Quinn's Formula and Abelian 3-Cocycles for Quadratic Forms

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Abstract

In pointed braided fusion categories knowing the self-symmetry braiding of simples is theoretically enough to reconstruct the associator and braiding on the entire category (up to twisting by a braided monoidal auto-equivalence). We address the problem to provide explicit associator formulas given only such input. This problem was solved by Quinn in the case of finitely many simples. We reprove and generalize this in various ways. In particular, we show that extra symmetries of Quinn's associator can still be arranged to hold in situations where one has infinitely many isoclasses of simples.

Keywords Braided categorical group · Picard groupoid · Strictification · Skeletalization · Associator

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1 Introduction

This note applies to both (a) pointed braided fusion categories as well as (b) braided categorical groups. Both are special types of braided monoidal categories. Both settings are closely related, yet a little different. We hope that we have found a way to formulate the introduction so that it is clear, irrespective of which of these applications the reader might have in mind. Among our four results below, Theorem B and Theorem C mainly reprove known results differently, while Theorem A and Theorem D appear to be new.

The problem motivating this note is the following: Suppose $(C, \otimes)$ is a braided monoidal category of type either as in (a) or (b). Then this category has an associator

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

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and a braiding

$$s_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

as part of its braided monoidal structure. One may find a different associator/braiding for the same bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ such that the identity functor $\text{id} : \mathcal{C} \rightarrow \mathcal{C}$ can be promoted to a braided monoidal functor¹. The “orbit” of all associators/braidings which can be obtained by twisting with such braided monoidal self-equivalences is pinned down by very little data. In concrete terms: In the setting of (a) suppose X is a simple object, resp. in the setting of (b) an arbitrary object. It has the self-symmetry braiding $s_{X,X} : X \otimes X \xrightarrow{\sim} X \otimes X$, and moreover X is invertible in $\mathcal{C}$, so we have the functor of tensoring with its inverse, $(-) \mapsto (-) \otimes X^{-1}$. Thus, by functoriality, we get a well-defined automorphism

$$s_{X,X} \otimes X^{-1} \otimes X^{-1} : 1_{\mathcal{C}} \xrightarrow{\sim} 1_{\mathcal{C}} \quad (1.1)$$

of the monoidal unit $1_{\mathcal{C}}$. Hence, this is an element of the abelian group $\pi_1(\mathcal{C}, \otimes) = \text{Aut}_{\mathcal{C}}(1_{\mathcal{C}})$. It turns out that this element only depends on the class of the object X in $\pi_0(\mathcal{C}_{\text{simp}}, \otimes)$, where in the setting of (a) $\mathcal{C}_{\text{simp}}$ is the groupoid of simple objects in $\mathcal{C}$, and for (b) let $\mathcal{C}_{\text{simp}} := \mathcal{C}$ be the entire category. Thus, we get a map

$$q : \pi_0(\mathcal{C}_{\text{simp}}, \otimes) \longrightarrow \pi_1(\mathcal{C}_{\text{simp}}, \otimes).$$

One can show that this map is a quadratic form. On the one hand, this form does not change under the aforementioned braided monoidal self-equivalences of $\mathcal{C}$ (which induce the identity map on π_0 and π_1). On the other hand, it also distinguishes different such orbits, i.e. it is a complete invariant.

This note is concerned with the problem to provide an explicit formula for the associator and braiding for a given quadratic form q , i.e. if we only know the self-symmetry braidings of Eq. 1.1, but have possibly no clue and no candidates what the associator and braiding should do on general objects X, Y, Z . This is a kind of integration problem: Find a valid choice of associators and braidings for the entire category such that, restricted to self-symmetries, it agrees with the given quadratic form.

Even stronger: One may ask whether there is a “simplest choice” of associators, e.g., an associator with additional symmetries (we propose a possible solution in Theorem D).

The problem can be attacked in concrete form as follows: First, since we only work up to braided monoidal equivalences inducing the identity map on π_0 and π_1 , it suffices to work with a skeleton of the category. Here the datum of an associator and braiding is encoded in an abelian 3-cocycle

$$[(h, c)] \in H_{ab}^3(G, M)$$

for $G := \pi_0(\mathcal{C}_{\text{simp}}, \otimes)$, $M := \pi_1(\mathcal{C}_{\text{simp}}, \otimes)$. In concrete terms, a cocycle representative has the form of maps

$$h : G \times G \times G \longrightarrow M \quad \text{and} \quad c : G \times G \longrightarrow M$$

corresponding to the associator and braiding. Note that in general h or c cannot be taken multilinear. It is more complicated than that.

For general abelian groups G, M , Eilenberg and Mac Lane have constructed an isomorphism

$$\text{tr} : H_{ab}^3(G, M) \xrightarrow{\sim} \text{Quad}(G, M), \quad (1.2)$$

showing that this cohomology group is isomorphic to the group of quadratic forms on G with values in M , [6, 15]. This isomorphism underlies the above reconstruction procedure.

¹in the setting (a) we tacitly assume these to be k -linear, where k is the base field of the fusion category.

Given only the “self-symmetries”, i.e. only the right side, finding an associator and braiding amounts to finding a preimage under this isomorphism. (In particular, this paper provides a new proof of the surjectivity of the map in Eq. 1.2 as a side result, see Section 11)

1.1 Application to Explicit Associator Formulas

Unfortunately, solving the integration problem is not entirely trivial. The classical proof for the isomorphism in Eq. 1.2 goes as follows: Do the cases $G = \mathbb{Z}$ and $G = \mathbb{Z}/n\mathbb{Z}$ individually, then use that both sides of Eq. 1.2 are quadratic functors in G , and exploit that any abelian group is a colimit of finitely generated ones. This sounds deceptively simple, but note that if one wants an explicit formula, one needs a cocycle formula, i.e. a lift

$$\begin{array}{ccc} Z_{ab}^3(G, M) & & \\ \downarrow & \nwarrow ? & \\ H_{ab}^3(G, M) & \xleftarrow[\cong]{\text{tr}^{-1}} & \text{Quad}(G, M) \end{array} \quad (1.3)$$

and once you consider such lifts, the nice functoriality properties break (as an illustration: A compatible family of cohomology classes in some directed system underlying the colimit need not come from a compatible family of cocycles, because there is no a priori control of the coboundaries along the system). We solve this as follows:

In Sections 3–5 we shall introduce the concept of ‘*optimal admissible presentations*’ and ‘*admissible liftings*’, and this can be thought of as machines to produce explicit cocycle formulas. Since these are in general necessarily non-linear maps and highly non-unique, it should not be surprising that admissible presentations and liftings also allow for a lot of variation.

Theorem A *Let G, M be abelian groups and $q \in \text{Quad}(G, M)$. Suppose*

- *(F_0, π, C) is an optimal admissible presentation (see Section 3), and*
- *$(-)$ is an admissible lifting (see Section 5).*

Then

$$h(x, y, z) := -C(\tilde{x}, L(y, z)), \quad c(x, y) := C(\tilde{x}, \tilde{y}),$$

with the non-linear function $L(x, y) := \widetilde{(x + y)} - \tilde{x} - \tilde{y}$, defines an abelian 3-cocycle whose attached quadratic form is q . This gives a concrete lift as in Eq. 1.3 for the inverse Eilenberg–Mac Lane isomorphism.

