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The Simple Connectedness of Tame Algebras with Separating Almost Cyclic Coherent Auslander–Reiten Components

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Abstract
We study the simple connectedness of the class of finite-dimensional algebras over an algebraically closed field for which the Auslander–Reiten quiver admits a separating family of almost cyclic coherent components. We show that a tame algebra in this class is simply connected if and only if its first Hochschild cohomology space vanishes.

Keywords Simply connected algebra · Hochschild cohomology · Auslander–Reiten quiver · Tame algebra · Generalized multicoil algebra

Mathematics Subject Classification (2010) Primary 16G70 · Secondary 16G20

1 Introduction and the Main Results

Throughout the paper k will denote a fixed algebraically closed field. By an algebra is meant an associative finite-dimensional k-algebra with an identity, which we shall assume (without loss of generality) to be basic. Then such an algebra has a presentation $A \cong kQ_A/I$, where $Q_A = (Q_0, Q_1)$ is the ordinary quiver of A with the set of vertices Q_0 and the set of arrows Q_1 and I is an admissible ideal in the path algebra kQ_A of Q_A. If the quiver Q_A has no oriented cycles, the algebra A is said to be *triangular*. For an algebra A, we denote by mod A the category of finitely generated right A-modules, and by ind A a full subcategory of mod A consisting of a complete set of representatives of the isomorphism classes of indecomposable modules. We shall denote by rad_A the Jacobson radical of mod A, and by rad_A^∞ the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. Moreover, we denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^- the Auslander–Reiten translations

Dedicated to Claus Michael Ringel on the occasion of his 75th birthday

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$D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$, respectively. We will not distinguish between a module in ind A and the vertex of Γ_A corresponding to it. Following [45], a family $\mathscr{C}$ of components is said to be *generalized standard* if $\mathrm{rad}_A^{\infty}(X, Y) = 0$ for all modules X and Y in $\mathscr{C}$. We note that different components in a generalized standard family $\mathscr{C}$ are orthogonal, and all but finitely many τ_A-orbits in $\mathscr{C}$ are τ_A-periodic (see [45, (2.3)]). We refer to [37] for the structure and homological properties of arbitrary generalized standard Auslander–Reiten components of algebras.

Following Assem and Skowroński [7], a triangular algebra A is called *simply connected* if, for any presentation $A \cong kQ_A/I$ of A as a bound quiver algebra, the fundamental group $\pi_1(Q_A, I)$ of (Q_A, I) is trivial (see Section 2). The importance of these algebras follows from the fact that often we may reduce (using techniques of Galois coverings) the study of the module category of an algebra to that for the corresponding simply connected algebras. Let us note that to prove that an algebra is simply connected seems to be a difficult problem, because one has to check that various fundamental groups are trivial. Therefore, it is worth looking for a simpler characterization of simple connectedness. In [44, Problem 1] Skowroński has asked, whether it is true that a tame triangular algebra A is simply connected if and only if the first Hochschild cohomology space $H^1(A)$ of A vanishes. This equivalence is true for representation-finite algebras [3, Proposition 3.7] (see also [12] for the general case), for tilted algebras (see [5] for the tame case and [25] for the general case), for quasitilted algebras (see [3] for the tame case and [26] for the general case), for piecewise hereditary algebras of type any quiver [25], and for weakly shod algebras [4].

A prominent role in the representation theory of algebras is played by the algebras with separating families of Auslander–Reiten components. A concept of a separating family of tubes has been introduced by Ringel in [40, 41] who proved that they occur in the Auslander–Reiten quivers of hereditary algebras of Euclidean type, tubular algebras, and canonical algebras. In order to deal with wider classes of algebras, the following more general concept of a separating family of Auslander–Reiten components was proposed by Assem, Skowroński and Tomé in [10] (see also [33]). A family $\mathscr{C} = (\mathscr{C}_i)_{i\in I}$ of components of the Auslander–Reiten quiver Γ_A of an algebra A is called *separating* in mod A if the components of Γ_A split into three disjoint families $\mathcal{P}^A$, $\mathscr{C}^A = \mathscr{C}$ and $\mathcal{Q}^A$ such that:

(S1) $\mathscr{C}^A$ is a sincere generalized standard family of components;

(S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathscr{C}^A) = 0$, $\mathrm{Hom}_A(\mathscr{C}^A, \mathcal{P}^A) = 0$;

(S3) any homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in mod A factors through the additive category $\mathrm{add}(\mathscr{C}^A)$ of $\mathscr{C}^A$.

Then we say that $\mathscr{C}^A$ separates $\mathcal{P}^A$ from $\mathcal{Q}^A$ and write $\Gamma_A = \mathcal{P}^A \cup \mathscr{C}^A \cup \mathcal{Q}^A$. We note that then $\mathcal{P}^A$ and $\mathcal{Q}^A$ are uniquely determined by $\mathscr{C}^A$ (see [10, (2.1)] or [41, (3.1)]). Moreover, $\mathscr{C}^A$ is called *sincere* if any simple A-module occurs as a composition factor of a module in $\mathscr{C}^A$. We note that if A is an algebra of finite representation type that $\mathscr{C}^A = \Gamma_A$ is trivially a unique separating component of Γ_A, with $\mathcal{P}^A$ and $\mathcal{Q}^A$ being empty. Frequently, we may recover A completely from the shape and categorical behavior of the separating family $\mathscr{C}^A$ of components of Γ_A. For example, the tilted algebras [24, 41], or more generally double tilted algebras [39](the strict shod algebras in the sense of [15]), are determined by their (separating) connecting components. Further, it was proved in [28] that the class of algebras with a separating family of stable tubes coincides with the class of concealed canonical algebras. This was extended in [29] to a characterization of all quasitilted algebras of canonical type, for which the Auslander–Reiten quiver admits a separating family of semiregular tubes. Then, the latter has been extended in [33] to a characterization of algebras with a separating family of almost cyclic coherent Auslander–Reiten components. Recall that a component Γ of an Auslander–Reiten quiver Γ_A is called *almost cyclic* if all but finitely many modules in Γ lie on oriented cycles contained entirely in Γ. Moreover, a component Γ of Γ_A is said to be *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path $P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots$ (that is, $X_i \neq \tau_A X_{i+2}$ for any $i \geq 1$) in Γ;

(C2) For each injective module I in Γ there is an infinite sectional path $\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I$ (that is, $Y_{j+2} \neq \tau_A Y_j$ for any $j \geq 1$) in Γ.

We are now in position to formulate the first main result of the paper, which answers positively the above mentioned question of Skowroński [44, Problem 1] for tame algebras with separating almost cyclic coherent Auslander–Reiten components.

Theorem 1.1 *Let A be a tame algebra with a separating family of almost cyclic coherent components in Γ_A. Then A is simply connected if and only if $H^1(A) = 0$.*

It has been proved in [33, Theorem A] that the Auslander–Reiten quiver Γ_A of an algebra A admits a separating family $\mathscr{C}^A$ of almost cyclic coherent components if and only if A is a generalized multicoil enlargement of a finite product of concealed canonical algebras $C_1, \ldots, C_m$ by an iterated application of admissible algebra operations of types (ad 1)–(ad 5) and their duals. These algebras are called *generalized multicoil algebras* (see Section 3 for details). Note that for such an algebra A, we have that A is triangular, $\operatorname{gl.dim} A \leq 3$, and $\operatorname{pd}_A M \leq 2$ or $\operatorname{id}_A M \leq 2$ for any module M in ind A (see [33, Corollary B and Theorem E]). Moreover, let $\Gamma_A = \mathcal{P}^A \cup \mathscr{C}^A \cup \mathcal{Q}^A$ be the induced decomposition of Γ_A. Then, by [33, Theorem C], there are uniquely determined quotient algebras $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ and $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of A which are the quasitilted algebras of canonical type such that $\mathcal{P}^A = \mathcal{P}^{A^{(l)}}$ and $\mathcal{Q}^A = \mathcal{Q}^{A^{(r)}}$.

Let A be a generalized multicoil algebra obtained from a concealed canonical algebra $C = C_1 \times \cdots \times C_m$ and $C = A_0, A_1, \ldots, A_n = A$ be an admissible sequence for A (see Section 3). In order to formulate the next result we need one more definition. Namely, if the sectional paths occurring in the definitions of the operations (ad 4), (fad 4), (ad 4*), (fad 4*) come from a component or two components of the same connected algebra A_i, $i \in \{0, \ldots, n-1\}$, then we will say that $\Gamma_{A_{i+1}}$ contains an *exceptional configuration of modules*.

The following theorem is the second main result of the paper.

Theorem 1.2 *Let A be a generalized multicoil algebra obtained from a family $C_1, \ldots, C_m$ of simply connected concealed canonical algebras. Assume moreover that Γ_A does not contain exceptional configurations of modules. Then there are quotient algebras $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ and $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of A such that the following statements are equivalent:*

(i) *A is simply connected.*
(ii) *$A_i^{(l)}$ and $A_i^{(r)}$ are simply connected, for any $i \in \{1, \ldots, m\}$.*
(iii) *$H^1(A) = 0$.*
(iv) *$H^1(A_i^{(l)}) = 0$ and $H^1(A_i^{(r)}) = 0$, for any $i \in \{1, \ldots, m\}$.*
(v) *A is strongly simply connected.*

This paper is organized as follows. In Section 2 we recall some concepts and facts from representation theory, which are necessary for further considerations. Section 3 is devoted to describing some properties of almost cyclic coherent components of the Auslander–Reiten quivers of algebras, applied in the proofs of the preliminary results and the main theorems. In Section 4 we present and prove several results applied in the proof of the first main result of the paper. Sections 5 and 6 are devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively. The aim of the final Section 7 is to present examples illustrating the main results of the paper.

For basic background on the representation theory of algebras we refer to the books [6, 41–43], for more information on simply connected algebras we refer to the survey article [2], and for more details on algebras with separating families of Auslander–Reiten components and their representation theory to the survey article [35].

2 Preliminaries

2.1 Let A be an algebra and $A \cong kQ_A/I$ be a presentation of A as a bound quiver algebra. Then the algebra $A = kQ_A/I$ can equivalently be considered as a k-linear category, of which the object class A_0 is the set of points of Q_A, and the set of morphisms $A(x, y)$ from x to y is the quotient of the k-vector space $kQ_A(x, y)$ of all formal linear combinations of paths in Q_A from x to y by the subspace $I(x, y) = kQ_A(x, y) \cap I$ (see [11]). A full subcategory B of A is called *convex* (in A) if any path in A with source and target in B lies entirely in B. For each vertex v of Q_A we denote by S_v the corresponding simple A-module, and by P_v (respectively, I_v) the projective cover (respectively, the injective envelope) of S_v.

2.2 One-point Extensions and Coextensions Frequently an algebra A can be obtained from another algebra B by a sequence of one-point extensions and one-point coextensions. Recall that the *one-point extension* of an algebra B by a B-module M is the matrix algebra

$$B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $B[M]$ contains Q_B as a convex subquiver and there is an additional (extension) point which is a source. $B[M]$-modules are usually identified with triples (V, X, φ), where V is a k-vector space, X a B-module and $\varphi : V \to \mathrm{Hom}_B(M, X)$ a k-linear map. A $B[M]$-linear map $(V, X, \varphi) \to (V', X', \varphi')$ is then identified with a pair (f, g), where $f : V \to V'$ is k-linear, $g : X \to X'$ is B-linear and $\varphi' f = \mathrm{Hom}_B(M, g)\varphi$. One defines dually the *one-point coextension* $[M]B$ of B by M (see [41]).

2.3 Tameness and Wildness Let A be an algebra and $K[x]$ the polynomial algebra in one variable x. Following [17], the algebra A is said to be *tame* if, for any positive integer d, there exists a finite number of $K[x] - A$-bimodules M_i, $1 \le i \le n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isoclasses of indecomposable A-modules of dimension d are of the form $K[x]/(x - \lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some $i \in \{1, \dots, n_d\}$. Recall that, following [17], the algebra A is *wild* if there is a $k\langle x, y\rangle$-A-bimodule M, free of finite rank as left $k\langle x, y\rangle$-module, and the functor $-\otimes_{k\langle x,y\rangle} M : \mathrm{mod}\, k\langle x, y\rangle \to \mathrm{mod}\, A$ preserves the indecomposability of modules and sends nonisomorphic modules to nonisomorphic modules. From Drozd's Tame and Wild Theorem [17] the class of algebras may be divided into two disjoint classes. One class consists of the tame algebras and the second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional algebras over k.

Hence, a classification of the finite dimensional modules is only feasible for tame algebras. It has been shown by Crawley-Boevey [16] that, if A is a tame algebra, then, for any positive integer $d \geq 1$, all but finitely many isomorphism classes of indecomposable A-modules of dimension d are invariant on the action of τ_A, and hence, by a result due to Hoshino [23], lie in stable tubes of rank one in Γ_A.

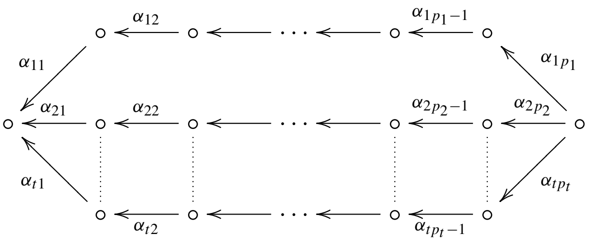
2.4 Hochschild Cohomology of Algebras Let A be an algebra. Denote by $C^\bullet A$ the Hochschild complex $C^\bullet = (C^i, d^i)_{i\in\mathbb{Z}}$ defined as follows: $C^i = 0$, $d^i = 0$ for $i < 0$, $C^0 = {}_AA_A$, $C^i = \mathrm{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product over k of A with itself, $d^0 : A \to \mathrm{Hom}_k(A, A)$ with $(d^0x)(a) = ax - xa$ for $x, a \in A$, $d^i : C^i \to C^{i+1}$ with

$$(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1})$$

$$+ (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i)a_{i+1}$$

for $f \in C^i$ and $a_1, a_2, \ldots, a_{i+1} \in A$. Then $H^i(A) = H^i(C^\bullet A)$ is called the i-th *Hochschild cohomology space* of A (see [14, Chapter IX]). Recall that the first Hochschild cohomology space $H^1(A)$ of an algebra A is isomorphic to the space $\mathrm{Der}(A, A)/\mathrm{Der}^0(A, A)$ of outer derivations of A, where $\mathrm{Der}(A, A) = \{\delta \in \mathrm{Hom}_k(A, A) \mid \delta(ab) = a\delta(b) + \delta(a)b, \text{ for } a, b \in A\}$ is the space of k-linear derivations of A and $\mathrm{Der}^0(A, A)$ is the subspace $\{\delta_x \in \mathrm{Hom}_k(A, A) \mid \delta_x(a) = ax - xa, \text{ for } a \in A\}$ of inner derivations of A.

