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Counting the Number of τ-Exceptional Sequences over Nakayama Algebras

Dixy Msapato[1]

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Abstract
The notion of a τ-exceptional sequence was introduced by Buan and Marsh in (2018) as a generalisation of an exceptional sequence for finite dimensional algebras. We calculate the number of complete τ-exceptional sequences over certain classes of Nakayama algebras. In some cases, we obtain closed formulas which also count other well known combinatorial objects, and exceptional sequences of path algebras of Dynkin quivers.

Keywords τ-Exceptional sequence · Exceptional sequence · Nakayama algebras · τ-Perpendicular category · Restricted Fubini numbers

Mathematics Subject Classification (2020) 16G10 · 16G20 (Primary), 05A10 · 05A15 · 05A18 · 05A19 (Secondary)

1 Introduction

Let A be a finite dimensional algebra over a field $\mathbb{F}$, where $\mathbb{F}$ is algebraically closed. Let mod A be the category of finitely generated left A-modules. A left A-module M is called *exceptional* if $\mathrm{Hom}(M, M) \cong \mathbb{F}$ and $\mathrm{Ext}^i_A(M, M) = 0$ for $i \geq 1$. A sequence of indecomposable modules $(M_1, M_2, \ldots, M_r)$ is called an *exceptional sequence* if for each pair (M_l, M_j) with $1 \leq l < j \leq r$, we have that $\mathrm{Hom}(M_j, M_l) = \mathrm{Ext}^i_A(M_j, M_l) = 0$ for $i \geq 1$, and each M_k is exceptional for $1 \leq k \leq r$. Exceptional sequences were first introduced in the context of algebraic geometry by [6], [15] and [14].

Exceptional sequences exhibit some interesting behaviours. It was shown by Crawley-Boevey [10] and Ringel [25] that there is a transitive braid group action on the set of

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Presented by: Henning Krause

✉ Dixy Msapato
mmdmm@leeds.ac.uk

[1] School of Mathematics, University of Leeds, Leeds, LS2 9JT, UK

exceptional sequences. Igusa and Schiffler give [16] a characterisation of exceptional sequences for hereditary algebras using the fact that the product of the corresponding reflections is the inverse Coxeter element of the Weyl group. The exceptional sequences for mod $\mathbb{A}_r$, where $\mathbb{A}_r$ is the path algebra of a Dynkin type A quiver are classified in [12] using combinatorial objects called strand diagrams. The exceptional sequences over path algebras of type A, were also characterised using non-crossing spanning trees in [3]. A natural question for exceptional sequences is to ask how many there are. The number of them has been computed for all the hereditary Dynkin algebras in [28] and [22].

Exceptional sequences have been subject to a number of generalisations. Igusa and Todorov introduced the *signed exceptional sequences* in [17]. More recently, *weak exceptional sequences* were introduced and studied by Sen in [30]. Finally, Buan and Marsh introduced in [7] the *signed* τ*-exceptional sequences* and τ*-exceptional sequences*. It is τ-exceptional sequences which are the subject of this paper. An A-module M is called τ*-rigid* if $\mathrm{Hom}(M, \tau M) = 0$, see Definition 0.1 in [2]. The τ*-perpendicular category* of M in mod A is the subcategory $J(M) = M^{\perp} \cap {}^{\perp}(\tau M)$, see Definition 3.3 in [18]. A sequence of indecomposable modules $(M_1, M_2, \ldots, M_r)$ in mod A is called a τ-exceptional sequence if M_r is τ-rigid in mod A and $(M_1, M_2, \ldots, M_{r-1})$ is a τ-exceptional sequence in $J(M_r)$.

Our main results are derivations of closed formulas for the number of complete τ-exceptional sequences in the module categories of certain Nakayama algebras. Most notably, we see that the complete τ-exceptional sequences over the linear radical square zero Nakayama algebras Γ_n^2 are counted by the *restricted Fubini numbers* $F_{n,\leqslant 2}$ [21]. The numbers $F_{n,\leqslant 2}$ count the number of ordered set partitions of the set $\{1, 2, \ldots, n\}$ with blocks of size at most two. In the case for the cyclic Nakayama algebra Λ_n^n, we get that the complete τ-exceptional sequences are counted by the sequence n^n. We remark that this sequence also counts the number of complete exceptional sequences for the hereditary Dykin algebras of quivers of type B and C, as shown in [22], and full weak exceptional sequences over Λ_n^n, see [29, Theorem 3.5]. In fact, we show that the complete τ-exceptional sequences over Λ_n^n coincide with the full weak exceptional sequences over Λ_n^n, see Corollary 6.13.

We remark that Buan and Marsh showed in [7] that there is a bijection between complete signed τ-exceptional sequences and basic ordered support τ-tilting modules over a finite dimensional algebra. So, one way of counting signed τ-exceptional sequences would be to count ordered support τ-tilting modules, but this would not give the number of (unsigned) τ-exceptional sequences, which is what we consider here. In this direction, Asai [4] gave a recurrence relation for the number of support τ-tilting modules over Nakayama algebras with a linearly oriented type A quiver. Adachi [1] also gave a recurrence relation for the number of τ-tilting modules over the same algebras as Asai. More recently Gao and Schiffler [11] have extended the recurrence relations of Adachi and Asai to τ-tilting modules and support τ-tilting modules over Nakayama algebras whose quiver is an oriented cycle. In a paper of Sen [29], the number of exceptional sequences over the linear radical square zero Nakayama algebras Γ_n^2 are counted. However, to date the number of exceptional sequences for other classes of Nakayama algebras have not been counted.

This paper is organised as follows: In Section 2 we fix some notation and recall definitions. In Section 3, we state and prove preliminary results which we use in the latter sections to prove our main results of the paper. Our main results of this section state that under certain assumptions, the τ-exceptional sequences of mod A are obtained by interleaving τ-exceptional sequences of certain subcategories of mod A. In Section 4, we count the number of complete τ-exceptional sequences for the linear radical square zero Nakayama algebras Γ_n^2. We derive a recurrence and closed formula for the number of complete τ-exceptional sequences in this case. In Section 5, we deal with the case of cyclic radical square zero Nakayama algebras Λ_n^2. We derive a closed formula for the number of complete τ-exceptional sequences in this case. We also derive a formula for the number of complete τ-exceptional sequences of Λ_n^2 in terms of the number of complete τ-exceptional sequences of Γ_n^2. In Section 6, we count the number of complete τ-exceptional sequences over the cyclic Nakayama algebras Λ_n^n. We derive a recurrence and closed formula for the number of complete τ-exceptional sequences in this case. Section 7 deals with the linear Nakayama algebras Γ_n^{n-1}. We derive a recurrence relation for the number of complete τ-exceptional sequences in this case, and show that the corresponding exponential generating function satisfies a certain first order linear ordinary differential equation involving Lambert's W function. Section 8 concludes the paper by giving a justification of why we only consider the above the classes of Nakayama algebras.

2 Definitions and Notation

Let A be a basic finite dimensional algebra over a field $\mathbb{F}$ which is algebraically closed. Let mod A be the category of finite dimensional left A-modules. Denote by $\mathcal{P}(A)$ the full subcategory of projective objects in mod A. If $\mathcal{T}$ is a subcategory of mod A, we say an A-module M in $\mathcal{T}$ is Ext-*projective* in $\mathcal{T}$ if $\mathrm{Ext}^1_A(M, \mathcal{T}) = 0$; that is to say $\mathrm{Ext}^1_A(M, T) = 0$ for all $T \in \mathcal{T}$. We will then write $\mathcal{P}(\mathcal{T})$ to denote the direct sum of the indecomposable Ext-projective modules in $\mathcal{T}$. In everything that follows, we make the assumption that all subcategories are full, and closed under isomorphism. We will also take all objects to be basic where possible, and they will be considered up to isomorphism.

For an additive category $\mathcal{C}$, and an object X in $\mathcal{C}$, we denote by add X the additive subcategory of $\mathcal{C}$ generated by X. This is the subcategory of $\mathcal{C}$ with objects the direct summands of direct sums of copies of X. For a subcategory $\mathcal{X} \subseteq \mathcal{C}$, we define ${}^{\perp}\mathcal{X} := \{Y \in \mathcal{C} : \mathrm{Hom}(Y, X) = 0 \text{ for all } X \in \mathcal{X}\}$ and we similarly define $\mathcal{X}^{\perp}$. If $\mathcal{C}$ is skeletally small and Krull-Schmidt, we denote by $\mathrm{ind}(\mathcal{C})$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. For any basic object X in $\mathcal{C}$, let $\delta(X)$ denote the number of indecomposable direct summands of X. We fix $\delta(A)$ to be n, where $n \geq 1$ is a positive integer.

Let τ denote the Auslander-Reiten translate of mod A.

Definition 2.1 **τ-rigid and τ-tilting** [2, Definition 0.1]. A left A-module M is said to be τ-*rigid* if $\mathrm{Hom}(M, \tau M)$=0. If furthermore $\delta(M) = n$, we say that M is τ-*tilting*.

Definition 2.2 **τ-perpendicular category** [18, Definition 3.3]. Let M be a basic τ-rigid left A-module. The τ-*perpendicular category* associated to M is the subcategory of mod A given by $J_{\mathrm{mod}\,A}(M) := M^{\perp} \cap {}^{\perp}(\tau M)$. If there is no risk of ambiguity, we will write $J(M)$ for the subcategory $J_{\mathrm{mod}\,A}(M)$.

Definition 2.3 **τ-exceptional sequence** [7, Definition 1.3]. Let k be a positive integer. A sequence of indecomposable modules $(M_1, M_2, \ldots, M_k)$ in mod A is called a τ-*exceptional sequence* in mod A if M_k is τ-rigid in mod A and $(M_1, M_2, \ldots, M_{k-1})$ is a τ-exceptional sequence in $J(M_k)$. If $k = n$ we say that the sequence is a *complete* τ-exceptional sequence.

Let Q be a finite quiver on n vertices labelled by the set $\{1, 2, \ldots, n\}$. A path p in Q from the vertex v_1 to the vertex v_m is a sequence of vertices $p = (v_1, v_2, v_3, \ldots, v_{m-1}, v_m)$

such that (v_j, v_{j+1}) is an arrow in Q for all $1 \leq j \leq m-1$. The positive integer m is called the length of p and it is denoted by $l(p)$. The path algebra $\mathbb{F}Q$ of the quiver Q is the $\mathbb{F}$-algebra with basis all paths of Q, and multiplication is defined by concatenation of paths. The the arrow ideal R_Q of $\mathbb{F}Q$ is defined to be the two-sided ideal generated by all arrows in Q. The arrow ideal has a vector space decomposition given by,

$$R_Q = \mathbb{F}Q_1 \oplus \mathbb{F}Q_2 \oplus \cdots \oplus \mathbb{F}Q_l \oplus \dots$$

where $\mathbb{F}Q_l$ is the subspace of $\mathbb{F}Q$ with basis the set Q_l of paths of length l. The l^{th} power of the arrow ideal, denoted by R^l_Q is given by,

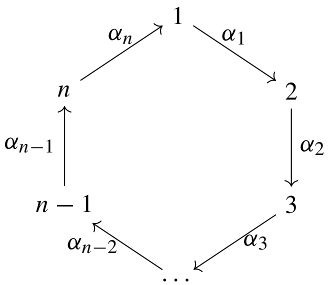
$$R^l_Q = \bigoplus_{m \geq l} \mathbb{F}Q_m,$$

it has a basis consisting of all paths of length greater than or equal to l.

For a positive integer $n \geq 1$, let A_n denote the linearly oriented quiver with n vertices,

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n.$$

Let C_n be the linearly oriented n-cycle.



We denote by Γ^t_n the Nakayama algebra $\mathbb{F}A_n/R^t_Q$ and by Λ^t_n the self injective Nakayama algebra $\mathbb{F}C_n/R^t_Q$ where $2 \leq t \leq n$. Throughout the text, we will write P_i for the indecomposable projective module at vertex i of the underlying quiver of the algebra A in question. Likewise we will write S_i for the simple A-module at vertex i.

Definition 2.4 [27, Definition 3.1]. Let Q be a finite quiver.

1. Two paths $p = (v_1, v_2, \dots, v_m)$ and $p' = (v'_1, v'_2, \dots, v'_{m'})$ in Q are called *parallel* if $v_1 = v'_1$ and $v_m = v'_{m'}$.
2. A *relation* ρ in Q is an $\mathbb{F}$-linear combination $\rho = \sum_c \lambda_c c$ of parallel paths with $l(c) \geq 2$, and $\lambda_c \in \mathbb{F}$.

For a positive integer $n \geq 1$, we will write $(a)_n$ to stand for a modulo n. We will also write $[i, j]_n$ for the set $\{(i)_n, (i+1)_n, \dots, (j-1)_n, (j)_n\}$.

3 Preliminary Results

In this section, we will state and prove results which will be used in later sections to calculate the number of τ-exceptional sequences over the algebras Γ^t_n and Λ^t_n. However, our main results are much more general and they apply to other finite dimensional algebras. For this section, we fix an arbitrary finite dimensional $\mathbb{F}$-algebra A.

Proposition 3.1 [2, Theorem 2.10]. *Let M be a τ-rigid A-module. Then the following holds:*

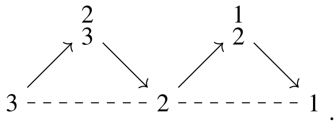
1. *The module M is Ext-projective in ${}^{\perp}(\tau M)$, which is to say that M is in add($\mathcal{P}({}^{\perp}(\tau M))$).*
2. *The module $T_M := \mathcal{P}({}^{\perp}(\tau M))$ is a τ-tilting A-module.*

The A-module T_M is called the *Bongartz completion* of M in mod A.

Example 3.2 Let A be the algebra Γ_3^2 given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3,$$

subject to the relation $\alpha\beta = 0$. The Auslander-Reiten quiver of mod Γ_3^2 is as follows,



For the Γ_3^2-module $M = 1$, $\text{ind}({}^{\perp}(\tau 1)) = \text{ind}({}^{\perp}2) = \{3, \frac{1}{2}, 1\}$. Therefore it is easy to see that $T_1 = \mathcal{P}({}^{\perp}(\tau 1)) = 3 \oplus \frac{1}{2} \oplus 1$. It is also easy to observe that T_1 is indeed a τ-tilting Γ_3^2-module.

Proposition 3.3 [2, Lemma 2.1]. *Let I be an ideal of A, and let M, N be A/I-modules. Then we have the following:*

1. *If* $\text{Hom}_A(M, \tau N) = 0$ *then* $\text{Hom}_{A/I}(M, \tau_{A/I} N) = 0$.
2. *If $I = \langle e \rangle$ for some idempotent $e \in A$, then it is the case that* $\text{Hom}_A(M, \tau N) = 0$ *if and only if* $\text{Hom}_{A/I}(M, \tau_{A/I} N) = 0$.

The following lemma is well known and it will be important in this paper.

Lemma 3.4 *Let Q be a finite simple quiver with vertex set $\{1, 2, \ldots, n\}$. Let I be the ideal of $\mathbb{F}Q$ generated by relations on Q where each relation is a path in Q and take $A = \mathbb{F}Q/I$. For some $j \in \{1, 2, \ldots, n\}$ let $Q^{(j)}$ be the quiver obtained from Q by removing the vertex j and any arrows incident to j. Let $I^{(j)} \subset I$ be the ideal of $\mathbb{F}Q$ generated by the generating relations of I defined by paths of Q not containing the vertex j and take $B = \mathbb{F}Q^{(j)}/I^j$. Then $B \cong A/\langle e_j \rangle$ as an $\mathbb{F}$-algebra , where e_j is the idempotent at vertex j of $\mathbb{F}Q$.*

Theorem 3.5 [18, Theorem 3.8]. *Let A be a finite dimensional algebra and M a basic τ-rigid A-module. Let T_M be the Bongartz completion of M in mod A. Let $E_M = \text{End}_A(T_M)$ and $D_M = E_M/\langle e_M \rangle$, where e_M is the idempotent corresponding to the projective E_M-module $Hom_A(T_M, M)$. Then there is an additive exact equivalence of categories between the category $J(M)$, (the τ-perpendicular category of M in mod A) and the category mod D_M. Moreover, if M is indecomposable we have that $\delta(D_M) = \delta(A) - 1$.*

We now prove some results which will be crucial in our strategy for calculating the number of τ-exceptional sequences in mod A.

