U(\mathfrak{g}) = I(\lambda_k) \cap U(\mathfrak{g})$ for all $j, k \in X$.

Therefore, using Lemma 6.2, this means that for every $j, k \in X$, there exists $a_{j,k} \in \mathbb{G}(\overline{K})$ such that $a_{j,k} \cdot \lambda_j = \lambda_k$, i.e. all λ_j lie in the same coadjoint orbit.

Proposition 6.9 *Let $\lambda : \mathfrak{g} \rightarrow \overline{K}$ be a K -linear map such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$. Then there exists an integer $N \geq 0$ such that for any linear form $\mu : \mathfrak{g} \rightarrow \overline{K}$ in the $\mathbb{G}$ -coadjoint orbit of λ with $\mu(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$, $\mu = \exp(\text{ad}(u)) \cdot \lambda$ for some $u \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$.*

Proof Let Y be the coadjoint orbit in $\mathfrak{g}^* = \text{Hom}_K(\mathfrak{g}, \overline{K})$ containing λ , and recall from Lemma 2.14 that there exists an affine algebraic subgroup S of $\mathbb{G}$ such that $\mathbb{G} \cong S \times Y$ as varieties, where the natural morphism $\mathbb{G} \rightarrow Y, g \mapsto g \cdot \lambda$ is just the projection to the second factor. Consider the following sequence in the category of $\overline{K}$ -varieties defined over K :

$$\text{ad}(\mathfrak{g}) \xrightarrow{\exp} \mathbb{G} \cong S \times Y \rightarrow Y. \quad (11)$$

Let U be the set of all $\mu \in Y$ such that $\mu(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$. Then U is an affinoid subdomain of Y (when Y is considered as a rigid variety) isomorphic to $\text{Sp } \widehat{S(\mathfrak{g})}$, where $\widehat{S(\mathfrak{g})}$ is the π -adic completion of the symmetric algebra $S(\mathfrak{g})$ with respect to the lattice $S(\mathcal{L})$. Since $\exp$ is an isomorphism, we may take the inverse image $V := \exp^{-1}(1 \times U)$ of $1 \times U$, which will be an affinoid subdomain of $\text{ad}(\mathfrak{g})$.

But $\text{ad}(\mathfrak{g}) \cong \mathfrak{g}/Z(\mathfrak{g})$ is a union of open discs containing $p^{-n}(\mathcal{L}/Z(\mathcal{L}))$ for $n \in \mathbb{N}$. So since V is affinoid, it follows that V is contained in $p^{-N}(\mathcal{L}/Z(\mathcal{L}) \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$ for some $N \in \mathbb{N}$.

Therefore, for any $\mu \in U$, we can choose $u \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$ such that μ is the image of $\text{ad}(u)$ under the composition $\text{ad}(\mathfrak{g}) \rightarrow \mathbb{G} \rightarrow Y$, i.e. $\mu = \exp(\text{ad}(u)) \cdot \lambda$ as required. $\square$

Now we can finally prove our main theorem:

Proof of Theorem A. Let P be a weakly rational ideal of $\widehat{U(\mathcal{L})}_K$, then since P is prime, we see using Corollary 5.9 that $P = \bigcap_{j \in X} I(\lambda_j)$ for linear forms λ_j all lying in the same coadjoint orbit.

Since $\lambda_j(\mathcal{L}) \subseteq \mathcal{O}_{\overline{K}}$ for each j , it follows from Proposition 6.9 that we can choose $N \in \mathbb{N}$, $u_{j,k} \in p^{-N}(\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_{\overline{K}})$ for each $j, k \in X$ such that $\lambda_k = \exp(\text{ad}(u_{j,k})) \cdot \lambda_j$.

Therefore, let c be the nilpotency class of $\mathfrak{g}$, and choose $n \in \mathbb{N}$ with $n \geq 2Nc + 2v(c!)$. Then using Theorem 6.8, we see that $I(\lambda_j) \cap \widehat{U(p^n \mathcal{L})}_K = I(\lambda_k) \cap \widehat{U(p^n \mathcal{L})}_K$ for each $j, k \in X$.

Therefore, $P \cap \widehat{U(p^n \mathcal{L})}_K = \bigcap_{j \in X} I(\lambda_j) \cap \widehat{U(p^n \mathcal{L})}_K = I(\lambda_j) \cap \widehat{U(p^n \mathcal{L})}_K$ for any $j \in X$.

Hence $P \cap \widehat{U(p^n \mathcal{L})}_K = \text{Ann}_{\widehat{U(p^n \mathcal{L})}_K} \widehat{D(\lambda_j)}$ is a Dixmier annihilator as required. $\square$

We suspect that we can always take $n = 0$ in the statement of Theorem A, but we do not have a proof of this.

7 Special Dixmier Annihilators

As outlined in the introduction, our aim is to prove the *deformed Dixmier-Moeglin equivalence* for $\widehat{U(\mathcal{L})}_K$, and using Proposition 5.3, we see that this just means proving that if P is a weakly rational ideal of $\widehat{U(\mathcal{L})}_K$ then $P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is locally closed in $\widehat{U(\pi^n \mathcal{L})}_K$ for n sufficiently high. In fact, in the case where $\mathcal{L}$ is nilpotent, we suspect moreover that $P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is maximal.

So now that we have established that weakly rational ideals become Dixmier annihilators after passing to a suitably high deformation $\widehat{U(\pi^n \mathcal{L})}_K$, it remains to prove that Dixmier annihilator ideals $I(\lambda)$ are maximal. In this section, we will establish this in some important cases, and in particular prove Theorem B.

7.1 Base Change

Fix F/K a finite, Galois extension, with $G := \text{Gal}(F/K)$, let R be a K -algebra and let $S := R \otimes_K F$. There is a natural action of G on S by automorphisms via $\sigma \cdot (r \otimes \alpha) = r \otimes \sigma(\alpha)$, and clearly R is invariant under this action.

We will make a further assumption on S , namely that if $\alpha_1, \dots, \alpha_n \in F$ are linearly independent over K and $r_1 \otimes \alpha_1 + \dots + r_n \otimes \alpha_n = 0$ in S then $r_i = 0$ for all i . Using Lemma 2.1, it is easily verified that this property is satisfied if we take R to be the affinoid enveloping algebra $\widehat{U(\mathcal{L})}_K$.

Proposition 7.1 *Let I be a G -invariant ideal of S , and let $J := I \cap R$. Then $I = J \otimes_K F$. In fact, if $r_1 \otimes \alpha_1 + \dots + r_n \otimes \alpha_n \in I$ with $\alpha_1, \dots, \alpha_n$ linearly independent over K , then $r_1, \dots, r_n \in J$.*

Proof For each $s \in S$, let $l(s)$ be the minimal number of simple tensors required to sum to s , so $l(s) = 0$ if and only if $s = 0$. If $s \in I$ and $l(s) = 1$ then $s = r \otimes \alpha$ is a simple tensor, with $\alpha \neq 0$, thus $r = \alpha^{-1}s \in I \cap R = J$. Hence $s \in J \otimes_K F$.

So, we will assume, for induction, that for all $s \in I$ with $l(s) < n$, $s \in J \otimes_K F$, and moreover that if $s = r_1 \otimes \alpha_1 + \dots + r_m \otimes \alpha_m$ with $m < n$ and $\alpha_1, \dots, \alpha_m$ linearly independent over K , then $r_i \in J$ for all i .

Now, suppose $s \in I$ with $l(s) = n > 1$. Then choose a subfield $F' \subseteq F$ with $K \subseteq F'$ such that F' is minimal with respect to the property that there exist $r_1, \dots, r_n \in R$, $\alpha_1, \dots, \alpha_n \in F'$ such that $s = r_1 \otimes \alpha_1 + \dots + r_n \otimes \alpha_n$. Since $l(s) = n$, it follows immediately that $\alpha_1, \dots, \alpha_n$ are linearly independent over K , and note that since $\alpha_1^{-1}s \in I$ and $l(s) = l(\alpha_1^{-1}s)$, we may assume that $\alpha_1 = 1$.

If $F' = K$ then $s \in R$ so $l(s) = 1$, a contradiction. Hence $F' \neq K$, so since F/K is a Galois extension, there exists $\sigma \in G$ such that the set $\mathcal{J} := \{\alpha \in F' : \sigma(\alpha) = \alpha\} \neq F'$. Clearly F'' is a subfield of F' with $K \subseteq F''$.