2.5 Concealed Canonical Algebras An important role in our considerations will be played by certain tilts of canonical algebras introduced by Ringel [41]. Let $p_1, p_2, \ldots, p_t$ be a sequence of positive integers with $t \geq 2$, $1 \leq p_1 \leq p_2 \leq \ldots \leq p_t$, and $p_1 \geq 2$ if $t \geq 3$. Denote by $\Delta(p_1, \ldots, p_t)$ the quiver of the form



For $t \geq 3$, consider a $(t + 1)$-tuple of pairwise different elements of $\mathbb{P}_1(k) = k \cup \{\infty\}$, normalized such that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, and the admissible ideal $I(\lambda_1, \lambda_2, \ldots, \lambda_t)$ in the path algebra $k\Delta(p_1, \ldots, p_t)$ of $\Delta(p_1, \ldots, p_t)$ generated by the elements

$$\alpha_{ip_i} \ldots \alpha_{i2}\alpha_{i1} + \alpha_{2p_2} \ldots \alpha_{22}\alpha_{21} + \lambda_i \alpha_{1p_1} \ldots \alpha_{12}\alpha_{11}, 3 \leq i \leq t.$$

Then the bound quiver algebra $\Lambda(p, \lambda) = k\Delta(p_1, \ldots, p_t)/I(\lambda_1, \lambda_2, \ldots, \lambda_t)$ is said to be the *canonical algebra of type* $p = (p_1, \ldots, p_t)$. Moreover, for $t = 2$, the path algebra $\Lambda(p) = k\Delta(p_1, p_2)$ is said to be the *canonical algebra of type* $p = (p_1, p_2)$. It has been proved in [41, Theorem 3.7] that if Λ is a canonical algebra of type $(p_1, \ldots, p_t)$ then $\Gamma_\Lambda = \mathcal{P}^\Lambda \cup \mathcal{T}^\Lambda \cup \mathcal{Q}^\Lambda$ for a $\mathbb{P}_1(k)$-family $\mathcal{T}^\Lambda$ of stable tubes of tubular type $(p_1, \ldots, p_t)$, separating $\mathcal{P}^\Lambda$ from $\mathcal{Q}^\Lambda$. Following [27], a connected algebra C is called a *concealed canonical algebra of type* $(p_1, \ldots, p_t)$ if C is the endomorphism algebra $\mathrm{End}_\Lambda(T)$, for some canonical algebra Λ of type $(p_1, \dots, p_t)$ and a tilting Λ-module T whose indecomposable direct summands belong to $\mathcal{P}^\Lambda$. Then the images of modules from $\mathcal{T}^\Lambda$ via the functor $\mathrm{Hom}_\Lambda(T, -)$ form a separating family $\mathcal{T}^C$ of stable tubes of Γ_C, and in particular we have a decomposition $\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C$. It has been proved by Lenzing and de la Peña [28, Theorem 1.1] that the class of (connected) concealed canonical algebras coincides with the class of all connected algebras with a separating family of stable tubes. It is also known that the class of concealed canonical algebras of type (p_1, p_2) coincides with the class of hereditary algebras of Euclidean types $\widetilde{\mathbb{A}}_m$, $m \geq 1$ (see [22]). Recall also that the canonical algebras of types $(2, 2, 2, 2)$, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ are called the *tubular canonical algebras*, and an algebra which is tilting-cotilting equivalent to a tubular canonical algebra is called a *tubular algebra* (see [18, 21, 41]).

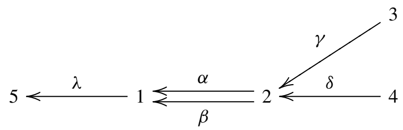
2.6 Simple Connectedness Let (Q, I) be a connected bound quiver. A relation $\varrho = \sum_{i=1}^m \lambda_i w_i \in I(x, y)$ is *minimal* if $m \geq 2$ and, for any nonempty proper subset $J \subset \{1, \dots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. We denote by α^{-1} the formal inverse of an arrow $\alpha \in Q_1$. A *walk* in Q from x to y is a formal composition $\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2} \dots \alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{-1, 1\}$ for all i) with source x and target y. We denote by e_x the trivial path at x. Let $\sim$ be the *homotopy relation* on (Q, I), that is, the smallest equivalence relation on the set of all walks in Q such that:

(a) If $\alpha : x \to y$ is an arrow, then $\alpha^{-1}\alpha \sim e_y$ and $\alpha\alpha^{-1} \sim e_x$.
(b) If $\varrho = \sum_{i=1}^m \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j.
(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions make sense.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes $\widetilde{u}$ of closed walks u starting and ending at u has a group structure defined by the operation $\widetilde{u} \cdot \widetilde{v} = \widetilde{uv}$. Since Q is connected, $\pi_1(Q, I, x)$ does not depend on the choice of x. We denote it by $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I).

Let $A \cong kQ_A/I$ be a presentation of a triangular algebra A as a bound quiver algebra. The fundamental group $\pi_1(Q_A, I)$ depends essentially on I, so is not an invariant of A. A triangular algebra A is called *simply connected* if, for any presentation $A \cong kQ_A/I$ of A as a bound quiver algebra, the fundamental group $\pi_1(Q_A, I)$ of (Q_A, I) is trivial [7].

Example 2.7 Let $A = kQ/I$ be the bound quiver algebra given by the quiver Q of the form



and I the ideal in the path algebra kQ of Q over k generated by the elements $\gamma\beta$, $\delta\alpha - a\delta\beta$, $\alpha\lambda$, where $a \in k \setminus \{0\}$. Then $\pi_1(Q, I)$ is trivial. Moreover, the triangular algebra A is simply connected. Indeed, any choice of a basis of $\mathrm{rad}_A / \mathrm{rad}_A^2$ will lead to at least one minimal relation with target 1 and source $i \in \{3, 4\}$ or with target 5 and source 2.

Remark 2.8 It is known, for example, that the following important classes of algebras are simply connected: the iterated tilted algebras of Dynkin type (see [1, Proposition 3.5]), the iterated tilted algebras of Euclidean types $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_p$, $n \geq 4$, $p = 6, 7, 8$, the tubular algebras (see [7, Corollary 1.4]), and the pg-critical algebras (see [38, Corollary 3.3]).

3 Almost Cyclic Coherent Auslander–Reiten components

3.1 Generalized Multicoil Algebras It has been proved in [32, Theorem A] that a connected component Γ of an Auslander–Reiten quiver Γ_A of an algebra A is almost cyclic and coherent if and only if Γ is a generalized multicoil, that is, can be obtained, as a translation quiver, from a finite family of stable tubes by a sequence of operations called *admissible*. We recall briefly the generalized multicoil enlargements of algebras from [33, Section 3].

Given a generalized standard component Γ of Γ_A, and an indecomposable module X in Γ, the *support* $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X,-)|_\Gamma$ is the k-linear category defined as follows [9]. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X,M)\neq 0$, and $\mathcal{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f: M\to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X,f)=0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathcal{I}_X$. Following the above convention, we usually identify the k-linear category $\mathcal{S}(X)$ with its quiver.

Recall that a module X in mod A is called a *brick* if $\mathrm{End}_A(X)\cong k$.

Let A be an algebra and Γ be a family of generalized standard infinite components of Γ_A. For an indecomposable brick X in Γ, called the *pivot*, five admissible operations are defined, depending on the shape of the support $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X,-)|_\Gamma$. These admissible operations yield in each case a modified algebra A' such that the modified translation quiver Γ' is a family of generalized standard infinite components in the Auslander–Reiten quiver $\Gamma_{A'}$ of A' (see [32, Section 2] or [35, Section 4] for the figures illustrating the modified translation quiver Γ').

(ad 1) Assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \cdots$$

Let $t\geq 1$ be a positive integer, D be the full $t\times t$ lower triangular matrix algebra, and $Y_1,\ldots, Y_t$ denote the indecomposable injective D-modules with $Y=Y_1$ the unique indecomposable projective-injective D-module. We set $A'=(A\times D)[X\oplus Y]$. In this case, Γ' is obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij}=\left(k, X_i\oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i\geq 0$, $1\leq j\leq t$, and $X_i'=(k,X_i,1)$ for $i\geq 0$. If $t=0$ we set $A'=A[X]$ and the rectangle reduces to the sectional path consisting of the modules X_i', $i\geq 0$.

(ad 2) Suppose that $\mathcal{S}(X)$ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

where $t\geq 1$. In particular, X is necessarily injective. We set $A'=A[X]$. In this case, Γ' is obtained by inserting in Γ the rectangle consisting of the modules $Z_{ij}=\left(k, X_i\oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i\geq 1$, $1\leq j\leq t$, and $X_i'=(k,X_i,1)$ for $i\geq 0$.

(ad 3) Assume $\mathcal{S}(X)$ is the mesh-category of two parallel sectional paths:

$$\begin{array}{ccccccccc} Y_1 & \to & Y_2 & \to & \cdots & \to & Y_t & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ X=X_0 & \to & X_1 & \to & \cdots & \to & X_{t-1} & \to & X_t \to \cdots \end{array}$$

with the upper sectional path finite and $t\geq 2$. In particular, X_{t-1} is necessarily injective. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i\to \tau_A^{-1}Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the modules $\tau_A^{-1}Y_{i-1}$, $2\leq i\leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. We set $A'=A[X]$. In this case, Γ' is obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 1$, $1 \leq j \leq i$, and $X_i' = (k, X_i, 1)$ for $i \geq 0$.

(ad 4) Suppose that $\mathcal{S}(X)$ consists of an infinite sectional path, starting at X

$$X = X_0 \to X_1 \to X_2 \to \cdots \quad \text{and} \quad Y = Y_1 \to Y_2 \to \cdots \to Y_t$$

with $t \geq 1$, is a finite sectional path in Γ_A. Let r be a positive integer. Moreover, we consider the translation quiver $\overline{\Gamma}$ of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$. We assume that the union $\widehat{\Gamma}$ of connected components of $\overline{\Gamma}$ containing the vertices $\tau_A^{-1} Y_{i-1}$, $2 \leq i \leq t$, is a finite translation quiver. Then $\overline{\Gamma}$ is a disjoint union of $\widehat{\Gamma}$ and a cofinite full translation subquiver Γ^*, containing the pivot X. For $r = 0$ we set $A' = A[X \oplus Y]$. In this case, Γ' is obtained from Γ^* by inserting the rectangle consisting of the modules $Z_{ij} = \left(k, X_i \oplus Y_j, \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\right)$ for $i \geq 0$, $1 \leq j \leq t$, and $X_i' = (k, X_i, 1)$ for $i \geq 0$.

For $r \geq 1$, let G be the full $r \times r$ lower triangular matrix algebra, $U_{1,t+1}$, $U_{2,t+1}$, ..., $U_{r,t+1}$ denote the indecomposable projective G-modules, $U_{r,t+1}$, $U_{r,t+2}$, ..., $U_{r,t+r}$ denote the indecomposable injective G-modules, with $U_{r,t+1}$ the unique indecomposable projective-injective G-module. We define the matrix algebra

$$A' = \begin{bmatrix} A & 0 & 0 & \dots & 0 & 0 \\ Y & k & 0 & \dots & 0 & 0 \\ Y & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & k & k & \dots & k & 0 \\ X \oplus Y & k & k & \dots & k & k \end{bmatrix}$$

with $r + 2$ columns and rows. In this case, Γ' is obtained from Γ^* by inserting the following modules

$$U_{sl} = \begin{cases} (k, Y_l, 1) & \text{for } s = 1, 1 \leq l \leq t, \\ (k, U_{s,l-1}, 1) & \text{for } 2 \leq s \leq r, 1 \leq l < t + s, \\ (k, 0, 0) & \text{for } 2 \leq s \leq r, l = t + s, \end{cases} \qquad Z_{ij} = \left(k, X_i \oplus U_{rj}, \begin{bmatrix}1\\1\end{bmatrix}\right) \begin{array}{l}\text{for } i \geq 0 \text{ and}\\ 1 \leq j \leq t + r,\end{array}$$

and $X_i' = (k, X_i, 1)$ for $i \geq 0$. In the above formulas U_{sl} is treated as a module over the algebra $A_s = A_{s-1}[U_{s-1,1}]$, where $A_0 = A$ and $U_{01} = Y$ (in other words A_s is an algebra consisting of matrices obtained from the matrices belonging to A' by choosing the first $s + 1$ rows and columns).

We note that the quiver $Q_{A'}$ of A' is obtained from the quiver of the double one-point extension $A[X][Y]$ by adding a path of length $r + 1$ with source at the extension vertex of $A[X]$ and sink at the extension vertex of $A[Y]$.

For the definition of the next admissible operation we need also the finite versions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), which we denote by (fad 1), (fad 2), (fad 3) and (fad 4), respectively. In order to obtain these operations we replace all infinite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots$ (in the definitions of (ad 1), (ad 2), (ad 3), (ad 4)) by the finite sectional paths of the form $X_0 \to X_1 \to X_2 \to \cdots \to X_s$. For the operation (fad 1) $s \geq 0$, for (fad 2) and (fad 4) $s \geq 1$, and for (fad 3) $s \geq t - 1$. In all above operations X_s is injective (see the figures for (fad 1)–(fad 4) in [32, Section 2] or [35, Section 4]).

(ad 5) We define the modified algebra A' of A to be the iteration of the extensions described in the definitions of the admissible operations (ad 1), (ad 2), (ad 3), (ad 4), and their finite versions corresponding to the operations (fad 1), (fad 2), (fad 3) and (fad 4). In this case, Γ' is obtained in the following three steps: first we are doing on Γ one of the operations (fad 1), (fad 2) or (fad 3), next a finite number (possibly zero) of the operation (fad 4) and finally the operation (ad 4), and in such a way that the sectional paths starting from all the new projective modules have a common cofinite (infinite) sectional subpath. By an (ad 5)-pivot we mean an indecomposable brick X from the last (ad 4) operation used in the whole process of creating (ad 5).

Moreover, together with each of the admissible operations (ad 1)–(ad 5), we consider its dual, denoted by (ad 1*)–(ad 5*). These dual operations are also called admissible. Following [32], a connected translation quiver Γ is said to be a *generalized multicoil* if Γ can be obtained from a finite family $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s$ of stable tubes by an iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3), (ad 3*), (ad 4), (ad 4*), (ad 5) or (ad 5*). If $s = 1$, such a translation quiver Γ is said to be a *generalized coil*. The admissible operations of types (ad 1)–(ad 3), (ad 1*)–(ad 3*) have been introduced in [8–10], and the admissible operations (ad 4) and (ad 4*) for $r = 0$ in [30].