Definition 3.6 Interleaving. Let $X = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_t)$ be sequences. An interleaved sequence of X and Y is a sequence $Z = (Z_1, Z_2, \ldots, Z_{s+t})$ with

$Z_i \in \{X_j : 1 \leq i \leq s\} \cup \{Y_j : 1 \leq j \leq t\}$ such that the subsequence of Z containing only elements X or Y is precisely X or Y respectively.

Example 3.7 Let $X = \left(5, \frac{4}{5}, 6\right)$ and $Y = \left(2, \frac{1}{2}\right)$ be sequences in $\operatorname{mod}\Gamma_6^2$. The sequence $Z = \left(2, 5, \frac{4}{5}, \frac{1}{2}, 6\right)$ is an interleaved sequence of X and Y. However $W = \left(\frac{4}{5}, 5, 2, 6, \frac{1}{2}\right)$ is not an interleaved sequence of X and Y because the subsequence containing only elements of X is not equal to X.

Let A and B be finite-dimensional $\mathbb{F}$-algebras and let $\operatorname{mod} A$ and $\operatorname{mod} B$ be the categories of finitely generated left A-modules and left B-modules respectively. We may consider the category $\operatorname{mod} A \oplus \operatorname{mod} B$, the direct product category of $\operatorname{mod} A$ and $\operatorname{mod} B$. The objects of $\operatorname{mod} A \oplus \operatorname{mod} B$ are pairs (M, N) with $M \in \operatorname{mod} A$ and $N \in \operatorname{mod} B$. A morphism between a pair of objects, (M_1, N_1) and (M_2, N_2) in $\operatorname{mod} A \oplus \operatorname{mod} B$ is a pair of morphisms $(f : M_1 \to M_2, g : N_1 \to N_2)$ where $f \in \operatorname{mod} A$ and $g \in \operatorname{mod} B$. The indecomposable objects of $\operatorname{mod} A \oplus \operatorname{mod} B$ are pairs $(M, 0)$ and $(0, N)$ where M and N are indecomposable in their respective categories. The category $\operatorname{mod} A \oplus \operatorname{mod} B$ is an abelian category, in fact, there is an exact, additive equivalence to $\operatorname{mod}(A \times B)$. The category $\operatorname{mod} A \oplus \operatorname{mod} B$ also has an Auslander-Reiten translate $\tau_{A,B}$ which acts in the obvious way i.e. $\tau_{A,B}(M, 0) = (\tau_A M, 0)$ and $\tau_{A,B}(0, N) = (0, \tau_B N)$. It is easy to see that the above exact equivalence preserves the Auslander-Reiten translations, since irreducible morphisms, left minimal almost split and right minimal almost split morphisms are preserved under equivalence of categories. Let M be an A-module, we identify M with the object $(M, 0)$ in $\operatorname{mod} A \oplus \operatorname{mod} B$. We like wise identify the B-module N with the object $(0, N)$ in $\operatorname{mod} A \oplus \operatorname{mod} B$. It is easy to observe that $(M, 0)$ is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$ if and only if M is τ-rigid in $\operatorname{mod} A$. The similar statement for $(0, N)$ and N is also true.

Theorem 3.8 *Let A and B be finite dimensional $\mathbb{F}$-algebras. Suppose $X = (X_1, X_2, \ldots, X_s)$ is a τ-exceptional sequence in $\operatorname{mod} A$ and $Y = (Y_1, Y_2, \ldots, Y_t)$ is a τ-exceptional sequence in $\operatorname{mod} B$. Suppose $Z = (Z_1, Z_2, \ldots, Z_{s+t})$ is an interleaved sequence of X and Y. Then Z is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$.*

Proof We prove this by induction on $s + t$. For the base case, suppose $s + t = 1$. Without loss of generality suppose $t = 0$, so $Z = (X_1)$. By assumption, X_1 is τ-rigid in $\operatorname{mod} A$, so it is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$. This completes the base case.

Suppose the statement is true for $s+t = m$. We consider the $s+t = m+1$ case. Suppose the sequence $Z = (Z_1, Z_2, \ldots, Z_{m+1})$ is an interleaved sequence of $X = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_t)$, where X is a τ-exceptional sequence in $\operatorname{mod} A$ and Y is a τ-exceptional sequence in $\operatorname{mod} B$. Suppose without loss of generality that Z_{m+1} is in X i.e. $Z_{m+1} = X_s$. To show that Z is a τ-exceptional sequence in $\operatorname{mod} A \oplus \operatorname{mod} B$, we need to show that Z_{m+1} is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$ and that $(Z_1, Z_2, \ldots, Z_m)$ is a τ-exceptional sequence in $J_{(A,B)}(Z_{m+1})$, the τ-perpendicular category of Z_{m+1} in $\operatorname{mod} A \oplus \operatorname{mod} B$. By assumption, Z_{m+1} is τ-rigid in $\operatorname{mod} A$, so it is τ-rigid in $\operatorname{mod} A \oplus \operatorname{mod} B$. Observe that $\operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(X_s, N) = \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(N, \tau X_s) = 0$ for all $N \in \operatorname{mod} B$, so it follows that

$$J_{(A,B)}(Z_{m+1}) = \{U \in \operatorname{mod} A \oplus \operatorname{mod} B : \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(X_s, U) = \operatorname{Hom}_{\operatorname{mod} A \oplus \operatorname{mod} B}(U, \tau_A X_s) = 0\}$$

$$= J_{\operatorname{mod} A}(X_s) \oplus \operatorname{mod} B,$$

where $J_{\text{mod}\,A}(X_s)$ is the τ-perpendicular category of X_s in mod A. By Theorem 3.5, $J_{\text{mod}\,A}(X_s)$ is equivalent to a category of modules over some finite dimensional $\mathbb{F}$-algebra. By assumption, X is a τ-exceptional sequence in mod A, thus $X' = (X_1, X_2, \ldots, X_{s-1})$ is a τ-exceptional sequence in $J_{\text{mod}\,A}(X_s)$. Moreover, $Z' = (Z_1, Z_2, \ldots, Z_m)$ is an interleaved sequence of X' and Y, so it follows by the inductive hypothesis that Z' is a τ-exceptional sequence in $J_{\text{mod}\,A}(X_s) \oplus \text{mod}\,B = J_{(A,B)}(Z_{m+1})$, hence Z is a τ-exceptional sequence in mod $A \oplus$ mod B. This completes the proof. □

We now prove the converse statement.

Theorem 3.9 *Let A and B be finite dimensional $\mathbb{F}$-algebras. Suppose $Z = (Z_1, Z_2, \ldots, Z_m)$ is a τ-exceptional sequence in mod $A \oplus$ mod B. Then Z is an interleaved sequence of some $X = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_{m-s})$, where X is a τ-exceptional sequence in mod A and Y is a τ-exceptional sequence in mod B.*

Proof We prove this by induction on m.

For the base case, suppose $m = 1$, so $Z = (Z_1)$ is a τ-exceptional sequence in mod $A \oplus$ mod B. The module Z_1 either lies in mod A or mod B. Suppose without loss of generality that $Z_1 \in$ mod A. So we define the sequence $X := (Z_1)$ and the sequence Y to be the empty sequence. The sequence Z is trivially an interleaved sequence of X and Y. As Z is a τ-exceptional sequence in mod $A \oplus$ mod B, by definition Z_1 is τ-rigid in mod $A \oplus$ mod B, so Z_1 is τ-rigid in mod A. This completes the base case.

Now suppose the statement is true for $m = k$. We consider the $m = k + 1$ case. The sequence $Z = (Z_1, Z_2, \ldots, Z_{k+1})$ is a τ-exceptional sequence in mod $A \oplus$ mod B, so by definition, Z_{k+1} is τ-rigid in mod $A \oplus$ mod B and the sequence $Z' = (Z_1, Z_2, \ldots, Z_k)$ is a τ-exceptional sequence in $J_{(A,B)}(Z_{k+1})$, the τ-perpendicular category of Z_{k+1} in mod $A \oplus$ mod B. Suppose without loss of generality that $Z_{k+1} \in$ mod A. We then observe that $\text{Hom}_{\text{mod}\,A \oplus \text{mod}\,B}(Z_{k+1}, N) = \text{Hom}_{\text{mod}\,A \oplus \text{mod}\,B}(N, \tau Z_{k+1}) = 0$ for all $N \in$ mod B, so it follows that

$$J_{(A,B)}(Z_{m+1}) = \{U \in \text{mod}\,A \oplus \text{mod}\,B : \text{Hom}_{\text{mod}\,A \oplus \text{mod}\,B}(X_s, U) = \text{Hom}_{\text{mod}\,A \oplus \text{mod}\,B}(U, \tau_A X_s) = 0\}$$
$$= J_{\text{mod}\,A}(Z_{k+1}) \oplus \text{mod}\,B,$$

where $J_{\text{mod}\,A}(Z_{k+1})$ is the τ-perpendicular category of Z_{k+1} in mod A. By theorem 3.5 we have that $J_{\text{mod}\,A}(Z_{k+1})$ is equivalent to a category of modules over some finite dimensional $\mathbb{F}$-algebra. So we may apply the inductive hypothesis to Z', hence Z' is an interleaved sequence of some $X' = (X_1, X_2, \ldots, X_s)$ and $Y = (Y_1, Y_2, \ldots, Y_{k-s})$, where X' is a τ-exceptional sequence in $J_{\text{mod}\,A}(Z_{k+1})$ and Y is a τ-exceptional sequence in mod B. Since Z_{k+1} is τ-rigid in mod $A \oplus$ mod B, it is also τ-rigid mod A, hence $X = (X_1, X_2, \ldots, X_s, Z_{k+1})$ is a τ-exceptional sequence in mod A. Clearly Z is an interleaved sequence X and Y, so this completes the proof by induction. □

We will now recall some standard definitions from [5] which we require for the rest of this paper. Recall that the radical of an A-module M, denoted by rad(M), is defined to be the intersection of all maximal submodules of M. The quotient $M/\text{rad}(M)$ is known as the top of M and is denoted top(M). The socle of an A-module M denoted soc(M) is the sum of the simple submodules of M.

Definition 3.10 Radical Series [5, V.1]. Let M be an A-module. The radical series of M is defined to be the following sequence of submodules,

$$0 \subset \cdots \subset \mathrm{rad}^2(M) \subset \mathrm{rad}(M) \subset M.$$

Since the left A-modules M are finite dimensional as $\mathbb{F}$-vector spaces, there exists a least positive integer m such that $\mathrm{rad}^m(M) = 0$. The integer m is called the length of the radical series and we denote it by $l(M) = m$. We will also refer to $l(M)$ as the length of the module M.

Proposition 3.11 [5, V.3.5, V.4.1, V.4.2]. *Let A be a basic connected Nakayama algebra and let M be an indecomposable A-module. Then there exists some $1 \leq i \leq n$ and $1 \leq j \leq l(P_i)$, such that $M \cong P_i/\mathrm{rad}^j(P_i)$ and $j = l(M)$. Moreover, if M is not projective, we have that $\tau M \cong \mathrm{rad}(P_i)/\mathrm{rad}^{j+1}(P_i)$ and $l(\tau M) = l(M)$.*

So we see that modules M of Nakayama algebras are uniquely determined by their top, $\mathrm{top}(M)$ and their length $l(M)$.

Proposition 3.12 [1, Lemma 2.4]. *Let $M = P_j/\mathrm{rad}^l(P_j)$ and $N = P_i/\mathrm{rad}^k(P_i)$ for $1 \leq i, j, k, l \leq n$. Then the following conditions are equivalent,*

1. *$Hom(M, N) \neq 0$*
2. *$j \in [i, (i + k - 1)]_n$ and $(i + k - 1)_n \in [j, (j + l - 1)]_n$*

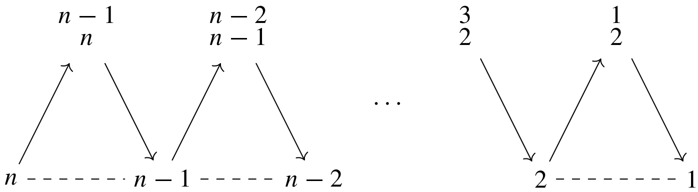
4 The Γ_n^2 Case

Let $n \geq 1$ be a positive integer. In this section we will derive a closed formula for the number of complete τ-exceptional sequences in $\mathrm{mod}\,\Gamma_n^2$. Recall that we denote by A_n the linearly oriented quiver with n vertices,

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \ldots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n.$$

The algebra Γ_n^2 is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}A_n/R_Q^2$. This is the path algebra of the quiver A_n modulo the relations $\alpha_i\alpha_{i+1} = 0$ for $1 \leq i \leq n-2$.

The category $\mathrm{mod}\,\Gamma_n^2$ has the following Auslander-Reiten quiver.



Our strategy for calculating the number of τ-exceptional sequences is straightforward. For each M in $\mathrm{ind}(\mathrm{mod}\,\Gamma_n^2)$, we will calculate the number of complete τ-exceptional sequences ending in M. If M is indecomposable, then either $M = P_i$, the projective at vertex i of A_n, or $M = S_i$, the simple at vertex i of A_n (notice that $S_n = P_n$). In the former case $\tau P_i = 0$ for $1 \leq i \leq n$ and in the latter case $\tau S_j = S_{j+1}$ for $1 \leq j \leq n-1$. In both cases we see that M is τ-rigid i.e. every indecomposable M in $\mathrm{mod}\,\Gamma_n^2$ is τ-rigid. We recall that a sequence of indecomposable modules $(M_1, M_2, \ldots, M_{n-1}, M)$ is a τ-exceptional sequence in $\mathrm{mod}\,\Gamma_n^2$ if M is τ-rigid, and $(M_1, M_2, \ldots, M_{n-1})$ is a τ-exceptional sequence in $J(M)$.

Having seen that every indecomposable module M is τ-rigid, what is left to do is to calculate $J(M)$ for each indecomposable module. Theorem 3.5 and Lemma 3.4 are the main tools for these calculations.