Since $\alpha_1 = 1 \in F''$, choose $m \geq 1$ maximal such that there exists $t_1, \dots, t_n \in R$, $\beta_1, \dots, \beta_n \in F'$ with $\beta_1, \dots, \beta_m \in F''$ and $s = t_1 \otimes \beta_1 + \dots + t_n \otimes \beta_n$. Clearly $m < n$ by minimality of F' and the fact that $F'' \subsetneq F'$.

Now, since I is G -invariant, $\sigma(s) - s \in I$. But $\sigma(s) - s = t_1 \otimes (\sigma(\beta_1) - \beta_1) + \dots + t_n \otimes (\sigma(\beta_n) - \beta_n) = t_{m+1} \otimes (\sigma(\beta_{m+1}) - \beta_{m+1}) + \dots + t_n \otimes (\sigma(\beta_n) - \beta_n)$.

If $t_i \in J$ for all $i > m$ then $t_1 \otimes \beta_1 + \dots + t_m \otimes \beta_m \in I$, and hence since $m < n$ and $\beta_1, \dots, \beta_m$ are linearly independent over K , $t_i \in J$ for all i by induction, and hence $s \in J \otimes_K F$.

Therefore, we will assume for contradiction that $t_i \notin J$ for some $i > m$, so since $\sigma(s) - s \in I$, it follows from induction that $\sigma(\beta_{m+1}) - \beta_{m+1}, \dots, \sigma(\beta_n) - \beta_n$ are linearly dependent over K .

So, there exist $\gamma_{m+1}, \dots, \gamma_n \in K$, not all zero, such that $\sigma(\gamma_{m+1}\beta_{m+1} + \dots + \gamma_n\beta_n) = \gamma_{m+1}\beta_{m+1} + \dots + \gamma_n\beta_n$. So let $\beta' := \gamma_{m+1}\beta_{m+1} + \dots + \gamma_n\beta_n \in F'$, and clearly $\beta' \in F''$ by the definition of F'' . We may assume without loss of generality that $\gamma_{m+1} \neq 0$. Thus:

$$\begin{aligned} s &= t_1 \otimes \beta_1 + \dots + t_n \otimes \beta_n \\ &= t_1 \otimes \beta_1 + \dots + t_m \otimes \beta_m + \gamma_{m+1}^{-1} t_{m+1} \otimes (\beta' - \gamma_{m+2}\beta_{m+2} - \dots - \gamma_n\beta_n) + t_{m+2} \otimes \beta_{m+2} + \dots + t_n \otimes \beta_n \\ &= t_1 \otimes \beta_1 + \dots + t_m \otimes \beta_m + \gamma_{m+1}^{-1} t_{m+1} \otimes \beta' + (t_{m+2} - \gamma_{m+1}^{-1} \gamma_{m+2} t_{m+1}) \otimes \beta_{m+2} + \dots + (t_n - \gamma_{m+1}^{-1} \gamma_n t_n) \otimes \beta_n. \end{aligned}$$

But since $\beta_1, \dots, \beta_m, \beta' \in F''$, this contradicts the maximality of m . Therefore $s \in J \otimes_K F$.

Finally, to complete the induction, suppose that $s = r_1 \otimes \alpha_1 + \dots + r_n \otimes \alpha_n \in I$ with $\alpha_1, \dots, \alpha_n$ linearly independent over K . Then $l(s) \leq n$ so we have proved that $s \in J \otimes_K F$, so there exist $t_1, \dots, t_m \in J$, $\beta_1, \dots, \beta_m \in F$ such that $s = t_1 \otimes \beta_1 + \dots + t_m \otimes \beta_m$.

So, extend $\{\alpha_1, \dots, \alpha_n\}$ to a basis $\{\alpha_1, \dots, \alpha_d\}$ of F/K , and set $r_{n+1} = \dots = r_d = 0$. Then for each $i = 1 \dots, m$, $\beta_i = \sum_{1 \leq j \leq d} \gamma_{i,j} \alpha_j$ for some $\gamma_{i,j} \in K$, and:

$$s = \sum_{1 \leq i \leq m} t_i \otimes \beta_i = \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq d} \gamma_{i,j} t_i \otimes \alpha_j = \sum_{1 \leq j \leq d} \left(\sum_{1 \leq i \leq m} \gamma_{i,j} t_i \right) \otimes \alpha_j.$$

But we know that $s = \sum_{1 \leq j \leq d} r_j \otimes \alpha_j$, and hence $\sum_{1 \leq j \leq d} \left(r_j - \sum_{1 \leq i \leq m} \gamma_{i,j} t_i \right) \otimes \alpha_j = 0$, and this implies that $r_j = \sum_{1 \leq i \leq m} \gamma_{i,j} t_i \in J$ for each j as required. $\square$

Corollary 7.2 *If we assume S has finite left and right Krull dimension, and P be is prime ideal of S , then $\{\sigma(P) : \sigma \in G\}$ form a complete set of minimal prime ideals above $(P \cap R) \otimes_K F$.*

Proof Let $Q = P \cap R$. Then for every $\sigma \in G$, $\sigma(Q) = Q$, and hence $Q \subseteq \sigma(P)$. So, let $I := \bigcap_{\sigma \in G} \sigma(P)$, then I is a G -invariant two-sided ideal of S , and $I \cap R = Q$, thus $I = Q \otimes_K F$ by Proposition 7.1.

If P' is a prime ideal of S with $I \subseteq P' \subseteq P$, then $\prod_{\sigma \in G} \sigma(P) \subseteq I \subseteq P'$, which implies that $\sigma(P) \subseteq P' \subseteq P$ for some $\sigma \in G$, so $P \subseteq \sigma^{-1}(P)$.

But S has finite Krull dimension, so it follows from [16, Lemma 6.4.5] that S has finite classical Krull dimension, and hence there exists $n \geq 0$ maximal such that there exists a chain of prime ideals $P = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$ of S . But $P \subseteq \sigma^{-1}(P) \subsetneq \sigma^{-1}(P_1) \subsetneq \dots \subsetneq \sigma^{-1}(P_n)$ is also a chain of prime ideals, and thus $P = \sigma^{-1}(P)$ and $\sigma(P) = P$.

But since $\sigma(P) \subseteq P' \subseteq P$, this implies that $P = P'$, and hence P is minimal prime above I as required. The same argument shows that $\sigma(P)$ is minimal prime above I for every $\sigma \in G$. $\square$

In particular, we could take $R = \widehat{U(\mathcal{L})}_K$, in which case $S = \widehat{U(\mathcal{L})}_F$ has finite Krull dimension as required. Therefore, we can apply this result in the case where P is a Dixmier annihilator.

So, let $\lambda : \mathfrak{g} \rightarrow F$ be a linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$, and $P := \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\lambda)}_F$, then for every $\sigma \in G$, $\sigma(P) = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)}_F$ by Lemma 6.1.

Proposition 7.3 *Let $\lambda, \mu : \mathfrak{g} \rightarrow F$ be linear forms such that $\lambda(\mathcal{L}), \mu(\mathcal{L}) \subseteq \mathcal{O}_F$. Let $P = I(\lambda) \trianglelefteq \widehat{U(\mathcal{L})}_K$, $Q = I(\mu) \trianglelefteq \widehat{U(\mathcal{L})}_K$.*

Then if $P \subseteq Q$, there exists $\sigma \in G$ such that if $P_\sigma := \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)}_F$ and $Q_1 := \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\mu)}_F$ then $P_\sigma \subseteq Q_1$.

Proof Firstly, let $P_1 = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\lambda)}_F$. Then since P_1, Q_1 are primitive ideals of $\text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\lambda)}_F$ by Theorem 5.8 and Theorem 4.4, it follows from Corollary 7.2 that the set $\{\sigma(P_1) : \sigma \in G\}$ consists of all minimal prime ideals above $(P_1 \cap \widehat{U(\mathcal{L})}_K) \otimes_K F = P \otimes_K F$, and also that Q_1 is minimal prime above $Q \otimes_K F$.