Finally, let C be a (not necessarily connected) concealed canonical algebra and $\mathcal{T}^C$ a separating family of stable tubes of Γ_C. Following [33] we say that an algebra A is a *generalized multicoil enlargement* of C using modules from $\mathcal{T}^C$ if there exists a sequence of algebras $C = A_0, A_1, \ldots, A_n = A$ such that A_{i+1} is obtained from A_i by an admissible operation of one of the types (ad 1)–(ad 5), (ad 1*)–(ad 5*) performed either on stable tubes of $\mathcal{T}^{A_i}$, or on generalized multicoils obtained from stable tubes of $\mathcal{T}^{A_i}$ by means of operations done so far. The sequence $C = A_0, A_1, \ldots, A_n = A$ is then called an *admissible sequence* for A. Observe that this definition extends the concept of a coil enlargement of a concealed canonical algebra introduced in [10]. We note that a generalized multicoil enlargement A of C invoking only admissible operations of type (ad 1) (respectively, of type (ad 1*)) is a tubular extension (respectively, tubular coextension) of C in the sense of [41]. An algebra A is said to be a *generalized multicoil algebra* if A is a connected generalized multicoil enlargement of a product C of connected concealed canonical algebras.

Proposition 3.2 [33, Proposition 3.7] *Let C be a concealed canonical algebra, $\mathcal{T}^C$ a separating family of stable tubes of Γ_C, and A a generalized multicoil enlargement of C using modules from $\mathcal{T}^C$. Then Γ_A admits a generalized standard family $\mathscr{C}^A$ of generalized multicoils obtained from the family $\mathcal{T}^C$ of stable tubes by a sequence of admissible operations corresponding to the admissible operations leading from C to A.*

The following theorem, proved in [33, Theorem A], will be crucial for our further considerations.

Theorem 3.3 *Let A be an algebra. The following statements are equivalent:*

(i) Γ_A admits a separating family of almost cyclic coherent components.
(ii) A is a generalized multicoil enlargement of a concealed canonical algebra C.

Remark 3.4 The concealed canonical algebra C is called the *core* of A and the number m of connected summands of C is a numerical invariant of A. We note that m can be arbitrary large, even if A is connected. Let us also note that the class of algebras with generalized standard almost cyclic coherent Auslander–Reiten components is large (see [34, Proposition 2.9] and the following comments).

We note that the class of tubular extensions (respectively, tubular coextensions) of concealed canonical algebras coincides with the class of algebras having a separating family of ray tubes (respectively, coray tubes) in their Auslander–Reiten quiver (see [27, 29]). Moreover, these algebras are quasitilted algebras of canonical type.

We recall also the following theorem on the structure of the module category of an algebra with a separating family of almost cyclic coherent Auslander–Reiten components proved in [33, Theorems C and F].

Theorem 3.5 *Let A be an algebra with a separating family $\mathscr{C}^A$ of almost cyclic coherent components in Γ_A, and $\Gamma_A = \mathcal{P}^A \cup \mathscr{C}^A \cup \mathcal{Q}^A$ the associated decomposition of Γ_A. Then the following statements hold.*

- *(i) There is a unique full convex subcategory $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ of A which is a tubular coextension of a concealed canonical algebra $C = C_1 \times \ldots \times C_m$ such that $\Gamma_{A^{(l)}} = \mathcal{P}^{A^{(l)}} \cup \mathcal{T}^{A^{(l)}} \cup \mathcal{Q}^{A^{(l)}}$ for a separating family $\mathcal{T}^{A^{(l)}}$ of coray tubes in $\Gamma_{A^{(l)}}$, $\mathcal{P}^A = \mathcal{P}^{A^{(l)}}$, and A is obtained from $A^{(l)}$ by a sequence of admissible operations of types* (ad 1)–(ad 5) *using modules from $\mathcal{T}^{A^{(l)}}$.*
- *(ii) There is a unique full convex subcategory $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of A which is a tubular extension of a concealed canonical algebra $C = C_1 \times \ldots \times C_m$ such that $\Gamma_{A^{(r)}} = \mathcal{P}^{A^{(r)}} \cup \mathcal{T}^{A^{(r)}} \cup \mathcal{Q}^{A^{(r)}}$ for a separating family $\mathcal{T}^{A^{(r)}}$ of ray tubes in $\Gamma_{A^{(r)}}$, $\mathcal{Q}^A = \mathcal{Q}^{A^{(r)}}$, and A is obtained from $A^{(r)}$ by a sequence of admissible operations of types* (ad 1*)–(ad 5*) *using modules from $\mathcal{T}^{A^{(r)}}$.*
- *(iii) A is tame if and only if $A^{(l)}$ and $A^{(r)}$ are tame.*

In the above notation, the algebras $A^{(l)}$ and $A^{(r)}$ are called the *left* and *right quasitilted algebras* of A. Moreover, the algebras $A^{(l)}$ and $A^{(r)}$ are tame if and only if $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean type or tubular algebras.

Recall that an algebra A is *strongly simply connected* if every convex subcategory of A is simply connected (see [44]). Clearly, if A is strongly simply connected then A is simply connected. We need the following result proved in [31, Theorem 1.1].

Theorem 3.6 *Let A be an algebra with a separating family of almost cyclic coherent components in Γ_A without exceptional configurations of modules. Then there are quotient algebras $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ and $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of A such that the following statements are equivalent:*

- *(i) A is strongly simply connected.*
- *(ii) $A_i^{(l)}$ and $A_i^{(r)}$ are strongly simply connected, for any $i \in \{1, \ldots, m\}$.*

4 Preliminary Results

4.1 Branch Extensions and Coextensions Let A be an algebra and $A \cong kQ_A/I$ be a presentation of A as a bound quiver algebra. For a given vertex v in Q_A, we denote by $v^{\rightarrow}$ (respectively, by ${}^{\rightarrow}v$) the set of all arrows of the quiver Q_A starting at v (respectively, terminating at v). Let now K be a branch at a vertex $v \in Q_A$ and $E \in \operatorname{mod} A$. Recall that the *branch extension* $A[E, K]$ by the branch K [41, (4.4)] is constructed in the following way: to the one-point extension $A[E]$ with extension vertex w (that is, $\operatorname{rad} P_w = E$) we add the branch K by identifying the vertices v and w. If $E_1, \ldots, E_n \in \operatorname{mod} A$ and $K_1, \ldots, K_n$ is a set of branches, then the branch extension $A[E_i, K_i]_{i=1}^n$ is defined inductively as: $A[E_i, K_i]_{i=1}^n = (A[E_i, K_i]_{i=1}^{n-1})[E_n, K_n]$. The concept of *branch coextension* is defined dually.

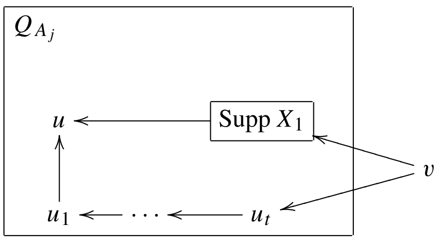
Lemma 4.2 *Let A be a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$. Moreover, let $C = A_0, \ldots, A_p = A^{(l)}, A_{p+1}, \ldots, A_n = A$ be an admissible sequence for A, $j \geq p$, $X \in \operatorname{ind} A_j$ be an* (ad 2) *or* (ad 3)*-pivot, and A_{j+1} be the modified algebra of A_j. If v is the corresponding extension point then there is a unique vertex $u \in A^{(l)} \setminus A^{(r)}$ that satisfies:*

(i) Each $\alpha \in v^{\rightarrow}$ is the starting point of a nonzero path $\omega_\alpha \in A(v, u)$.

(ii) There are at least two different arrows in $v^{\rightarrow}$. Moreover, if $\alpha, \beta \in v^{\rightarrow}$, and $\alpha \neq \beta$, then $\omega_\alpha - \omega_\beta \in I$.

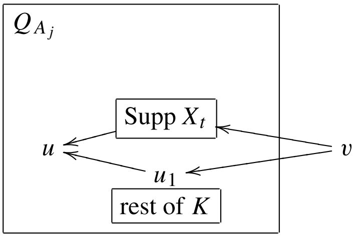
Proof We know from [33, Section 4] that $A^{(l)}$ is a unique maximal convex branch coextension of $C = C_1 \times \cdots \times C_m$ inside A, that is, $A^{(l)} = B_1^{(l)} \times \cdots \times B_m^{(l)}$, where $B_i^{(l)}$ is a unique maximal convex branch coextension of C_i inside A, $i \in \{1, \ldots, m\}$. More precisely, $B_i^{(l)} = {}_{j=1}^{t_i}[K_j, E_j]C_i$, where $K_1, \ldots, K_{t_i}$ are branches, $i \in \{1, \ldots, m\}$. Then there exists $s \in \{1, \ldots, m\}$ such that $u \in B_s^{(l)}$ and $A_{j+1} = A_j[X]$. If X is an (ad 2)-pivot (respectively, (ad 3)-pivot), then in the sequence of earlier admissible operations, there is an operation of type (ad 1*) or (ad 5*) which contains an operation (fad 1*) which gives rise to the pivot X of (ad 2) (respectively, to the pivot X of (ad 3) and to the modules $Y_1, \ldots, Y_t$ in the support of $\operatorname{Hom}_A(X, -)$ restricted to the generalized multicoil containing X - see definition of (ad 3)). The operations done after must not affect the support of $\operatorname{Hom}_A(X, -)$ restricted to the generalized multicoil containing X. Note that in general, in the sequence of earlier admissible operations, there can be an operation of type (ad 5) which contains an operation (fad 4) which gives rise to the pivot X of (ad 2) (respectively, to the pivot X of (ad 3)) but from Lemma [33, Lemma 3.10] this case can be reduced to (ad 5*) which contains an operation (fad 1*).

Let X be an (ad 2)-pivot, $A_{j+1} = A_j[X]$, and $u, u_1, \ldots, u_t$ (where $X = I_u$, $Y_i = I_{u_i}$ for $i \in \{1, \ldots, t\}$ - see definition of (ad 2)) be the points in the quiver Q_{A_j} of A_j corresponding to the new indecomposable injective A_j-modules obtained after performing the above admissible operation (ad 1*) or the operation (fad 1*). Then $u, u_1, \ldots, u_t \in A^{(l)}$. Since $X = \operatorname{rad} P_v$, there must be a nonzero path from v to each vertex w which is a predecessor of u. Hence, each $\alpha \in v^{\rightarrow}$ is the starting arrow of a nonzero path from v to u, and there are at least two arrows in $v^{\rightarrow}$, namely: one from v to u_t and one from v to a point in $\operatorname{Supp} X_1$, where X_1 is the immediate successor of X on the infinite sectional path in $\mathcal{S}(X)$ (see definition of (ad 2)). Moreover, since $P_v(u) = X(u) = k$, all paths from v to u are congruent modulo I_{j+1}. The bound quiver $Q_{A_{j+1}}$ of A_{j+1} is of the form



where $A_{j+1}(v, u)$ is one-dimensional. From the proofs of [33, Theorems A and C], we have $u \in A^{(l)} \setminus A^{(r)}$, $v \in A^{(r)} \setminus A^{(l)}$, and $u_1, \ldots, u_t \in A^{(l)} \cap A^{(r)}$.

Let now X be an (ad 3)-pivot, $A_{j+1} = A_j[X]$, and assume that we had r consecutive admissible operations of types (ad 1*) or (fad 1*), the first of which had X_t as a pivot, and these admissible operations built up a branch K in A_j with points $u, u_1, \ldots, u_t$ in Q_{A_j}, so that X_{t-1} and Y_t are the indecomposable injective A_j-modules corresponding respectively to u and u_1, and both Y_1 and $\tau_{A_j}^{-1}Y_1$ are coray modules in the generalized multicoil containing the (ad 3)-pivot X (where X, X_{t-1}, X_t, Y_1 and Y_t are as in the definition of (ad 3)). Then $u, u_1 \in A^{(l)}$ and X is the indecomposable A_j-module given by: $X(w) = 0$ if $w < u_1$, $X(w) = k$ if $u_1 < w$, and $X(w) = X_{t-1}(w)$ in any other case. Since $X = \operatorname{rad} P_v$, there must be a nonzero path from v to each vertex w which is a predecessor of u, but those which are predecessors of u_1. Hence, each $\alpha \in v^{\rightarrow}$ is the starting arrow of a nonzero path from v to u, and there are at least two arrows in $v^{\rightarrow}$, namely: one from v to u_1 and one from v to a point in $\operatorname{Supp} X_t$, where X_t is the immediate successor of X_{t-1} on the infinite sectional path in $\mathcal{S}(X)$ (see definition of (ad 3)). Moreover, since $P_v(u) = X_{t-1}(u) = k$, all paths from v to u are congruent modulo I_{j+1}. The bound quiver $Q_{A_{j+1}}$ of A_{j+1} is of the form



where $A_{j+1}(v, u)$ is one-dimensional. Again, from the proofs of [33, Theorems A and C], we have $u \in A^{(l)} \setminus A^{(r)}$, $v \in A^{(r)} \setminus A^{(l)}$, u_1 and the vertices of the branch K belong to $A^{(l)} \cap A^{(r)}$. □

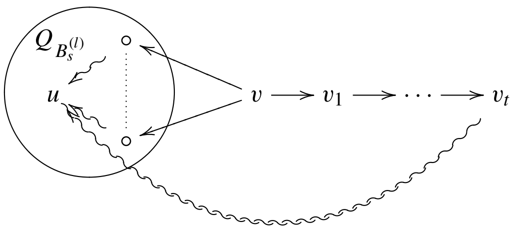
Lemma 4.3 *Let A be a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$. Moreover, let $C = A_0, \ldots, A_p = A^{(l)}, A_{p+1}, \ldots, A_n = A$ be an admissible sequence for A, $j \geq p$, $X \in \operatorname{ind} A_j$ be an* (ad 1)*-pivot, A_{j+1} be the modified algebra of A_j, and v be the corresponding extension point. Then the following statements hold.*

(i) If there is a vertex $u \in A^{(l)} \setminus A^{(r)}$ such that each $\alpha \in v^{\rightarrow}$ is the starting point of a nonzero path $\omega_\alpha \in A(v, u)$, then:
  (a) The vertex u is unique.
  (b) There are at least two different arrows in $v^{\rightarrow}$.
  (c) If $\alpha, \beta \in v^{\rightarrow}$, and $\alpha \neq \beta$, then $\omega_\alpha - \omega_\beta \in I$.