Proposition 4.1 *Let P_i be an indecomposable projective module in mod Γ_n^2 for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in mod Γ_n^2 is $J(P_i) \cong$ mod $\Gamma_{i-1}^2 \oplus$mod Γ_{n-i}^2.*

Proof By definition $T_{P_i} = \mathcal{P}(^{\perp}(\tau P_i))$. Since $\tau P_i = 0$, we have that $^{\perp}(\tau P_i) = \operatorname{mod}\Gamma_n^2$. As a result the Ext-projectives of $^{\perp}(\tau P_i)$ are just the projectives of mod Γ_n^2, hence

$$T_{P_i} = \mathcal{P}(^{\perp}(\tau P_i)) = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \operatorname{End}_{\Gamma_n^2}(T_{P_i})$ is precisely given by the path algebra of A_n^{op},

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_4} \dots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_i} i \xleftarrow{\alpha_{i+1}} i+1 \xleftarrow{\alpha_{i+2}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n$$

modulo the relations $\alpha_j\alpha_{j-1} = 0$ for $3 \leq j \leq n$. Let $A_n^{\text{op}(i)}$ be the quiver obtained from A_n^{op} by removing the vertex i and any arrows incident to i,

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_4} \dots \xleftarrow{\alpha_{i-1}} i-1 \qquad\qquad i+1 \xleftarrow{\alpha_{i+2}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n.$$

The quiver $A_n^{\text{op}(i)}$ has relations $\alpha_j\alpha_{j-1} = 0$ for $3 \leq j \leq i-1$ and $i+3 \leq j \leq n$. By Lemma 3.4, $D_{P_i} = E_{P_i}/\langle e_{P_i}\rangle$ is the path algebra of $A_n^{\text{op}(i)}$ modulo its relations. So it follows that $J(P_i) \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i}^2$ by Theorem 3.5. □

Proposition 4.2 *Let S_i be a simple non-projective module in mod Γ_n^2 for some $1 \leq i \leq n-1$. Then the τ-perpendicular category of S_i in mod Γ_n^2 is $J(S_i) \cong$ mod $\Gamma_{i-1}^2 \oplus$mod $\Gamma_{n-i-1}^2 \oplus$ mod Γ_1^2.*

Proof By definition $T_{S_i} = \mathcal{P}(^{\perp}(\tau S_i))$. Since S_i is a simple non-projective indecomposable module S_i, we have that $\tau S_i = S_{i+1}$. Note that the only indecomposable Γ_n^2-modules not in $^{\perp}(\tau S_i)$ are S_{i+1} and P_{i+1}. Observe also that $\operatorname{Ext}_{\Gamma_n^2}(P_j, {}^{\perp}(\tau S_i)) = 0$ if $j \neq i+1$, $1 \leq j \leq n$. We also have that $\operatorname{Ext}_{\Gamma_n^2}(S_j, {}^{\perp}(\tau S_i)) \neq 0$ for $j \neq i$, and $1 \leq j \leq n$ because S_{j+1} is in $^{\perp}(\tau S_i)$ in these cases. By Proposition 3.1, S_i is Ext-projective in $^{\perp}(\tau S_i)$. Therefore

$$T_{S_i} = \mathcal{P}(^{\perp}(\tau S_i)) = S_i \oplus \bigoplus_{j \neq i+1} P_j.$$

The $\mathbb{F}$-algebra $E_{S_i} = \operatorname{End}_{\Gamma_n^2}(T_{S_i})$ is the path algebra of the following quiver,

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \dots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_{v_{S_i}}} v_{S_i} \xleftarrow{\alpha_i} i \qquad i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n$$

modulo the relations $\alpha_j\alpha_{j-1} = 0$ for $3 \leq j \leq i-1$ and $i+4 \leq j \leq n$. Here the vertex v_{S_i} is the one corresponding to the simple non-projective module S_i and the rest correspond to the projective modules P_j. Consider the following quiver obtained from the one above by removing the vertex v_{S_i} and any arrows incident to v_{S_i},

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} \dots \xleftarrow{\alpha_{i-1}} i-1 \qquad i \qquad i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n,$$

it has the relations $\alpha_j\alpha_{j-1} = 0$ for $3 \leq j \leq i-1$ and $i+4 \leq j \leq n$. By Lemma 3.4, $D_{S_i} = E_{S_i}/\langle e_{S_i}\rangle$ is the path algebra of this quiver modulo its relations. So it follows that $\operatorname{mod} D_{S_i} \cong \operatorname{mod}\Gamma_{i-1}^2 \oplus \operatorname{mod}\Gamma_{n-i-1}^2 \oplus \operatorname{mod}\Gamma_1^2$. By Theorem 3.5, the statement of this Proposition follows. □

Let us denote by G_n the number of complete τ-exceptional sequences of mod Γ_n^2. When $n = 0, 1, 2$ the τ-exceptional sequences coincide with the "classical" exceptional sequences since the algebra Γ_n^2 is the hereditary Dynkin type A algebra $\mathbb{A}_n$ in this case. Hence, $G_0 = G_1 = 1$ and $G_2 = 3$.

Lemma 4.3 *Let P_i be the indecomposable projective module in mod Γ_n^2 at the vertex i of A_n for some $1 \leq i \leq n$. The number of complete τ-exceptional sequences in mod Γ_n^2 ending in P_i is,*

$$\binom{n-1}{n-i, i-1} G_{n-i} G_{i-1}.$$

Proof Let $(X_1, X_2, \dots, X_{n-1}, P_i)$ be a complete τ-exceptional sequence in mod Γ_n^2 ending in P_i. Then by definition and the fact that $\delta(J(P_i)) = n-1$, the sequence $(X_1, X_2, \dots, X_{n-1})$ is a τ-exceptional sequence in $J(P_i)$. So to count the number of complete τ-exceptional sequences in mod Γ_n^2 ending in P_i, we just need to count the number of complete τ-exceptional sequences in $J(P_i)$. By Lemma 4.1, $J(P_i) \cong \text{mod}\,\Gamma_{i-1}^2 \oplus \text{mod}\,\Gamma_{n-i}^2$. By Theorem 3.8 and 3.9, the τ-exceptional sequences of $J(P_i)$ are interleavings of τ-exceptional sequences of mod Γ_{i-1}^2 and mod Γ_{n-i}^2. The number of interleaved sequences coming from a sequence of length $i-1$ and a sequence of length $n-i$ is precisely $\binom{n-1}{n-i,i-1}$. Thus the number of complete τ-exceptional sequences ending in P_i is $\binom{n-1}{n-i,i-1} G_{n-i} G_{i-1}$. □

Lemma 4.4 *Let S_i be the indecomposable simple non-projective module in mod Γ_n^2 at the vertex i of A_n for some $1 \leq i \leq n-1$. The number of τ-exceptional sequences in mod Γ_n^2 ending in S_i is,*

$$\binom{n-1}{n-i-1, i-1} G_{n-i-1} G_{i-1}.$$

Proof Let $(X_1, X_2, \dots, X_{n-1}, S_i)$ be a complete τ-exceptional sequence in mod Γ_n^2 ending in S_i. Then by definition and the fact that $\delta(J(S_i)) = n-1$, the sequence $(X_1, X_2, \dots, X_{n-1})$ is a complete τ-exceptional sequence in $J(S_i)$. Hence to count the number of complete τ-exceptional sequences in mod Γ_n^2 ending in S_i, we just need to count the number of complete τ-exceptional sequences in $J(S_i)$. By Lemma 4.2, $J(S_i) \cong \text{mod}\,\Gamma_{i-1}^2 \oplus \text{mod}\,\Gamma_{n-i-1}^2 \oplus \text{mod}\,\Gamma_1^2$. The number of interleaved sequences coming from a sequence of length $i-1$, a sequence of length $n-i-1$ and a sequence of length 1 is precisely $\binom{n-1}{n-i-1,i-1,1} = \binom{n-1}{n-i-1,i-1}$. Thus the number of complete τ-exceptional sequences ending in S_i is $\binom{n-1}{n-i-1,i-1} G_{n-i-1} G_{i-1}$. □

Theorem 4.5 *Let G_n denote the number of complete τ-exceptional sequences in mod Γ_n^2. Then G_n satisfies the recurrence relation,*

$$G_n = \sum_{i=1}^{n} \binom{n-1}{n-i, i-1} G_{n-i} G_{i-1} + \sum_{i=1}^{n-1} \binom{n-1}{n-i-1, i-1} G_{n-i-1} G_{i-1},$$

with initial conditions $G_0 = G_1 = 1$.

Proof Let M be an indecomposable in mod Γ_n^2, then either M is projective or M simple non-projective. There are n projective indecomposable modules in mod Γ_n^2 denoted by P_i for $1 \leq i \leq n$. There are $n-1$ simple non-projective indecomposable modules in mod Γ_n^2 denoted by

S_i for $1 \le i \le n-1$. Therefore by Lemma 4.3 and 4.4, $G_n = \sum_{i=1}^{n} \binom{n-1}{n-i,i-1} G_{n-i} G_{i-1} + \sum_{i=1}^{n-1} \binom{n-1}{n-i-1,i-1} G_{n-i-1} G_{i-1}$. □

Theorem 4.5 allows us to calculate the first ten terms of the sequence $(G_n)_{n=0}^{\infty}$ as:

1, 1, 3, 12, 66, 450, 3690, 35280, 385560, 4740120, 6475140.

An ordered set partition of $\{1, 2, \dots, n\}$ is a partition of the set $\{1, 2, \dots, n\}$ together with a total order on the sets in the partition. We refer to the sets in an ordered partition as blocks. The *restricted Fubini* number $F_{n,\leqslant m}$ counts the number of ordered set partitions of $\{1, 2, \dots, n\}$ with blocks of size at most m. The *restricted Stirling number of the second kind*, denoted by $\left\{ {n \atop k} \right\}_{\leqslant m}$, is the number of (unordered) partitions of $\{1, 2, \dots, n\}$ into k subsets with the restriction that each block contains at most m elements. Therefore

$$F_{n,\leqslant m} = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\leqslant m}.$$

It is shown in [21, Section 5.4] that the restricted Fubini numbers satisfy the recurrence:

$$F_{n,\leqslant m} = \sum_{l=1}^{m} \binom{n}{l} F_{n-l,\leqslant m}.$$

The sequence $(F_{n,\leqslant 2})$ is listed on the On-line Encyclopedia of Integer Sequences (OEIS) as the sequence A080599. The first terms of this sequence coincide with the first terms we calculated for (G_n) so we would like to prove that it is the case that $F_{n,\leqslant 2} = G_n$.

When $m = 2$ the recurrence for $F_{n,\leqslant m}$ is given as $F_{n,\leqslant 2} = nF_{n-1,\leqslant 2} + \binom{n}{2} F_{n-2,\leqslant 2}$. In the paper [13, Theorem 3.7], the authors derive the closed formula

$$F_{n,\leqslant 2} = \frac{n!}{\sqrt{3}} \left((\sqrt{3}-1)^{-n-1} - (-\sqrt{3}-1)^{-n-1} \right).$$

An exponential generating function for $F_{n,\leqslant m}$ is given in [20, Theorem 4]:

$$\sum_{n=0}^{\infty} F_{n,\leqslant m} \frac{x^n}{n!} = \frac{1}{1 - x - \frac{x^2}{2!} - \dots \frac{x^m}{m!}}.$$

We will show that $G_n = F_{n,\leqslant 2}$ by showing that the exponential generating functions for G_n and $F_{n,\leqslant 2}$ coincide.

Theorem 4.6 *Let G_n denote the number of complete τ-exceptional sequences in* mod Γ_n^2. *The exponential generating function of G_n is as follows,*

$$\sum_{n=0}^{\infty} G_n \frac{x^n}{n!} = \frac{1}{1 - x - \frac{x^2}{2!}}.$$

Therefore $G_n = F_{n,\leqslant 2}$ and

$$G_n = \frac{n!}{\sqrt{3}} \left((\sqrt{3}-1)^{-n-1} - (-\sqrt{3}-1)^{-n-1} \right).$$

Proof First let us recall the recurrence relation for G_n.

$$G_n = \sum_{i=1}^{n} \binom{n-1}{n-i, i-1} G_{n-i} G_{i-1} + \sum_{i=1}^{n-1} \binom{n-1}{n-i-1, i-1} G_{n-i-1} G_{i-1}.$$

$$= \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!(i-1)!} G_{n-i} G_{i-1} + \sum_{i=1}^{n-1} \frac{(n-1)!}{(n-i-1)!(i-1)!} G_{n-i-1} G_{i-1}.$$

Therefore

$$G_{n+1} = \sum_{i=1}^{n+1} \frac{n!}{(n+1-i)!(i-1)!} G_{n+1-i} G_{i-1} + \sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} G_{n-i} G_{i-1}.$$

Let

$$g(x) = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!} \text{ with } g(0) = 1,$$

be the exponential generating function of G_n. We then have that the first derivative of $g(x)$ is $g'(x) = \sum_{n=0}^{\infty} G_{n+1} \frac{x^n}{n!}$. Expanding G_{n+1} in $g'(x)$ by the recurrence relation above we obtain the following.

$$g'(x) = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n+1} \frac{n!}{(n+1-i)!(i-1)!} G_{n+1-i} G_{i-1} \right) \frac{x^n}{n!} + \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} G_{n-i} G_{i-1} \right) \frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n+1} \frac{G_{n+1-i} G_{i-1}}{(n+1-i)!(i-1)!} \right) x^n + \sum_{n=0}^{\infty} \left(\sum_{i=1}^{n} \frac{G_{n-i} G_{i-1}}{(n-i)!(i-1)!} \right) x^n.$$

Recall the Cauchy product of formal power series is as follows,

$$\left(\sum_{s=0}^{\infty} a_s x^s \right) \left(\sum_{t=0}^{\infty} b_t x^t \right) = \sum_{k=0}^{\infty} c_k x^k \text{ where } c_k = \sum_{l=0}^{k} a_l b_{k-l}.$$

By performing a change of variable in $g'(x)$ by setting $j = i - 1$ and factorising x from the right summand we write,

$$g'(x) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \frac{G_{n-j} G_j}{(n-j)!j!} \right) x^n + x \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n-1} \frac{G_{n-j-1} Gj}{(n-j-1)!j!} \right) x^{n-1}.$$

Using the Cauchy product of formal power series, we obtain the following first order non-linear ordinary differential equation.

$$g'(x) = (g(x))^2 + x(g(x))^2 = (1+x)(g(x))^2 \text{ with initial conditions } g(0) = 1.$$

It is easy to check that the unique solution to this ODE is given by,

$$g(x) = \frac{-2}{-2 + x(x+2)} = \frac{1}{1 - x - \frac{x^2}{2}}.$$

This completes the proof. □

Remark 4.7 Here we focused on τ-exceptional sequences, but it's natural to ask what is known about the more classical exceptional sequences. It is shown in [29] that the number of complete exceptional sequences of mod Γ_n^2 are equal to the sum, $\sum_{j=1}^{n} \binom{n}{j} j^{n-j}$. The first ten terms of the sequence $(\sum_{j=1}^{n} \binom{n}{j} j^{n-j})_{n=1}^{\infty}$ are,

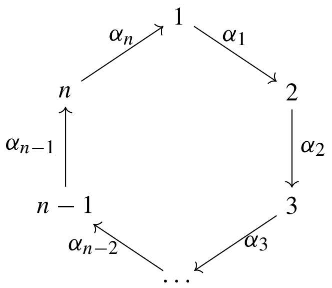
1, 3, 10, 41, 196, 1057, 6322, 41393, 293608, 2237921.

For comparison the number of complete τ-exceptional sequences of mod Γ_n^2 are given by G_n, the first ten terms of the sequence $(G_n)_{n=1}^{\infty}$ are,

1, 3, 12, 66, 450, 3690, 35280, 385560, 4740120, 6475140.

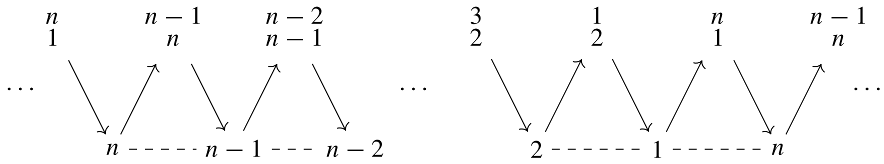
5 The Λ_n^2 Case

Let $n \geq 1$ be a positive integer. In this section we will derive a closed formula for the number of complete τ-exceptional sequences in mod Λ_n^2. Recall that we denote by C_n the linearly oriented n-cycle.



The algebra Λ_n^2 is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}C_n/R_Q^2$. This is the path algebra of the quiver C_n modulo the relations $\alpha_j\alpha_{(j+1)_n} = 0$ for $1 \leq j \leq n$.

The category mod Λ_n^2 has the following Auslander-Reiten quiver.



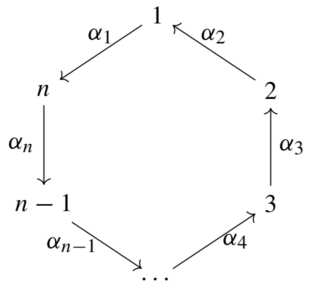
We will use the same approach for calculating the number of complete τ-exceptional sequences for mod Λ_n^2 as we did for mod Γ_n^2. If M is indecomposable in mod Λ_n^2, then $M = P_i$, the projective at vertex i of C_n, or $M = S_i$, the simple at vertex i of C_n. In the former case $\tau P_i = 0$ and in the latter case $\tau S_i = S_{(i+1)_n}$. In both cases M is τ-rigid i.e. every indecomposable M in mod Λ_n^2 is τ-rigid. We recall that a sequence of indecomposable modules $(M_1, M_2, \ldots, M_{n-1}, M)$ is a τ-exceptional sequence in mod Λ_n^2 if M is τ-rigid, and $(M_1, M_2, \ldots, M_{n-1})$ is a τ-exceptional sequence in $J(M)$. Having seen that every indecomposable module is τ-rigid, what is left to do is to calculate $J(M)$ for each indecomposable module. Theorem 3.5 and Lemma 3.4 are the main tools for these calculations.

Proposition 5.1 *Let P_i be an indecomposable projective module in mod Λ_n^2 for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in mod Λ_n^2 is $J(P_i) \cong$ mod Γ_{n-1}^2.*

Proof By definition $T_{P_i} = \mathcal{P}(^{\perp}(\tau P_i))$. Since P_i is projective, we have that $\tau P_i = 0$, therefore $^{\perp}(\tau P_i) = \text{mod}\,\Lambda_n^2$. As a result the Ext-projectives of $^{\perp}(\tau P_i)$ are just the projectives of mod Λ_n^2, hence

$$T_{P_i} = \mathcal{P}(^{\perp}(\tau P_i)) = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \mathrm{End}_{\Lambda_n^2}(T_{P_i})$ is precisely given by the path algebra of the quiver C_n^{op},



modulo the relations $\alpha_i \alpha_{(i-1)_n} = 0$ for $1 \leq i \leq n$.