This will be Theorem 6.2. It underlies all other results of the paper. Quinn [17] has given an explicit formula in the case G finite abelian and $M := R^\times$ for R a commutative ring. We give a new proof for his formula as an application of Theorem A. In our generalized version M is arbitrary and G is allowed to be of a more general form. In particular, all finitely generated abelian groups are covered by our version:

Theorem B (Generalized Quinn formula) *Let M be any abelian group. Suppose*

$$G = \left(\bigoplus_{j \in J_1} \mathbb{Z} \right) \oplus \left(\bigoplus_{j \in J_2} \mathbb{Z}/n_j \mathbb{Z} \right)$$

for J_1, J_2 any index sets, and $n_j \geq 1$ integers. Fix a total order on the disjoint union $J := J_1 \dot{\cup} J_2$, say with $J_1 < J_2$. Write $(\mathbf{e}_j)_{j \in J}$ for the generator 1 in the j -th cyclic group. Let $q \in \text{Quad}(G, M)$ be a quadratic form and $b(x, y) := q(x + y) - q(x) - q(y)$ its polarization. Define

$$\sigma_{i,j} := \begin{cases} b(\mathbf{e}_i, \mathbf{e}_j) & \text{if } i < j \\ q(\mathbf{e}_i) & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases}$$

Then the pair (h, c) with

$$h(x, y, z) := \sum_{\substack{j \in J_2 \\ \text{with } y_j + z_j \geq n_j}} x_j n_j \sigma_{j,j} \quad \text{and} \quad c(x, y) := \sum_{\substack{i, j \in J \\ \text{with } i \leq j}} x_i y_j \sigma_{i,j}$$

defines an abelian 3-cocycle such that the trace map of Eq. 1.2 sends it to the given quadratic form q . Here x_j (resp. y_j, z_j) refers to coordinates with values $x_j \in \mathbb{Z}$ for $j \in J_1$ resp. $x_j \in \{0, 1, 2, \dots, n_j - 1\}$ for $j \in J_2$. The map $q \mapsto (h, c)$ is linear, so it provides a group homomorphism $\text{Quad}(G, M) \rightarrow Z_{ab}^3(G, M)$, which makes Diagram (1.3) commute.

This will be Theorem 7.1. If G is finite abelian (so $J_1 = \emptyset$ and $\#J_2 < \infty$) and $M := R^\times$, then up to rewriting the formula for h and c in the multiplicative notation customary for elements in the units $R^\times$, we recover precisely Quinn's formula given in [17, §2.5.2].

For some applications, especially when wanting to do explicit computations in the setting of fusion categories over $\mathbb{C}$, the following formulation might be more useful. We provide it with full details so that it can easily be referenced whenever needed:

Theorem C (Exponential format 3-cocycle formula) *Suppose*

$$G = \bigoplus_{k \in J} \mathbb{Z}/n_k \mathbb{Z} \tag{1.4}$$

for $n_k \geq 1$ and J some (possibly infinite) totally ordered index set. Write $(e_k)_{k \in J}$ for the generator 1 of the k -th summand. Then there is a bijection between the following three sets:

(1) All possible choices of values

- $p^{(k)} \in \{0, 1, \dots, \gcd(n_k^2, 2n_k) - 1\}$ for every $k \in J$,
- $q^{(k,l)} \in \{0, 1, \dots, \gcd(n_k, n_l) - 1\}$ for all $k < l$ with $k, l \in J$.

(2) All quadratic forms $q \in \text{Quad}(G, \mathbb{C}^\times)$, uniquely described by the following properties

$$\begin{aligned} q(e_k) &= \exp\left(\frac{2\pi i}{\gcd(n_k^2, 2n_k)} p^{(k)}\right), \\ b(e_k, e_l) &= \exp\left(\frac{2\pi i}{\gcd(n_k, n_l)} q^{(k,l)}\right) \quad (\text{for } k < l), \end{aligned}$$

where b is the polarization of q (and furthermore we necessarily then have $b(e_k, e_l) = b(e_l, e_k)$ for $k > l$ and $b(e_k, e_k) = 2q(e_k)$ as well).

(3) All abelian 3-cocycles $(h, c) \in H_{ab}^3(G, \mathbb{C}^\times)$, uniquely pinned down by the cocycle representative

$$\begin{aligned} c(x, y) &= \prod_{k < l} \exp\left(\frac{2\pi i q^{(k,l)}}{\gcd(n_k, n_l)} x_k y_l\right) \\ &\quad \cdot \prod_k \exp\left(\frac{2\pi i p^{(k)}}{\gcd(2n_k, n_k^2)} x_k y_k\right), \end{aligned}$$

and

$$h(x, y, z) = \prod_k \exp \left(\frac{2\pi i p^{(k)}}{\gcd(2n_k, n_k^2)} (x_k ([y_k]_{n_k} + [z_k]_{n_k} - [y_k + z_k]_{n_k})) \right),$$

where x_k (resp. y_k, z_k) denotes the coordinates of vectors $x, y, z \in G$ according to Eq. 1.4. Here $[-]_{n_k}$ refers to the remainder of division by n_k , expressed as an element in $\{0, 1, \dots, n_k - 1\}$.

Really, $\text{Quad}(G, \mathbb{C}^\times)$ and $H_{ab}^3(G, \mathbb{C}^\times)$ are abelian groups and the above bijections are abelian group isomorphisms, given in terms of the parameters $p^{(k)}, q^{(k,l)}$ by elementwise addition in the quotient groups (i.e. $\mathbb{Z}/(n_k^2, 2n_k)$ for $p^{(k)}$ etc.).

The map $q \mapsto (h, c)$ is linear, so it provides a group homomorphism $\text{Quad}(G, M) \rightarrow Z_{ab}^3(G, M)$, which makes Diagram (1.3) commute.

See Theorem 8.1. This result appears to readily imply various counting and enumeration problems in the literature regarding small examples of pointed braided fusion categories for a given G , see Section 10.

1.2 Application to Normal Forms of Associators

In the setting of (a), i.e. pointed braided fusion categories, Quinn's formula is sufficient to describe an associator and braiding in all situations. This is because in this setting $G := \pi_0(\mathcal{C}_{\text{simp}}, \otimes)$ is a finite abelian group and thus safely covered by both Theorem B and Theorem C.

However, Quinn's formula has some special shape (e.g., far more symmetry than one might a priori expect!). Let us broaden the question: Suppose that $(\mathcal{C}, \otimes)$ is a pointed braided fusion category, but drop the assumption that there are only finitely many isomorphism classes of simple objects. You could think of finite-dimensional G -graded vector spaces Vect_k^G with some associator and braiding, but where the grading comes from any abelian group G , and not just a finite one. We will properly define this later and call it a *big* fusion category.