Therefore, if $P \subseteq Q$ then $P \otimes_K F \subseteq Q \otimes_K F \subseteq Q_1$, and hence there exists $\sigma \in G$ such that $\sigma(P_1) \subseteq Q_1$. But using Lemma 6.1 we see that $\sigma(P_1) = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)}_F = P_\sigma$ as required. $\square$

7.2 Dimension Theory

Now, fix linear forms $\lambda, \mu : \mathfrak{g} \rightarrow K$ such that $\lambda(\mathcal{L}), \mu(\mathcal{L}) \subseteq \mathcal{O}$, and suppose that $I(\lambda) \subseteq I(\mu)$. Let us assume further that $\mathcal{L}$ is powerful, i.e. $[\mathcal{L}, \mathcal{L}] \subseteq p\mathcal{L}$.

Let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{g}$, and recall from Section 3.2 that we define $\mathfrak{a}^\perp = \{u \in \mathfrak{g} : \lambda([u, \mathfrak{a}]) = 0\}$. Also, using Proposition 2.4, we may fix a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ such that $\mathfrak{a} \subseteq \mathfrak{b}$, and construct the affinoid Dixmier module $\widehat{D(\lambda)}_{\mathfrak{b}}$ with respect to this polarisation. Note that since $\lambda([\mathfrak{a}, \mathfrak{b}]) = 0$, $\mathfrak{b} \subseteq \mathfrak{a}^\perp$.

Suppose $\mathfrak{a}^\perp$ has codimension s in $\mathfrak{g}$, and fix a basis $\{u_1, \dots, u_s\}$ for $\mathcal{L}/(\mathfrak{a}^\perp \cap \mathcal{L})$. Then if $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$, it follows from Corollary 3.4 and Proposition 3.5 that the action $\widehat{U(\mathcal{A})}_K \rightarrow \text{End}_K \widehat{D(\lambda)}$ has image contained in $K\langle \partial_1, \dots, \partial_s \rangle$, where $\partial_i = \frac{d}{du_i}$, and this image contains $\partial_1, \dots, \partial_s$.

Proposition 7.4 *The quotient $\frac{\widehat{U(\mathcal{A})}_K}{I(\lambda) \cap \widehat{U(\mathcal{A})}_K}$ has Krull dimension s .*

Proof We will prove that the image R of $\widehat{U(\mathcal{A})}_K$ under the action $\rho : \widehat{U(\mathcal{A})}_K \rightarrow \text{End}_K \widehat{D(\lambda)}_{\mathfrak{b}}$ has Krull dimension s . Since the Dixmier annihilator $I(\lambda)$ does not depend on the choice of polarisation by Theorem 4.4, the result will follow.

Note that since $\mathcal{A}$ is abelian, $\widehat{U(\mathcal{A})}_K$ is isomorphic to a commutative Tate-power series ring, and hence R is a commutative affinoid algebra. Also, since $\partial_1, \dots, \partial_s \in \rho(\widehat{U(\mathcal{A})}_K)$, it follows that there exist $m \geq 0$ such that $\pi^m \partial_1, \dots, \pi^m \partial_s \in \rho(\widehat{U(\mathcal{A})}_K)$, and hence every power series in $K\langle \pi^m \partial_1, \dots, \pi^m \partial_s \rangle$ lies in $\rho(\widehat{U(\mathcal{A})}_K) = R$.

Therefore we have inclusions of commutative affinoid algebras, $K\langle \pi^m \partial_1, \dots, \pi^m \partial_s \rangle \hookrightarrow R \hookrightarrow K\langle \partial_1, \dots, \partial_s \rangle$, which gives rise to a chain of open embeddings of the associated affinoid spectra: $\text{Sp } K\langle \partial_1, \dots, \partial_s \rangle \hookrightarrow \text{Sp } R \hookrightarrow \text{Sp } K\langle \pi^m \partial_1, \dots, \pi^m \partial_s \rangle$.

Now, in [9] the notion of the analytic dimension $\dim \mathcal{X}$ of a rigid variety $\mathcal{X}$ is defined, and it is proved to be equal to the supremum of the Krull dimensions of every affinoid algebra A such that $\text{Sp } A$ is an affinoid subdomain of $\mathcal{X}$. In particular, if $\text{Sp } A$ is an affinoid subdomain of $\text{Sp } B$ in the sense of Definition 2.1, then $\text{K.dim}(A) \leq \text{K.dim}(B)$.

Therefore, since the Tate algebras $K\langle \partial_1, \dots, \partial_s \rangle$ and $K\langle \pi^m \partial_1, \dots, \pi^m \partial_s \rangle$ both have dimension s , it remains to prove that the embeddings $\text{Sp } R \rightarrow \text{Sp } K\langle \pi^m \partial_1, \dots, \pi^m \partial_s \rangle$ and $\text{Sp } K\langle \partial_1, \dots, \partial_s \rangle \rightarrow \text{Sp } R$ define affinoid subdomains.

For convenience, set $D := \text{Sp } K\langle \partial_1, \dots, \partial_s \rangle$ and $D_1 := \text{Sp } K\langle \pi^m \partial_1, \dots, \pi^m \partial_s \rangle$. Then D can be realised as the unit disc in s -dimensional rigid K -space, while D_1 is a larger disc containing D , so explicitly $D = \{(x_1, \dots, x_s) \in D_1 : |x_i| \leq 1 \text{ for all } i\}$.

But since $\text{Sp } R$ contains D , we could instead write $D = \{(x_1, \dots, x_s) \in \text{Sp } R : |x_i| \leq 1 \text{ for all } i\}$, and this is a Weierstrass subdomain of $\text{Sp } R$ in the sense of [5, Definition

3.3.7], and hence D is an affinoid subdomain of $\mathrm{Sp} R$ by [5, Proposition 3.3.11]. Therefore $\mathrm{K.dim}(R) \geq \dim D = s$.

Now, choose a basis $\{v_1, \dots, v_r\}$ for $\mathcal{A}$, so that $\widehat{U(\mathcal{A})}_K \cong K\langle v_1, \dots, v_r \rangle$. Using Proposition 3.5 we see that for each i , $\rho(v_i) = f_i(\partial_1, \dots, \partial_s)$ for some polynomial f_i . So, let $T := K\langle \pi^{m_1} \partial_1, \dots, \pi^{m_s} \partial_s \rangle$, then since $\rho(\widehat{U(\mathcal{A})}_K)$ contains T , it follows that the affinoid algebra $B := T\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_i - f_i(\partial_1, \dots, \partial_s))$ surjects onto $R = \rho(\widehat{U(\mathcal{A})}_K)$, where each $a \in T$ is sent to a , and each ζ_i is sent to $\rho(v_i)$. Therefore, $\mathrm{K.dim}(R) \leq \mathrm{K.dim}(B)$.

But clearly there is a map $T \rightarrow B$, inducing a map of affinoid varieties $\mathrm{Sp} B \rightarrow \mathrm{Sp} T$, and the proof of [5, Proposition 3.3.11] shows that this corresponds to the embedding of the Weierstrass subdomain $\{x \in \mathrm{Sp} T : |f_i(x)| \leq 1 \text{ for all } i\}$, into $\mathrm{Sp} T$, and hence $\mathrm{Sp} B$ is an affinoid subdomain of $\mathrm{Sp} T = D_1$ by [5, Proposition 3.3.11], and hence $\mathrm{K.dim}(B) \leq \dim(D_1) = s$.

Therefore $s \leq \mathrm{K.dim}(R) \leq \mathrm{K.dim}(B) \leq s$, forcing equality, so $\mathrm{K.dim}(R) = s$ as required. $\square$

Corollary 7.5 *Suppose that $\mathcal{L}$ is a powerful Lie lattice in $\mathfrak{g}$, $\lambda, \mu : \mathfrak{g} \rightarrow K$ are linear forms with $\lambda(\mathcal{L}), \mu(\mathcal{L}) \subseteq \mathcal{O}$, and suppose that $I(\lambda) \subseteq I(\mu)$. Then for any abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$, if $\mathcal{A} := \mathfrak{a} \cap \mathcal{L}$ then $I(\lambda) \cap \widehat{U(\mathcal{A})}_K = I(\mu) \cap \widehat{U(\mathcal{A})}_K$.*

Proof Let $\mathfrak{a}_1^\perp = \{u \in \mathfrak{g} : \lambda([u, \mathfrak{a}]) = 0\}$, and let $\mathfrak{a}_2^\perp = \{u \in \mathfrak{g} : \mu([u, \mathfrak{a}]) = 0\}$. Then if s_i is the codimension of $\mathfrak{a}_i^\perp$ in $\mathfrak{g}$, for $i = 1, 2$, then it follows from Proposition 7.4 that $\widehat{U(\mathcal{A})}_K / I(\lambda) \cap \widehat{U(\mathcal{A})}_K$ has Krull dimension s_1 and $\widehat{U(\mathcal{A})}_K / I(\mu) \cap \widehat{U(\mathcal{A})}_K$ has Krull dimension s_2 .