Let $C_n^{\mathrm{op}(i)}$ be the quiver obtained from C_n^{op} by removing the vertex at i and any arrows incident to i.

$$i+1 \xleftarrow{\alpha_{i+2}} i+2 \xleftarrow{\alpha_{i+3}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{i-2}} i-2 \xleftarrow{\alpha_{i-1}} i-1$$

It has the relations $\alpha_1 \alpha_n = 0$ and $\alpha_j \alpha_{j-1} = 0$ for $i+2 \leq j \leq n$ and $2 \leq j \leq i-2$. By Lemma 3.4, $D_{P_i} = E_{P_i}/\langle e_{P_i} \rangle$ is the path algebra of the quiver $C_n^{\mathrm{op}(i)}$ modulo relations. It is easy to see that in fact D_{P_i} is isomorphic to Γ^2_{n-1}. Hence by Theorem 3.5, the τ-perpendicular category $J(M) \cong \mathrm{mod}\,\Gamma^2_{n-1}$. □

Proposition 5.2 *Let S_i be a simple module in mod Λ_n^2 for some $1 \leq i \leq n$. Then the τ-perpendicular category of S_i in mod Λ_n^2 is $J(S_i) \cong$ mod $\Gamma^2_{n-2} \oplus$ mod Γ^2_1.*

Proof By definition $T_{S_i} = \mathcal{P}({}^{\perp}(\tau S_i))$. Since S_i is a simple Λ_n^2-module, we have that $\tau S_i = S_{(i+1)_n}$. Note that the only Λ_n^2-modules not in ${}^{\perp}(\tau S_i)$ are $S_{(i+1)_n}$ and $P_{(i+1)_n}$. Observe also that $\mathrm{Ext}_{\Lambda_n^2}(P_j, {}^{\perp}(\tau S_i)) = 0$ for $j \neq (i+1)_n$ and $1 \leq j \leq n$. However for $j \neq i, i+1$, $\mathrm{Ext}_{\Lambda_n^2}(S_j, {}^{\perp}(\tau S_i)) \neq 0$ because $S_{(j+1)_n}$ is in ${}^{\perp}(\tau M)$. By Proposition 3.1, S_i is Ext-projective in ${}^{\perp}(\tau S_i)$. Hence

$$T_{S_i} = \mathcal{P}({}^{\perp}(\tau S_i)) = S_i \oplus \bigoplus_{j \neq (i+1)_n} P_j$$

is the Bongartz completion of S_i.

The $\mathbb{F}$-algebra $E_{S_i} = \mathrm{End}_{\Lambda_n^2}(T_{S_i})$ is given by the path algebra of the quiver,

$$i+2 \xleftarrow{\alpha_{i+3}} i+3 \xleftarrow{\alpha_{i+4}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_{v_{S_i}}} v_{S_i} \xleftarrow{\alpha i} i$$

modulo the relations $\alpha_{v_{S_i}} \alpha_{i-1} = 0 = \alpha_1 \alpha_n$ and $\alpha_j \alpha_{j-1} = 0$ for $i+4 \leq j \leq n$ and $2 \leq j \leq i-1$. Here the vertex v_{S_i} is the one corresponding to the simple module S_i and the rest correspond to the projective modules P_j. By Lemma 3.4, $D_{S_i} = E_{S_i}/\langle e_{S_i} \rangle$ is the path algebra of the quiver obtained from the one above by removing the vertex v_{S_i},

$$i+2 \xleftarrow{\alpha_{i+3}} i+3 \xleftarrow{\alpha_{i+4}} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \dots \xleftarrow{\alpha_{i-1}} i-1 \qquad\qquad i$$

modulo the relations $\alpha_1 \alpha_n = 0$ and $\alpha_j \alpha_{j-1} = 0$ for $i+4 \leq j \leq n$ and $2 \leq j \leq i-1$. So it follows that mod $D_{S_i} \cong$ mod $\Gamma^2_{n-2} \oplus$ mod Γ^2_1. By Theorem 3.5 the statement of this Proposition follows. □

Denote by L_n the number of complete τ-exceptional sequences in mod Λ_n^2.

Theorem 5.3 *Let L_n be the number of complete τ-exceptional sequences in mod Λ_n^2. Then L_n satisfies the relation,*

$$L_n = nG_{n-1} + n(n-1)G_{n-2},$$

with initial conditions $L_1 = 1$ and $L_2 = 4$, *and where G_m denotes the number of complete τ-exceptional sequences in mod Γ_m^2.*

Proof Suppose M is an indecomposable projective Λ_n^2-module, then by Lemma 5.1, the τ-perpendicular category $J(M) \cong \operatorname{mod}\Gamma_{n-1}^2$. Suppose $(X_1, X_2, \ldots, X_{n-1}, M)$ is a complete τ-exceptional sequence ending in M in mod Λ_n^2 . Then by the fact that $\delta(J(M)) = n-1$ and by definition, the sequence $(X_1, X_2, \ldots, X_{n-1})$ is a complete τ-exceptional sequence in $J(M) \cong \operatorname{mod}\Gamma_{n-1}^2$. Hence the number of complete τ-exceptional sequences ending in M is G_{n-1}, which is the number of complete τ-exceptional sequences in mod Γ_{n-1}^2.

Now suppose M is a simple Λ_n^2-module. By Lemma 5.2, the τ-perpendicular category $J(M) \cong \operatorname{mod}\Gamma_{n-2}^2 \oplus \operatorname{mod}\Gamma_1^2$. Arguing as above the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $J(M)$. Since $J(M) \cong \operatorname{mod}\Gamma_{n-2}^2 \oplus \operatorname{mod}\Gamma_1^2$, by Theorem 3.8 and 3.9, the τ-exceptional sequences of $J(M)$ are interleavings of τ-exceptional sequences of $\operatorname{mod}\Gamma_{n-2}^2$ and $\operatorname{mod}\Gamma_1^2$. The number of interleaved sequences coming from a sequence of length $n-2$ and a sequence of length 1 is precisely $\binom{n-1}{n-2,1} = (n-1)$. Thus the number of complete τ-exceptional sequences ending in M is $(n-1)G_{n-2}G_1 = (n-1)G_{n-2}$.

An arbitrary indecomposable Λ_n^2-module is either projective or simple. There are n projective modules and n simple modules up to isomorphism in mod Λ_n^2, hence the number of complete τ-exceptional sequences in mod Λ_n^2 is $L_n = nG_{n-1} + n(n-1)G_{n-2}$. It then follows easily that $L_1 = 1$ and $L_2 = 4$. □

In the previous section we found the exponential generating function and closed formula for G_n. Using the above theorem, we can immediately do the same for L_n.

Theorem 5.4 *Let L_n denote the number of complete τ-exceptional sequences in mod Λ_n^2. The exponential generating function of L_n is as follows,*

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = \frac{x + x^2}{1 - x - \frac{x^2}{2}} .$$

Proof Let $h(x) = \sum_{n=0}^{\infty} L_n \frac{x^n}{n!}$ be the exponential generating function of L_n. Let

$$g(x) = \sum_{n=0}^{\infty} G_n \frac{x^n}{n!}$$

be the exponential generating function of G_n. We then recall the recurrence relation of L_n,

$$L_n = nG_{n-1} + n(n-1)G_{n-2}.$$

Therefore the exponential generating function of L_n is,

$$\sum_{n=0}^{\infty} L_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} nG_{n-1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} n(n-1)G_{n-2} \frac{x^n}{n!}.$$

$$= \sum_{n=0}^{\infty} G_{n-1} \frac{x^n}{(n-1)!} + \sum_{n=0}^{\infty} G_{n-2} \frac{x^n}{(n-2)!}$$

$$= x\sum_{n=0}^{\infty} G_{n-1}\frac{x^{n-1}}{(n-1)!} + x^2\sum_{n=0}^{\infty} G_{n-2}\frac{x^{n-2}}{(n-2)!}.$$

Therefore

$$h(x) = xg(x) + x^2(g(x)) = (x+x^2)g(x).$$

By Theorem 4.6,

$$g(x) = \frac{1}{1-x-\frac{x^2}{2}},$$

hence

$$h(x) = \frac{x+x^2}{1-x-\frac{x^2}{2!}}.$$

□

Theorem 5.5 *Let L_n denote the number of complete τ-exceptional sequences in mod Λ_n^2. Then L_n is given by the closed formula,*

$$L_n = \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-2} - (-\sqrt{3}-1)^{-n-2}\right) + \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-3} - (-\sqrt{3}-1)^{-n-3}\right).$$

Proof It is immediate from the recurrence relation for L_n and Theorem 4.6 that,

$$L_n = n\frac{(n-1)!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-2} - (-\sqrt{3}-1)^{-n-2}\right) + n(n-1)\frac{(n-2)!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-3} - (-\sqrt{3}-1)^{-n-3}\right)$$
$$= \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-2} - (-\sqrt{3}-1)^{-n-2}\right) + \frac{n!}{\sqrt{3}}\left((\sqrt{3}-1)^{-n-3} - (-\sqrt{3}-1)^{-n-3}\right).$$

□

We calculate the first 10 terms of the sequence $(L_n)_{n=0}^{\infty}$ to be,

$$1, 4, 15, 84, 570, 4680, 44730, 488880, 6010200, 82101600.$$

In comparison to τ-exceptional sequences, there are no complete exceptional sequences in mod Λ_n^2, as we will show. In general, not much is known about exceptional sequences over the Nakayama algebras Λ_n^2.

Proposition 5.6 *There are no complete exceptional sequences in mod Λ_n^2 when $n > 1$.*

Proof Suppose $M = (M_1, M_2, \ldots, M_n)$ is a complete exceptional sequence. Recall that an indecomposable module in mod Λ_n^2 is either projective or simple. Since $\mathrm{Hom}(P_i, P_{(i+1)_n}) \neq 0$ for $1 \leq i \leq n$, the sequence M cannot consist entirely of just indecomposable projective modules, so M must contain at least one simple module.

Consider the simple module S_i for some $1 \leq i \leq n$. Then S_i has the following infinite exact sequence as its projective resolution.

$$\ldots \longrightarrow P_1 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_{i+1} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0$$

We observe that the projective resolution of S_i contains every projective indecomposable module of mod Λ_n^2. We also observe that the only projective module P such that $\mathrm{Hom}(P, S_i) \neq 0$ is $P = P_i$. Hence applying the functor $\mathrm{Hom}(-, S_i)$ to the above projective resolution we get the following sequence.

$$0 \longrightarrow \mathrm{Hom}(S_i, S_i) \xrightarrow{f_0} \mathrm{Hom}(P_i, S_i) \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} 0 \xrightarrow{f_n} \mathrm{Hom}(P_i, S_i) \xrightarrow{f_{n+1}} 0 \xrightarrow{f_{n+2}} \ldots$$

We then observe that $\mathrm{Ext}^n(S_i, S_i) = \ker(f_{n+1})/\mathrm{im}(f_n) \neq 0$, so no simple module in mod Λ_n^2 is exceptional. From this we conclude M cannot contain simple modules, a contradiction. □

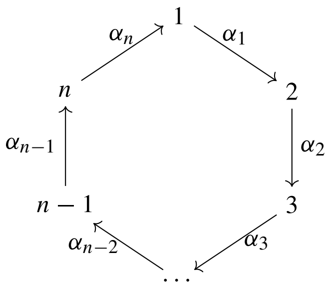
Unlike exceptional sequences, weak exceptional sequences number over Λ_n^2 have been studied. Weak exceptional sequences are defined in [30] as follows.

Definition 5.7 [30, Definition 1.1]. Let A be a finite dimensional algebra over a field $\mathbb{F}$, where $\mathbb{F}$ is algebraically closed. A left A-module M is called *weak exceptional* if $\mathrm{Hom}(M, M) \cong \mathbb{F}$ and $\mathrm{Ext}_A^1(M, M) = 0$. A sequence of indecomposable modules $(M_1, M_2, \ldots, M_r)$ is called a *weak exceptional sequence* if, for each pair (M_i, M_j) with $1 \leq i < j \leq r$, we have that $\mathrm{Hom}(M_j, M_i) = \mathrm{Ext}_A^1(M_j, M_i) = 0$ and each M_k is weak exceptional for $1 \leq k \leq r$.

It turns out that for weak exceptional sequences over Λ_n^2, the maximum length need not be n and in fact can exceed n. According to [30, Theorem 1.6], if $n = 2m + 1$ is odd, the maximum length of a weak exceptional sequence over Λ_n^2 is equal to $3m + 1$. On the other hand, if $n = 2m$ is even, then the maximum length of a weak exceptional sequence over Λ_n^2 is $3m - 1$. A weak exceptional sequence with maximum length is called *full*. Again by [30, Theorem 1.6], when $n = 2m$, the number of full weak exceptional sequences is given by $2m\left(\frac{8^m}{12} - \frac{(-1)^m}{3} + 1\right)$, and when $n = 2m + 1$, the number of full weak exceptional sequences is given by n.

6 The Λ_n^n Case

Let $n \geq 1$ be a positive integer. In this section we will derive a closed formula for the number of complete τ-exceptional sequences in mod Λ_n^n. Recall that we denote by C_n the linearly oriented n-cycle.



The algebra Λ_n^n is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}C_n/R_Q^n$. This is the path algebra of the quiver C_n modulo the relations $\alpha_i\alpha_{(i+1)_n}\ldots\alpha_{(i+(n-1))_n} = 0$ for $1 \leq j \leq n$.

Proposition 6.1 [1, Proposition 2.5]. *Let A be a Nakayama algebra. Let M be an indecomposable non-projective module in mod A. Then M is rigid if and only if $l(M) < n$ holds.*

For our purposes, the following Proposition is a more convenient restatement of Proposition 3.12.

Proposition 6.2 *Let M be an indecomposable Λ_n^n-module with length $1 \le l(M) \le n-1$. Then Hom$(X, \tau M) \ne 0$ if and only if top$(X) \cong$ top(rad$^k(\tau M)$) for some $0 \le k \le l(M)-1$ and $l(\mathrm{rad}^k(\tau M)) \le l(X)$.*

Proof All indecomposable modules in mod Λ_n^n have simple tops. By Proposition 3.11, for a Λ_n^n-module X, we have that $X = P_j/\mathrm{rad}^{l(X)}(P_j)$ hence $\mathrm{top}(X) = S_j$. Let $M = P_{i-1}/\mathrm{rad}^{l(M)}P_{i-1}$, then $\tau M = P_{(i)_n}/\mathrm{rad}^{l(M)}(P_{(i)_n})$ by Proposition 3.11 as well. Observe that for $0 \le k \le l(M)-1$, $\mathrm{rad}^k(\tau M) = P_{(i+k)_n}/\mathrm{rad}^{(l(M)-k)}(P_{(i+k)_n})$ thus $\mathrm{top}(\mathrm{rad}^k(\tau M)) = S_{(i+k)_n}$ and $l(\mathrm{rad}^k(\tau M)) = l(M) - k$. By Proposition 3.12 we have that,

$$\mathrm{Hom}(X, \tau M) \ne 0 \text{ if and only if } j \in [i, (i+l(M)-1)]_n \text{ and } (i+l(M)-1)_n \in [j, (j+l(X)-1)]_n.$$

Suppose $\mathrm{Hom}(X, \tau M) \ne 0$, this implies that $j = (i+k)_n$ for some $0 \le k \le l(M)-1$, and $(i + l(M) - 1)_n = (j + a)_n$ for some $0 \le a \le l(X) - 1$. It then immediately follows $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^k(\tau M))$ and $l(\mathrm{rad}^k(\tau M)) \le l(X)$.