(ii) If $X|_{C_i} = 0$ for any $i \in \{1, \ldots, m\}$, then X is uniserial.

Proof Since X is an (ad 1)-pivot, the support $\mathcal{S}(X)$ consists of an infinite sectional path $X = X_0 \to X_1 \to X_2 \to \cdots$ starting at X. Let $t \geq 1$ be a positive integer, D be the full $t \times t$ lower triangular matrix algebra, and $Y_1, \ldots, Y_t$ be the indecomposable injective D-modules with Y_1 the unique indecomposable projective-injective D-module (see definition of (ad 1)).

(i) Again, we know from [33, Section 4] that $A^{(l)}$ is a unique maximal convex branch coextension of $C = C_1 \times \cdots \times C_m$ inside A, that is, $A^{(l)} = B_1^{(l)} \times \cdots \times B_m^{(l)}$, where $B_i^{(l)}$ is a unique maximal convex branch coextension of C_i inside A, $i \in \{1, \ldots, m\}$. More precisely, $B_i^{(l)} =_{j=1}^{t_i} [K_j, E_j] C_i$, where $K_1, \ldots, K_{t_i}$ are branches, $i \in \{1, \ldots, m\}$. Assume that there is a vertex $u \in A^{(l)} \setminus A^{(r)}$ such that each $\alpha \in v^{\rightarrow}$ is the starting point of a nonzero path $\omega_\alpha \in A(v, u)$. Then there exists $s \in \{1, \ldots, m\}$ such that $u \in B_s^{(l)}$. Moreover, $A_{j+1} = (A_j \times D)[X \oplus Y_1]$ and the bound quiver $Q_{A_{j+1}}|_{\mathrm{Supp}\, X}$ is of the form



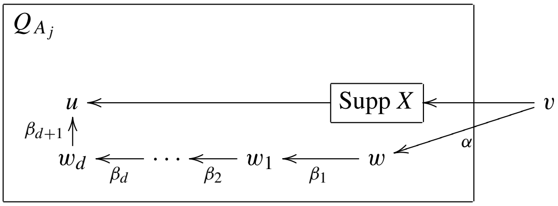
where $v_1, \ldots, v_t$ are the points in the quiver $Q_{A_{j+1}}$ of A_{j+1} corresponding to the new indecomposable projective A_{j+1}-modules. Then A_{j+1} is the extension of $B_s^{(l)}$ at X by the extension branch K consisting of the points $v, v_1, \ldots, v_t$, that is, we have $A_{j+1} = A_j[X, K]$. Since u does not belong to $A^{(r)}$ and for any $\alpha \in v^{\rightarrow}$ it is the starting point of a nonzero path $\omega_\alpha \in A(v, u)$, we get that u is the coextension point of the admissible operation (ad 2*) or (ad 3*). By [10, Lemma 3.1] the admissible operations (ad 2*) and (ad 3*) commute with (ad 1), so we can apply (ad 2*) after (ad 1) (respectively, (ad 3*) after (ad 1)). Using now [10, Lemma 3.3] (respectively, [10, Lemma 3.4]), we are able to replace (ad 1) followed by (ad 2*) (respectively, (ad 1) followed by (ad 3*)) by an operation of type (ad 1*) followed by an operation of type (ad 2) (respectively, (ad 1*) followed by an operation of type (ad 3)). Therefore, the statements (a), (b) and (c) follow from Lemma 4.2.

(ii) A case by case inspection (which admissible operation gives rise to the (ad 1)-pivot X) shows that X is either simple module or the support of X is a linearly ordered quiver of type $\mathbb{A}_t$. □

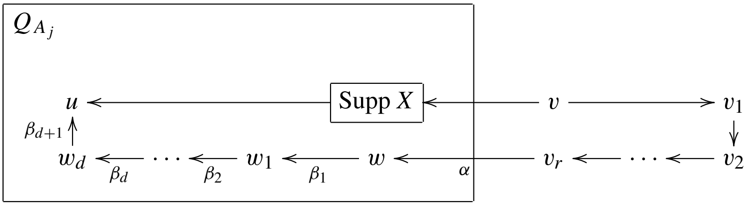
Lemma 4.4 *Let A be a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times \cdots \times C_m$. Moreover, let $C = A_0, \ldots, A_p = A^{(l)}, A_{p+1}, \ldots, A_n = A$ be an admissible sequence for A, $j \geq p$, $X \in \operatorname{ind} A_j$ be an* (ad 4) *or* (ad 5)*-pivot, A_{j+1} be the modified algebra of A_j, and v be the corresponding extension point. If there is a vertex $u \in A^{(l)} \setminus A^{(r)}$ such that for pairwise different arrows $\alpha_1, \ldots, \alpha_q \in v^{\rightarrow}$, $q \geq 2$ there are paths $\omega_{\alpha_1}, \ldots, \omega_{\alpha_q} \in A(v, u)$, then for arbitrary $f, g \in \{1, \ldots, q\}$, $f \neq g$, one of the following cases holds:*

(i) At least one of $\omega_{\alpha_f}, \omega_{\alpha_g}$ is zero path.
(ii) The paths $\omega_{\alpha_f}, \omega_{\alpha_g}$ are nonzero and $\omega_{\alpha_f} - \omega_{\alpha_g} \in I$.

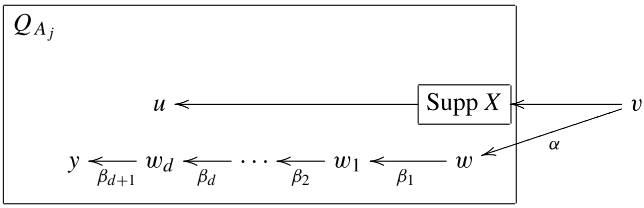
Proof It follows from [33, Section 4] that $A^{(l)}$ is a unique maximal convex branch coextension of $C = C_1 \times \cdots \times C_m$ inside A, that is, $A^{(l)} = B_1^{(l)} \times \cdots \times B_m^{(l)}$, where $B_i^{(l)}$ is a unique maximal convex branch coextension of C_i inside A, $i \in \{1, \ldots, m\}$. More precisely, $B_i^{(l)} =_{j=1}^{t_i} [K_j, E_j] C_i$, where $K_1, \ldots, K_{t_i}$ are branches, $i \in \{1, \ldots, m\}$. Assume that there is a vertex $u \in A^{(l)} \setminus A^{(r)}$ such that for pairwise different arrows $\alpha_1, \ldots, \alpha_q \in v^{\rightarrow}$, $q \geq 2$, there are paths $\omega_{\alpha_1}, \ldots, \omega_{\alpha_q} \in A(v, u)$. Then there exists $s \in \{1, \ldots, m\}$ such that $u \in B_s^{(l)}$. Let X be an (ad 4)-pivot and $Y_1 \to Y_2 \to \cdots \to Y_t$ with $t \geq 1$, be a finite sectional path in Γ_{A_j} (as in the definition of (ad 4)). Note that this finite sectional path is the linearly oriented quiver of type $\mathbb{A}_t$ and its support algebra Λ (given by the vertices corresponding to the simple composition factors of the modules $Y_1, Y_2, \ldots, Y_t$) is a tilted algebra of the path algebra D of the linearly oriented quiver of type $\mathbb{A}_t$. From [41, (4.4)(2)] we know that Λ is a bound quiver algebra given by a branch in x, where x corresponds to the unique projective-injective D-module. Let Γ be a generalized multicoil of $\Gamma_{A_{j+1}}$ obtained by applying the admissible operation (ad 4), where X is the pivot contained in the generalized multicoil Ω_1, and Y_1 is the starting vertex of a finite sectional path contained in the generalized multicoil Ω_1 or Ω_2. So, Γ is obtained from Ω_1 or from the disjoint union of two generalized multicoils Ω_1, Ω_2 by the corresponding translation quiver admissible operations. In general, Ω_1 and Ω_2 are components of the same connected algebra or two connected algebras. Hence, we get two cases. In the first case $X, Y_1 \in \Omega_1$ or $X \in \Omega_1$, $Y_1 \in \Omega_2$ and Ω_1, Ω_2 are two components of the same connected algebra. In the second case $X \in \Omega_1$, $Y_1 \in \Omega_2$ and Ω_1, Ω_2 are two components of two connected algebras. Therefore, the bound quiver $Q_{A_{j+1}}$ of A_{j+1} in the first case is of the form



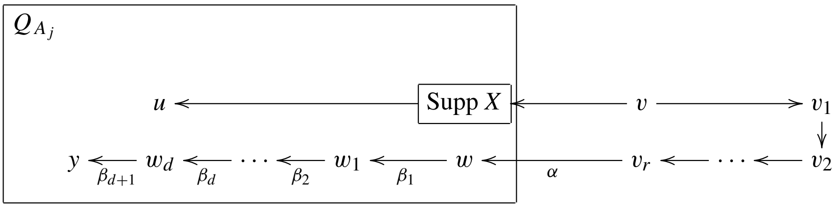
for $r = 0$ and



for $r \geq 1$, where the index r is as in the definition of (ad 4), v is the extension point of $A_j[X]$, w is the extension point of $A_j[Y_1]$, $w_1, \ldots, w_d$ belong to the branch in w generated by the support of $Y_1 \oplus \cdots \oplus Y_t$, and $\alpha\beta_1 \ldots \beta_h = 0$ for some $h \in \{1, \ldots, d+1\}$. In the second case the bound quiver $Q_{A_{j+1}}$ of A_{j+1} is of the form



for $r = 0$ and



for $r \geq 1$, where the index r is as in the definition of (ad 4), v is the extension point of $A_j[X]$, w is the extension point of $A_j[Y_1]$, $w_1, \dots, w_d$ belong to the branch in w generated by the support of $Y_1 \oplus \cdots \oplus Y_t$, $\alpha\beta_1 \dots \beta_h = 0$ for some $h \in \{1, \dots, d+1\}$, and y is the coextension point of A_j such that $y \in A^{(l)} \setminus A^{(r)}$. More precisely, $y \in B_{s'}^{(l)}$, where $s' \in \{1, \dots, m\}$ and $s' \neq s$. Moreover in both cases, we have $P_v(u) = X(u) = k$ or $P_v(u) = X(u) = 0$, and hence all nonzero paths from v to u are congruent modulo I_{j+1}. So, $A_{j+1}(v, u)$ is at most one-dimensional. We note that in the first case, the definition of (ad 4) (see the shape of the bound quiver $Q_{A_{j+1}}$ of A_{j+1}) implies that if the paths $\omega_{\alpha_f}, \omega_{\alpha_g} \in A_{j+1}(v, u)$ are nonzero and $\omega_{\alpha_f} - \omega_{\alpha_g} \in I$, then there is also a zero path $\omega_{\alpha_h} \in A_{j+1}(v, u)$ for some $h \in \{1, \dots, q\}$, $h \neq f \neq g$.

Let X be an (ad 5)-pivot and Γ be a generalized multicoil of $\Gamma_{A_{j+1}}$ obtained by applying this admissible operation with pivot X. Then Γ is obtained from the disjoint union of the finite family of generalized multicoils $\Omega_1, \Omega_2, \dots, \Omega_e$ by the corresponding translation quiver admissible operations, $1 \leq e \leq l$, where l is the number of stable tubes of Γ_C used in the whole process of creating Γ. Since in the definition of admissible operation (ad 5) we use the finite versions (fad 1)–(fad 4) of the admissible operations (ad 1)–(ad 4) and the admissible operation (ad 4), we conclude that the required statement follows from the above considerations. □

Remark 4.5 Let A be a generalized multicoil enlargement of a concealed canonical algebra C. We know from Theorems 3.3 and 3.5 that A can be obtained from $A^{(l)}$ by a sequence of admissible operations of types (ad 1)–(ad 5) or A can be obtained from $A^{(r)}$ by a sequence of admissible operations of types (ad 1*)–(ad 5*). We note that all presented above lemmas can be formulated and proved for dual operations (ad 1*)–(ad 5*) in a similar way.

4.6 The Separating Vertex Let A be a triangular algebra. Recall that a vertex v of Q_A is called *separating* if the radical of P_v is a direct sum of pairwise nonisomorphic indecomposable modules whose supports are contained in different connected components of the subquiver $Q(v)$ of Q_A obtained by deleting all those vertices u of Q_A being the source of a path with target v (including the trivial path from v to v).

We have the following lemma which follows from the proof of [44, Proposition 2.3] (see also [2, Lemma 2.3]).

Lemma 4.7 *Let A be a triangular algebra and assume that $A = B[X]$, where v is the extension vertex and $X = \operatorname{rad}_A P_v$. If B is simply connected and v is separating, then A is simply connected.*

Let D be the same as in the definition of (ad 1), that is, the full $t \times t$ lower triangular matrix algebra. Denote by $Y_1, \dots, Y_t$ the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective D-module.

Lemma 4.8 *Let A be a triangular algebra and assume that $A = (B \times D)[X \oplus Y]$, where v is the extension vertex and $X \oplus Y = \operatorname{rad}_A P_v$. If B is simply connected and v is separating, then A is simply connected.*

Proof Since the module P_v is a sink in the full subcategory of ind A consisting of projectives, the vertex v is a source in Q_A. Moreover, $A = (B \times D)[X \oplus Y]$, where X is the indecomposable direct summand of $\operatorname{rad}_A P_v$ that belongs to mod B and Y is a directing module (that is, an indecomposable module which does not lie on a cycle in ind A) such that $\mathrm{rad}_A P_v = X \oplus Y$. Therefore, the proof follows from the proof of [44, Proposition 2.3] (see also the proof of Lemma 2.3 in [2]). □

4.9 The Pointed Bound Quiver In order to carry out the construction of the free product of two fundamental groups of bound quivers, and in analogy with algebraic topology where pointed spaces are considered, one can define a *pointed bound quiver* (Q, I, x), that is, a bound quiver (Q, I) together with a distinguished vertex x (see [13, Section 3]). Given two pointed bound quivers $Q' = (Q', I', x')$ and $Q'' = (Q'', I'', x'')$, we can assume, without loss of generality, that $Q'_0 \cap Q''_0 = Q'_1 \cap Q''_1 = \emptyset$. We define the quiver $Q = Q' \amalg Q''$ as follows: Q_0 is $Q'_0 \cup Q''_0$ in which we identify x' and x'' to a single new vertex x, and $Q_1 = Q'_1 \cup Q''_1$. Then, Q' and Q'' are identified to two full convex subquivers of Q, so walks on Q' or Q'' can be considered as walks on Q. Thus, I' and I'' generate two-sided ideals of kQ which we denote again by I' and I''. We define I to be the ideal $I' + I''$ of kQ. It follows from this definition that the minimal relations of I are precisely the minimal relations of I' together with the minimal relations of I'' give the minimal relations needed to determine the homotopy relation on (Q, I). Moreover, we can consider an element $\widetilde{w} \in \pi_1(Q', I', x')$ as an element $\widetilde{w} \in \pi_1(Q, I, x)$ (we denote by $\widetilde{w}$ the homotopy class of a walk w). Conversely, any (reduced) walk w in Q has a decomposition $w = w'_1 w''_1 w'_2 w''_2 \dots w'_n w''_n$, where w'_i and w''_i are walks in Q' and Q'' for $i \in \{1, \dots, n\}$, respectively. Moreover, this decomposition is unique, up to reduced walk, and compatible with the homotopy relations involved. This leads us to the following proposition.