Since $I(\lambda) \subseteq I(\mu)$, clearly there is a surjection $\widehat{U(\mathcal{A})}_K / I(\lambda) \cap \widehat{U(\mathcal{A})}_K \rightarrow \widehat{U(\mathcal{A})}_K / I(\mu) \cap \widehat{U(\mathcal{A})}_K$, and by Corollary 3.4 both these algebras are commutative domains. Therefore $s_2 \leq s_1$, with equality if and only if $I(\lambda) \cap \widehat{U(\mathcal{A})}_K = I(\mu) \cap \widehat{U(\mathcal{A})}_K$, so we will prove that $s_1 = s_2$.

Since $I(\lambda) \subseteq I(\mu)$, it follows from Lemma 6.2 that there exists $g \in \mathbb{G}(K)$ such that $\mu = g \cdot \lambda$. We will prove that the image of $\mathfrak{a}_1^\perp$ under g is $\mathfrak{a}_2^\perp$, and hence $\mathfrak{a}_1^\perp$ and $\mathfrak{a}_2^\perp$ must have the same dimension, and hence the same codimension in $\mathfrak{g}$ as required.

Firstly, if $u \in \mathfrak{a}_1^\perp$, then $\lambda([u, \mathfrak{g}]) = 0$, so given $v \in \mathfrak{g}$, $\mu([g(u), v]) = (g \cdot \lambda)([g(u), g(g^{-1}(v))]) = \lambda(g^{-1}g[u, g^{-1}(v)]) = \lambda([u, g^{-1}(v)]) = 0$, and hence $\mu([g(u), \mathfrak{g}]) = 0$ and $g(u) \in \mathfrak{a}_2^\perp$. Thus $g(\mathfrak{a}_1^\perp) \subseteq \mathfrak{a}_2^\perp$.

Conversely, since $\lambda = g^{-1} \cdot \mu$, the same argument shows that $g^{-1}(\mathfrak{a}_2^\perp) \subseteq \mathfrak{a}_1^\perp$, and hence $g(\mathfrak{a}_1^\perp) = \mathfrak{a}_2^\perp$ as required. $\square$

7.3 Control Theorem

A particularly useful technique, which we hope should ultimately help us complete the proof of the deformed Dixmier-Moeglin equivalence in full generality, is to look for a suitable generating set for Dixmier annihilators.

Specifically, recall that an ideal I of $\widehat{U(\mathcal{L})}_K$ is *controlled* by a closed ideal $\mathcal{A}$ of $\mathcal{L}$ if $I = \widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A})}_K)$, i.e. I is generated as a left ideal by a subset of $\widehat{U(\mathcal{A})}_K$. If $\mathcal{A} = \mathfrak{a} \cap \mathcal{L}$ for some ideal $\mathfrak{a}$ of $\mathfrak{g}$, we may also say that I is controlled by $\mathfrak{a}$ for convenience.

Lemma 7.6 *Let I be a two-sided ideal of $\widehat{U(\mathcal{L})}_K$, and let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$:*

- If I is controlled by $\mathfrak{a}$, then I is controlled by any ideal $\mathfrak{a}'$ of $\mathfrak{g}$ with $\mathfrak{a} \subseteq \mathfrak{a}'$.
- If $I = \bigcap_{i \in X} J_i$ for some two-sided ideals J_i of $\widehat{U(\mathcal{L})}_K$, and each J_i is controlled by $\mathfrak{a}$, then I is controlled by $\mathfrak{a}$.
- If F/K is a finite extension, and J is a two-sided ideal of $\widehat{U(\mathcal{L})}_F$ such that $J \cap \widehat{U(\mathcal{L})}_K = I$, then if J is controlled by $\mathfrak{a} \otimes_K F$, I is controlled by $\mathfrak{a}$.

Proof For the first statement, it is clear that if I is controlled by $\mathcal{A} = \mathfrak{a} \cap \mathcal{L} \subseteq \mathfrak{a}' \cap \mathcal{L} = \mathcal{A}'$, then $I = \widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A})}_K) \subseteq \widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A}')}_K) \subseteq I$, forcing equality.

Now, let $\mathcal{A} = \mathfrak{a} \cap \mathcal{L}$, and let $\{u_1, \dots, u_d\}$ be a basis for $\mathcal{L}$ such that $\{u_{r+1}, \dots, u_d\}$ is a basis for $\mathcal{A}$ for some $r \geq 0$. Using Lemma 2.1, we see that every element $s \in \widehat{U(\mathcal{L})}_K$ can be written as $s = \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha u_1^{\alpha_1} \cdots u_d^{\alpha_d}$ for some unique $\lambda_\alpha \in K$ with $\lambda_\alpha \rightarrow 0$ and

$\alpha \rightarrow \infty$. For each $\beta \in \mathbb{N}^r$, set $s_\beta = \sum_{\gamma \in \mathbb{N}^{d-r}} \lambda_{(\beta, \gamma)} u_{r+1}^{\gamma_{r+1}} \cdots u_d^{\gamma_d} \in \widehat{U(\mathcal{A})}_K$, and it follows that

$$s = \sum_{\beta \in \mathbb{N}^r} u_1^{\beta_1} \cdots u_r^{\beta_r} s_\beta.$$

We will first prove that I is controlled by $\mathcal{A}$ if and only if for every $s \in I$, $s_\beta \in I$ for all $\beta \in \mathbb{N}^r$. One direction is clear since if $I = \widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A})}_K)$ then every element of I is a sum of elements of the form su with $u \in I \cap \widehat{U(\mathcal{A})}_K$, so $su = \sum_{\beta \in \mathbb{N}^r} u_1^{\beta_1} \cdots u_r^{\beta_r} s_\beta u$ and $s_\beta u \in I$ as required.

Conversely, if $s_\beta \in I$ for all $s \in I$ then $u_1^{\beta_1} \cdots u_r^{\beta_r} s_\beta \in \widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A})}_K)$ for each β , so since $\widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A})}_K)$ is closed in $\widehat{U(\mathcal{L})}_K$, it follows that $s \in \widehat{U(\mathcal{L})}_K(I \cap \widehat{U(\mathcal{A})}_K)$ and hence I is controlled by $\mathcal{A}$.

Now, if $I = \bigcap_{i \in X} J_i$ for some two-sided ideals J_i of $\widehat{U(\mathcal{L})}_K$, then given $s = \sum_{\beta \in \mathbb{N}^r} u_1^{\beta_1} \cdots u_r^{\beta_r} s_\beta \in I$, we know that $s \in J_i = \widehat{U(\mathcal{L})}_K(J_i \cap \widehat{U(\mathcal{A})}_K)$ for every i , and hence $s_\beta \in J_i$ for every i, β , i.e. $s_\beta \in \bigcap_{i \in X} J_i = I$ for all β , and hence I is controlled by $\mathcal{A}$.

For the final statement, it is clear that $\{u_{r+1}, \dots, u_d\}$ is an F -basis for $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}_F$. So if J is controlled by $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{O}_F$ then given $s \in I \subseteq J$, $s_\beta \in J$ for every β . Therefore $s_\beta \in J \cap \widehat{U(\mathcal{L})}_K = I$ and hence I is controlled by $\mathcal{A}$ as required. $\square$

Now, given a linear form $\lambda : \mathfrak{g} \rightarrow K$, recall that we define $\mathfrak{g}^\lambda := \{u \in \mathfrak{g} : \lambda([u, \mathfrak{g}]) = 0\}$.