For the converse, suppose that $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^k(\tau M))$ and $l(\mathrm{rad}^k(\tau M)) \le l(X)$. Then $j = (i+k)_n$ for some $0 \le k \le l(M)-1$. Moreover, $l(\mathrm{rad}^k(\tau M)) = l(M)-k \le l(X)$ which implies $i+l(M)-1 \le (i+k)+l(X)-1$ therefore $(i+l(M)-1)_n \in [j, (j+l(X)-1)]_n$. Hence by Proposition 3.12, $\mathrm{Hom}(X, \tau M) \ne 0$. □

By Proposition 6.1, every indecomposable module M of mod Λ_n^n is τ-rigid in mod Λ_n^n since it is either projective or has length $l(M) < n$. Hence, we once again adopt the same strategy for calculating the number of complete τ-exceptional sequences in mod Λ_n^n as we have done thus far. For each M in ind(mod Λ_n^n), we will calculate the number of complete τ-exceptional sequences ending in M. By definition a sequence of indecomposable modules $(M_1, M_2, \ldots, M_{n-1}, M)$ is a τ-exceptional sequence in mod Λ_n^n if M is τ-rigid and $(M_1, M_2, \ldots, M_{n-1})$ is a τ-exceptional sequence in $J(M)$. Having seen that every indecomposable Λ_n^n-module M is τ-rigid, what is left to do is to calculate $J(M)$ for each indecomposable module. Theorem 3.5 and Lemma 3.4 are once again the main tools these calculations.

Proposition 6.3 *Let M be an indecomposable Λ_n^n-module with length $1 \le l(M) \le n-1$ and top$(M) = S_i$. Then for all $1 \le k \le l(M)-1$,*

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{\substack{1\le j\le n \\ j\notin[i+1, i+l(M)]_n}} P_j.$$

Proof Suppose the Λ_n^n-module M has top equal to $\mathrm{top}(M) = S_i$ and has length $1 \le l(M) \le n-1$ i.e. M is not projective. By Proposition 3.11, $M = P_i/\mathrm{rad}^{l(M)}(P_i)$ and $\tau M = \mathrm{rad}(P_i)/\mathrm{rad}^{l(M)+1}(P_i)$ with $l(M) = l(\tau M)$. It is easy to see that $\mathrm{top}(\tau M) = S_{(i+1)_n}$ hence $\tau M = P_{(i+1)_n}/\mathrm{rad}^{l(M)}(P_{(i+1)_n})$.

By Proposition 6.2, a Λ_n^n-module X is not in ${}^{\perp}(\tau M)$ if and only if $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^k(\tau M))$ for some $0 \le k \le l(M)-1$ and $l(\mathrm{rad}^k(\tau M)) \le l(X)$. Let $X = P_j/\mathrm{rad}^{l(X)}(P_j)$ for some $1 \le j \le n$. The statement $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^k(\tau M))$ for some $0 \le k \le l(M)-1$ means that $j = (i+1+k)_n$ for some $0 \le k \le l(M)-1$. With this we are able to determine the Ext-projectives in ${}^{\perp}(\tau M)$.

Let $Y = P_l$ be the indecomposable project at the vertex l with $l \ne (i+1+k)_n$ for some $0 \le k \le l(M)-1$. Then P_l is in ${}^{\perp}(\tau M)$ by Proposition 6.2. Moreover $\mathrm{Ext}_{\Lambda_n^n}(P_l, {}^{\perp}(\tau M)) = 0$ since P_l is a projective Λ_n^n-module. Hence P_l is Ext-projective in ${}^{\perp}(\tau M)$.

Let $Y = \mathrm{rad}^s(M)$ for some $1 \leq s \leq l(M) - 1$. Then observe that $Y = P_{(i+s)_n}/\mathrm{rad}^{(l(M)-s)}(P_{(i+s)_n})$ meaning $l(Y) = l(M) - s$. Recall a Λ_n^n-module X is not in ${}^{\perp}(\tau Y)$ if and only if $\mathrm{top}(X) \cong \mathrm{top}(\mathrm{rad}^r(\tau Y))$ for some $0 \leq r \leq l(M) - s - 1$ and $l(\mathrm{rad}^r(\tau Y)) \leq l(X)$. Therefore if $X = P_j/\mathrm{rad}^{l(X)}(P_j)$, then $j = (i + 1 + s + r)_n$ for some $0 \leq r \leq l(M) - s - 1$. This implies that $\{X : \underline{\mathrm{Hom}}(X, \tau Y) \neq 0\} \subset \{X : \mathrm{Hom}(X, \tau M) \neq 0\}$, which further implies that $\mathrm{Ext}_{\Lambda_n^n}(Y, N) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Lambda_n^n}(N, \tau Y) = 0$ for all N in ${}^{\perp}(\tau M)$ by the Auslander-Reiten formula. Hence $Y = \mathrm{rad}^s(M)$ is an Ext-projective in ${}^{\perp}(\tau M)$.

By Proposition 3.1, M is Ext-projective in ${}^{\perp}(\tau M)$. For every other indecomposable Λ_n^n-module Y, we have that τY is in ${}^{\perp}(\tau M)$, therefore $\mathrm{Ext}_{\Lambda_n^n}(Y, \tau Y) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Lambda_n^n}(\tau Y, \tau Y) \neq 0$ i.e. they are not Ext-projective in ${}^{\perp}(\tau M)$. By definition, $T_M = \mathcal{P}({}^{\perp}(\tau M))$, hence by the above arguments,

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{\substack{1 \leq j \leq n \\ j \notin [i+1, i+l(M)]_n}} P_j.$$

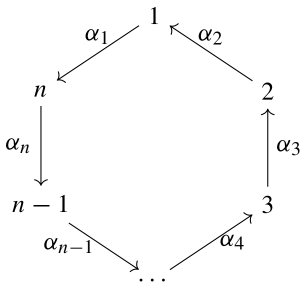
□

Proposition 6.4 *Let P_i be an indecomposable projective module in mod Λ_n^n for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in mod Λ_n^n is $J(P_i) \cong$ mod $\mathbb{A}_{n-1}$, where $\mathbb{A}_{n-1}$ is the Dynkin type A hereditary algebra.*

Proof By definition $T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i))$. Since P_i is projective, we have that $\tau P_i = 0$ therefore ${}^{\perp}(\tau P_i) = \mathrm{mod}\, \Lambda_n^n$. As a result the Ext-projectives of ${}^{\perp}(\tau P_i)$ are just the projectives of mod Λ_n^n therefore

$$T_{P_i} = \mathcal{P}({}^{\perp}(\tau P_i)) = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \mathrm{End}_{\Lambda_n^2}(T_{P_i})$ is precisely given by the path algebra of the quiver C_n^{op},



modulo the relations $\alpha_i \alpha_{(i+1)_n} \ldots \alpha_{(i+(n-1))_n} = 0$ for $1 \leq j \leq n$.

By Lemma 3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $C_n^{\mathrm{op}(i)}$ which is the quiver obtained from C_n^{op} by removing the vertex i. More precisely, $C_n^{\mathrm{op}(i)}$ is the quiver,

$$i+1 \xleftarrow{\alpha_{i+2}} i+2 \xleftarrow{\alpha_{i+3}} \ldots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \xleftarrow{\alpha_1} 1 \xleftarrow{\alpha_2} \ldots \xleftarrow{\alpha_{i-2}} i-2 \xleftarrow{\alpha_{i-1}} i-1$$

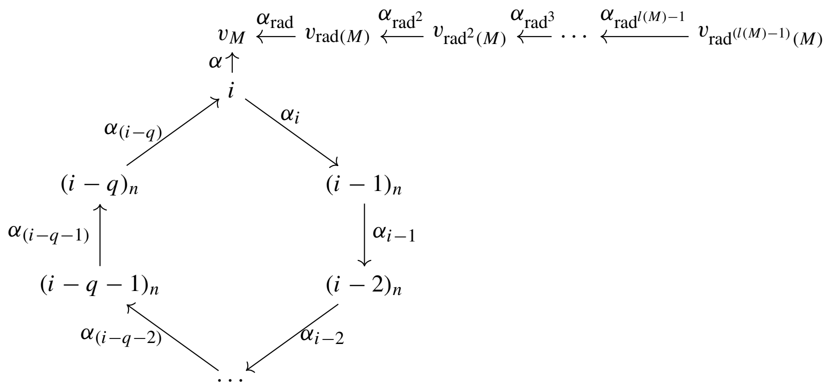
with no relations. It is easy to see that the path algebra $\mathbb{F}C_n^{\mathrm{op}(i)}$ is isomorphic to $\mathbb{A}_{n-1}$. Hence the Proposition follows by Theorem 3.5. □

Proposition 6.5 *Let M be an indecomposable Λ_n^n-module with length $1 \leq l(M) \leq n - 1$ and top$(M) = S_i$. Which is to say that $M = P_i/\mathrm{rad}^{l(M)}(P_i)$. Then the τ-perpendicular* *category of M in mod Λ_n^n is $J(M) \cong$ mod $\mathbb{A}_{l(M)-1} \oplus \mathrm{mod}\Lambda_{n-l(M)}^{n-l(M)}$, where $\mathbb{A}_m$ is the Dynkin type A hereditary algebra.*

Proof By Proposition 6.3 the Bongartz completion of M is,

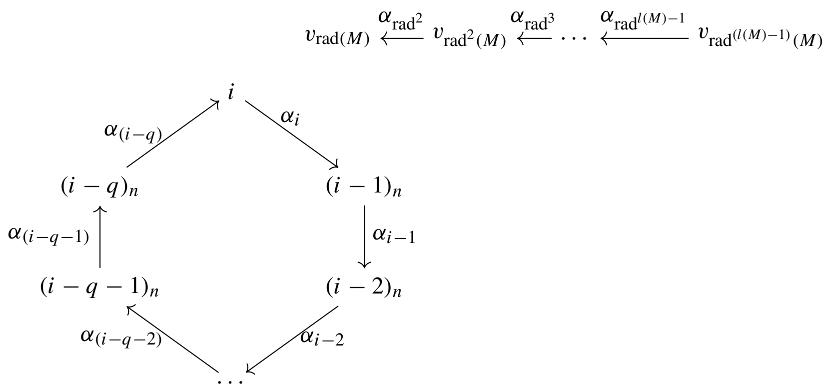
$$T_M = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{\substack{1 \le j \le n \\ j \notin [i+1, i+l(M)]_n}} P_j.$$

Hence the $\mathbb{F}$-algebra $E_M = \mathrm{End}_{\Lambda_n^n}(T_M)$ is the path algebra of the following quiver Q_n,



modulo the relations $\alpha_r\alpha_{r-1}\ldots\alpha_{r-(n-l(M)-1)} = 0$ for $r \in [i, i-(n-l(M)-1)]_n$ and where $q = n - l(M) - 1$.

Let $Q_n^{(v_M)}$ be the quiver obtained from Q_n by removing the vertex v_M and any arrows incident to v_M. More precisely, $Q_n^{(v_M)}$ is the following quiver with two connected components,



and with relations $\alpha_r\alpha_{r-1}\ldots\alpha_{r-(n-l(M)-1)} = 0$ for $r \in [i, i-(n-l(M)-1)]_n$. By Lemma 3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of $Q_n^{(v_M)}$ modulo relations. So it follows that mod $D_M \cong$ mod $\mathbb{A}_{l(M)-1} \oplus \mathrm{mod}\Lambda_{n-l(M)}^{n-l(M)}$. So by Theorem 3.5, the statement of the Proposition follows. $\square$

Theorem 6.6 *Let H_n denote the number of complete τ-exceptional sequences in mod Λ_n^n. Then H_n satisfies the recurrence relation,*

$$H_n = n\sum_{i=1}^{n}\binom{n-1}{i-1} i^{i-2} H_{n-i},$$

with $H_0 = 1$.

Proof Let M be an indecomposable Λ_n^n-module. Suppose $(M_1, M_2, \ldots, M_{n-1}, M)$ is a complete τ-exceptional sequence in mod Λ_n^n ending in M. Then by definition and the fact that $\delta(J(M)) = n-1$, the sequence $(M_1, M_2, \ldots, M_{n-1})$ is a complete τ-exceptional sequence in $J(M)$. It then follows that the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $J(M)$.

The length of M is $1 \leq l(M) \leq n$. For each possible value of $l(M)$, there are n indecomposable Λ_n^n-modules of that length. If $l(M) = n$ then M is projective and by Proposition 6.4, $J(M) \cong \text{mod}\,\mathbb{A}_{n-1}$. The number of τ-exceptional sequences in mod $\mathbb{A}_{n-1}$ was shown in [[28] [Proposition 1.1]] to be $n^{n-2} = \binom{n-1}{n-1} n^{n-2} H_0$, where $H_0 = 1$.

If $1 \leq l(M) \leq n-1$, then by Proposition 6.5, the τ-perpendicular category of M is $J(M) \cong \text{mod}\,\mathbb{A}_{l(M)-1} \oplus \text{mod}\,\Lambda_{n-l(M)}^{n-l(M)}$. Arguing as above, the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in mod $\mathbb{A}_{l(M)-1} \oplus \text{mod}\,\Lambda_{n-l(M)}^{n-l(M)}$. By Theorems 3.8 and 3.9, this is equal to

$$\binom{n-1}{n-l(M), l(M)-1} l(M)^{(l(M)-2)} H_{n-l(M)} = \binom{n-1}{l(M)-1} l(M)^{(l(M)-2)} H_{n-l(M)}.$$

So it follows that,

$$H_n = \sum_{l(M)=1}^{n} n\binom{n-1}{l(M)-1} l(M)^{l(M)-2} H_{n-l(M)} = n\sum_{i=1}^{n}\binom{n-1}{i-1} i^{i-2} H_{n-i}.$$

It is trivial to see that $H_1 = 1$. Using the recurrence we obtain $H_1 = \binom{0}{0} 1^{-1} H_0 = 1$, therefore $H_0 = 1$. □

We are now in a position to derive the exponential generating function of H_n. First we state the following results and definitions which will be useful in deriving the exponential generating function.

Lemma 6.7 [31, Section 2.3 Rule 3′]. *Let $f = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $g = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$ be the generating functions of the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ respectively. Then the series fg is the exponential generating function of the sequence,*

$$\left\{\sum_{k=0}^{n}\binom{n}{k} a_k b_{n-k}\right\}_{n=0}^{\infty}.$$

The Lambert W function is defined to be the function $W(z)$ satisfying $W(z)e^{W(z)} = z$. The *tree function* $T(z)$ is defined by the equation $T(z) = -W(-z)$. The functions W and T have many applications in mathematics. For example, they appear in the enumeration of trees and the calculation of water-wave heights. The reader is referred to [9] for more on Lambert's W function.

Lemma 6.8 [9, Section 2, Equation 2.36]. *Let $a \geq 1$ and $n \geq 0$ be integers. Let $N(a, n) := a(a+n)^{n-1}$ be a function of two variables. For a fixed positive integer a, the exponential generating function of the sequence $N(a, n)$ is given by,*

$$\sum_{n=0}^{\infty} a(a+n)^{n-1}\frac{x^n}{n!} = e^{-aW(-x)},$$

where W is Lambert's W function.

Theorem 6.9 *The exponential generating function of H_n is,*

$$\sum_{n=0}^{\infty} H_n \frac{x^n}{n!} = \frac{1}{1+W(-x)},$$

where W is Lambert's W function and H_n is given by the closed formula,

$$H_n = n^n.$$

Proof Let a_n be the sequence $a_n = (n+1)^{n-1}$. Let $h(x) = \sum_{n=0}^{\infty} H_n \frac{x^n}{n!}$ and $g(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ be exponential generating functions of H_n and a_n respectively. Recall the recurrence relation of H_n is given by,

$$H_n = n\sum_{k=1}^{n}\binom{n-1}{k-1}k^{k-2}H_{n-k},$$

so,

$$\frac{H_n}{n} = \sum_{k=1}^{n}\binom{n-1}{k-1}k^{k-2}H_{n-k}.$$

We make the change of variable $j = k-1$ in $\frac{H_n}{n}$ to obtain the following.

$$\frac{H_n}{n} = \sum_{j=0}^{n-1}\binom{n-1}{j}(j+1)^{j-1}H_{n-(j+1)},$$

thus

$$\frac{H_{n+1}}{n+1} = \sum_{j=0}^{n}\binom{n}{j}(j+1)^{j-1}H_{n-j}.$$

We now study the exponential generating function of $\frac{H_{n+1}}{n+1}$,

$$\sum_{n=0}^{\infty}\frac{H_{n+1}}{n+1}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(j+1)^{j-1}H_{n-j}\right)\frac{x^n}{n!}.$$

By Lemma 6.7, the right hand side is given the product $g(x)h(x)$. So we have,

$$\sum_{n=0}^{\infty}\frac{H_{n+1}}{n+1}\frac{x^n}{n!} = g(x)h(x).$$

We can manipulate the right hand side so that the exponent of x matches the factorial, hence

$$\frac{1}{x}\sum_{n=0}^{\infty}H_{n+1}\frac{x^{n+1}}{(n+1)!} = g(x)h(x),$$

so we can write the left hand side in terms of $h(x)$ as follows,

$$\frac{1}{x}(h(x) - H_0) = g(x)h(x).$$

Since $H_0 = 1$,

$$h(x) - 1 = xh(x)g(x).$$

By Lemma 6.8, $g(x) = e^{-W(-x)}$ therefore,

$$h(x) = \frac{1}{1 - xg(x)} = \frac{1}{1 - xe^{-W(-x)}}.$$

Recall that Lambert's W function is defined by the equation $x = W(x)e^{W(x)}$ (See [9] for more on Lambert's W function), thus $-xe^{-W(-x)} = W(-x)$, giving us that,

$$h(x) = \frac{1}{1 + W(-x)} = \frac{1}{1 - T(x)},$$

where $T(x) = -W(-x)$ is called Euler's tree function, again see [9]. This exponential generating function is precisely the exponential generating function of the sequence n^n, see; [19] Section 2 equation 2.7 and [26]. □

It is interesting to note that n^n is also the number of complete exceptional sequences over the hereditary algebras of type B and C; see section 5 of [22]. On a more interesting note, n^n also counts the number of full weak exceptional sequences (see Definition 5.7) over Λ_n^n [30, Theorem 1.4]. The full weak exceptional sequences over Λ_n^n also have length n, so a natural question to ask is whether the complete τ-exceptional sequences over Λ_n^n coincide with the full weak exceptional sequences (see Definition 5.7) over Λ_n^n. We answer this question in the affirmative. First, we state the following well known result.