Suppose we want to bring $(\mathcal{C}, \otimes)$ into some particularly nice “normal form” under braided monoidal equivalence. As before, replace $(\mathcal{C}, \otimes)$ by a skeleton. Then the associator and braiding are merely automorphisms. If X, Y, Z are simple objects, we may read $a_{X,Y,Z}$ and $s_{X,Y}$ as elements of $k^\times$ canonically. Now, the simplest conceivable normal form would be, through a braided monoidal equivalence, to make all associators and the braiding trivial. This is, however, an unrealistic hope (it cannot be achieved). Perhaps the following is the best possible normal form one can expect in general.²

Theorem D (Extra symmetries) *Suppose k is an algebraically closed field of any characteristic. Let $(\mathcal{C}, \otimes)$ be a k -linear pointed braided big fusion category. Then $(\mathcal{C}, \otimes)$ is braided monoidal equivalent to a skeletal big fusion category such that*

$$a_{X,Y,Z} = \frac{s_{X,Y} \cdot s_{X,Z}}{s_{X,Y \otimes Z}} \quad \text{and} \quad a_{Z,X,Y} = \frac{s_{X \otimes Y, Z}}{s_{X,Z} \cdot s_{Y,Z}}$$

hold for all simple objects X, Y, Z .

²by “in general” we mean: for any abelian group. Note that, for example, if we restrict G to free abelian groups, our generalized Quinn formula directly shows that one can always make the associator zero (because then $J_2 = \emptyset$).

See Theorem 9.13.

These properties are “extra symmetries” which are not visibly forced by the hexagon and pentagon axioms. I do not have a philosophical interpretation why such extra symmetries always exist (e.g., note that it follows from them that $a_{X,Y,Z} = a_{X,Z,Y}$).

To restate the result in other words: The associator only measures the lack of “ $\otimes$ -linearity” of the braiding, in either variable. A tool to memorize the formulas: the first argument of the associator is the one argument which appears in all three factors on the other side of the equality sign.

The above result follows from Quinn’s formula if $(C, \otimes)$ is an ordinary pointed braided fusion category with G finite. We believe the above observation is new in the case of arbitrary G . It does not follow by a “colimit argument” from the case of finite G by the same problem as discussed around (1.3). And at any rate our argument takes a different path and circumvents Quinn’s formula or its siblings.

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2 Abelian Cohomology

Let us recall Eilenberg and MacLane’s theory of abelian cohomology. We refrain from giving a careful motivation how this formalism arose. Instead, we may refer to [3, §3.1] for some more background.

Let G, M be abelian groups. While there are elegant and systematic definitions of group cohomology and abelian cohomology, we will just work with an explicit presentation here, namely normalized inhomogeneous cochains. Also, we shall only need H^3 .

Write $G^n := G \times \cdots \times G$ for the n -fold product of abelian groups. A *group 3-cocycle* is a map of sets

$$h : G^3 \longrightarrow M$$

such that the identity

$$h(x, y, z) + h(u, x + y, z) + h(u, x, y) = h(u, x, y + z) + h(u + x, y, z). \quad (2.1)$$

holds for all $x, y, z \in G$ (corresponding in tensor category language to the “pentagon axiom for associators”, see Section 9, e.g., the proof of Theorem 9.2).

A group 3-cocycle is called *normalized* if $h(x_1, x_2, x_3) = 0$ as soon as $x_i = 0$ for some $i \in \{1, 2, 3\}$. A *normalized group 3-coboundary* is a group 3-cocycle of the shape

$$h(x, y, z) = k(y, z) - k(x + y, z) + k(x, y + z) - k(x, y) \quad (2.2)$$

for some map of sets $k : G^2 \rightarrow M$ such that $k(x, 0) = 0$ and $k(0, y) = 0$. It is easy to check that this is a normalized group 3-cocycle. These explicit expressions can directly be unravelled from [16, Chapter I, §2] for example.

An *abelian 3-cocycle* is a pair (h, c) consisting of a group 3-cocycle $h : G^3 \rightarrow M$ such that

$$h(x, 0, z) = 0 \quad (2.3)$$

and a map $c : G^2 \rightarrow M$ which satisfies

$$h(y, z, x) + c(x, y + z) + h(x, y, z) = c(x, z) + h(y, x, z) + c(x, y) \quad (\text{A})$$

$$-h(z, x, y) + c(x + y, z) - h(x, y, z) = c(x, z) - h(x, z, y) + c(y, z) \quad (\text{A}')$$

for all $x, y, z \in G$ (corresponding to the two “hexagon axioms” in the dictionary with tensor categories). Equation 2.3 implies that h is normalized [3, Remark 3.5]. An *abelian*

3-coboundary is a pair (h, c) , where h is a normalized group 3-coboundary coming from $k : G^2 \rightarrow M$, and

$$c(x, y) := k(x, y) - k(y, x) \quad (2.4)$$

for the same k . Write Z_{grp}^3 (resp. Z_{ab}^3) to denote the group of normalized group 3-cocycles (resp. abelian 3-cocycles), resp. B_{grp}^3 and B_{ab}^3 for coboundaries.

Definition 2.1 We have third *group cohomology*

$$H_{grp}^3(G, M) = \frac{Z_{grp}^3(G, M)}{B_{grp}^3(G, M)} = \frac{\{(\text{normalized}) \text{ group 3-cocycles}\}}{\{(\text{normalized}) \text{ group 3-coboundaries}\}}$$

and third *abelian cohomology*

$$H_{ab}^3(G, M) = \frac{Z_{ab}^3(G, M)}{B_{ab}^3(G, M)} = \frac{\{\text{abelian 3-cocycles}\}}{\{\text{abelian 3-coboundaries}\}}$$

For both definitions we use normalized inhomogeneous chains, cf. [16, Chapter I, §2, Exercise 5].

By a *quadratic form* $q : G \rightarrow M$ (also known as ‘quadratic map’ or ‘quadratic function’ in various texts, depending on the taste of the various authors) we mean a map of sets such that $q(x) = q(-x)$ and

$$b(x, y) = q(x + y) - q(x) - q(y) \quad (2.5)$$

is $\mathbb{Z}$ -bilinear for all $x, y \in G$. The map b is known as the *polarization form*. Write $\text{Quad}(G, M)$ for the set of all quadratic forms. This is an abelian group under pointwise addition of maps.

Example 2.2 This definition may encompass more types of maps than a casual reader might expect. For example if G and M happen to be $\mathbb{F}_2$ -vector spaces, every *linear* map $G \rightarrow M$ is a quadratic form. Concretely, the map

$$q : \mathbb{F}_8[X, Y] \longrightarrow \mathbb{F}_8[X, Y], \quad q(x) = x + x^2 + x^4$$

might not ‘look quadratic’ as an algebraic expression, but it is a quadratic form. Thanks to $(a + b)^2 = a^2 + b^2$ in characteristic two rings, the polarization vanishes.