Lemma 7.7 *Let σ be a Lie automorphism of $\mathfrak{g}$, and let $\sigma \cdot \lambda$ be the linear form defined by $(\sigma \cdot \lambda)(u) = \lambda(\sigma^{-1}(u))$. Then $\mathfrak{g}^{\sigma \cdot \lambda} = \sigma(\mathfrak{g}^\lambda)$.*

Proof Firstly, $u \in \mathfrak{g}^{\sigma \cdot \lambda}$ if and only if $\sigma \cdot \lambda([u, \mathfrak{g}]) = \lambda(\sigma^{-1}([u, \mathfrak{g}])) = 0$. But since σ is a Lie automorphism, this is true if and only if $\lambda([\sigma^{-1}(u), \mathfrak{g}]) = 0$, i.e. $\sigma^{-1}(u) \in \mathfrak{g}^\lambda$ and $u \in \sigma(\mathfrak{g}^\lambda)$ as required. $\square$

Proposition 7.8 *Suppose that the subalgebra $\mathfrak{g}^\lambda$ is an ideal in $\mathfrak{g}$. Then $P = I(\lambda)$ is controlled by $\mathfrak{g}^\lambda$.*

Proof We proceed by induction on $n := \dim_K \mathfrak{g}$. If $n = 1$ then $\mathfrak{g}^\lambda = \mathfrak{g}$ and the statement is obvious, so the base case holds.

For the inductive step, if there exists a non-zero ideal $\mathfrak{t}$ of $\mathfrak{g}$ such that $\lambda(\mathfrak{t}) = 0$ then clearly $\mathfrak{t} \subseteq \mathfrak{g}^\lambda$. Let $\mathfrak{g}_0 := \mathfrak{g}/\mathfrak{t}$, let λ_0 be the linear form of $\mathfrak{g}_0$ induced by λ , and let $\mathcal{L}_0 := \mathcal{L}/\mathfrak{t} \cap \mathcal{L}$. Then $\mathfrak{g}_0^{\lambda_0} = \mathfrak{g}^\lambda/\mathfrak{t}$ is an ideal in $\mathfrak{g}_0$, so using induction, the Dixmier annihilator $I(\lambda_0)$ of $\widehat{U(\mathcal{L}_0)}_K$ is controlled by $\mathfrak{g}_0^{\lambda_0} = \mathfrak{g}^\lambda/\mathfrak{t}$.

But using Lemma 3.1 we see that $\widehat{D(\lambda_0)} \cong \widehat{D(\lambda)}$ as $\widehat{U(\mathcal{L})}_K$ -modules, and thus $I(\lambda_0) = I(\lambda)/\mathfrak{t}\widehat{U(\mathcal{L})}_K$ and it follows immediately that $I(\lambda)$ is controlled by $\mathfrak{g}^\lambda$.

Therefore, we may assume that $\lambda(\mathfrak{t}) \neq 0$ for all non-zero ideals $\mathfrak{t}$ of $\mathfrak{g}$. Note that since $\mathfrak{g}^\lambda$ is an ideal in $\mathfrak{g}$, it follows that $\mathfrak{t} = \mathfrak{g}^\lambda \cap \ker(\lambda)$ is an ideal in $\mathfrak{g}$, and clearly $\lambda(\mathfrak{t}) = 0$, so $\mathfrak{t} = 0$. Thus λ is injective when restricted to $\mathfrak{g}^\lambda$, and hence $\mathfrak{g}^\lambda$ is one-dimensional over K .

Write $\mathfrak{g}^\lambda = Kz$ for some $z \in Z(\mathfrak{g})$ with $\lambda(z) \neq 0$, and since $Z(\mathfrak{g}) \subseteq \mathfrak{g}^\lambda$, it follows that $Z(\mathfrak{g}) = Kz$. Naturally we may assume that $z \in \mathcal{L}$.

Using Proposition 2.7, $\mathfrak{g}$ has a reducing quadruple $(x, y, z, \mathfrak{g}')$, with $y, z \in \mathcal{L}$, and we can choose a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ with $\mathfrak{b} \subseteq \mathfrak{g}'$. Since $\mathfrak{b}$ is also a polarisation of $\mathfrak{g}'$ at $\mu := \lambda|_{\mathfrak{g}'}$, it follows from Lemma 2.3 that $\dim_K \mathfrak{b} = \frac{1}{2}(\dim_K \mathfrak{g} + \dim_K \mathfrak{g}^\lambda) = \frac{1}{2}(\dim_K \mathfrak{g}' + \dim_K \mathfrak{g}^{\mu})$. But $\dim_K \mathfrak{g}' = \dim_K \mathfrak{g} - 1$, so this means that $\dim_K \mathfrak{g}^{\mu} = \dim_K \mathfrak{g}^\lambda + 1 = 2$.

But clearly $y, z \in \mathfrak{g}^{\mu}$, and hence $\mathfrak{g}^{\mu} = Kz \oplus Ky$, and clearly this is an ideal in $\mathfrak{g}$, and therefore in $\mathfrak{g}'$.

Now, using Theorem 4.2, we see that $I(\lambda)$ is controlled by $\mathcal{L}' = \mathfrak{g}' \cap \mathcal{L}$, so let $Q := I(\lambda) \cap \widehat{U(\mathcal{L}')}_{\mathcal{L}'}$ – a semiprime ideal of $\widehat{U(\mathcal{L}')}_{\mathcal{L}'}$ such that $I(\lambda) = Q\widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K Q$. We will prove that Q is controlled by Kz , and it will follow that $I(\lambda)$ is controlled by Kz .

Using Corollary 5.9, we know that $Q = \bigcap_{i \in I} \text{Ann}_{\widehat{U(\mathcal{L}')}_{\mathcal{L}'}} \widehat{D(\mu_i)}_{F_i}$ for some finite extensions F_i/K , linear forms $\mu_i : \mathfrak{g}' \rightarrow F$ such that $\mu_i(\mathcal{L}') \subseteq \mathcal{O}_F$. Thus $I(\lambda) = \widehat{U(\mathcal{L})}_K Q \subseteq \widehat{U(\mathcal{L})}_K \text{Ann}_{\widehat{U(\mathcal{L}')}_{\mathcal{L}'}} \widehat{D(\mu_i)}_{F_i}$ for each i .

Setting λ_i as any extension of μ_i to $\mathcal{L}$, it follows that $\widehat{D(\lambda_i)}_{F_i} = \widehat{U(\mathcal{L})}_{F_i} \otimes_{\widehat{U(\mathcal{L}')}_{F_i}} \widehat{D(\mu_i)}_{F_i}$. Since $I(\lambda) \otimes_K F_i$ is a two-sided ideal of $\widehat{U(\mathcal{L})}_{F_i}$ contained in $\widehat{U(\mathcal{L})}_{F_i} \text{Ann}_{\widehat{U(\mathcal{L}')}_{F_i}} \widehat{D(\mu_i)}_{F_i}$, it follows from Lemma 4.1 that $I(\lambda) \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_i)}_{F_i} = I(\lambda_i)$.

Using Lemma 6.2, this means that λ and λ_i lie in the same $\mathbb{G}$ -coadjoint orbit for all i , i.e. $\lambda_i = g_i \cdot \lambda$ for some $g_i \in \mathbb{G}(F)$.

So, fixing i , let $F = F_i$ for convenience, and let $\mu = \lambda|_{\mathfrak{g}'}$, then μ extends to an F -linear form on $\mathfrak{g}'_F = \mathfrak{g}' \otimes_K F$, and $\mathfrak{g}^{\mu}_F = (\mathfrak{g}^{\mu}) \otimes_K F$. Let σ be the Lie automorphism of $\mathfrak{g}'_F$ given by the restriction of g_i to $\mathfrak{g}'_F$, then $\mu_i = \sigma \cdot \mu$, so it follows from Lemma 7.7 that $\mathfrak{g}^{\mu_i}_F = \sigma(\mathfrak{g}^{\mu}_F) = \mathfrak{g}^{\mu} \otimes F$ since $\mathfrak{g}^{\mu}$ is an ideal in $\mathfrak{g}$, and hence is preserved by σ .

It follows from induction that $\text{Ann}_{\widehat{U(\mathcal{L}')}_{F_i}} \widehat{D(\mu_i)}$ is controlled by $\mathfrak{g}^{\mu} \otimes_K F_i$, and hence $\text{Ann}_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\mu_i)}$ is controlled by $\mathfrak{g}^{\mu}$ by Lemma 7.6. Since this is true for all i , it also follows from Lemma 7.6 that $Q = \bigcap_{i \in I} \text{Ann}_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\mu_i)}$ is controlled by $\mathfrak{g}^{\mu} = Kz \oplus Ky$.