Lemma 6.10 [8, Lemma 10.20] *Let $\mathcal{A}$ be an exact category and let $\mathcal{B}$ be a full additive subcategory of $\mathcal{A}$. Then if $\mathcal{B}$ is extension-closed in $\mathcal{A}$, the exact sequences $A \to B \to C$ in $\mathcal{A}$ with $A, B,$ and $C \in \mathcal{B}$ form an exact structure on $\mathcal{B}$. In particular for $X, Y \in \mathcal{B}$, we have that $Ext^1_{\mathcal{A}}(X, Y) = \mathrm{Ext}^1_{\mathcal{B}}(X, Y)$.*

Lemma 6.11 [18, Proposition 3.6] *Let A be a finite dimensional $\mathbb{F}$ algebra and let M be a basic τ-rigid left A-module. Then the τ-perpendicular category $J(M)$ is extension-closed in mod A.*

Proposition 6.12 *Let $M = (M_1, M_2, \ldots, M_n)$ be a complete τ-exceptional sequence in mod Λ_n^n. Then M is also a full weak exceptional sequence in mod Λ_n^n.*

Proof We will argue by induction on n. In the case of $n = 1$, there is only one indecomposable module which is both weak exceptional and τ-rigid, so the statement follows trivially. Suppose that the statement is true for all $1 \leq n \leq k$. Let us consider the $k+1$ case. Suppose $M = (M_1, M_2, \ldots, M_{k+1})$ is a τ-exceptional sequence in mod Λ_{k+1}^{k+1}.

Let $l = l(M_{k+1})$ be the length of M_{k+1}, then $1 \leq l \leq k+1$. By Propositions 6.4 and 6.5, the τ-perpendicular category

$$J(M_{k+1}) \cong \mathrm{mod}\, \mathbb{A}_{l-1} \oplus \Lambda_{k+1-l}^{k+1-l}.$$

By definition, the sequence $(M_1, M_2, \ldots, M_k)$ is τ-exceptional in $J(M_{k+1})$. By Theorems 3.8 and 3.9, we have that the sequence $(M_1, M_2, \ldots, M_k)$ is an interleaving of a complete exceptional sequence $X = (X_1, X_2, \ldots, X_{l-1})$ in mod $\mathbb{A}_{l-1}$ with a complete

τ-exceptional sequence $Y = (Y_1, Y_2, \ldots, Y_{k+1-l})$ in mod Λ_{k+1-l}^{k+1-l}. By the inductive hypothesis, the sequence Y is a full weak exceptional sequence in mod Λ_{k+1-l}^{k+1-l}. Moreover, we have that

$$\mathrm{Hom}_{J(M_{k+1})}(X_i, Y_j) = 0 = \mathrm{Hom}_{J(M_{k+1})}(Y_j, X_i),$$

where $1 \leq i \leq l-1$ and $1 \leq j \leq k+1-l$. Therefore, since the τ-perpendicular category $J(M_{k+1})$ is a full subcategory of mod Λ_{k+1}^{k+1}, we also have that

$$\mathrm{Hom}_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(X_i, Y_j) = 0 = \mathrm{Hom}_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(Y_j, X_i),$$

where $1 \leq i \leq l-1$ and $1 \leq j \leq k+1-l$. By a similar argument, we also have that

$$\mathrm{Hom}_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(X_j, X_i) = 0$$

for $1 \leq i < j \leq l-1$ and

$$\mathrm{Hom}_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(Y_j, Y_i) = 0$$

for $1 \leq i < j \leq k+1-l$. By Lemma 6.10, since $J(M_{k+1})$ is an extension-closed subcategory of Λ_{k+1}^{k+1}, we can argue in a similar way that

$$\mathrm{Ext}^1_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(X_i, Y_j) = \mathrm{Ext}^1_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(Y_j, X_i) = 0,$$

where $1 \leq i \leq l-1$ and $1 \leq j \leq k+1-l$. By another similar argument,

$$\mathrm{Ext}^1_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(X_j, X_i) = 0$$

for $1 \leq i \leq j \leq l-1$ and

$$\mathrm{Ext}^1_{\mathrm{mod}\,\Lambda_{k+1}^{k+1}}(Y_j, Y_i) = 0$$

for $1 \leq i \leq j \leq k+1-l$. So we can conclude that the sequence $(M_1, M_2, \ldots, M_k)$ is weak exceptional in mod Λ_{k+1}^{k+1}, hence $M = (M_1, M_2, \ldots M_{k+1})$ is a full weak exceptional sequence in mod Λ_{k+1}^{k+1}. This completes the proof. □

Corollary 6.13 *The complete τ-exceptional sequences of mod Λ_n^n and the full weak exceptional sequences of mod Λ_n^n coincide.*

Proof Let us denote by T_n the set of complete τ-exceptional sequences in mod Λ_n^n and denote by W_n the set of full weak exceptional sequences in mod Λ_n^n. Using Proposition 6.12, we can construct the following map, $f\colon T_n \to W_n$, where by $f(M) = M$. The map f is clearly injective and since $\mid T_n \mid=\mid W_n \mid= n^n$, the map f is bijective, but more precisely the complete τ-exceptional sequences of mod Λ_n^n and the full weak exceptional sequences of mod Λ_n^n coincide. □

Unlike τ-exceptional sequences, there are no complete exceptional sequences in mod Λ_n^n, as we will show. In general, not much is known about exceptional sequences over the Nakayama algebras Λ_n^n.

Proposition 6.14 *There are no exceptional sequences $(M_1, M_2, \ldots, M_l)$ in mod Λ_n^n of length $l > 1$. In particular, there are no complete exceptional sequences in mod Λ_n^n where $n > 1$.*

Proof Suppose $M = (M_1, M_2, \ldots, M_l)$ is an exceptional sequence of length $l > 1$. Every indecomposable projective module in mod Λ_n^n has length n, so by Proposition 3.12 we have that $\mathrm{Hom}(P_i, P_j) \neq 0$ for all $1 \leq i, j \leq n$. As a consequence of this M cannot contain more than one indecomposable projective module, in particular, if $l > 1$, then M must contain non-projective indecomposable modules.

Let N be a non-projective indecomposable module in $\mathrm{mod}\,\Lambda_n^n$. Observe that $N = \mathrm{rad}^k(P_i)$ for some $1 \leq i \leq n$ and $1 \leq k \leq n-1$. We can further observe that the length of N is $l(N) = n-k$ and that $\mathrm{top}(N) = S_{(i+k)_n}$. Further observe that N has the following infinite sequence as its projective resolution.

$$\ldots \longrightarrow P_i \longrightarrow P_{(i+k)_n} \longrightarrow P_i \longrightarrow P_{(i+k)_n} \longrightarrow P_i \longrightarrow P_{(i+k)_n} \longrightarrow N \longrightarrow 0$$

Since N has length $l(N) = n-k$ and $\mathrm{top}(N) = S_{(i+k)_n}$, we can write $N = P_{(i+k)_n}/\mathrm{rad}^{n-k}(P_{(i+k)_n})$. By Proposition 3.12, we can observe that $\mathrm{Hom}(P_i, N) = 0$ and $\mathrm{Hom}(P_{(i+k)_n}, N) \neq 0$. Therefore by applying the functor $\mathrm{Hom}(-, N)$ to the above projective resolution, we obtain the following sequence.

$$0 \longrightarrow \mathrm{Hom}(N,N) \xrightarrow{f_0} \mathrm{Hom}(P_{(i+k)_n}, N) \xrightarrow{f_1} 0 \xrightarrow{f_2} \mathrm{Hom}(P_{(i+k)_n}, N) \xrightarrow{f_3} 0 \xrightarrow{f_4} \ldots$$

So we have that $\mathrm{Ext}^2(N,N) = \ker(f_3)/\mathrm{im}(f_2) \neq 0$. Which is to say any non-projective module in $\mathrm{mod}\,\Lambda_n^n$ is not exceptional. From this we conclude that M cannot contain non-projective modules. A contradiction. □

7 The Γ_n^{n-1} Case

Let $n \geq 1$ be a positive integer. In this section we will study the combinatorics for the number of complete τ-exceptional sequence in $\mathrm{mod}\,\Gamma_n^{n-1}$. Recall that we denote by A_n the linearly oriented quiver with n vertices,

$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \ldots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n$. The algebra Γ_n^{n-1} is defined to be the $\mathbb{F}$-algebra, $\mathbb{F}A_n/R_Q^{n-1}$. This is the path algebra of the quiver A_n modulo the relation $\alpha_1\alpha_2\ldots\alpha_{n-1} = 0$.

Observe the following. Let M be an indecomposable module in $\mathrm{mod}\,\Gamma_n^{n-1}$, then M belongs to one of the following disjoint sets. The first set contains the indecomposable projective modules P_j for $1 \leq j \leq n$. The second set contains non-projective modules of the form $M = \mathrm{rad}^i(P_1)$ where $1 \leq i \leq n-2$ and P_1 is the indecomposable projective at vertex 1. The third set contains indecomposable modules which are neither projective or of the form $M = \mathrm{rad}^i(P_1)$ for $1 \leq i \leq n-2$. Any indecomposable module M in $\mathrm{mod}\,\Gamma_n^{n-1}$ has length $l(M) < n$, therefore by Proposition 6.1, every indecomposable module of $\mathrm{mod}\,\Gamma_n^{n-1}$ is τ-rigid.

Proposition 7.1 *Let P_i be an indecomposable projective module in mod Γ_n^{n-1} for some $1 \leq i \leq n$. Then the τ-perpendicular category of P_i in mod Γ_n^{n-1} is $J(P_i) \cong$ mod $\mathbb{A}_{n-i} \oplus$ mod $\mathbb{A}_{i-1}$, where $\mathbb{A}_j$ is the hereditary type A hereditary algebra.*

Proof Let P_i be an indecomposable projective with length $1 \leq l(P_i) \leq n-1$. By definition the Bongartz completion $T_{P_i} = \mathcal{P}(^{\perp}(\tau P_i))$. Since P_i is projective, $\tau P_i = 0$ therefore $^{\perp}(\tau P_i) = \mathrm{mod}\,\Gamma_n^{n-1}$, hence the Bongartz completion

$$T_{P_i} = \bigoplus_{j=1}^{n} P_j.$$

Thus the $\mathbb{F}$-algebra $E_{P_i} = \mathrm{End}_{\Gamma_n^{n-1}}(T_{P_i})$ is precisely the algebra Γ_n^{n-1}, the path algebra of the quiver A_n^{op},

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_4} \ldots \xleftarrow{\alpha_{i-1}} i-1 \xleftarrow{\alpha_i} i \xleftarrow{\alpha_{i+1}} i+1 \xleftarrow{\alpha_{i+2}} \ldots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n$$

modulo the relation $\alpha_n\alpha_{n-1}\ldots\alpha_1 = 0$. Let $A_n^{\mathrm{op}(i)}$ be the quiver obtained from A_n^{op} by removing the vertex i and all arrows incident to i.

$$1 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_3} 3 \xleftarrow{\alpha_4} \ldots \xleftarrow{\alpha_{i-1}} i-1 \qquad\qquad i+1 \xleftarrow{\alpha_{i+2}} \ldots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n$$

The quiver $A_n^{\mathrm{op}(i)}$ has no relations. By Lemma 3.4, $D_M = E_M/\langle e_M\rangle$ is the path algebra of the quiver $A_n^{\mathrm{op}(i)}$. Since $D_M = \mathbb{F}A_n^{\mathrm{op}(i)}$, it follows that $J(M) \cong \operatorname{mod}\mathbb{A}_{n-i} \oplus \operatorname{mod}\mathbb{A}_{i-1}$ by Theorem 3.5. □

Proposition 7.2 *Let M be an indecomposable module in mod Γ_n^{n-1} of the form $M = \operatorname{rad}^i(P_1)$ for some $1 \le i \le n-2$ with length $1 \le l(M) \le n-2$. Then the τ-perpendicular category of M in mod Γ_n^{n-1} is*

$$J(M) \cong \begin{cases} \operatorname{mod}\mathbb{A}_{l(M)-1} \oplus \operatorname{mod}\mathbb{A}_1 \oplus \operatorname{mod}\mathbb{A}_1 & i = 1 \\ \operatorname{mod}\mathbb{A}_{n-l(M)} \oplus \operatorname{mod}\mathbb{A}_{l(M)-1} & i \neq 1 \end{cases},$$

where $\mathbb{A}_j$ is the hereditary type A hereditary algebra.

Proof Consider P_j the indecomposable projective module at the vertex j in mod Γ_n^{n-1} with $j \neq 1$. Then it is easy to see that $P_j = \operatorname{rad}^{j-2}(P_2)$ and that P_j has length $l(P_j) = n-j+1$. From this it follows that $\operatorname{rad}^q(P_j) = P_{j+q}$ where $0 \le q \le n-j$.

Let $M = \operatorname{rad}^i(P_1)$ for some $1 \le i \le n-1$. We observe that $l(M) = n-i-1$ and $\operatorname{top}(M) = S_{i+1}$. In accordance to Proposition 3.11, M may in fact be written as $M = P_{i+1}/\operatorname{rad}^{n-i-1}(P_{i+1})$. Using Proposition 3.11 again, we can see that Auslander-Reiten translate of M is given by $\tau M = \operatorname{rad}(P_{i+1})/\operatorname{rad}^{n-i}(P_{i+1}) = P_{i+2}$ because $\operatorname{rad}(P_{i+1}) = P_{i+2}$ and $l(P_{i+1}) = n-(i+1)+1 = n-i$, hence $\operatorname{rad}^{n-i}(P_{i+1}) = 0$. So we see that the only indecomposable Γ_n^{n-1}-modules not in ${}^{\perp}(\tau M)$ are the projectives $P_j = \operatorname{rad}^s(P_{i+2})$ for $0 \le s < n-i-1$, in other words $i+2 \le j \le n$ since $l(P_{i+2}) = n-i-1$.

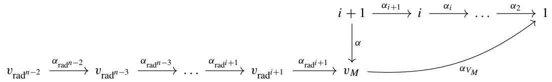
We are now in the position to determine the Ext-projectives of ${}^{\perp}(\tau M)$ where $M = \operatorname{rad}^i(P_1)$. By the above calculation, we can say that for $1 \le j \le i+1$, the projective P_j is in ${}^{\perp}(\tau M)$ hence $\operatorname{Ext}_{\Gamma_n^{n-1}}(P_j, {}^{\perp}(\tau M)) = 0$.