Example 2.3 If q is any quadratic form, we have $q(nx) = n^2q(x)$ for any $x \in G$ and $n \in \mathbb{Z}$. To see this, note that

$$\begin{aligned} b(x, x) &= q(2x) - 2q(x) \\ b(x, -x) &= q(0) - q(x) - q(-x) = -2q(x) \end{aligned} \quad (2.6)$$

both follow from Eq. 2.5. The case $n = 0$ is clear. By induction, assuming the case n to be done,

$$b(nx, x) = q(nx + x) - q(nx) - q(x) = q((n + 1)x) - n^2q(x) - q(x)$$

and by the $\mathbb{Z}$ -bilinearity of b and Eq. 2.6, $b(nx, x) = -nb(x, -x) = 2nq(x)$, and then

$$(n^2 + 2n + 1)q(x) = q((n + 1)x),$$

proving the claim for all $n \geq 0$. It follows for negative n by $q(-x) = q(x)$.

The key connection between abelian 3-cocycles and quadratic forms is the following theorem.

Theorem 2.4 (Eilenberg–Mac Lane) *Let G, M be abelian groups. The so-called trace*

$$\begin{aligned} \mathrm{tr} : H_{ab}^3(G, M) &\longrightarrow \mathrm{Quad}(G, M) \\ (h, c) &\longmapsto (x \mapsto c(x, x)) \end{aligned}$$

is an isomorphism of abelian groups.

We give an outline how this is proven in Section 11, including a new proof of surjectivity.

3 Admissible Presentations

Let G, M be abelian groups. Suppose $q \in \mathrm{Quad}(G, M)$ is a quadratic form. We write b for the polarization form of q as given in Eq. 2.5.

Definition 3.1 *A pre-admissible presentation for q is a triple (F_0, π, C) , where*

- (1) F_0 is an abelian group and π a surjective group homomorphism

$$\pi : F_0 \twoheadrightarrow G;$$

and write $F_1 := \ker(\pi)$;

- (2) C is a $\mathbb{Z}$ -bilinear form $C : F_0 \otimes_{\mathbb{Z}} F_0 \rightarrow M$ such that

$$b(\pi x, \pi y) = C(x, y) + C(y, x) \quad (3.1)$$

holds for all $x, y \in F_0$.

- (3) For all $x \in F_1$ we have $C(x, x) = 0$.

We speak of an *admissible presentation* when instead of (3) we have the stronger property that $C(x, y) = 0$ holds for all $x, y \in F_1$.

Axiom (3) just demands that the restriction $C|_{F_1}$ is an alternating form. For an admissible presentation, F_1 is an isotropic subgroup.

Given a pre-admissible presentation, we can lift the quadratic form from G to F_0 . To this end, we define

$$Q(x) := q(\pi(x)) \quad \text{for } x \in F_0. \quad (3.2)$$

Then $Q \in \mathrm{Quad}(F_0, M)$ is indeed a quadratic form. Here and henceforth write

$$B(x, y) := Q(x + y) - Q(x) - Q(y) \quad (3.3)$$

for its polarization form. Note that

$$B(x, y) = b(\pi x, \pi y). \quad (3.4)$$

We have chosen our notation so that the uppercase letters refer to the lifts of their lowercase letter counterpart.

There is a slightly more refined property one can demand (and always arrange) to hold:

Definition 3.2 We call a (pre-)admissible presentation *optimal* if we have

$$Q(x) = C(x, x) \quad (3.5)$$

for all $x \in F_0$.

Example 3.3 The simplest example of a non-optimal admissible presentation is for the quadratic form $q \in \mathrm{Quad}(\mathbb{F}_2, \mathbb{F}_2)$ given by $q(x) = x^2$. For this form $(\mathbb{F}_2, \mathrm{id}_{\mathbb{F}_2}, C)$ with

$C(x, y) := 0$ is a non-optimal admissible presentation with $F_1 = 0$. An optimal presentation is given by $C(x, y) := xy$.

Starting with any pre-admissible presentation (F_0, π, C) , in order to achieve optimality, one only needs to change the bilinear form C , while F_0 and π can remain the same. We prove this now.

Proposition 3.4 *Let G, M be abelian groups and $q \in \text{Quad}(G, M)$ a quadratic form.*

- (1) *If (F_0, π, C) is a pre-admissible presentation, one can find an optimal pre-admissible presentation (F_0, π, C') .*
- (3) *If (F_0, π, C) is an admissible presentation, one can find an optimal admissible presentation (F_0, π, C') .*

Below, we write ${}_nM := \{m \in M \mid nm = 0\}$ for the n -torsion subgroup.

Proof (Step 1) We first prove the first claim. It is immediate to see that $Q'(x) := C(x, x)$ is a quadratic form on F_0 . Its polarization form is

$$B'(x, y) = C(x + y, x + y) - C(x, x) - C(y, y) = C(x, y) + C(y, x)$$

and by Eq. 3.1 this is $b(\pi x, \pi y)$, so Q' has the same polarization as Q by Eqs. 3.3–3.4. Thus,

$$L := Q' - Q \tag{3.6}$$

is a quadratic form in $\text{Quad}(F_0, M)$ whose polarization vanishes. This means that L satisfies

$$L(x + y) - L(x) - L(y) = 0 \tag{3.7}$$

for all $x, y \in F_0$, so L is a semigroup homomorphism. Take $x = y = 0$ to obtain $L(0) = 0$, and $y = -x$ to obtain $L(-x) = -L(x)$, showing that L is a morphism of abelian groups,

$$L : F_0 \longrightarrow M.$$

As a quadratic form, it also satisfies $L(x) = L(-x)$, i.e. $2L(x) = 0$ holds for all $x \in F_0$. Thus, L descends to a group homomorphism $L : F_0/2F_0 \rightarrow {}_2M$. The $\mathbb{Z}$ -module structure of either side now induces an $\mathbb{F}_2$ -vector space structure, rendering L an $\mathbb{F}_2$ -linear map. We next choose a special basis of $F_0/2F_0$. To this end, pick a direct sum splitting

$$F_0/2F_0 \simeq \text{im}(F_1) \oplus (\text{rest}), \tag{3.8}$$

where we refer to the image coming from the inclusion $F_1 \subseteq F_0$. Pick $(\gamma_i)_{i \in I}$ as a basis of $F_0/2F_0$ first by picking a basis on the subspace $\text{im}(F_1)$, say with indices in a subset $I_{\text{im } F_1} \subseteq I$ of the index set, and then prolong it to all of $F_0/2F_0$. Define a symmetric bilinear form on $F_0/2F_0$ by

$$J(x, y) := \sum_{i \in I} \bar{x}_i \bar{y}_i L(\gamma_i), \tag{3.9}$$

where $\bar{x}_i \in \mathbb{F}_2$ denotes the $\mathbb{F}_2$ -coordinates of the vector $x \in F_0/2F_0$ (resp. $\bar{y}_i$ for y) with respect to the basis $(\gamma_i)_{i \in I}$. As each $L(x)$ lies in the 2-torsion group ${}_2M$, the scalar multiplication with elements from $\mathbb{F}_2$ is well-defined and indeed linear. For an arbitrary $x \in F_0/2F_0$ we compute

$$J(x, x) = \sum_i \bar{x}_i^2 L(\gamma_i) \equiv \sum_i \bar{x}_i L(\gamma_i) = L\left(\sum_i \bar{x}_i \gamma_i\right) = L(x)$$

since in $\mathbb{F}_2$ we have $\alpha^2 \equiv \alpha \pmod{2}$. Under the linear surjection $F_0 \twoheadrightarrow F_0/2F_0$ we can now lift J to a symmetric bilinear form

$$J : F_0 \otimes_{\mathbb{Z}} F_0 \longrightarrow {}_2M.$$

We keep the same name J for this lift. Returning to our definition of L in Eq. 3.6, we now find

$$Q(x) = Q'(x) - L(x) = C(x, x) - J(x, x).$$

Since both C and J are $\mathbb{Z}$ -bilinear forms, so is $C - J$. Define

$$C' := C - J. \quad (3.10)$$

We thus have $Q(x) = C'(x, x)$, so Eq. 3.5 holds, showing that C' is a promising candidate to satisfy the optimality property.