Finally, consider the subalgebra $\mathfrak{h} = \text{Span}_K \{x, y, z\}$ of $\mathfrak{g}$, let $\mathcal{H} := \mathfrak{h} \cap \mathcal{L}$, $\mathcal{A} := \mathfrak{g}^{\mu} \cap \mathcal{L} = \mathcal{O}z \oplus \mathcal{O}y$, and let $I := (Q \cap \widehat{U(\mathcal{A})}_K) \widehat{U(\mathcal{H})}_K$. Then I is a proper two-sided ideal of $\widehat{U(\mathcal{H})}_K$ containing $z - \lambda(z)$, thus $I = (z - \lambda(z))\widehat{U(\mathcal{H})}_K$ by Lemma 4.3.

Therefore, $Q \cap \widehat{U(\mathcal{A})}_K = I \cap \widehat{U(\mathcal{A})}_K = (z - \lambda(z))\widehat{U(\mathcal{A})}_K$, and hence $Q = (Q \cap \widehat{U(\mathcal{A}')}_K)\widehat{U(\mathcal{L}')}_K = (z - \lambda(z))\widehat{U(\mathcal{L}')}_K$, i.e. Q is controlled by $\mathcal{O}z$.

So since $\mathfrak{g}^\lambda = Kz$, it follows that Q is controlled by $\mathfrak{g}^\lambda$, and thus $I(\lambda)$ is controlled by $\mathfrak{g}^\lambda$ as required. $\square$

Proposition 7.9 *Suppose that $\mathfrak{a}$ is any ideal of $\mathfrak{g}$ containing $\mathfrak{g}^\lambda$. Then $I(\lambda)$ is controlled by $\mathfrak{a}$.*

Proof This proof is very similar to the proof of Proposition 7.8, but uses it as a key step. Again, we proceed by induction on $n := \dim_K \mathfrak{g}$. If $n = 1$ then $\mathfrak{g}^\lambda = \mathfrak{g} = \mathfrak{a}$ and the statement is obvious, so the base case holds.

For the inductive step, if there exists a non-zero ideal $\mathfrak{t}$ of $\mathfrak{g}$ such that $\lambda(\mathfrak{t}) = 0$ then clearly $\mathfrak{t} \subseteq \mathfrak{g}^\lambda \subseteq \mathfrak{a}$. Let $\mathfrak{g}_0 := \mathfrak{g}/\mathfrak{t}$, let $\mathfrak{a}_0 := \mathfrak{a}/\mathfrak{t}$, let λ_0 be the linear form of $\mathfrak{g}_0$ induced by λ , and let $\mathcal{L}_0 = \mathcal{L}/\mathfrak{t} \cap \mathcal{L}$. Then $\mathfrak{g}_0^{\lambda_0} = \mathfrak{g}^\lambda/\mathfrak{t}$, and $\mathfrak{a}_0$ is an ideal in $\mathfrak{g}_0$ containing $\mathfrak{g}_0^{\lambda_0}$. So, using induction, the Dixmier annihilator $I(\lambda_0)$ of $\widehat{U(\mathcal{L}_0)}_K$ is controlled by $\mathfrak{a}_0$.

But using Lemma 3.1, $\widehat{D(\lambda)} \cong \widehat{D(\lambda)}$ as $\widehat{U(\mathcal{L})}_K$ -modules, and thus $I(\lambda_0) = I(\lambda)/\mathfrak{t}\widehat{U(\mathcal{L})}_K$ and it follows immediately that $I(\lambda)$ is controlled by $\mathfrak{a}$.

Therefore, we may assume that $\lambda(\mathfrak{t}) \neq 0$ for all non-zero ideals $\mathfrak{t}$ of $\mathfrak{g}$, and it follows that $Z(\mathfrak{g}) = Kz$ for some $z \in Z(\mathcal{L})$ such that $\lambda(z) \neq 0$. Furthermore, we may assume that $\mathfrak{a}$ is the ideal of $\mathfrak{g}$ generated by $\mathfrak{g}^\lambda$, i.e. the subspace $\mathfrak{g}^\lambda + [\mathfrak{g}, \mathfrak{g}^\lambda] + [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}^\lambda]] + \cdots$, since if $I(\lambda)$ is controlled by this ideal, then it will be controlled by any larger ideal by Lemma 7.6.

If $\mathfrak{g}^\lambda = \mathfrak{a}$ then $\mathfrak{g}^\lambda$ is an ideal of $\mathfrak{g}$, and the result follows from Proposition 7.8. So we may assume that $\mathfrak{g}^\lambda \neq \mathfrak{a}$, and hence $\mathfrak{a}$ is not central in $\mathfrak{g}$. Therefore, since $\mathfrak{g}$ is nilpotent, there exists $y \in \mathfrak{a} \cap \mathcal{L}$ with $y \notin Z(\mathfrak{g})$ such that $[y, \mathfrak{g}] \subseteq Z(\mathfrak{g}) = Kz$. So setting $\mathfrak{g}' := \ker(\text{ad}(y))$, it follows that $\mathfrak{g} = \mathfrak{g}' \oplus Kx$ and $(x, y, z, \mathfrak{g}')$ is a reducing quadruple.

Moreover, $\lambda([y, \mathfrak{g}^\lambda]) = 0$ and since $[y, \mathfrak{g}] \subseteq Kz$ and $\lambda(z) \neq 0$, this means that $[y, \mathfrak{g}^\lambda] = 0$. Also, $[y, [\mathfrak{g}, \mathfrak{g}]] = 0$, so $[y, [\mathfrak{g}, [\mathfrak{g}, \cdots], [\mathfrak{g}, \mathfrak{g}^\lambda] \cdots]] = 0$, and hence $[y, \mathfrak{a}] = 0$ by the construction of $\mathfrak{a}$, and thus $\mathfrak{a} \subseteq \mathfrak{g}'$.

Now, using Theorem 4.2, we see that $I(\lambda)$ is controlled by $\mathcal{L}' = \mathfrak{g}' \cap \mathcal{L}$, so let $Q := I(\lambda) \cap \widehat{U(\mathcal{L}')}_K$ – a semiprime ideal of $\widehat{U(\mathcal{L}')}_K$ such that $I(\lambda) = Q\widehat{U(\mathcal{L})}_K = \widehat{U(\mathcal{L})}_K Q$. We will prove that Q is controlled by $\mathfrak{a}$, and it will follow that $I(\lambda)$ is controlled by $\mathfrak{a}$.

Using Proposition 2.7, we can choose a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ with $\mathfrak{b} \subseteq \mathfrak{g}'$. Since $\mathfrak{b}$ is also a polarisation of $\mathfrak{g}'$ at $\mu = \lambda|_{\mathfrak{g}'}$, it follows from Lemma 2.3 that $\dim_K \mathfrak{b} = \frac{1}{2}(\dim_K \mathfrak{g} + \dim_K \mathfrak{g}^\lambda) = \frac{1}{2}(\dim_K \mathfrak{g}' + \dim_K \mathfrak{g}'^\mu)$. But $\dim_K \mathfrak{g}' = \dim_K \mathfrak{g} - 1$, so this means that $\dim_K \mathfrak{g}'^\mu = \dim_K \mathfrak{g}^\lambda + 1$.

So since $\mathfrak{g}'^\mu$ contains $\mathfrak{g}^\lambda$ and y , and since $\lambda([x, y]) \neq 0$, $y \notin \mathfrak{g}^\lambda$. Therefore $\mathfrak{g}^\lambda \oplus Ky \subseteq \mathfrak{g}'^\mu$ and $\dim_K \mathfrak{g}^\lambda \oplus Ky = \dim_K \mathfrak{g}'^\mu$, so $\mathfrak{g}'^\mu = \mathfrak{g}^\lambda \oplus Ky \subseteq \mathfrak{a}$.