Let $N = \operatorname{rad}^j(P_1)$ for some $j > i$. Arguing as above we can see that the only indecomposable Γ_n^{n-1}-modules not in ${}^{\perp}(\tau N)$ are the indecomposable projectives P_m where $j+2 \le m \le n$, so it follows that $\{X : \operatorname{Hom}(X, \tau N) \neq 0\} \subset \{X : \operatorname{Hom}(X, \tau M) \neq 0\}$. This implies that $\operatorname{Ext}_{\Gamma_n^{n-1}}(N, X) \cong D\overline{\operatorname{Hom}}_{\Gamma_n^{n-1}}(X, \tau N) = 0$ for all X in ${}^{\perp}(\tau M)$ by the Auslander-Reiten formula. Hence $N = \operatorname{rad}^j(P_1)$ is an Ext-projective in ${}^{\perp}(\tau M)$.

For every other indecomposable Γ_n^{n-1} module Y, we have that τY is in ${}^{\perp}(\tau M)$, therefore since $\operatorname{Ext}_{\Gamma_n^{n-1}}(Y, \tau Y) \cong D\overline{\operatorname{Hom}}_{\Gamma_n^{n-1}}(X, \tau N) \neq 0$. Therefore these modules are not Ext-projective in ${}^{\perp}(\tau M)$. By definition $T_M = \mathcal{P}({}^{\perp}(\tau M))$, so by the above arguments,

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=i+1}^{n-2} \operatorname{rad}^s(P_1) \oplus \bigoplus_{j=1}^{i+1} P_j.$$

In the case when $i \neq 1$ the $\mathbb{F}$-algebra $E_M = \operatorname{End}_{\Gamma_n^{n-1}}(T_M)$ is the path algebra of the quiver Q_n,



By Lemma 3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n^{(v_M)}$ which is the quiver obtained from Q_n by removing the vertex v_M and all arrows incident to v_M. The quiver $Q_n^{(v_M)}$ has two connected components.

$$i+1 \xrightarrow{\alpha_{i+1}} i \xrightarrow{\alpha_i} \dots \xrightarrow{\alpha_2} 1$$

$$v_{\mathrm{rad}^{n-2}} \xrightarrow{\alpha_{\mathrm{rad}^{n-2}}} v_{\mathrm{rad}^{n-3}} \xrightarrow{\alpha_{\mathrm{rad}^{n-3}}} \dots \xrightarrow{\alpha_{\mathrm{rad}^{i+1}}} v_{\mathrm{rad}^{i+1}}$$

Since $D_M = \mathbb{F}Q_n^{(v_M)}$, it follows that $J(M) \cong \mathrm{mod}\,\mathbb{A}_{i+1} \oplus \mathrm{mod}\,\mathbb{A}_{n-i-2}$ by Theorem 3.5. Recall that $l(M) = n - i - 1$, hence $J(M) \cong \mathrm{mod}\,\mathbb{A}_{n-l(M)} \oplus \mathrm{mod}\,\mathbb{A}_{l(M)-1}$.

When $i = 1$ however, the $\mathbb{F}$-algebra $E_M = \mathrm{End}_{\Gamma_n^{n-1}}(T_M)$ is the path algebra of the quiver Q'_n,

$$\begin{array}{ccccccccccc} & & & & & & & & 2 & & \\ & & & & & & & & \downarrow{\scriptstyle\alpha} & & \\ v_{\mathrm{rad}^{n-2}} & \xrightarrow{\alpha_{\mathrm{rad}^{n-2}}} & v_{\mathrm{rad}^{n-3}} & \xrightarrow{\alpha_{\mathrm{rad}^{n-3}}} & \dots & \xrightarrow{\alpha_{\mathrm{rad}^{i+1}}} & v_{\mathrm{rad}^{i+1}} & \xrightarrow{\alpha_{\mathrm{rad}^{i+1}}} & v_M & \xrightarrow{\alpha_{v_M}} & 1 \end{array}$$

with no relations. By Lemma, 3.4 $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n^{\prime(v_M)}$ which is the quiver obtained from Q'_n by removing the vertex v_M and all arrows incident to v_M. This quiver has three connected components.

$$\begin{array}{ccccccccc} & & & & & & & 2 & \\ v_{\mathrm{rad}^{n-2}} & \xrightarrow{\alpha_{\mathrm{rad}^{n-2}}} & v_{\mathrm{rad}^{n-3}} & \xrightarrow{\alpha_{\mathrm{rad}^{n-3}}} & \dots & \xrightarrow{\alpha_{\mathrm{rad}^{i+1}}} & v_{\mathrm{rad}^{i+1}} & & 1 \end{array}$$

Since $D_M = \mathbb{F}Q_n^{\prime(v_M)}$, it follows that $J(M) \cong \mathrm{mod}\,\mathbb{A}_{n-3} \oplus \mathrm{mod}\,\mathbb{A}_1 \oplus \mathrm{mod}\,\mathbb{A}_1$ by Theorem 3.5. Since $l(\mathrm{rad}^1(P_1)) = n-2$ then $J(M) \cong \mathrm{mod}\,\mathbb{A}_{l(M)-1} \oplus \mathrm{mod}\,\mathbb{A}_1 \oplus \mathrm{mod}\,\mathbb{A}_1$. □

Proposition 7.3 *Let M be an indecomposable Γ_n^{n-1}-module such that $M \neq \mathrm{rad}^k(P)$ for some indecomposable projective P and positive integer k. Suppose M has length $1 \leq l(M) \leq n-2$, then the τ-perpendicular category of M in mod Γ_n^{n-1} is $J(M) \cong$ mod $\mathbb{A}_{l(M)-1} \oplus$ mod $\Gamma_{n-l(M)}^{n-l(M)-1}$.*

Proof By Proposition 3.11, we can write $M = P_i/\mathrm{rad}^{l(M)}(P_i)$ for some $1 \leq i \leq n-1$ and $\tau M = \mathrm{rad}(P_i)/\mathrm{rad}^{l(M)+1}(P_i)$. We will consider the case where $i \neq 1$ and $i = 1$ separately.

Suppose $i \neq 1$, then we have that $P_i = \mathrm{rad}^{i-2}(P_2)$ and $l(P_i) = n - i + 1$. Therefore, $\mathrm{rad}^q(P_i) = P_{i+q}$, in particular we have that $\mathrm{rad}(P_i) = P_{i+1}$, hence $\tau M = P_{i+1}/\mathrm{rad}^{l(M)}(P_{i+1})$. Now suppose that $i = 1$, hence $M = P_1/\mathrm{rad}^{l(M)}(P_1)$, then $\tau M = \mathrm{rad}(P_1)/\mathrm{rad}^{l(M)+1}(P_1)$. Observe that $\mathrm{top}(\tau M) = S_2$ and $l(\tau M) = l(M)$, hence $\tau M = P_2/\mathrm{rad}^{l(M)}(P_2)$. In either case of i, we have that $\tau M = P_{i+1}/\mathrm{rad}^{l(M)}(P_{i+1})$.

Now let $X = P_j/\mathrm{rad}^l(P_j)$ be an arbitrary indecomposable Γ_n^{n-1} module. By Proposition 3.12, $\mathrm{Hom}(X, \tau M) \neq 0$ if and only $j \in [i+1, i+l(M)]_n$ and $i+l(M) \in [j, j+l-1]_n$. From this it follows that P_j is not in ${}^{\perp}(\tau M)$ if $i+1 \leq j \leq i+l(M)$. Hence $\mathrm{Ext}_{\Gamma_n^{n-1}}(P_j, {}^{\perp}(\tau M)) = 0$ if $j \notin [i+1, i+l(M)]$.

Consider the module $\mathrm{rad}^s(M)$ for $1 \leq s \leq l(M)-1$. The length of $\mathrm{rad}^s(M)$ is given by $l(\mathrm{rad}^s(M)) = l(M) - s$. Moreover, $\mathrm{rad}^s(M) = P_{i+s}/\mathrm{rad}^{l(M)-s}(P_{i+s})$, from which it follows that $\tau\mathrm{rad}^s(M) = P_{i+s+1}/\mathrm{rad}^{l(M)-s}(P_{i+s+1})$. Again let $X = P_j/\mathrm{rad}^l(P_j)$ be an arbitrary indecomposable Γ_n^{n-1} module. By Proposition 3.12, $\mathrm{Hom}(X, \tau\ \mathrm{rad}^s(M)) \neq 0$ if and only $j \in [i+s+1, i+l(M)]_n$ and $i + l(M) \in [j, j+l-1]_n$. Therefore $\{X : \mathrm{Hom}(X, \tau\mathrm{rad}^s(M)) \neq 0\} \subset \{X : \mathrm{Hom}(X, \tau M) \neq 0\}$, which implies that $\mathrm{Ext}_{\Gamma_n^{n-1}}(\mathrm{rad}^s(M), Y) \cong \mathrm{D}\overline{\mathrm{Hom}}_{\Gamma_n^{n-1}}(Y, \tau\mathrm{rad}^s(M)) = 0$ for all Y in ${}^{\perp}(\tau M)$. In other words, $\mathrm{rad}^s(M)$ is Ext-projective in ${}^{\perp}(\tau M)$.

By Proposition 3.1, M is Ext-projective in ${}^{\perp}(\tau M)$, so

$$\mathcal{P}({}^{\perp}(\tau M)) = M \oplus \bigoplus_{s=1}^{l(M)-1} \mathrm{rad}^s(M) \oplus \bigoplus_{j \notin [i+1, i+l(M)]} P_j.$$

By definition, the Bongartz completion $T_M = \mathcal{P}({}^{\perp}(\tau M))$, so the $\mathbb{F}$-algebra $E_M = \mathrm{End}_{\Gamma_n^{n-1}}(T_M)$ is the path algebra of the quiver Q_n modulo relations (set $l(M) := m$),

$$\begin{array}{ccccccccccccc}
n & \xrightarrow{\alpha_n} & n-1 & \xrightarrow{\alpha_{n-1}} & \dots & \xrightarrow{\alpha_{i+m+2}} & i+m+1 & \xrightarrow{\alpha_{i+m+1}} & i & \xrightarrow{\alpha_i} & i-1 & \xrightarrow{\alpha_{i-1}} \dots \xrightarrow{\alpha_2} & 1 \\
 & & & & & & & & \downarrow{\scriptstyle \alpha} & & & & \\
v_{\mathrm{rad}^{m-1}} & \xrightarrow{\alpha_{\mathrm{rad}^{m-1}}} & v_{\mathrm{rad}^{m-2}} & \xrightarrow{\alpha_{\mathrm{rad}^{m-2}}} & \dots & \xrightarrow{\alpha_{\mathrm{rad}^2}} & v_{\mathrm{rad}^1} & \xrightarrow{\alpha_{\mathrm{rad}^1}} & v_M & & & &
\end{array}.$$

Since the vertices of the top row of the quiver correspond to the indecomposable projectives of $\mathrm{mod}\,\Gamma_n^{n-1}$ and the arrows reflect the relations the corresponding maps between the projectives, we see that we have the relation $\alpha_n\alpha_{n-1}\dots\alpha_{i+m+1}\alpha_i\alpha_{i-1}\dots\alpha_1 = 0$. Let $Q_n^{(v_M)}$ be the quiver obtained from Q_n by removing the vertex v_M and all the arrows incident to v_M,

$$\begin{array}{ccccccccccccc}
n & \xrightarrow{\alpha_n} & n-1 & \xrightarrow{\alpha_{n-1}} & \dots & \xrightarrow{\alpha_{i+m+2}} & i+m+1 & \xrightarrow{\alpha_{i+m+1}} & i & \xrightarrow{\alpha_i} & i-1 & \xrightarrow{\alpha_{i-1}} \dots \xrightarrow{\alpha_2} & 1 \\
 & & & & & & & & & & & & \\
v_{\mathrm{rad}^{m-1}} & \xrightarrow{\alpha_{\mathrm{rad}^{m-1}}} & v_{\mathrm{rad}^{m-2}} & \xrightarrow{\alpha_{\mathrm{rad}^{m-2}}} & \dots & \xrightarrow{\alpha_{\mathrm{rad}^2}} & v_{\mathrm{rad}^1} & & & & & &
\end{array}$$

with the relation $\alpha_n\alpha_{n-1}\dots\alpha_{i+m+1}\alpha_i\alpha_{i-1}\dots\alpha_1 = 0$. By Lemma 3.4, $D_M = E_M/\langle e_M\rangle$ is the path algebra of the quiver $Q_n^{(v_M)}$ modulo the relation $\alpha_n\alpha_{n-1}\dots\alpha_{i+m+1}\alpha_i\alpha_{i-1}\dots\alpha_1 = 0$. It follows that $J(M) \cong \mathrm{mod}\,\mathbb{A}_{l(M)-1} \oplus \mathrm{mod}\,\Gamma_{n-l(M)}^{n-l(M)-1}$ by Theorem 3.5. □

Theorem 7.4 *Let K_n denote the number of complete τ-exceptional sequences in $\mathrm{mod}\,\Gamma_n^{n-1}$. Then K_n satisfies the recurrence relation;*

$$K_n = (n-1)(n-2)^{(n-3)} + \sum_{i=1}^{n} \binom{n-1}{i-1}(n-i+1)^{(n-i-1)} i^{i-2} + \sum_{i=1}^{n-3} \binom{n-1}{i-1}(n-i+1)^{(n-i-1)} i^{i-2}$$
$$+ \sum_{i=1}^{n-2} \binom{n-1}{i-1}(n-i-1) i^{i-2} K_{n-i}$$

with $K_1 = 1$.

Proof Let M be an indecomposable module in $\mathrm{mod}\,\Gamma_n^{n-1}$. Suppose $(X_1, X_2, \dots, X_{n-1}, M)$ is a τ-exceptional sequence in $\mathrm{mod}\,\Gamma_n^{n-1}$. Then by definition and the fact that $\delta(J(M)) = n-1$, the sequence $(X_1, X_2, \dots, X_{n-1})$ is a complete τ-exceptional sequence in $J(M)$. Hence the number of complete τ-exceptional sequences ending in M is equal to the number of complete τ-exceptional sequences in $J(M)$.

Suppose M is projective, hence $M = P_i$ for some $1 \leq i \leq n$, then by Proposition 7.1 the τ-perpendicular category $J(M) \cong \mathrm{mod}\,\mathbb{A}_{n-i} \oplus \mathrm{mod}\,\mathbb{A}_{i-1}$. The number of complete τ-exceptional sequences in $\mathrm{mod}\,\mathbb{A}_l$ is precisely the number of complete exceptional sequence in mod $\mathbb{A}_l$ which is shown in [[28] [Proposition 1.1]] to be $(l+1)^{(l-1)}$. Therefore by Theorems 3.8 and 3.9 the number of complete τ-exceptional sequence ending in $M = P_i$ is $\binom{n-1}{i-1}(n-i+1)^{n-i-1}i^{i-2}$.

Suppose $M = \text{rad}^i(P_1)$ for some $1 \leq i \leq n-2$. If $i = 1$ then we saw in Proposition 7.2 that $J(M) \cong \text{mod}\,\mathbb{A}_{n-3} \oplus \text{mod}\,\mathbb{A}_1 \oplus \text{mod}\,\mathbb{A}_1$. Arguing as above it follows that the number of complete τ-exceptional sequences ending in $\text{rad}^1(P_1)$ is $\binom{n-1}{n-3,1,1}(n-2)^{(n-4)}2^0 2^0 = (n-1)(n-2)^{(n-3)}$. If it is the case that $2 \leq i \leq n-2$, then $J(M) \cong \text{mod}\,\mathbb{A}_{n-l(M)} \oplus \text{mod}\,\mathbb{A}_{l(M)-1}$. Therefore the number of complete τ-exceptional sequences ending in $M = \text{rad}^i(P_1)$ for some $2 \leq i \leq n-2$ is $\binom{n-1}{l(M)-1}(n-l(M)+1)^{n-l(M)-1}l(M)^{l(M)-2}$, where $l(M)$ is the length of M.

Finally suppose that M is not of the form $\text{rad}^i(P)$ for some indecomposable projective module P. By Proposition 7.3, $J(M) \cong \text{mod}\,\mathbb{A}_{l(M)-1} \oplus \text{mod}\,\Gamma_{n-l(M)}^{n-l(M)-1}$ where $l(M)$ is the length of M. Therefore the number of complete τ-exceptional sequences ending in M is $\binom{n-1}{l(M)-1}K_{n-l(M)}l(M)^{l(M)-2}$. Observe that in this case the length of M is $1 \leq l(M) \leq n-2$ and for each fixed value of $l(M)$ there are $n-l(M)-1$ indecomposable modules M such that $M \neq \text{rad}^i(P)$.