(Step 2) We need to check that (F_0, π, C') is a pre-admissible presentation: Axiom (1) is clear; nothing about F_1 or π has changed. For axiom (2) we find

$$C'(x, y) + C'(y, x) = C(x, y) - J(x, y) + C(y, x) - J(y, x) = B(x, y) - 2J(x, y)$$

since C satisfies axiom (2) by assumption and J is a symmetric form. However, J by construction takes values in the 2-torsion elements ${}_2M$, so $2J(x, y) = 0$ for any x, y . Hence, axiom (2) is satisfied. Axiom (3): From Eqs. 3.6 and 3.2 we get

$$L(x) = Q'(x) - Q(x) = C(x, x) - q(\pi x).$$

Thus, if $x \in F_1$ then $C(x, x) = 0$ since C satisfies axiom (3), and $q(\pi x) = 0$ since $F_1 = \ker(\pi_1)$. We deduce that $L|_{F_1} = 0$. Next, recall from Eq. 3.9 that J was defined by

$$J(x, y) = \sum_{i \in I} \bar{x}_i \bar{y}_i L(\gamma_i),$$

where $\bar{x}_i, \bar{y}_i$ were the respective $\mathbb{F}_2$ -coordinates. If $x, y \in F_1$, then thanks to our special choice of basis from Eq. 3.8, we have $\bar{x}_i = 0$ for all $i \in I \setminus I_{\text{im } F_1}$, and the same for $\bar{y}_i$. Thus, for $x, y \in F_1$ we have

$$J(x, y) = \sum_{i \in I_{\text{im } F_1}} \bar{x}_i \bar{y}_i L(\gamma_i) = 0$$

since for $i \in I_{\text{im } F_1}$ we know that γ_i lies in the image of F_1 inside $F_0/2F_0$, but we had found above that $L|_{F_1} = 0$. Thus, $J(x, y) = 0$ for all $x, y \in F_1$. Now suppose $x \in F_1$. Then by Eq. 3.10 we have

$$C'(x, x) = C(x, x) - J(x, x) \quad (3.11)$$

and this vanishes since $C(x, x) = 0$ (as axiom (3) holds for C) and we had already observed $J|_{F_1} = 0$. This shows that axiom (3) holds for C' . This shows that (F_0, π, C') is an optimal pre-admissible presentation (we had shown optimality in Step 1). This finishes the proof of the first claim. Finally, we want to show that if (F_0, π, C) was an admissible presentation to start with, so is (F_0, π, C') . We already know the latter is optimal and pre-admissible. Now, as a direct variation of Eq. 3.11, for $x, y \in F_1$ we get

$$C'(x, y) = C(x, y) - J(x, y)$$

and since C was admissible, $C(x, y) = 0$, and we had already observed $J(x, y) = 0$ above. $\square$

4 Existence Theorems for Admissible Presentations

We begin with the principal construction mechanism for pre-admissible presentations. This is a good construction whenever the quadratic form comes from a bilinear form, even if this is perhaps not possible on G , but only on a bigger group.

Lemma/Construction 4.1 *Suppose G and M are arbitrary abelian groups.*

- (1) *Assume one finds an abelian group F_0 with a surjection*

$$\pi : F_0 \twoheadrightarrow G$$

such that on F_0 one can exhibit a bilinear form C such that $q(\pi x) = C(x, x)$ (“the lift of q comes from a bilinear form”). Then (F_0, π, C) is an optimal pre-admissible presentation.

- (2) *If one can take $F_0 = G$ and $\pi = \text{id}_G$, then (G, id_G, C) is an optimal admissible presentation and $F_1 = 0$.*

Proof We begin with the first claim. Axiom (1) is immediate. The polarization form b of q , written in terms of images of elements x, y from F_0 , is

$$\begin{aligned} b(\pi x, \pi y) &= q(\pi(x + y)) - q(\pi x) - q(\pi y) \\ &= C(x + y, x + y) - C(x, x) - C(y, y) = C(x, y) + C(y, x), \end{aligned}$$

proving axiom (2). Finally, if $x \in F_1$, then since $F_1 = \ker(\pi)$ we have $C(x, x) = q(\pi x) = 0$, so axiom (3) holds. Optimality holds by construction. For the second claim, note that $F_1 = 0$, so the strong form of axiom (3) is clear. $\square$

We should also mention a trivial case, where nothing much needs to be done at all.

Lemma/Construction 4.2 *Suppose G is arbitrary and M is a $\mathbb{Z}[\frac{1}{2}]$ -module. Then pick $\pi := \text{id}_G$, i.e. use the presentation*

$$0 \longrightarrow G \xrightarrow{\pi} G \longrightarrow 0$$

with

$$C(x, y) := \frac{1}{2}b(\pi x, \pi y).$$

This is an optimal admissible presentation.

Proof Immediate. Since b is symmetric, so is C , take $F_0 := G$ and then $F_1 = 0$. In particular, there is nothing to check for axiom (3). For $x \in G$ the polarization form yields

$$b(x, x) = q(2x) - q(x) - q(x) = 4q(x) - 2q(x) = 2q(x),$$

i.e. $q(x) = \frac{1}{2}b(x, x)$. Thus, for $x \in F_0$ we have $C(x, x) = q(\pi x)$, so Eq. 3.5 holds. $\square$

The hypothesis of M being a $\mathbb{Z}[\frac{1}{2}]$ -module is virtually never satisfied in applications though.

Example 4.3 Suppose $(\mathcal{C}, \otimes)$ is any k -linear braided fusion category. Then the group which appears for M in applications (see Section 9) is $M := \pi_1(\mathcal{C}, \otimes) = k^\times$ since the tensor unit $1_{\mathcal{C}}$ is a simple object. For this group to be 2-divisible one needs that k is closed under all quadratic field extensions. This is for example satisfied if k is algebraically closed. However,

to be free from 2-torsion one also needs that $x^2 = 1$ implies $x = 1$, i.e. $+1 = -1$. This forces k to be of characteristic two.

Instead, a much more realistic hypothesis is that M is a divisible module. If k is any algebraically closed field, $k^\times$ is a divisible group.