Proposition 4.10 [13, Proposition 3.1] *With the notations above we have:*

(i) (Q, I, x) *is the coproduct of* (Q', I', x') *and* (Q'', I'', x'') *in the category of pointed bound quivers.*
(i) $\pi_1(Q, I, x) \cong \pi_1(Q', I', x') \amalg \pi_1(Q'', I'', x'')$.

5 Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1 and recall the relevant facts.

We know from Theorem 3.3 that the Auslander–Reiten quiver Γ_A of A admits a separating family of almost cyclic coherent components if and only if A is a generalized multicoil enlargement of a concealed canonical algebra C. Let $C = C_1 \times C_2 \times \dots \times C_l \times C_{l+1} \times \dots \times C_m$ be a decomposition of C into product of connected algebras such that $C_1, C_2, \dots, C_l$ are of type (p_1, p_2) and $C_{l+1}, C_{l+2}, \dots, C_m$ are of type $(p_1, \dots, p_t)$ with $t \geq 3$. Following [36], by h_i we denote the number of all stable tubes of rank one from Γ_{C_i} with $1 \leq i \leq l$, used in the whole process of creating A from C, and $h_i = 0$, if $l+1 \leq i \leq m$. Moreover, let

$$e_i = \begin{cases} 0 \ \text{if } C_i \text{ is of type } (p_1, \dots, p_t) \text{ with } t \geq 3 \\ 1 \ \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1, p_2 \geq 2 \\ 2 \ \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1 = 1, p_2 \geq 2 \\ 3 \ \text{if } C_i \text{ is of type } (p_1, p_2) \text{ with } p_1 = p_2 = 1, \end{cases}$$

for $i \in \{1, \dots, m\}$. We define also $f_{C_i} = \max\{e_i - h_i, 0\}$, for $i \in \{1, \dots, m\}$ and set $f_A = \sum_{i=1}^{m} f_{C_i} = \sum_{i=1}^{l} f_{C_i}$. Note that we can apply the operations (ad 4), (fad 4), (ad 4*), (fad 4*) in two ways. The first way is when the sectional paths occurring in the definitions of these operations come from a component or two components of the same connected algebra. The second one is, when these sectional paths come from two components of two connected algebras. By d_A we denote the number of all operations (ad 4), (fad 4), (ad 4*) or (fad 4*) which are of the first type, used in the whole process of creating A from C.

The Hochschild cohomology of a connected generalized multicoil algebra A has been described in [36, Theorem 1.1] using the numerical invariants of A (f_A, d_A and the others), depending on the types of admissible operations (ad 1)–(ad 5) and their duals, leading from a product C of concealed canonical algebras to A. Here, we will only need information about the first Hochschild cohomology of A, namely from [36, Theorem 1.1(iii)] we have:

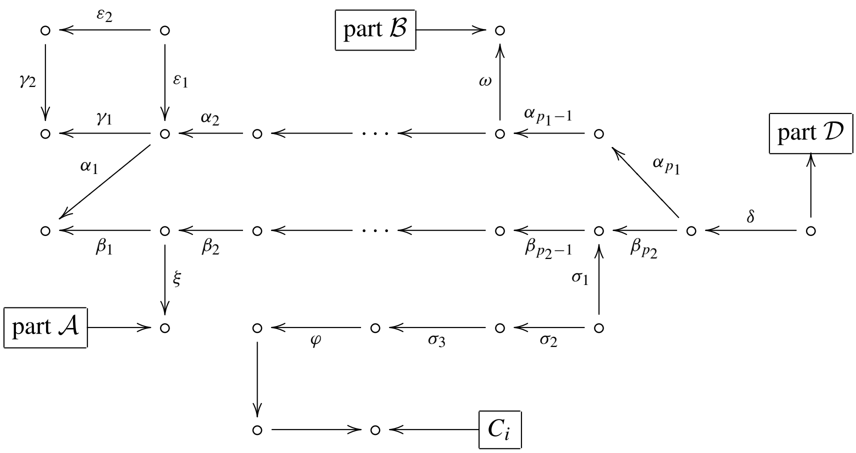
Theorem 5.1 *Let A be a connected generalized multicoil algebra. Then* $\dim_k H^1(A) = d_A + f_A$.

We are now able to complete the proof of Theorem 1.1.

Since A is tame, we may restrict to the generalized multicoil enlargements of tame concealed algebras. Namely, we have the following consequence of Theorem 3.3 and [33, Theorem F]: A is tame and Γ_A admits a separating family of almost cyclic coherent components if and only if A is a tame generalized multicoil enlargement of a finite family $C_1, \ldots, C_m$ of tame concealed algebras (concealed canonical algebras of Euclidean type).

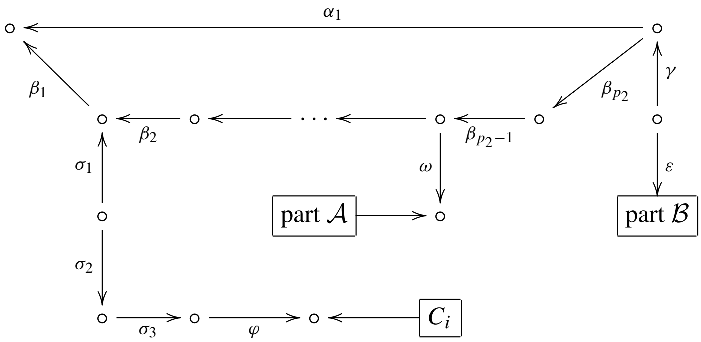
We first show the necessity. Suppose that A is simply connected. We must show that the first Hochschild cohomology $H^1(A)$ of A vanishes. Assume to the contrary that $H^1(A) \neq 0$. Then by Theorem 5.1, $d_A + f_A \neq 0$. If $d_A \neq 0$, then it follows from the proof of Lemma 4.4 (and its dual version) that A is not simply connected, a contradiction. Therefore, we may assume that $d_A = 0$ and $f_A \neq 0$. Since $f_A = \sum_{i=1}^{l} \max\{e_i - h_i, 0\} \neq 0$, we get that $\max\{e_j - h_j, 0\} \neq 0$ for some $j \in \{1, \ldots, l\}$. Note that, from Lemmas 4.2, 4.3, 4.4 and their proofs (and also from their dual versions - see Remark 4.5), we know how the bound quiver algebra changes after applying a given admissible operation. We have three cases to consider:

(1) Assume that the algebra C_j is of type (p_1, p_2) with $p_1, p_2 \geq 2$. Then $e_j = 1$ and $h_j = 0$. The bound quiver algebra $A = kQ/I$ is given by the quiver Q which can be visualized as follows:

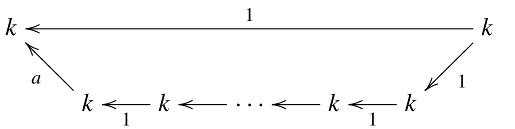


where I the ideal in the path algebra kQ of Q over k generated by the elements $\varepsilon_1\alpha_1$, $\alpha_2\gamma_1$, $\varepsilon_1\gamma_1 - \varepsilon_2\gamma_2$, $\beta_2\xi$, $\alpha_{p_1-1}\omega$, $\delta\alpha_{p_1}$, $\sigma_1\beta_{p_2-1}$, $\sigma_2\sigma_3\varphi$, elements from parts $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ of Q, and elements from C_i. Therefore, $\pi_1(Q, I)$ is not trivial and so A is not simply connected. More precisely, it follows from Proposition 4.10 that $\pi_1(Q, I) = \mathbb{Z} \amalg \pi_1(\mathcal{A}) \amalg \pi_1(\mathcal{B}) \amalg \pi_1(\mathcal{D}) \amalg \pi_1(C_i)$.

(2) Assume that the algebra C_j is of type (p_1, p_2) with $p_1 = 1$, $p_2 \geq 2$. Then $e_j = 2$, $h_j = 0$ or $h_j = 1$ and we have two subcases to consider. If $e_j = 2$ and $h_j = 0$, then the bound quiver algebra $A = kQ/I$ is given by the quiver Q which can be visualized as follows:



where I the ideal in the path algebra kQ of Q over k generated by the elements $\gamma\beta_{p_2}$, $\beta_{p_2-1}\omega$, $\sigma_1\beta_1$, $\sigma_2\sigma_3\varphi$, elements from parts $\mathcal{A}$, $\mathcal{B}$ of Q, and elements from C_i. Therefore, $\pi_1(Q, I)$ is not trivial and so A is not simply connected. More precisely, it follows from Proposition 4.10 that $\pi_1(Q, I) = \mathbb{Z} \amalg \pi_1(\mathcal{A}) \amalg \pi_1(\mathcal{B}) \amalg \pi_1(C_i)$. If $e_j = 2$ and $h_j = 1$, then the bound quiver algebra $A = kQ/J$ is given by the quiver Q which can be visualized as in the previous subcase with the ideal J of kQ generated by the elements $\gamma\alpha_1 - a\gamma\beta_{p_2} \dots \beta_2\beta_1$, $\beta_{p_2-1}\omega$, $\sigma_1\beta_1$, $\sigma_2\sigma_3\varphi$, elements from parts $\mathcal{A}$, $\mathcal{B}$ of Q, and elements from C_i, where $a \in k\backslash\{0\}$. Note that in general, we can apply to a stable tube $\mathcal{T}$ of one of the following admissible operations: (ad 1), (ad 4), (ad 5) or their dual versions (with an infinite sectional path belonging to $\mathcal{T}$). Since $h_j = 1$, we applied (in the above visualization) an admissible operation from the set $\mathcal{S} = \{(\text{ad } 1), (\text{ad } 4), (\text{ad } 5)\}$ to the algebra C_j with pivot the regular C_j-module corresponding to the indecomposable representation of the form



lying in a stable tube of rank 1 in Γ_{C_j} (see [42, XIII.2.4(c)]), where $a \in k \setminus \{0\}$. More precisely, if we apply (ad 1) with parameter $t = 0$, then we have to remove the arrow ε and the part $\mathcal{B}$. Observe also that A is not simply connected, because A is isomorphic to the algebra $A' = kQ/J'$, where the ideal J' of kQ is generated by the elements of $J \setminus \{\gamma\alpha_1 - a\gamma\beta_{p_2} \dots \beta_2\beta_1\} \cup \{\gamma\alpha_1\}$ and $\pi_1(Q, J')$ is not trivial. Again, it follows from Proposition 4.10 that $\pi_1(Q, J') = \mathbb{Z} \amalg \pi_1(\mathcal{A}) \amalg \pi_1(\mathcal{B}) \amalg \pi_1(C_i)$. If we apply an admissible operation from the set $\mathcal{S}^* = \{(\text{ad } 1^*), (\text{ad } 4^*), (\text{ad } 5^*)\}$ to the algebra C_j, the proof follows by dual arguments.

(3) Assume that the algebra C_j is of type (p_1, p_2) with $p_1 = p_2 = 1$. Then $e_j = 3$, $h_j = 0$, $h_j = 1$ or $h_j = 2$ and we have three subcases to consider. Note that in this case all stable tubes in Γ_{C_j} have ranks equal to 1. Now, if $e_j = 3$ and $h_j = 0$, then $j = l = 1$ and the path algebra $A = kQ$ is given by the Kronecker quiver Q: $\circ \overset{\alpha}{\underset{\beta}{\leftleftarrows}} \circ$. Therefore,

$\pi_1(Q) \cong \mathbb{Z}$ and so A is not simply connected. If $e_j = 3$ and $h_j = 1$, then the bound quiver algebra $A = kQ/J$ is given by the quiver Q which can be visualized as follows:

$$\circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ \xleftarrow{\ \gamma\ } \circ \xrightarrow{\ \varepsilon\ } \boxed{\text{part } \mathcal{A}}$$

with the ideal J in the path algebra kQ of Q over k generated by the element $\gamma\alpha - a\gamma\beta$ and elements from part $\mathcal{A}$ (the rest of Q), where $a \in k \setminus \{0\}$. Since $h_j = 1$, we applied (in the above visualization) an admissible operation from the set $\mathcal{S}$ to the algebra C_j with pivot the regular C_j-module corresponding to the indecomposable representation of the form $k \underset{a}{\overset{1}{\leftleftarrows}} k$ lying in a stable tube of rank 1 in Γ_{C_j} (see [42, XIII.2.4(c)]), where $a \in k \setminus \{0\}$. More precisely, if we apply (ad 1) with parameter $t = 0$, then we have to remove the arrow ε and the part $\mathcal{A}$. Observe also that A is not simply connected, because A is isomorphic to the algebra $A' = kQ/J'$, where the ideal J' of kQ is generated by the elements of $J \setminus \{\gamma\alpha - a\gamma\beta\} \cup \{\gamma\alpha\}$ and $\pi_1(Q, J')$ is not trivial. Again, it follows from Proposition 4.10 that $\pi_1(Q, J') = \mathbb{Z} \amalg \pi_1(\mathcal{A})$. Moreover, if we apply an admissible operation from the set $\mathcal{S}^*$ to the algebra C_j, the proof follows by dual arguments. If $e_j = 3$ and $h_j = 2$, then the bound quiver algebra $A = kQ/L$ is given by the quiver Q which can be visualized as follows:

$$\boxed{\text{part } \mathcal{A}} \xrightarrow{\ \lambda\ } \circ \xleftarrow{\ \delta\ } \circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ \xleftarrow{\ \gamma\ } \circ \xrightarrow{\ \varepsilon\ } \boxed{\text{part } \mathcal{B}}$$

with the ideal L of kQ generated by the elements $\gamma\alpha - a\gamma\beta$, $\alpha\delta - b\beta\delta$, $\gamma\alpha\delta$ and elements from parts $\mathcal{A}$, $\mathcal{B}$ of Q, where $a, b \in k \setminus \{0\}$ and $a \neq b$. Since $h_j = 2$, we applied (in the above visualization) one admissible operation from the set $\mathcal{S}$ and one from the set $\mathcal{S}^*$ to the algebra C_j with pivots the regular C_j-modules corresponding to the indecomposable representations of the form

$$k \underset{a}{\overset{1}{\leftleftarrows}} k \text{ and } k \underset{b}{\overset{1}{\leftleftarrows}} k$$

lying in different stable tubes of rank 1 in Γ_{C_j} (see [42, XIII.2.4(c)]), where $a, b \in k \setminus \{0\}$ and $a \neq b$. More precisely, if we apply (ad 1) (respectively, (ad 1*)) with parameter $t = 0$, then we have to remove the arrow ε and the part $\mathcal{B}$ (respectively, the arrow λ and the part $\mathcal{A}$). Observe also that A is not simply connected, because A is isomorphic to the algebra $A' = kQ/L'$, where the ideal L' of kQ is generated by the elements of $L \setminus \{\gamma\alpha - a\gamma\beta, \alpha\delta - b\beta\delta\} \cup \{\gamma\alpha, \alpha\delta\}$ and $\pi_1(Q, L')$ is not trivial. Again, it follows from Proposition 4.10 that $\pi_1(Q, L') = \mathbb{Z} \amalg \pi_1(\mathcal{A}) \amalg \pi_1(\mathcal{B})$. In a similar way, one can show all the cases of applying two admissible operations from the set $\mathcal{S} \cup \mathcal{S}^*$ to any two stable tubes of rank one from the Auslander–Reiten quiver of the Kronecker algebra.