Using Corollary 5.9, we know that $Q = \bigcap_{i \in I} \text{Ann}_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\mu_i)}_{F_i}$ for some finite extensions F_i/K , linear forms $\mu_i : \mathfrak{g}' \rightarrow F$ such that $\mu_i(\mathcal{L}') \subseteq \mathcal{O}_F$. Thus $I(\lambda) = \widehat{U(\mathcal{L})}_K Q \subseteq \widehat{U(\mathcal{L})}_K \text{Ann}_{\widehat{U(\mathcal{L}')}_K} \widehat{D(\mu_i)}_{F_i}$ for each i .

Setting λ_i as any extension of μ_i to $\mathcal{L}$, it follows that $\widehat{D(\lambda_i)}_{F_i} = \widehat{U(\mathcal{L})}_{F_i} \otimes_{\widehat{U(\mathcal{L}')}_{F_i}} \widehat{D(\mu_i)}_{F_i}$. Since $I(\lambda) \otimes_K F_i$ is a two-sided ideal of $\widehat{U(\mathcal{L})}_{F_i}$ contained in $\widehat{U(\mathcal{L})}_{F_i} \text{Ann}_{\widehat{U(\mathcal{L}')}_{F_i}} \widehat{D(\mu_i)}_{F_i}$, it follows from Lemma 4.1 that $I(\lambda) \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\lambda_i)}_{F_i} = I(\lambda_i)$.

Using Lemma 6.2, this means that λ and λ_i lie in the same $\mathbb{G}$ -coadjoint orbit for all i , i.e. $\lambda_i = g_i \cdot \lambda$ for some $g_i \in \mathbb{G}(F)$

So, fixing i , let $F = F_i$ for convenience, and let $\mu = \lambda|_{\mathfrak{g}'}$, then μ extends to an F -linear form on $\mathfrak{g}'_F = \mathfrak{g}' \otimes_K F$, and $\mathfrak{g}''_F = (\mathfrak{g}'^\mu) \otimes_K F$. Let σ be the Lie automorphism of $\mathfrak{g}'_F$ given by the restriction of g_i to $\mathfrak{g}'_F$, then $\mu_i = \sigma \cdot \mu$, so it follows from Lemma 7.7 that $\mathfrak{g}''^{\mu_i}_F = \sigma(\mathfrak{g}''^\mu_F) \subseteq \sigma(\mathfrak{a} \otimes F) \subseteq \mathfrak{a} \otimes F$.

Therefore, using induction, $\text{Ann}_{\widehat{U(\mathcal{L})}_{F_i}} \widehat{D(\mu_i)_{F_i}}$ is controlled by $\mathfrak{a} \otimes F_i$, and hence $\text{Ann}_{\widehat{U(\mathcal{L})}_K} \widehat{D(\mu_i)_{F_i}}$ is controlled by $\mathfrak{a}$ using Lemma 7.6. Since this holds for all i and $\mathcal{Q} = \bigcap_{i \in I} \text{Ann}_{\widehat{U(\mathcal{L}')_K}} \widehat{D(\mu_i)_{F_i}}$, it also follows from Lemma 7.6 that $\mathcal{Q}$ is controlled by $\mathfrak{a}$, and hence $I(\lambda)$ is controlled by $\mathfrak{a}$ as required. $\square$

7.4 Special Linear Forms

Definition 7.1 We call a linear form $\lambda : \mathfrak{g} \rightarrow K$ *special* if the ideal $\mathfrak{a}$ of $\mathfrak{g}$ generated by $\mathfrak{g}^\lambda$ satisfies $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$.

This condition is not universal, but as we will see, it is very useful. Using Proposition 7.9, we see that if λ is special then $P = I(\lambda)$ is controlled by an ideal $\mathfrak{a}$ of $\mathfrak{g}$ satisfying $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$. We will now study the Dixmier annihilators associated to special linear forms.

Lemma 7.10 *Let F/K be a finite extension, and let $\lambda : \mathfrak{g} \rightarrow F$ be a linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$. Then there exists a unique ideal $\mathfrak{t}$ of $\mathfrak{g}$, maximal with respect to the property that $\lambda(\mathfrak{t}) = 0$, and in fact $\mathfrak{t} = I(\lambda) \cap \mathfrak{g}$.*

Proof Firstly, it is clear that if $\mathfrak{t}_1, \mathfrak{t}_2 \trianglelefteq \mathfrak{g}$ with $\lambda(\mathfrak{t}_1) = \lambda(\mathfrak{t}_2) = 0$, then $\lambda(\mathfrak{t}_1 + \mathfrak{t}_2) = 0$, so it follows that there exists a unique ideal $\mathfrak{t}$ which is maximal with respect to the property that $\lambda(\mathfrak{t}) = 0$.

Moreover, by Lemma 2.3, $\mathfrak{t} \otimes F \subseteq \mathfrak{b}$ for any polarisation $\mathfrak{b}$ of $\mathfrak{g}_F$ at λ , hence $\mathfrak{t} \widehat{D(\lambda)}_F = \mathfrak{t} \widehat{U(\mathcal{L})}_F \otimes_{\widehat{U(\mathcal{B})}_F} F_\lambda = \widehat{U(\mathcal{L})}_F \mathfrak{t} \otimes_{\widehat{U(\mathcal{B})}_F} F_\lambda = \widehat{U(\mathcal{L})}_F \otimes_{\widehat{U(\mathcal{B})}_F} \mathfrak{t} F_\lambda = 0$, so $\mathfrak{t} \subseteq I(\lambda) \cap \mathfrak{g}$.

Conversely, let $\mathfrak{t}' := I(\lambda) \cap \mathfrak{g}$, then $\mathfrak{t}' \widehat{D(\lambda)}_F = 0$, so $\mathfrak{t}' \otimes F_\lambda = 0$, which implies that $\mathfrak{t}' \subseteq \mathfrak{b}$ and $\lambda(\mathfrak{t}') = 0$ as required. $\square$

Theorem 7.11 *Assume that $\mathcal{L}$ is a powerful Lie lattice, and $\lambda : \mathfrak{g} \rightarrow K$ is a special linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}$. Then the Dixmier annihilator $I(\lambda)$ is a maximal ideal of $\widehat{U(\mathcal{L})}_K$.*

Proof Let $\mathfrak{t} = I(\lambda) \cap \mathfrak{g}$, and let $\mathcal{L}_0 = \mathcal{L}/\mathcal{L} \cap \mathfrak{t}$, then to prove that $I(\lambda)$ is maximal in $\widehat{U(\mathcal{L})}_K$, we need only prove that $I(\lambda)/\mathfrak{t} \widehat{U(\mathcal{L})}_K$ is maximal in $\widehat{U(\mathcal{L}_0)}_K = \widehat{U(\mathcal{L})}_K / \mathfrak{t} \widehat{U(\mathcal{L})}_K$.

But $\lambda(\mathfrak{t}) = 0$ by Lemma 7.10, so $\mathfrak{t} \subseteq \mathfrak{g}^\lambda$ and λ induces a special linear form λ_0 of $\mathfrak{g}/\mathfrak{t}$, and it follows from Lemma 3.1 that $\widehat{D(\lambda)} \cong \widehat{D(\lambda_0)}$ as $\widehat{U(\mathcal{L})}_K$ -modules, and thus $\text{Ann}_{\widehat{U(\mathcal{L}_0)}_K} \widehat{D(\lambda_0)} = I(\lambda)/\mathfrak{t} \widehat{U(\mathcal{L})}_K$.

Therefore, after quotienting out by $\mathfrak{t}$, we may assume that $I(\lambda) \cap \mathfrak{g} = 0$, and hence $\lambda(\mathfrak{a}) \neq 0$ for all non-zero ideals $\mathfrak{a}$ of $\mathfrak{g}$.

Now, since λ is special, we may choose an ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{g}^\lambda \subseteq \mathfrak{a}$ and $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$. But $[\mathfrak{a}, \mathfrak{a}]$ is an ideal of $\mathfrak{g}$, and hence $[\mathfrak{a}, \mathfrak{a}] = 0$, i.e. $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. So suppose

that Q is a maximal ideal of $\widehat{U(\mathcal{L})}_K$ with $I(\lambda) \subseteq Q$, we will prove that $I(\lambda) = Q$, and hence Q is maximal.