Lemma 4.4 *Suppose G is an arbitrary abelian group and M a divisible abelian group. If (F_0, π, C) is a pre-admissible presentation with F_0 free abelian, then one can replace C by a bilinear form $\tilde{C}$ such that $(F_0, \pi, \tilde{C})$ is an admissible presentation. If the presentation was optimal to start with, the new $\tilde{C}$ can be taken optimal, too.*

Proof Consider the restriction $C|_{F_1}$. We have $C(x, x) = 0$ for all $x \in F_1$ by axiom (3). Thus, $C|_{F_1}$ is an alternating bilinear form³,

$$C|_{F_1} \in \text{Hom}_{\mathbb{Z}}(\text{Alt}^2(F_1), M).$$

Since F_0 is free, so is F_1 ($\mathbb{Z}$ is a hereditary ring). Then the injectivity $F_1 \hookrightarrow F_0$ implies that the top horizontal arrow in the diagram

$$\begin{array}{ccc} \text{Alt}^2(F_1) & \xrightarrow{\quad} & \text{Alt}^2(F_0) \\ & \searrow C|_{F_1} & \downarrow A \\ & & M \end{array} \quad (4.1)$$

is also injective (without too much harm it can be checked directly that exterior powers of free modules preserve injectivity, but a literature reference would be [7, Theorem 1]). Since M is divisible, it is injective as a $\mathbb{Z}$ -module, so the dashed arrow A above exists and makes the diagram commute. Define

$$\tilde{C}(x, y) := C(x, y) - A(x, y).$$

We claim that $(F_0, \pi, \tilde{C})$ is an admissible presentation. We compute for $x, y \in F_0$ that

$$\tilde{C}(x, y) + \tilde{C}(y, x) = C(x, y) + C(y, x) - A(x, y) - A(y, x) = B(x, y) + 0$$

since A is alternating, so Eq. 3.1 is still valid. This settles axioms (1) and (2). Moreover, for $x, y \in F_1$ we have

$$\tilde{C}(x, y) = C(x, y) - A(x, y) = 0$$

since $C|_{F_1} = A|_{F_1}$ by Diagram (4.1), so axiom (3) holds and we really have an admissible presentation. Regarding optimality, note that $A(x, x) = 0$, so $\tilde{C}(x, x) = C(x, x)$. $\square$

The next construction will be the concrete input needed for establishing a generalized form of Quinn's formula.

Lemma/Construction 4.5 *Suppose M is arbitrary. If G is a (possibly infinite) direct sum of various (possibly infinite) cyclic groups, an optimal admissible presentation for q exists. (A concrete construction is given in the proof)*

³just to be sure about nomenclature: *Alternating* means that $C(x, x) = 0$ for all x . This implies $C(x, y) = -C(y, x)$, but is strictly more restrictive than the latter property.

Proof and Construction (Step 1) A cyclic group is either of the form $\mathbb{Z}$ or $\mathbb{Z}/n$ for some $n \geq 1$. Thus, each direct summand in G has a presentation of the shape

$$0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow \mathbb{Z}/n \rightarrow 0 \quad \text{or} \quad 0 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0.$$

Take the direct sum of these, i.e. we have found a presentation

$$0 \longrightarrow \bigoplus_{i \in I} \mathbb{Z} \longrightarrow \bigoplus_{j \in J} \mathbb{Z} \xrightarrow{\pi} G \longrightarrow 0 \quad (4.2)$$

for suitable index sets $I \subseteq J$, where the first arrow is given by a diagonal matrix. We take this as $F_1 \hookrightarrow F_0 \rightarrow G$. This sets up (1) in an admissible presentation. Use the same notation Q and B as in Eqs. 3.2 and 3.3 now.

(Step 2) Write $(e_j)_{j \in J}$ for the basis vectors of F_0 . Fix a total order on J . Define

$$C(e_i, e_j) := \begin{cases} B(e_i, e_j) & \text{if } i < j \\ Q(e_i) & \text{if } i = j \\ 0 & \text{if } i > j \end{cases} \quad (4.3)$$

on this basis. Prolong it uniquely to all of F_0 by $\mathbb{Z}$ -bilinearity. For any $i, j \in J$ with $i \neq j$ we find

$$C(e_i, e_j) + C(e_j, e_i) = B(e_i, e_j)$$

since b is symmetric. If $i = j$ then $C(e_i, e_j) + C(e_j, e_i) = 2Q(e_i)$, while we also have

$$B(e_i, e_i) = Q(2e_i) - 2Q(e_i) = 2Q(e_i)$$

by using Eq. 3.3 (and Example 2.3). For the polarization B' of the quadratic form $Q'(x) := C(x, x)$ we compute

$$B'(x, y) = C(x + y, x + y) - C(x, x) - C(y, y) = C(x, y) + C(y, x)$$

and thus

$$B'(e_i, e_j) = \begin{cases} B(e_i, e_j) & \text{if } i \neq j \\ 2Q(e_i) & \text{if } i = j. \end{cases}$$

We find that the polarization forms of Q and Q' agree. Thus, $L := Q - Q'$ is a quadratic form whose polarization is zero. With the same computation as in Eq. 3.7 it follows that $L : F_0/2F_0 \rightarrow {}_2M$ is a group homomorphism. We compute

$$L(e_i) = Q(e_i) - Q'(e_i) = Q(e_i) - C(e_i, e_i) = 0,$$

just by unravelling the definition of Q' and using Eq. 4.3. Since the $(e_j)_{j \in J}$ form a basis of F_0 and L is linear, it follows that $L(x) = 0$ for all $x \in F_0$. Thus, $Q(x) = Q'(x) = C(x, x)$ for all $x \in F_0$. This shows that Q comes from a bilinear form, so we may invoke Lemma 4.1 and learn that (F_0, π, C) is an optimal pre-admissible presentation.

(Step 3) Finally, we need to show the strong form of axiom (3). Note that by our special construction of the presentation in Eq. 4.2, the group $F_1 = \ker(\pi)$ is generated by elements of the shape $(n_i e_i)_{i \in I}$ with $n_i \in \mathbb{Z}_{\geq 1}$, i.e. $\pi(n_i e_i) = 0$. We compute for arbitrary $i, j \in I$ (again using the fact from Example 2.3),

$$\begin{aligned} C(n_i e_i, n_j e_j) &= n_i n_j C(e_i, e_j) \\ &= \begin{cases} n_i n_j B(e_i, e_j) = B(n_i e_i, n_j e_j) & \text{if } i < j \\ n_i^2 Q(e_i) = Q(n_i e_i) & \text{if } i = j \\ 0 & \text{if } i > j, \end{cases} \end{aligned}$$

but of course by the very definition of Q and B , these terms all vanish since we have $\pi(n_i e_i) = 0$. This proves that (F_0, π, C) is admissible. $\square$

Finally, let us show that optimal (pre-)admissible presentations always exist under very general hypotheses. However, the constructions in the proof cannot be carried out constructively usually, so the following theorem will not help when wanting to develop explicit formulas.