We now show the sufficiency. We know from Theorem 3.5 that there is a unique full convex subcategory $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ of A which is a tubular coextension of the product $C_1 \times \ldots \times C_m = C$ of a family $C_1, \ldots, C_m$ of tame concealed algebras (see remarks immediately after Theorem 5.1) such that A is obtained from $A^{(l)}$ by a sequence of admissible operations of types (ad 1)–(ad 5). We shall prove our claim by induction on the number of admissible operations leading from $A^{(l)}$ to the algebra A. Note that we can apply an admissible operation (ad 2), (ad 3), (ad 4) or (ad 5) if the number of all successors of the module Y_i (which occurs in the definitions of the above admissible operations) is finite for each $1 \leq i \leq t$. Indeed, if this is not the case, then the family of generalized multicoils obtained after applying such admissible operation is not sincere, and then it is not separating. Let $C = A_0, \dots, A_p = A^{(l)}, A_{p+1}, \dots, A_n = A$ be an admissible sequence for A and assume that $A_p = A$. In this case A is tame quasitilted algebra and our claim follows from [3, Theorem A]. Let $k \geq p$, $A = A_{k+1}$ and assume that A_k is simply connected. Moreover, let v be the extension point of A_k and $X \in \operatorname{ind} A_k$ be the pivot of the admissible operation. Since $H^1(A) = 0$, the vertex v is separating, by [44, Lemma 3.2]. Note that if the admissible operation leading from A_k to A is of type (ad 1), (ad 2) or (ad 3), then A_k is a connected algebra.

If X is an (ad 1)-pivot, then $A = A_k[X]$ or $A = (A_k \times D)[X \oplus Y]$, where $\operatorname{rad}_A P_v = X$ or $\operatorname{rad}_A P_v = X \oplus Y$ respectively, D is the full $t \times t$ lower triangular matrix algebra over k for some $t \geq 1$, and Y is the unique indecomposable projective-injective D-module (see definition of (ad 1)). Applying Lemma 4.7 or Lemma 4.8 respectively, we conclude that A is simply connected.

If X is an (ad 2)-pivot or (ad 3)-pivot, then $A = A_k[X]$, where $\operatorname{rad}_A P_v = X$. Applying Lemma 4.7, we conclude that A is simply connected.

Let X be an (ad 4)-pivot and $Y = Y_1 \to Y_2 \to \cdots \to Y_t$ with $t \geq 1$ be a finite sectional path in Γ_{A_k}. Then, for $r = 0$, $A = A_k[X \oplus Y]$, and for $r \geq 1$,

$$A = \begin{bmatrix} A_k & 0 & 0 & \dots & 0 & 0 \\ Y & k & 0 & \dots & 0 & 0 \\ Y & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Y & k & k & \dots & k & 0 \\ X \oplus Y & k & k & \dots & k & k \end{bmatrix}$$

with $r+2$ columns and rows (see definition of (ad 4)). We note that Y_i is directing A-module for each $1 \leq i \leq t$. Indeed, since $H^1(A) = 0$, we get $d_A = 0$, and so A_k is not connected.

Now, if $r = 0$, then $A = A_k[X \oplus Y]$ and $\operatorname{rad}_A P_v = X \oplus Y$. Then it follows from Lemma 4.7 that A is simply connected.

If $r \geq 1$, then observe that the modified algebra A of A_k can be obtained by applying $r + 1$ one-point extensions in the following way: $A_k^{(0)} = A_k[U_{01}]$, $A_k^{(1)} = A_k^{(0)}[U_{11}]$, $A_k^{(2)} = A_k^{(1)}[U_{21}], \dots, A_k^{(r-1)} = A_k^{(r-2)}[U_{r-1,1}]$ and finally $A = A_k^{(r)} = A_k^{(r-1)}[X \oplus U_{r1}]$, where $U_{01} = Y$, U_{j1} is a projective $A_k^{(j-1)}$-module such that $\operatorname{rad}_{A_k^{(j-1)}} U_{j1} = U_{j-1,1}$, for $r \geq 1$, $1 \leq j \leq r$. We denote by v_j the extension vertex of $A_k^{(j-1)}$, for $1 \leq j \leq r$. Since the vertex v_1 of $Q_{A_k^{(0)}}$ is separating and $\operatorname{rad}_{A_k^{(0)}} P_{v_1} = U_{01}$, applying Lemma 4.7, we conclude that the algebra $A_k^{(0)}$ is simply connected. Further, since the vertex v_2 of $Q_{A_k^{(1)}}$ is separating, $\operatorname{rad}_{A_k^{(1)}} P_{v_2} = U_{11}$, and $A_k^{(0)}$ is simply connected, it follows from Lemma 4.7 that $A_k^{(1)}$ is simply connected. Iterating a finite number of times the same arguments, we get that $A_k^{(r-1)}$ is simply connected. Finally, since the vertex v of Q_A is separating and $\operatorname{rad}_A P_v = X \oplus U_{r1}$, applying again Lemma 4.7, we get that A is simply connected.

Let X be an (ad 5)-pivot. Since in the definition of admissible operation (ad 5) we use the finite versions (fad 1)–(fad 4) of the admissible operations (ad 1)–(ad 4) and the admissible operation (ad 4), we conclude that the required statement follows from the above considerations.

This finishes the proof of Theorem 1.1.

6 Proof of Theorem 1.2

Let A be a generalized multicoil algebra. Then A is a connected generalized multicoil enlargement of a concealed canonical algebra C. Let $C = C_1 \times C_2 \times \cdots \times C_l \times C_{l+1} \times \cdots \times C_m$ be a decomposition of C into product of connected algebras such that $C_1, C_2, \ldots, C_l$ are of type (p_1, p_2) and $C_{l+1}, C_{l+2}, \ldots, C_m$ are of type $(p_1, \ldots, p_t)$ with $t \geq 3$. Since C_i, $i \in \{1, \ldots, m\}$, are simply connected, we get $l = 0$. Moreover, by the assumption, the sectional paths occuring in the definitions of the operations (ad 4), (fad 4), (ad 4*), (fad 4*) come from two components of two connected algebras. Applying Theorems 3.3 and 3.5 we infer that there exists a unique factor algebra $A^{(l)} = A_1^{(l)} \times \cdots \times A_m^{(l)}$ of A which is a tubular coextension of a concealed canonical algebra $C = C_1 \times \ldots \times C_m$, and a unique factor algebra $A^{(r)} = A_1^{(r)} \times \cdots \times A_m^{(r)}$ of A which is a tubular extension of a concealed canonical algebra $C = C_1 \times \ldots \times C_m$. Since $A^{(l)}$ and $A^{(r)}$ are quasitilted algebras (of canonical types), the equivalence (ii) and (iv) follows from [26, Theorem 1]. Clearly, (v) implies (i).

We now show that (i) implies (iii). Since all algebras $C_1, \ldots, C_m$ are of type $(p_1, \ldots, p_t)$ with $t \geq 3$ $(l = 0)$, we get $f_A = 0$. Assume to the contrary that $H^1(A) \neq 0$. Then, by Theorem 5.1, $d_A + f_A \neq 0$. Therefore, $d_A \neq 0$ and it follows from the proof of Lemma 4.4 (and its dual version) that A is not simply connected, a contradiction with (i).

We show that (iii) implies (iv). Assume to the contrary that there exists $i \in \{1, \ldots, m\}$ such that $H^1(A_i^{(l)}) \neq 0$ or $H^1(A_i^{(r)}) \neq 0$. Without loss of generality, we may assume that $H^1(A_i^{(l)}) \neq 0$ for some $i \in \{1, \ldots, m\}$. Since $A_i^{(l)}$ is a tubular coextension of a concealed canonical algebra C_i, we have that $A_i^{(l)}$ is a generalized multicoil enlargement of C_i, and so, by Theorem 5.1, $\dim_k H^1(A_i^{(l)}) = d_{A_i^{(l)}} + f_{A_i^{(l)}}$. Moreover, by our assumption on C_i, we have $f_{A_i^{(l)}} = 0$. Hence $d_{A_i^{(l)}} \neq 0$. Since $d_A \geq d_{A_i^{(l)}}$, we get a contradiction with (iii).

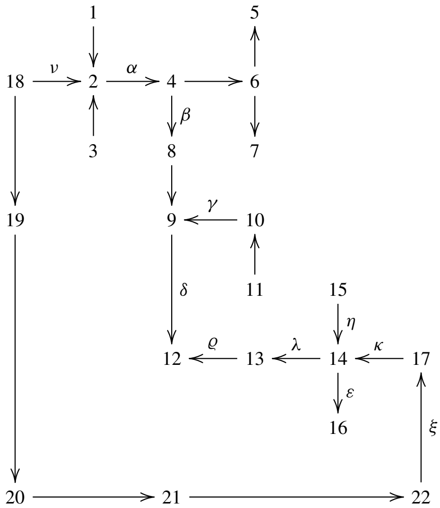
In order to finish the proof we will show that (iv) implies (v). Assume that $H^1(A_i^{(l)}) = 0$ and $H^1(A_i^{(r)}) = 0$, for any $i \in \{1, \ldots, m\}$. We know that for each $i \in \{1, \ldots, m\}$, $A_i^{(l)}$ (respectively, $A_i^{(r)}$) is a tubular coextension (respectively, extension) of a concealed canonical algebra C_i of type $(p_1, \ldots, p_t)$, $t \geq 3$ and $H^1(C_i) = 0$, by [20, Theorem 2.4]. Then $H^1(B) = 0$ for every full convex subcategory B of $A_i^{(l)}$ (respectively, $A_i^{(r)}$). Therefore, it follows from [44, Theorem 4.1] that $A_i^{(l)}$ and $A_i^{(r)}$ are strongly simply connected, for any $i \in \{1, \ldots, m\}$. Moreover, by our assumption on A, the Auslander–Reiten quiver Γ_A does not contain exceptional configurations of modules. Applying now Theorems 3.3 and 3.6 we infer that A is strongly simply connected.

7 Examples

We start this section with the following remark.

Remark 7.1 We can apply Theorem 1.1 to important classes of algebras. For example, to the cycle-finite algebras with separating families of almost cyclic coherent Auslander–Reiten components. Indeed, it is known (see [8]) that every cycle-finite algebra is tame.

Example 7.2 Let $A = kQ/I$ be the bound quiver algebra given by the quiver Q of the form



and I the ideal in the path algebra kQ of Q over k generated by the elements $\alpha\beta$, $\gamma\delta$, $\eta\varepsilon$, $\kappa\lambda\varrho$, $\xi\kappa\lambda$, $\nu\alpha$. Then A is a generalized multicoil enlargement of a concealed canonical algebra C, where C is the hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_6$ given by the vertices $1, 2, \dots, 7$. Indeed, consider the dimension-vectors

$$\mathbf{a_1} = \begin{array}{c} 0\ 0 \\ 010 \\ 0\ 0 \end{array}, \mathbf{a_2} = \begin{array}{c} 0\ 0 \\ 010 \\ 010 \end{array}, \mathbf{a_3} = \begin{array}{c} 0\ 0 \\ 010 \\ 010 \\ 10 \\ 0 \end{array}, \mathbf{a_4} = \begin{array}{c} 0\ 0 \\ 000 \\ 000 \\ 00 \\ 00 \\ 001 \end{array}, \mathbf{a_5} = \begin{array}{c} 0\ 0 \\ 000 \\ 000 \\ 00 \\ 00 \\ 011 \\ 1 \end{array}, \mathbf{a_6} = \begin{array}{c} 0\ 0 \\ 100 \\ 000 \\ 00 \\ 00 \\ 0000 \\ 0 \end{array}.$$

We apply (ad 1*) with pivot the simple regular C-module with vector $\mathbf{a_1}$, and with parameter $t = 0$. The modified algebra B_1 is given by the quiver with vertices $1, 2, \dots, 8$ bound by $\alpha\beta = 0$. Now, we apply (ad 1*) with pivot the indecomposable B_1-module with vector $\mathbf{a_2}$, and with parameter $t = 2$. The modified algebra B_2 is given by the quiver with vertices $1, 2, \dots, 11$ bound by $\alpha\beta = 0$. Next, we apply (ad 1*) with pivot the indecomposable B_2-module with vector $\mathbf{a_3}$, and with parameter $t = 3$. The modified algebra B_3 is given by the quiver with vertices $1, 2, \dots, 15$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$. In the next step we apply (ad 1*) with pivot the indecomposable B_3-module with vector $\mathbf{a_4}$, and with parameter $t = 0$. The modified algebra B_4 is given by the quiver with vertices $1, 2, \dots, 16$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$. Next, we apply the admissible operation (ad 5) in two steps. The first step: we apply the operation (fad 3) with pivot the indecomposable B_4-module with vector $\mathbf{a_5}$, and with parameters $t = 3$, $s = 2$. The modified algebra B_5 is given by the quiver with vertices $1, 2, \dots, 17$ bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\eta\varepsilon = 0$, $\kappa\lambda\varrho = 0$. The second step: we apply the operation (ad 4) with pivot the indecomposable B_5-module with vector $\mathbf{a_6}$, and with a finite sectional path consisting of the indecomposable B_5-modules with dimension-vectors

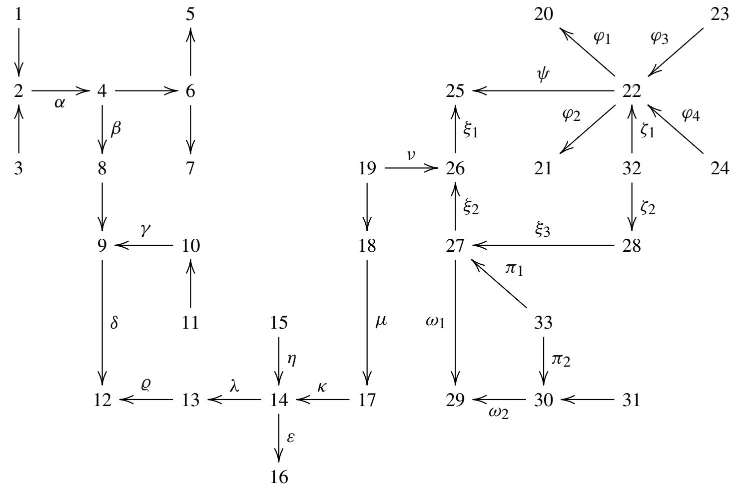
$$\begin{smallmatrix} 0\ 0 \\ 000 \\ 000 \\ 00 \\ 00 \\ 0011 \\ 1 \end{smallmatrix} \rightarrow \begin{smallmatrix} 0\ 0 \\ 000 \\ 000 \\ 00 \\ 00 \\ 0011 \\ 0 \end{smallmatrix} \rightarrow \begin{smallmatrix} 0\ 0 \\ 000 \\ 000 \\ 00 \\ 01 \\ 0011 \\ 0 \end{smallmatrix} \rightarrow \begin{smallmatrix} 0\ 0 \\ 000 \\ 000 \\ 00 \\ 00 \\ 0001 \\ 0 \end{smallmatrix}$$

and with parameter $r = 4$. The modified algebra is equal to A.