Since Q is maximal, it is locally closed, so by Theorem 5.8, $Q = I(\mu)$ for some finite extension F/K and some linear form $\mu : \mathfrak{g} \rightarrow F$ with $\mu(\mathcal{L}) \subseteq \mathcal{O}_F$, and after extending F if necessary, we may assume that F/K is a Galois extension.

Using Proposition 7.3, we see that there exists $\sigma \in \text{Gal}(F/K)$ such that $\text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)_F} \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\mu)}$, and note that $\text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)_F} \cap \widehat{U(\mathcal{L})}_K = I(\lambda)$ by Lemma 6.1.

Also, note that $\mathfrak{g}_F^{\sigma \cdot \lambda} = \sigma(\mathfrak{g}_F^\lambda)$ by Lemma 7.7. But since λ takes values in K , $\mathfrak{g}_F^\lambda = (\mathfrak{g}^\lambda) \otimes_K F$, so it follows that $\mathfrak{g}_F^{\sigma \cdot \lambda} = \sigma((\mathfrak{g}^\lambda) \otimes_K F)$ is contained in the abelian ideal $\sigma(\mathfrak{a} \otimes_K F) = \mathfrak{a} \otimes_K F$. Therefore $\sigma \cdot \lambda$ is a special linear form of $\mathfrak{g} \otimes_K F$.

So let us first suppose that $F = K$, and hence $\sigma = 1$, then since $I(\lambda) \subseteq I(\mu)$, we know that $\mu = g \cdot \lambda$ for some $g \in \mathbb{G}(K)$, and hence $\mathfrak{g}^\mu = g(\mathfrak{g}^\lambda)$ by Lemma 7.7. Hence $\mathfrak{g}^\mu \subseteq g(\mathfrak{a}) = \mathfrak{a}$ and hence μ is also a special linear form, and it follows from Proposition 7.9 that $I(\lambda)$ and $I(\mu)$ are both controlled by $\mathfrak{a}$, i.e. if $\mathcal{A} = \mathfrak{a} \cap \mathcal{L}$ then $I(\lambda) = \widehat{U(\mathcal{L})}_K(I(\lambda) \cap \widehat{U(\mathcal{A})}_K)$ and $I(\mu) = \widehat{U(\mathcal{L})}_K(I(\mu) \cap \widehat{U(\mathcal{A})}_K)$.

But since $\mathfrak{a}$ is abelian, we see using Corollary 7.5 that $I(\lambda) \cap \widehat{U(\mathcal{A})}_K = I(\mu) \cap \widehat{U(\mathcal{A})}_K$, and hence $I(\mu) = \widehat{U(\mathcal{L})}_K(I(\mu) \cap \widehat{U(\mathcal{A})}_K) = \widehat{U(\mathcal{L})}_K(I(\lambda) \cap \widehat{U(\mathcal{A})}_K) = I(\lambda)$ as required.

More generally, since $\text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)_F} \subseteq \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\mu)}$ and $\sigma \cdot \lambda$ is special, it follows from this argument that $\text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)_F} = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\mu)}$. Thus $I(\lambda) = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\sigma \cdot \lambda)_F} \cap \widehat{U(\mathcal{L})}_K = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\mu)_F} \cap \widehat{U(\mathcal{L})}_K = I(\mu) = Q$, and hence $I(\lambda) = Q$ is maximal as required. $\square$

Corollary 7.12 *Let F/K be a finite, Galois extension, $\mathcal{L}$ a powerful Lie lattice, $\lambda : \mathfrak{g} \rightarrow F$ a linear form such that $\lambda(\mathcal{L}) \subseteq \mathcal{O}_F$ and the extension of λ to $\mathfrak{g} \otimes_K F$ is special. Then $I(\lambda)$ is a maximal ideal of $\widehat{U(\mathcal{L})}_K$.*

Proof Using Theorem 7.11, we see that $P = \text{Ann}_{\widehat{U(\mathcal{L})}_F} \widehat{D(\lambda)}$ is a maximal ideal of $\widehat{U(\mathcal{L})}_F$. So let us suppose that Q is a maximal ideal of $\widehat{U(\mathcal{L})}_K$ with $I(\lambda) \subseteq Q$.

Choose a prime ideal Q' of $\widehat{U(\mathcal{L})}_F$ which is minimal prime above $Q \otimes_K F$. Then since $I(\lambda) \otimes_K F \subseteq Q \otimes_K F \subseteq Q'$, it follows that Q' contains a minimal prime above $I(\lambda) \otimes_K F$. But using Corollary 7.2, this means that $\sigma(P) \subseteq Q'$ for some $\sigma \in \text{Gal}(F/K)$, and since P is maximal, $\sigma(P)$ is maximal, which implies that $Q' = \sigma(P)$ is minimal prime above $I(\lambda) \otimes_K F$.

Therefore, $Q' \cap \widehat{U(\mathcal{L})}_K = \sigma(P) \cap \widehat{U(\mathcal{L})}_K = I(\lambda)$, but since $Q \subseteq Q' \cap \widehat{U(\mathcal{L})}_K$, it follows that $Q = I(\lambda)$ as required. $\square$

7.5 Proof of Theorem B

We are now ready to prove Theorem B, which now requires only the following result:

Lemma 7.13 *Assume $\mathfrak{g}$ is nilpotent and metabelian (i.e. $[\mathfrak{g}, \mathfrak{g}]$ is abelian). Then any linear form $\lambda : \mathfrak{g} \rightarrow K$ is special.*

Proof We want to prove that for any linear form $\lambda : \mathfrak{g} \rightarrow K$, $\mathfrak{g}^\lambda$ is contained in an ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\lambda([\mathfrak{a}, \mathfrak{a}]) = 0$. Firstly, since $[\mathfrak{g}, \mathfrak{g}]$ is an abelian ideal of $\mathfrak{g}$, using Lemma 2.4 we see that there exists a polarisation $\mathfrak{b}$ of $\mathfrak{g}$ at λ such that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{b}$.

But since $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$, it follows that $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Also, $\mathfrak{g}^\lambda \subseteq \mathfrak{b}$ by Lemma 2.3, and of course $\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$. Therefore λ is special. $\square$

Proof of Theorem B. As per Definition 1.2, we want to prove that for all prime ideals P of $\widehat{U(\mathcal{L})}_K$, there exist n sufficiently high such that $P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is weakly rational if and only if it is primitive if and only if it is locally closed.

We already know using Proposition 5.3 that if that if $P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is locally closed, then it is primitive and weakly rational, and also note if $P \cap \widehat{U(\pi^N \mathcal{L})}_K$ is weakly rational for any N , then $P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is weakly rational for all $n \geq N$.

So we can assume that there exists such an $N \geq 0$, and we will prove that $P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is locally closed, and in fact maximal, for sufficiently high $n \geq N$.

Since $P \cap \widehat{U(\pi^N \mathcal{L})}_K$ is weakly rational, it follows from Theorem A that for all sufficiently high $n \geq N$, there exists a finite extension F/K and a linear form $\lambda : \mathfrak{g} \rightarrow F$ such that $\lambda(\pi^n \mathcal{L}) \subseteq \mathcal{O}_F$ and $P \cap \widehat{U(\pi^n \mathcal{L})}_K = \text{Ann}_{\widehat{U(\pi^n \mathcal{L})}_K} \widehat{D(\lambda)}_F$. After extending F if necessary, we may assume that F/K is Galois. Since $\mathfrak{g}$ is metabelian, it is clear that $\mathfrak{g} \otimes_K F$ is also metabelian, so it follows from Lemma 7.13 that λ is special.

Choose n such that $\pi^n \mathcal{L}$ is a powerful Lie lattice, i.e. such that $\pi^n \in p\mathcal{O}$, which is true for all sufficiently high n . Then using Corollary 7.12 we see that $\text{Ann}_{\widehat{U(\pi^n \mathcal{L})}_K} \widehat{D(\lambda)}_F = P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is a maximal ideal of $\widehat{U(\pi^n \mathcal{L})}_K$ as required.

Finally, it follows from Corollary 3.4 that $\widehat{U(\pi^n \mathcal{L})}_K / \text{Ann}_{\widehat{U(\pi^n \mathcal{L})}_K} \widehat{D(\lambda)}_F$ is a domain, and hence $\widehat{U(\pi^n \mathcal{L})}_K / P \cap \widehat{U(\pi^n \mathcal{L})}_K$ is a simple domain.

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