By counting the number of complete τ-exceptional sequences ending in each indecomposable Γ_n^{n-1}-module M, the recurrence relation of K_n follows. It is also trivial to see that $K_1 = 1$. $\square$

Theorem 7.5 *Let $h(x) = \sum_{n=0}^{\infty} K_n \frac{x^n}{n!}$ be the exponential generating function of K_n. Then $h(x)$ satisfies the first order linear ODE,*

$$h'(x)(1 - xe^{-W(-x)}) + h(x)e^{-W(-x)} = 2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x).$$

Proof Let $h(x) = \sum_{n=0}^{\infty} K_n \frac{x^n}{n!}$ be the exponential generating function of K_n. Let $a(n) = (n+1)^{n-1}$. Let $g(x) = \sum_{n=0}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}$. Then $g(x) = e^{-W(-x)}$ by Lemma 6.8, where $W(x)$ is Lambert's W function. By the only Proposition in Section 6 of [22],

$$2(n+2)^{n-1} = \sum_{i=0}^{n}\binom{n}{i}a(i)a(n-i).$$

So it follows from Lemma 6.7 that

$$(g(x))^2 = \sum_{n=0}^{\infty} 2(n+2)^{n-1}\frac{x^n}{n!}. \tag{1}$$

We make the following observations about

$$\sum_{i=1}^{n}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)} \cdot i^{i-2}.$$

With the change of variable $j = i - 1$,

$$\sum_{i=1}^{n}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)} \cdot i^{i-2} = \sum_{j=0}^{n-1}\binom{n-1}{j}(n-j)^{(n-j-2)}(j+1)^{j-1} = 2(n+1)^{n-2},$$

as shown in the proof of the only Proposition in Section 6 of [22]. We also observe that

$$\sum_{i=1}^{n}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)}\cdot i^{i-2}=\sum_{i=1}^{n-3}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)}\cdot i^{i-2}+n^{n-2}+(n-1)^{n-2}+\frac{3}{2}(n-1)(n-2)^{n-3},$$

so,

$$\sum_{i=1}^{n-3}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)}\cdot i^{i-2}=2(n+1)^{n-2}-n^{n-2}-(n-1)^{n-2}-\frac{3}{2}(n-1)(n-2)^{n-3}. \tag{2}$$

As a result we can write the recurrence for K_{n+1} (from Theorem 7.4) in the following way ,

$$K_{n+1}=2\cdot 2(n+2)^{n-1}-(n+1)^{n-1}-n^{n-1}-\frac{1}{2}(n)(n-1)^{n-2}+\sum_{i=1}^{n-1}\binom{n}{i-1}(n-i)K_{n+1-i}\cdot i^{i-2}. \tag{3}$$

Making the change of variable $j=i-1$ we get,

$$K_{n+1}=2\cdot 2(n+2)^{n-1}-(n+1)^{n-1}-n^{n-1}-\frac{1}{2}(n)(n-1)^{n-2}+\sum_{j=0}^{n-2}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}.$$

We will now study the exponential generating function of K_{n+1}. To do this we look at the exponential generating function of each of the summands on the right hand side. We have already seen from Eq. 1 that

$$\sum_{n=0}^{\infty}2(n+2)^{n-1}\frac{x^n}{n!}=(g(x))^2 \tag{4}$$

To deal with the rest of the summands of K_{n+1} in Eq. 3 but the last one, we first re-organise them in the following way using Eq. 2. Let

$$\phi(n)=\sum_{i=1}^{n-2}\binom{n-1}{i-1}(n-i+1)^{(n-i-1)}\cdot i^{i-2}=2(n+2)^{n-1}-(n+1)^{n-1}-n^{n-1}-\frac{3}{2}n(n-1)^{n-2}.$$

The change of variable $j=i-1$ gives us

$$\phi(n)=\sum_{j=0}^{n-3}\binom{n}{j}(n-j)^{n-j-2}(j+1)^{j-1}.$$

We have $\phi(n)=0$ for $n=0,1,2$ since the sum is empty for these values of n. This further implies that,

$$\sum_{n=0}^{\infty}\phi(n)\frac{x^n}{n!}=\sum_{n=2}^{\infty}\phi(n)\frac{x^n}{n!}$$

$$=\sum_{n=2}^{\infty}2(n+2)^{n-1}\frac{x^n}{n!}-\sum_{n=2}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}-\sum_{n=2}^{\infty}n^{n-1}\frac{x^n}{n!}-\frac{3}{2}\sum_{n=2}^{\infty}n(n-1)^{n-2}\frac{x^n}{n!}$$

$$=\left((g(x))^2-1-2x\right)-\sum_{n=2}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}-\sum_{n=2}^{\infty}n^{n-1}\frac{x^n}{n!}-\frac{3}{2}\sum_{n=2}^{\infty}n(n-1)^{n-2}\frac{x^n}{n!},$$

by Eq. 1. Lemma 6.8 resolves the second summand. The third summand is resolved by [9] in Section 2, page 4. This was previously was done in [24]. This has been translated into English; see [23, Page 48]). To resolve the fourth summand we use the fact the exponential generating function is a right index shift and multiplication by n of the 3rd summand. Right index shifting is equivalent to formal integration and by Rule $2'$ in Section 2.3 page 41 of [31] multiplication by n is equivalent to differentiating and then multiplying the exponential generating function by x (This is also given on the OEIS A055541). Therefore.

$$\begin{aligned}\sum_{n=0}^{\infty}\phi(n)\frac{x^n}{n!} &= [e^{-2W(-x)}-1-2x]-[e^{-W(-x)}-1-x]-[-W(-x)-x]-\frac{3}{2}[-xW(-x)]\\ &= e^{-2W(-x)}-1-2x-e^{-W(-x)}+1+x+W(-x)+x+\frac{3}{2}xW(-x)\\ &= e^{-2W(-x)}-e^{-W(-x)}+W(-x)+\frac{3}{2}xW(-x). \qquad (5)\end{aligned}$$

Now let us study the final summand of Eq. 3

$$\sum_{j=0}^{n-2}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}.$$

Notice that the term $\binom{n}{n-1}(n-(n-1)-1)K_1 n^{n-2}=0$ and $\binom{n}{n}(n-n-1)K_0 n+1^{n-1}=0$ since $K_0=0$. Therefore,

$$\sum_{j=0}^{n-2}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}=\sum_{j=0}^{n}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}.$$

By Lemma 6.7,

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}\right)\frac{x^n}{n!}=\left(\sum_{n=0}^{\infty}(n-1)K_n\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}\right).$$

By Rule $2'$ in Section 2.3 page 41 of [31]

$$\left(\sum_{n=0}^{\infty}(n-1)K_n\frac{x^n}{n!}\right)=x\frac{d}{dx}\left(\sum_{n=0}^{\infty}K_n\frac{x^n}{n!}\right)-\left(\sum_{n=0}^{\infty}K_n\frac{x^n}{n!}\right)=xh'(x)-h(x).$$

By Lemma 6.8,

$$\left(\sum_{n=0}^{\infty}(n+1)^{n-1}\frac{x^n}{n!}\right)=e^{-W(-x)}.$$

Therefore

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}(n-j-1)K_{n-j}\cdot(j+1)^{j-1}\right)\frac{x^n}{n!}=(xh'(x)-h(x))e^{-W(-x)}. \qquad (6)$$

By Rule $1'$ in Section 2.3 page 41 of [31],

$$\sum_{n=0}^{\infty}K_{n+1}\frac{x^n}{n!}=h'(x).$$

We now write the exponential generating function of K_{n+1}, using the expression of K_{n+1} in Eq. 3 and the exponential generating functions of the summands of K_{n+1} obtained as in Eqs. 4, 5 and 6.

$$\sum_{n=0}^{\infty} K_{n+1}\frac{x^n}{n!} = 2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x) + (xh'(x) - h(x))e^{-W(-x)},$$

so

$$h'(x) = 2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x) + (xh'(x) - h(x))e^{-W(-x)},$$

Therefore we have the following first order linear ODE,

$$h'(x) + \frac{h(x)e^{-W(-x)}}{(1 - xe^{-W(-x)})} = \frac{2e^{-2W(-x)} - e^{-W(-x)} + W(-x) + \frac{1}{2}xW(-x)}{(1 - xe^{-W(-x)})}.$$

This ODE is of the form,

$$h'(x) + Q(x)h(x) = F(x),$$

so we may apply the integrating factor method and give a general solution for $h(x)$,

$$h(x) = e^{-V(x)}\int V(x)F(x)dx + C,$$

where $V(X)$ is the integrating factor,

$$V(X) = \int Q(x)dx = \int \frac{e^{-W(-x)}}{1 - xe^{-W(-x)}}dx.$$

Unfortunately, we are unable to evaluate $V(X)$ so we leave $h(x)$ as it is. □

8 Justification

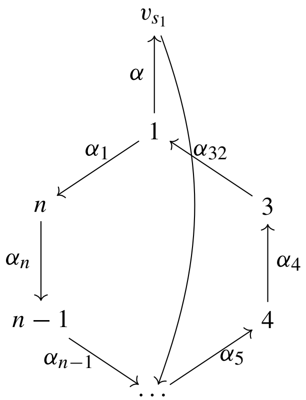
In this section we would like to justify why we only look at the four cases above. Our approach to counting the number of complete τ-exceptional sequences in the above module categories relied upon Theorems 3.8 and 3.9. We also took advantage of the fact that the τ-perpendicular categories of indecomposable modules M were of the form $J(M) \cong \mathcal{C} \oplus \mathcal{D}$ with $\mathcal{C}$ and $\mathcal{D}$ being module categories in the the two families Γ_n^t or Λ_n^t. It is our claim that these four cases, Γ_n^2, Γ_n^{n-1}, Λ_n^2, Λ_n^n are the only ones were all the τ-perpendicular categories $J(M)$ are of this form. In other words, our approach only works on these four cases.

Proposition 8.1 *Fix a positive integers $t \geq 3$. For $n \geq t+1$, let $A = \Lambda_n^t$. Then there exists an A-module M such that the τ-perpendicular category $J(M)$ is not a direct sum of module categories over algebras of the form $\Lambda_{n'}^{t'}$ or $\Gamma_{n'}^{t'}$ for $2 \leq t' \leq n' < n$.*

Proof We prove this by counter-example. Set $M = S_1$, the simple module at vertex 1 of the quiver C_n of A. Note that other simple modules also work, but for simplicity we choose S_1. The Auslander-Reiten translate of S_1 is $\tau S_1 = S_2$. Using Proposition 3.11 and 3.12, we can say that $\mathrm{Hom}(X, S_2) \neq 0$ if and only if $X = P_2/\mathrm{rad}^{l(X)}(P_2)$ where $l(X)$ is the length of X. It also follows that P_2 is the only projective with non-zero maps to S_2. Therefore all other indecomposable projective modules P_j with $j \neq 2$ are in ${}^{\perp}(\tau S_1)$, hence they are Ext-projectives in ${}^{\perp}(\tau S_1)$. By Proposition 3.1, the module S_1 is Ext-project in ${}^{\perp}(\tau S_1)$. We can thus conclude that,

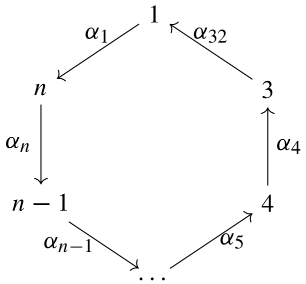
$$\mathcal{P}({}^{\perp}(\tau S_1)) = \bigoplus_{j\neq 2} P_j \oplus S_1.$$

By definition the Bongartz completion of M in mod A is $T_M = \mathcal{P}(\tau S_1)$. Let Q_n be the following quiver,



,

where the vertices labelled j correspond to the projective P_j and the vertex v_{s_1} corresponds to the simple S_1 and the arrows correspond to the irreducible maps between their respective modules. The $\mathbb{F}$-algebra $E_M = \mathrm{End}_A(T_M)$ is the path algebra of the quiver modulo relations. Let $Q_n^{v_{s_1}}$ be the quiver obtained from Q_n by removing the vertex v_{s_1} and any arrows incident to v_{s_1},



by Lemma 3.4, $D_M = E_M/\langle e_M\rangle$ is the path algebra of the quiver $Q_n^{v_{s_1}}$ modulo relations. We have the relation $\alpha_{t+1}\alpha_t \ldots \alpha_4\alpha_{32} = 0$ involving $t-1$ arrows because it corresponds to $\mathrm{Hom}_A(P_{t+1}, P_2) = 0$ since in mod Λ_n^t the composition of t maps between projectives is 0. However, at the same time we have that the composition of the $t'-1$ arrows $\alpha_1\alpha_n \ldots \alpha_{n-(t'-3)} \neq 0$ for $2 \le t' \le t$. Therefore as a module category $J(M)$ cannot be a direct sum of module categories of the form mod $\Lambda_{n'}^{t'}$ or mod $\Gamma_{n'}^{t'}$ as required. □

Proposition 8.2 *Fix a positive integers $t \ge 3$. For $n \ge t+2$, let $A = \Gamma_n^t$. Then there exists an A-module M such that the τ-perpendicular category $J(M)$ is not a direct sum of module categories over algebras of the form $\Gamma_{n'}^{t'}$ or $\Lambda_{n'}^{t'}$ for $2 \le t' \le n' < n$.*

Proof The argument is similar to that for the previous proposition. We prove this by counterexample. Set $M = S_1$, the simple module at vertex 1 of the quiver A_n of A. The Auslander-Reiten translate of S_1 is $\tau S_1 = S_2$. By Proposition 3.11 and 3.12, $\mathrm{Hom}(X, S_2) \neq 0$ if and only if $X = P_2/\mathrm{rad}^{l(X)}(P_2)$ where $l(X)$ is the length of X. It also follows that P_2 is the only projective with non-zero maps to S_2. Therefore all other indecomposable projective modules P_j with $j \neq 2$ are in ${}^{\perp}(\tau S_1)$, hence they are Ext-projectives in ${}^{\perp}(\tau S_1)$. By Proposition 3.1, the module S_1 is Ext-project in ${}^{\perp}(\tau S_1)$ We can thus conclude that,

$$\mathcal{P}(\tau S_1) = \bigoplus_{j \neq 2} P_j \oplus S_1.$$

By definition the Bongartz completion of M in mod A is $T_M = \mathcal{P}(\tau S_1)$. Let Q_n be the following quiver,

$$v_{s_1} \xleftarrow{\alpha} 1 \xleftarrow{\alpha_{32}} 3 \xleftarrow{\alpha_4} 4 \xleftarrow{\alpha_5} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \ ,$$

where the vertices labelled j correspond to the projective P_j and the vertex v_{s_1} corresponds to the simple S_1 and the arrows correspond to the irreducible maps between their respective modules. The $\mathbb{F}$-algebra $E_M = \mathrm{End}_A(T_M)$ is the path algebra of the quiver Q_n modulo relations. Let $Q_n^{v_{s_1}}$ be the quiver obtained from Q_n by removing the vertex v_{s_1} and any arrows incident to v_{s_1},

$$1 \xleftarrow{\alpha_{32}} 3 \xleftarrow{\alpha_4} 4 \xleftarrow{\alpha_5} \dots \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_n} n \ .$$

by Lemma 3.4, $D_M = E_M/\langle e_M \rangle$ is the path algebra of the quiver $Q_n^{v_{s_1}}$ modulo relations. We have the relation $\alpha_{t+1}\alpha_t \dots \alpha_4\alpha_{32} = 0$ involving $t-1$ arrows because it corresponds to $\mathrm{Hom}_A(P_{t+1}, P_2) = 0$ since in mod Γ_n^t the composition of t maps between projectives is 0. However, at the same time we have that the composition of the $t'-1$ arrows $\alpha_n\alpha_{n-1} \dots \alpha_{n-(t'-2)} \neq 0$. Therefore as a module category $J(M)$ cannot be a direct sum of module categories of the form mod $\Lambda_{n'}^{t'}$ or mod $\Gamma_{n'}^{t'}$ as required. □

So we have shown that our strategy for deriving recurrences for the number of complete τ-exceptional sequences over Nakayama algebras only works in the four cases we've studied. However, the statements of Theorems 3.8 and 3.9 are general enough that a similar strategy may be applied to other algebras, and may prove as effective for counting the τ-exceptional sequences for the module categories of those algebras.

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