Then the left quasitilted algebra $A^{(l)}$ of A is the convex subcategory of A being the bound quiver algebra $kQ^{(l)}/I^{(l)}$, where $Q^{(l)}$ is a full subquiver of Q given by the vertices $1, 2, \dots, 16$ and $I^{(l)} = kQ^{(l)} \cap I$ is the ideal in $kQ^{(l)}$. The right quasitilted algebra $A^{(r)}$ of A is the convex subcategory of A being the bound quiver algebra $kQ^{(r)}/I^{(r)}$, where $Q^{(r)}$ is a full subquiver of Q given by the vertices $1, 2, \dots, 7, 14, 15, \dots, 18$ and $I^{(r)} = kQ^{(r)} \cap I$ is the ideal in $kQ^{(r)}$. Note that $A^{(l)}$ and $A^{(r)}$ are tame.

It follows from Theorems 3.3, 3.5(iii) and the above construction that the Auslander–Reiten quiver Γ_A of the tame algebra $A = kQ/I$ admits a separating family of almost cyclic coherent components. Further, $\pi_1(Q, I) \cong \mathbb{Z}$ and hence A is not simply connected. Moreover, by Theorem 5.1, the first Hochschild cohomology space $H^1(A) \cong k$ ($d_A = 1$, $f_A = 0$). We also note that, since $A^{(l)}$ and $A^{(r)}$ are tame tilted algebras of Euclidean type $\widetilde{\mathbb{D}}$ such that $H^1(A^{(l)}) = 0$ and $H^1(A^{(r)}) = 0$, it follows from [5, Theorem] that $A^{(l)}$ and $A^{(r)}$ are simply connected (and even strongly simply connected from [5, Corollary]). We refer to [33, Example 4.1] (see also [35, Example 9.13]) for a more extensive example of the tame algebra with a separating family of almost cyclic coherent components which is not simply connected. Finally, we also mention that A is a generalized multicoil algebra such that Γ_A contains the exceptional configurations of modules.

Example 7.3 We borrow the following example from [31]. Let $A = kQ/I$ be the bound quiver algebra given by the quiver Q of the form

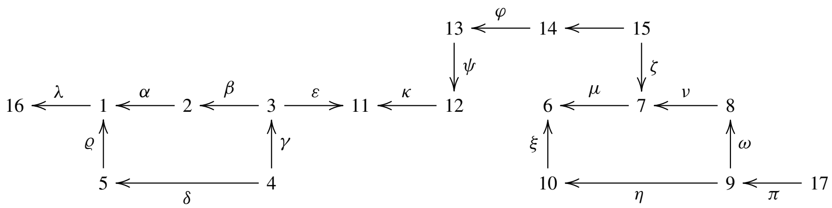


and I the ideal in the path algebra kQ of Q over k generated by the elements $\alpha\beta$, $\gamma\delta$, $\eta\varepsilon$, $\kappa\lambda\varrho$, $\varphi_3\psi$, $\varphi_4\psi$, $\xi_3\omega_1$, $\zeta_1\varphi_1$, $\zeta_1\varphi_2$, $\zeta_2\xi_3\xi_2\xi_1 - \zeta_1\psi$, $\pi_1\xi_2$, $\pi_1\omega_1 - \pi_2\omega_2$, $\mu\kappa\lambda$, $\nu\xi_1$. Then A is a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times C_2$, where C_1 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_6$ given by the vertices $1, 2, \dots, 7$, and C_2 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_5$ given by the vertices $20, 21, \dots, 24$. Indeed, we apply (ad 1*) to C_1 with pivot the simple regular C_1-module S_4, and with parameter $t=0$. The modified algebra B_1 is given by the quiver with the vertices $1,2,\dots,8$ bound by $\alpha\beta=0$. Next, we apply (ad 1*) to B_1 with pivot the indecomposable injective B_1-module I_8, and with parameter $t=2$. The modified algebra B_2 is given by the quiver with the vertices $1,2,\dots,11$ bound by $\alpha\beta=0$. Now, we apply (ad 1*) to B_2 with pivot the indecomposable B_2-module $\tau_{B_2}S_{10}$, and with parameter $t=3$. The modified algebra B_3 is given by the quiver with the vertices $1,2,\dots,15$ bound by $\alpha\beta=0$, $\gamma\delta=0$. Next, we apply (ad 1*) to B_3 with pivot the simple B_3-module S_{14}, and with parameter $t=0$. The modified algebra B_4 is given by the quiver with the vertices $1,2,\dots,16$ bound by $\alpha\beta=0$, $\gamma\delta=0$, $\eta\varepsilon=0$. In the next step we apply (ad 1*) to C_2 with pivot the simple regular C_2-module S_{22}, and with parameter $t=3$. The modified algebra B_5 is given by the quiver with the vertices $20,21,\dots,28$ bound by $\varphi_3\psi=0$, $\varphi_4\psi=0$. Now, we apply (ad 1*) to B_5 with pivot the simple B_5-module S_{27}, and with parameter $t=2$. The modified algebra B_6 is given by the quiver with the vertices $20,21,\dots,31$ bound by $\varphi_3\psi=0$, $\varphi_4\psi=0$, $\xi_3\omega_1=0$. Next, we apply (ad 2) to B_6 with pivot the indecomposable injective B_6-module I_{25}, and with parameter $t=3$. The modified algebra B_7 is given by the quiver with the vertices $20,21,\dots,32$ bound by $\varphi_3\psi=0$, $\varphi_4\psi=0$, $\xi_3\omega_1=0$, $\zeta_1\varphi_1=0$, $\zeta_1\varphi_2=0$, $\zeta_2\xi_3\xi_2\xi_1=\zeta_1\psi$. Now, we apply (ad 3) to B_7 with pivot the indecomposable B_7-module $\tau_{B_7}S_{30}$, and with parameter $t=2$. The modified algebra B_8 is given by the quiver with the vertices $20,21,\dots,33$ bound by $\varphi_3\psi=0$, $\varphi_4\psi=0$, $\xi_3\omega_1=0$, $\zeta_1\varphi_1=0$, $\zeta_1\varphi_2=0$, $\zeta_2\xi_3\xi_2\xi_1=\zeta_1\psi$, $\pi_1\xi_2=0$, $\pi_1\omega_1=\pi_2\omega_2$. Finally, we apply (ad 5) to $B_4\times B_8$ in two steps. The first step: we apply (fad 3) with pivot the indecomposable B_4-module $\tau_{B_4}S_{14}$, and with parameters $t=3$, $s=2$. The modified algebra B_9 is given by the quiver with the vertices $1,2,\dots,17$ bound by $\alpha\beta=0$, $\gamma\delta=0$, $\eta\varepsilon=0$, $\kappa\lambda\varrho=0$. The second step: we apply (ad 4) with pivot the simple B_8-module S_{26}, and with the finite sectional path $I_{16}\to\tau_{B_9}S_{15}\to I_{14}\to S_{17}$ consisting of the indecomposable B_9-modules, and with parameters $t=4$, $r=1$. The modified algebra is equal to A.

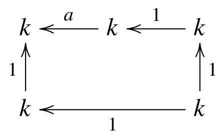
Then the left quasitilted algebra $A^{(l)}$ of A is the convex subcategory of A being the product $A^{(l)}=A_1^{(l)}\times A_2^{(l)}$, where $A_1^{(l)}=kQ_1^{(l)}/I_1^{(l)}$ is the branch coextension of the tame concealed algebra C_1, $Q_1^{(l)}$ is a full subquiver of Q given by the vertices $1,2,\dots,16$ and $I_1^{(l)}=kQ_1^{(l)}\cap I$ is the ideal in $kQ_1^{(l)}$, $A_2^{(l)}=kQ_2^{(l)}/I_2^{(l)}$ is the branch coextension of the tame concealed algebra C_2, $Q_2^{(l)}$ is a full subquiver of Q given by the vertices $20,21,\dots,31$ and $I_2^{(l)}=kQ_2^{(l)}\cap I$ is the ideal in $kQ_2^{(l)}$. The right quasitilted algebra $A^{(r)}$ of A is the convex subcategory of A being the product $A^{(r)}=A_1^{(r)}\times A_2^{(r)}$, where $A_1^{(r)}=C_1$, $A_2^{(r)}=kQ_2^{(r)}/I_2^{(r)}$ is the branch extension of the tame concealed algebra C_2, $Q_2^{(r)}$ is a full subquiver of Q given by the vertices $14,15,\dots,24,26,27,28,30,31,32,33$ and $I_2^{(r)}=kQ_2^{(r)}\cap I$ is the ideal in $kQ_2^{(r)}$. Note that $A_1^{(l)}$, $A_2^{(l)}$, $A_1^{(r)}$ and $A_2^{(r)}$ are tame.

It follows from Theorems 3.3, 3.5(iii) and the above construction that A is tame and Γ_A admits a separating family of almost cyclic coherent components. Moreover, by Theorem 5.1, the first Hochschild cohomology space $H^1(A)=0$ ($d_A=0$, $f_A=0$). Then, a direct application of Theorem 1.1 shows that the algebra A is simply connected. In fact, it follows from [31, Theorem 1.2] that A is strongly simply connected. We also note that, since $A_1^{(l)}$, $A_2^{(l)}$, $A_1^{(r)}$ and $A_2^{(r)}$ are tame tilted algebras of Euclidean type $\widetilde{\mathbb{D}}$ such that $H^1(A_1^{(l)})=0$, $H^1(A_2^{(l)})=0$, $H^1(A_1^{(r)})=0$ and $H^1(A_2^{(r)})=0$ it follows from [5, Theorem] that $A_1^{(l)}$, $A_2^{(l)}$, $A_1^{(r)}$ and $A_2^{(r)}$ are simply connected (and even strongly simply connected from [5, Corollary]). Finally, we mention that C_1, C_2 are simply connected, A is a generalized multicoil algebra, Γ_A does not contain exceptional configurations of modules, and so this example illustrates also Theorem 1.2.

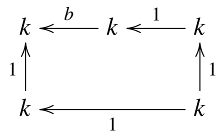
Example 7.4 Let $A = kQ/I$ be the bound quiver algebra given by the quiver Q of the form



and I the ideal in the path algebra kQ of Q over k generated by the elements $a\gamma\beta\alpha\lambda - \delta\varrho\lambda$, $\gamma\varepsilon$, $b\pi\omega\nu\mu - \pi\eta\xi$, $\zeta\mu$, $\varphi\psi\kappa$, where $a, b \in k \setminus \{0\}$. Then A is a generalized multicoil enlargement of a concealed canonical algebra $C = C_1 \times C_2$, where C_1 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_4$ given by the vertices $1, 2, \ldots, 5$, and C_2 is the hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_4$ given by the vertices $6, 7, \ldots, 10$. Indeed, we apply (ad 1*) to C_1 with pivot the simple regular C_1-module S_3, and with parameter $t = 2$. The modified algebra B_1 is given by the quiver with the vertices $1, 2, \ldots, 5, 11, 12, 13$ bound by $\gamma\varepsilon = 0$. Next, we apply (ad 4) to $B_1 \times C_2$ with pivot the simple regular C_2-module S_7 and with the finite sectional path $I_{12} \to S_{13}$ consisting of the indecomposable B_1-modules, and with parameters $t = 2$, $r = 1$. The modified algebra B_2 is given by the quiver with the vertices $1, 2, \ldots, 15$ bound by $\gamma\varepsilon = 0$, $\zeta\mu = 0$, $\varphi\psi\kappa = 0$. Now, we apply (ad 1*) with parameter $t = 0$ to the algebra B_2 with pivot the regular C_1-module corresponding to the indecomposable representation of the form



lying in a stable tube of rank 1 in Γ_{C_1} (see [42, XIII.2.4(c)]). The modified algebra B_3 is given by the quiver with the vertices $1, 2, \ldots, 16$ bound by $\gamma\varepsilon = 0$, $\zeta\mu = 0$, $\varphi\psi\kappa = 0$, $a\gamma\beta\alpha\lambda = \delta\varrho\lambda$, where $a \in k \setminus \{0\}$. Finally, we apply (ad 1) with parameter $t = 0$ to the algebra B_3 with pivot the regular C_2-module corresponding to the indecomposable representation of the form



lying in a stable tube of rank 1 in Γ_{C_2} (see [42, XIII.2.4(c)]). The modified algebra is then equal to A.