Theorem 4.6 (Abstract Existence) *Suppose G, M are abelian groups.*

- (1) *Then for any quadratic form $q : G \rightarrow M$ an optimal pre-admissible presentation exists.*
- (2) *If M is divisible, then for any quadratic form $q : G \rightarrow M$ an optimal admissible presentation exists.*

Proof

- (Step 1) We imitate the method of Lemma 4.5 for as long as possible. First, pick a free resolution of G ,

$$0 \longrightarrow \bigoplus_{i \in I} \mathbb{Z} \longrightarrow \bigoplus_{j \in J} \mathbb{Z} \xrightarrow{\pi} G \longrightarrow 0$$

for suitable index sets I, J . This always exists (and has length 2 either because $\mathbb{Z}$ is a ring of global dimension one, or, more down to earth, since the kernel of π has to be free abelian itself). As before, write $(e_j)_{j \in J}$ for the basis vectors of F_0 , fix a total order on J , and let Q and B denote the lifts of q and b to F_0 (as in Eqs. 3.2-3.3). This replaces Step 1 in the proof of Lemma 4.5.

- (Step 2) Define

$$C(e_i, e_j) := \begin{cases} B(e_i, e_j) & \text{if } i < j \\ Q(e_i) & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases} \quad (4.4)$$

Now repeat the same arguments as in Step 2 of the proof of Lemma 4.5. This all goes through and proves that (F_0, π, C) is an optimal pre-admissible presentation. This proves the first claim. Step 3 in the cited proof does not adapt to the present setting. We do something else:

- (Step 3) If M is divisible, we can invoke Lemma 4.4 and transform the construction from Step 2 into an admissible optimal presentation $(F_0, \pi, \tilde{C})$. This settles the second claim. $\square$

Problem 1 Does any quadratic form $q : G \rightarrow M$ admit an optimal admissible presentation without assuming M divisible?

Thanks to Proposition 3.4 one would only need to exhibit an admissible presentation; the optimality can be achieved afterwards.

Example 4.7 We illustrate that Step 1 in the above proof cannot be expected to give admissible presentations right away. Consider the needlessly complicated free resolution

$$\mathbb{Z}^{n-1} \longrightarrow \mathbb{Z}^n \xrightarrow{\pi} \mathbb{Z},$$

where $\pi(x_1, \dots, x_n) = \sum_{i=1}^n x_i$. For the quadratic form $x \mapsto x^2$ on $\mathbb{Z}$, the procedure in the proof of Theorem 4.6 yields that $(\mathbb{Z}^n, \pi, C)$ with

$$C(e_i, e_j) = \begin{cases} 2 & \text{if } i < j \\ 1 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

is an optimal pre-admissible presentation. All the vectors $e_i - e_k$ lie in the kernel $F_1 = \ker(\pi)$. For $i < j < k$ we compute $C(e_i - e_k, e_j - e_k) = 1$.

5 The Lifting Function

Suppose $q \in \text{Quad}(G, M)$ is a quadratic form and assume we have chosen an admissible presentation (F_0, π, C) as in Section 3.

Definition 5.1 For any non-zero element $x \in G$ pick once and for all a lift $\tilde{x} \in F_0$, i.e. some element such that $\pi(\tilde{x}) = x$. For the neutral element we pick the special lift

$$\tilde{0} := 0. \quad (5.1)$$

Call any such choice an *admissible lift*.

As π is surjective, it is clear that admissible lifts always exist.

Example 5.2 We stress that we have $\pi\tilde{x} = x$, but in general there is not much we can say about how $\widetilde{(-)}$ interacts with algebraic operations. For example, $\widetilde{2x} \neq 2\tilde{x}$, $\widetilde{(-x)} \neq -(\tilde{x})$ or $\widetilde{x+y} \neq \tilde{x} + \tilde{y}$ are all possible in suitably chosen examples, and in general π will not admit a splitting in terms of abelian groups, so in general we cannot avoid for these lifts to depend somewhat non-linearly on the input.

Having fixed an admissible lift, define a map

$$L : G \times G \longrightarrow M$$

$$L(x, y) := \widetilde{(x + y)} - \tilde{x} - \tilde{y}. \quad (5.2)$$

Note that there is *no reason* why L would have to be bilinear in any way. We can record a few useful facts about L nonetheless:

Lemma 5.3 Fix admissible lifts and suppose (F_0, π, C) is an admissible presentation. We have

1. $L(0, y) = 0$,
2. $L(x, y) = L(y, x)$,
3. $L(x + y, z) - L(x, y + z) = L(y, z) - L(x, y)$,
4. $C(L(u, x), L(y, z)) = 0$

for all $u, x, y, z \in G$.

For merely pre-admissible presentations property (4) can fail.

Proof (1) We find

$$L(0, y) = \tilde{y} - \tilde{0} - \tilde{y} = 0$$

using our special choice of lift in Eq. 5.1. (2) Obvious. (3) We find

$$\begin{aligned} L(x + y, z) - L(x, y + z) &= \widetilde{(x + y + z)} - \widetilde{(x + y)} - \tilde{z} - (\widetilde{(x + y + z)} + \tilde{x} + \widetilde{(y + z)}) \\ &= \tilde{x} - \widetilde{(x + y)} + \widetilde{(y + z)} - \tilde{z} = L(y, z) - L(x, y), \end{aligned}$$

where we have just used cancellations of terms. (4) Note that for all $x, y \in G$ we have $\pi(\widetilde{(x+y)} - \widetilde{x} - \widetilde{y}) = 0$, so $\widetilde{(x+y)} - \widetilde{x} - \widetilde{y} \in F_1$. This proves (4) since this applies to both arguments, so we can use the strong form of axiom (3) of an admissible presentation. $\square$

6 Constructing Abelian 3-Cocycles

Let $q \in \text{Quad}(G, M)$ be a quadratic form. Suppose we have fixed an admissible presentation (F_0, π, C) as in Definition 3.1, alongside a choice of admissible lifts as in Definition 5.1. We use the notation F_0, F_1, π, C, B, Q as explained in Section 3.

Define maps

$$h : G \times G \times G \longrightarrow M \quad \text{and} \quad c : G \times G \longrightarrow M$$

by

$$h(x, y, z) := -C(\widetilde{x}, L(y, z)) \quad \text{and} \quad c(x, y) := C(\widetilde{x}, \widetilde{y}), \quad (6.1)$$

where C is the bilinear form of the admissible presentation, $\widetilde{(-)}$ denotes the admissible lift and L is the (non-linear!) pairing of Eq. 5.2.

Again, note that there is no reason why h or c would be multilinear.

Lemma 6.1 (Key Lemma) *The datum (h, c) of Eq. 6.1 describes an abelian 3-cocycle.*

We stress that we only need the above assumptions, i.e. the admissible presentation (F_0, π, C) does not need to be optimal.