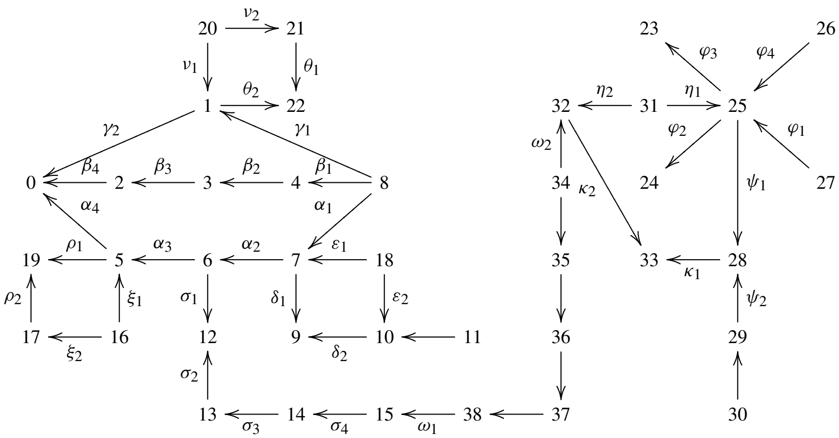
Then the left quasitilted algebra $A^{(l)}$ of A is the convex subcategory of A being the product $A^{(l)} = A_1^{(l)} \times A_2^{(l)}$, where $A_1^{(l)} = kQ_1^{(l)}/I_1^{(l)}$ is the branch coextension of C_1, $Q_1^{(l)}$ is a full subquiver of Q given by the vertices $1, 2, \ldots, 5, 11, 12, 13, 16$ and $I_1^{(l)} = kQ_1^{(l)} \cap I$ is the ideal in $kQ_1^{(l)}$, $A_2^{(l)} = C_2$. The right quasitilted algebra $A^{(r)}$ of A is the convex subcategory of A being the product $A^{(r)} = A_1^{(r)} \times A_2^{(r)}$, where $A_1^{(r)} = C_1$, $A_2^{(r)} = kQ_2^{(r)}/I_2^{(r)}$ is the branch extension of C_2, $Q_2^{(r)}$ is a full subquiver of Q given by the vertices

6, 7, ..., 10, 12, 13, 14, 15, 17 and $I_2^{(r)} = kQ_2^{(r)} \cap I$ is the ideal in $kQ_2^{(r)}$. Note that $A_1^{(l)}$, $A_2^{(l)}$, $A_1^{(r)}$ and $A_2^{(r)}$ are tame.

It follows from Theorems 3.3, 3.5(iii) and the above construction that A is tame and Γ_A admits a separating family of almost cyclic coherent components. Moreover, we have $h_1 = 1$, $e_1 = 1$, $h_2 = 1$, $e_2 = 1$, $f_{C_1} = 0$, $f_{C_2} = 0$, $f_A = f_{C_1} + f_{C_2} = 0$, and $d_A = 0$. Therefore, by Theorem 5.1, the first Hochschild cohomology space $H^1(A) = 0$. Then, a direct application of Theorem 1.1 shows that the algebra A is simply connected. We note that, by [19, Proposition 1.6], $H^1(A_2^{(l)}) \cong k$, $H^1(A_1^{(r)}) \cong k$. Since $A_1^{(l)}$ and $A_2^{(r)}$ are generalized multicoil algebras, we get by Theorem 5.1 that $H^1(A_1^{(l)}) = 0$, $H^1(A_2^{(r)}) = 0$. We also mention that $A_1^{(r)} = C_1$, $A_2^{(l)} = C_2$ are not simply connected, $A_1^{(l)}$, $A_2^{(r)}$ are simply connected, by [3, Theorem A], and so A is not strongly simply connected. Moreover, by the above construction we know that A is a generalized multicoil algebra, such that Γ_A does not contain exceptional configurations of modules. Therefore, this example shows that simple connectedness assumption imposed on the considered concealed canonical algebras is essential for the validity of Theorem 1.2.

We end this section with an example of a wild generalized multicoil algebra, illustrating Theorem 1.2.

Example 7.5 Let $A = kQ/I$ be the bound quiver algebra given by the quiver Q of the form



and I the ideal in the path algebra kQ of Q over k generated by the elements $\alpha_1\alpha_2\alpha_3\alpha_4 + \beta_1\beta_2\beta_3\beta_4 + \gamma_1\gamma_2$, $\alpha_1\delta_1$, $\alpha_2\sigma_1$, $\xi_1\alpha_4$, $\varepsilon_1\alpha_2$, $\varepsilon_1\delta_1 - \varepsilon_2\delta_2$, $\alpha_3\rho_1$, $\xi_1\rho_1 - \xi_2\rho_2$, $\nu_1\gamma_2$, $\gamma_1\theta_2$, $\nu_1\theta_2 - \nu_2\theta_1$, $\varphi_1\psi_1$, $\varphi_4\psi_1$, $\eta_1\varphi_2$, $\eta_1\varphi_3$, $\psi_2\kappa_1$, $\eta_2\kappa_2 - \eta_1\psi_1\kappa_1$, $\omega_2\kappa_2$, $\omega_1\sigma_4\sigma_3\sigma_2$. Then A is a generalized multicoil algebra. Indeed, A is a generalized multicoil enlargement of a canonical algebra $C = C_1 \times C_2$, where C_1 is the tubular canonical algebra of type (2, 4, 4) given by the vertices $0, 1, \ldots, 8$ bound by $\alpha_1\alpha_2\alpha_3\alpha_4 + \beta_1\beta_2\beta_3\beta_4 + \gamma_1\gamma_2 = 0$, and C_2 is the canonical algebra of Euclidean type $\widetilde{\mathbb{D}}_4$ given by the vertices $23, 24, \ldots, 27$. It is known that Γ_{C_1} admits an infinite family $\mathcal{T}_\lambda^{C_1}$, $\lambda \in \mathbb{P}_1(k)$, of pairwise orthogonal stable tubes, having a stable tube, say $\mathcal{T}_1^{C_1}$, of rank 4 with the mouth formed by the modules $S_5 = \tau_{C_1} S_6$, $S_6 = \tau_{C_1} S_7$, $S_7 = \tau_{C_1} E$, $E = \tau_{C_1} S_5$, where E is the unique indecomposable C_1-module with the dimension vector $\underline{\dim} E = {}^{\;1}_{11111}_{\;000}$, and a unique stable tube, say $\mathcal{T}_2^{C_1}$, of rank 2 with the mouth formed by the modules $S_1 = \tau_{C_1} F$, $F = \tau_{C_1} S_1$, where F is the unique indecomposable C_1-module with the dimension vector $\underline{\dim} F = \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & \end{smallmatrix}$ (see [41, (3.7)]). Moreover, Γ_{C_2} admits an infinite family $\mathcal{T}_\mu^{C_2}$, $\mu \in \mathbb{P}_1(k)$, of pairwise orthogonal stable tubes, having a stable tube, say $\mathcal{T}_1^{C_2}$, of rank 2 with the mouth formed by the modules $S_{25} = \tau_{C_2} G$, $G = \tau_{C_2} S_{25}$, where G is the unique indecomposable C_2-module with the dimension vector $\underline{\dim} G = \begin{smallmatrix} 1 & & 1 \\ & 1 & \\ 1 & & 1 \end{smallmatrix}$. We have the following sequence of the modified algebras. First, we apply (ad 1*) to C_1 with pivot the simple regular C_1-module S_7, and with parameter $t = 2$. The modified algebra B_1 is given by the quiver with the vertices $0, 1, \dots, 11$ bound by $\alpha_1\delta_1 = 0$. Next, we apply (ad 1*) to B_1 with pivot the simple B_1-module S_6, and with parameter $t = 3$. The modified algebra B_2 is given by the quiver with the vertices $0, 1, \dots, 15$ bound by $\alpha_1\delta_1 = 0$, $\alpha_2\sigma_1 = 0$. Now, we apply (ad 1) to B_2 with pivot the simple B_2-module S_5, and with parameter $t = 1$. The modified algebra B_3 is given by the quiver with the vertices $0, 1, \dots, 17$ bound by $\alpha_1\delta_1 = 0$, $\alpha_2\sigma_1 = 0$, $\xi_1\alpha_4 = 0$. Next, we apply (ad 3) to B_3 with pivot the indecomposable B_3-module $\tau_{B_3} I_{10}$, and with parameter $t = 2$. The modified algebra B_4 is given by the quiver with the vertices $0, 1, \dots, 18$ bound by $\alpha_1\delta_1 = 0$, $\alpha_2\sigma_1 = 0$, $\xi_1\alpha_4 = 0$, $\varepsilon_1\alpha_2 = 0$, $\varepsilon_1\delta_1 = \varepsilon_2\delta_2$. Further, we apply (ad 2*) to B_4 with pivot the indecomposable projective B_4-module P_{16}, and with parameter $t = 1$. The modified algebra B_5 is given by the quiver with the vertices $0, 1, \dots, 19$ bound by $\alpha_1\delta_1 = 0$, $\alpha_2\sigma_1 = 0$, $\xi_1\alpha_4 = 0$, $\varepsilon_1\alpha_2 = 0$, $\varepsilon_1\delta_1 = \varepsilon_2\delta_2$, $\alpha_3\rho_1 = 0$, $\xi_1\rho_1 = \xi_2\rho_2$. Now, we apply (ad 1) to B_5 with pivot the simple regular B_5-module S_1, and with parameter $t = 1$. The modified algebra B_6 is given by the quiver with the vertices $0, 1, \dots, 21$ bound by $\alpha_1\delta_1 = 0$, $\alpha_2\sigma_1 = 0$, $\xi_1\alpha_4 = 0$, $\varepsilon_1\alpha_2 = 0$, $\varepsilon_1\delta_1 = \varepsilon_2\delta_2$, $\alpha_3\rho_1 = 0$, $\xi_1\rho_1 = \xi_2\rho_2$, $\nu_1\gamma_2 = 0$. Next, we apply (ad 2*) to B_6 with pivot the indecomposable projective B_6-module P_{21}, and with parameter $t = 1$. The modified algebra B_7 is given by the quiver with the vertices $0, 1, \dots, 22$ bound by $\alpha_1\delta_1 = 0$, $\alpha_2\sigma_1 = 0$, $\xi_1\alpha_4 = 0$, $\varepsilon_1\alpha_2 = 0$, $\varepsilon_1\delta_1 = \varepsilon_2\delta_2$, $\alpha_3\rho_1 = 0$, $\xi_1\rho_1 = \xi_2\rho_2$, $\nu_1\gamma_2 = 0$, $\gamma_1\theta_2 = 0$, $\nu_1\theta_2 = \nu_2\theta_1$. Now, we apply (ad 1*) to C_2 with pivot the simple regular C_2-module S_{25}, and with parameter $t = 2$. The modified algebra B_8 is given by the quiver with the vertices $23, 24, \dots, 30$ bound by $\varphi_1\psi_1 = 0$, $\varphi_4\psi_1 = 0$. Next, we apply (ad 1) to B_8 with pivot the indecomposable B_8-module $\tau_{B_8} S_{29}$, and with parameter $t = 1$. The modified algebra B_9 is given by the quiver with the vertices $23, 24, \dots, 32$ bound by $\varphi_1\psi_1 = 0$, $\varphi_4\psi_1 = 0$, $\eta_1\varphi_2 = 0$, $\eta_1\varphi_3 = 0$. Now, we apply (ad 2*) to B_9 with pivot the indecomposable projective B_9-module P_{31}, and with parameter $t = 1$. The modified algebra B_{10} is given by the quiver with the vertices $23, 24, \dots, 33$ bound by $\varphi_1\psi_1 = 0$, $\varphi_4\psi_1 = 0$, $\eta_1\varphi_2 = 0$, $\eta_1\varphi_3 = 0$, $\psi_2\kappa_1 = 0$, $\eta_2\kappa_2 = \eta_1\psi_1\kappa_1$. Next, we apply (ad 4) to $B_7 \times B_{10}$ with pivot the simple B_{10}-module S_{32}, and with the finite sectional path $I_{13} \to I_{14} \to S_{15}$ consisting of the indecomposable B_7-modules, and with parameters $t = 3$, $r = 4$. The modified algebra is then equal to A.

Then the left quasitilted algebra $A^{(l)}$ of A is the convex subcategory of A being the product $A^{(l)} = A_1^{(l)} \times A_2^{(l)}$, where $A_1^{(l)} = kQ_1^{(l)}/I_1^{(l)}$ is the branch coextension of C_1, $Q_1^{(l)}$ is a full subquiver of Q given by the vertices $0, 1, \dots, 15, 17, 19, 21, 22$ and $I_1^{(l)} = kQ_1^{(l)} \cap I$ is the ideal in $kQ_1^{(l)}$, $A_2^{(l)} = kQ_2^{(l)}/I_2^{(l)}$ is the branch coextension of C_2, $Q_2^{(l)}$ is a full subquiver of Q given by the vertices $23, 24, \dots, 30, 33$ and $I_2^{(l)} = kQ_2^{(l)} \cap I$ is the ideal in $kQ_2^{(l)}$. The right quasitilted algebra $A^{(r)}$ of A is the convex subcategory of A being the product $A^{(r)} = A_1^{(r)} \times A_2^{(r)}$, where $A_1^{(r)} = kQ_1^{(r)}/I_1^{(r)}$ is the branch extension of C_1, $Q_1^{(r)}$ is a full subquiver of Q given by the vertices $0, 1, \dots, 8, 10, 11, 16, 17, 18, 20, 21$ and $I_1^{(r)} = kQ_1^{(r)} \cap I$ is the ideal in $kQ_1^{(r)}$, $A_2^{(r)} = kQ_2^{(r)}/I_2^{(r)}$ is the branch extension of C_2, $Q_2^{(r)}$ is a full subquiver of Q given by the vertices $13, 14, 15, 23, 24, \dots, 27, 31, 32, 34, 35, \dots, 38$ and $I_2^{(r)} = kQ_2^{(r)} \cap I$ is the ideal in $kQ_2^{(r)}$. Then, $A_1^{(l)}$ and $A_1^{(r)}$ are the quasitilted algebras of wild types $(4, 4, 13)$, $(4, 4, 9)$, respectively. Moreover, $A_2^{(l)}$ and $A_2^{(r)}$ are tame.

It follows from [7, Corollary 1.4] that C_1 is simply connected. Moreover, C_2 is also simply connected. By the above construction we know that A is a generalized multicoil algebra obtained from C_1, C_2 and Γ_A does not contain exceptional configurations of modules. Further, by Theorem 5.1, the first Hochschild cohomology space $H^1(A) = 0$ ($d_A = 0$, $f_A = 0$) and $H^1(A_1^{(l)}) = 0$, $H^1(A_2^{(l)}) = 0$, $H^1(A_1^{(r)}) = 0$, $H^1(A_2^{(r)}) = 0$. Then, a direct application of Theorem 1.2 shows that the algebras $A_1^{(l)}$, $A_2^{(l)}$, $A_1^{(r)}$, $A_2^{(r)}$ and A are simply connected.

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