Proof (Step 0) We have $h(x, 0, z) = -C(\widetilde{x}, L(0, z)) = 0$ by Lemma 5.3 and since C is $\mathbb{Z}$ -bilinear. *(Step 1)* We first check Eq. A. We unravel

$$\begin{aligned} h(y, z, x) + h(x, y, z) - h(y, x, z) &= -C(\widetilde{y}, L(z, x)) - C(\widetilde{x}, L(y, z)) + C(\widetilde{y}, L(x, z)) \\ &= -C(\widetilde{x}, L(y, z)) \end{aligned}$$

as the first and third term cancel each other out since L is symmetric (Lemma 5.3). Now unpack the definition of L and use that C is $\mathbb{Z}$ -bilinear on F_0 , giving

$$= -C(\widetilde{x}, \widetilde{y+z}) + C(\widetilde{x}, \widetilde{y}) + C(\widetilde{x}, \widetilde{z}) = -c(x, y+z) + c(x, y) + c(x, z),$$

which confirms Eq. A. *(Step 2)* Next, we check Eq. A', which is a little asymmetric in comparison to the previous computation: We unravel

$$\begin{aligned} h(x, z, y) - h(z, x, y) - h(x, y, z) &= -C(\widetilde{x}, L(z, y)) + C(\widetilde{z}, L(x, y)) + C(\widetilde{x}, L(y, z)) \\ &= C(\widetilde{z}, L(x, y)), \end{aligned}$$

again using that L is symmetric. Again, unpack L and use the bilinearity of C , giving

$$= C(\widetilde{z}, \widetilde{x+y}) - C(\widetilde{z}, \widetilde{x}) - C(\widetilde{z}, \widetilde{y}).$$

Next, by Eq. 3.1 we have $C(y, x) = B(x, y) - C(x, y)$ for all $x, y \in F_0$. Thus, rewrite the preceding equation as

$$\begin{aligned} &= B(\widetilde{x+y}, \widetilde{z}) - B(\widetilde{x}, \widetilde{z}) - B(\widetilde{y}, \widetilde{z}) \\ &\quad - C(\widetilde{x+y}, \widetilde{z}) + C(\widetilde{x}, \widetilde{z}) + C(\widetilde{y}, \widetilde{z}). \end{aligned} \quad (6.2)$$

However, we also have (3.4), namely $B(x, y) = b(\pi x, \pi y)$, so the first line simplifies to

$$b(\pi(\widetilde{x+y}), \pi(\widetilde{z})) - b(\pi\widetilde{x}, \pi\widetilde{z}) - b(\pi\widetilde{y}, \pi\widetilde{z})$$

but $\pi\tilde{x} = x$ for all $x \in G$, so this equals $b(x + y, z) - b(x, z) - b(y, z)$. Since b is the polarization form of q , Eq. 2.5, b is $\mathbb{Z}$ -bilinear, so this expression vanishes for all $x, y, z \in G$. Thus, Eq. 6.2 simplifies to

$$= -C(\widetilde{x + y}, \tilde{z}) + C(\tilde{x}, \tilde{z}) + C(\tilde{y}, \tilde{z}) = -c(x + y, z) + c(x, z) + c(y, z),$$

which confirms (A'). (Step 3) Finally, we need to check whether h is a group 3-cocycle, i.e. confirm whether

$$h(x, y, z) + h(u, x + y, z) + h(u, x, y) - h(u, x, y + z) - h(u + x, y, z) = 0 \quad (6.3)$$

holds for all $x, y, z, u \in G$. We first evaluate

$$h(u, x + y, z) = -C(\tilde{u}, L(x + y, z))$$

and relying on $L(x + y, z) = L(x, y + z) + L(y, z) - L(x, y)$ (an equality stemming from Lemma 5, (3), rearranged), the preceding equation can be rewritten as

$$\begin{aligned} h(u, x + y, z) &= -C(\tilde{u}, L(x, y + z) + L(y, z) - L(x, y)) \\ &= -C(\tilde{u}, L(x, y + z)) - C(\tilde{u}, L(y, z)) + C(\tilde{u}, L(x, y)) \\ &= h(u, x, y + z) + h(u, y, z) - h(u, x, y), \end{aligned}$$

where we have used that C is a $\mathbb{Z}$ -bilinear form on F_0 . Plug this into the left-hand side of Eq. 6.3, showing that it suffices to prove

$$h(u, y, z) + h(x, y, z) - h(u + x, y, z) = 0.$$

However, unravelling the definition of h , this simplifies to

$$= -C(\tilde{u} + \tilde{x} - \widetilde{(u + x)}, L(y, z)) = C(L(u, x), L(y, z))$$

since C is $\mathbb{Z}$ -bilinear. This proves the desired vanishing by using the last property shown in Lemma 5.3. $\square$

Finally, we are ready for our main result.

Theorem 6.2 *Let G, M be abelian groups and $q \in \text{Quad}(G, M)$. Suppose*

- (F_0, π, C) is an optimal admissible presentation, and
- $(-)$ is an admissible lifting.

Then

$$h(x, y, z) := -C(\tilde{x}, L(y, z)), \quad c(x, y) := C(\tilde{x}, \tilde{y})$$

with the non-linear function $L(x, y) := \widetilde{(x + y)} - \tilde{x} - \tilde{y}$, defines an abelian 3-cocycle whose attached quadratic form is q , i.e.

$$H_{ab}^3(G, M) \longrightarrow \text{Quad}(G, M), \quad (h, c) \longmapsto q$$

under the map of Theorem 2.4.

Recall that if one only is given a non-optimal admissible presentation, one can always change it into an optimal one by Proposition 3.4.

Proof This is easy now. Firstly, by Lemma 6.1 (h, c) is an abelian 3-cocycle. The trace maps it to the quadratic form

$$q'(x) \underset{(1)}{=} c(x, x) \underset{(2)}{=} C(\tilde{x}, \tilde{x}) \underset{(3)}{=} Q(\tilde{x}) = q(\pi\tilde{x}) = q(x)$$

for $x \in G$. Here (1) is just the definition of the trace map from abelian 3-cocycles to quadratic forms, (2) is the definition of $c(-, -)$, (3) is the optimality of the admissible presentation (3.5), and the rest unravels definitions. $\square$

7 Generalized Quinn formula

We can now use the tools of Section 6 to reprove Quinn's formula in a generalized format. In particular, this gives an alternative approach to the original proof for G finite abelian [17, §2.5.1-2.5.2].

Below, we intentionally stay close to the notation of Quinn's article so that the resulting formula has the same shape.

Theorem 7.1 (Generalized Quinn formula) *Let M be any abelian group. Suppose*

$$G = \left(\bigoplus_{j \in J_1} \mathbb{Z} \right) \oplus \left(\bigoplus_{j \in J_2} \mathbb{Z}/n_j \mathbb{Z} \right) \quad (7.1)$$

for J_1, J_2 any index sets, and $n_j \geq 1$ suitable integers. Fix a total order on the disjoint union $J := J_1 \dot{\cup} J_2$, say with $J_1 < J_2$. Write $(\mathbf{e}_j)_{j \in J}$ for the standard generators (i.e. the element $1_{\mathbb{Z}}$ resp. $1_{\mathbb{Z}/n_j \mathbb{Z}}$ in the corresponding summand). Let $q \in \text{Quad}(G, M)$ be a quadratic form and

$$b(x, y) := q(x + y) - q(x) - q(y) \quad (7.2)$$

its polarization. Define

$$\sigma_{i,j} := \begin{cases} b(\mathbf{e}_i, \mathbf{e}_j) & \text{if } i < j \\ q(\mathbf{e}_i) & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases}$$

Then the pair (h, c) with

$$h(x, y, z) := \sum_{\substack{j \in J_2 \\ \text{with } y_j + z_j \geq n_j}} x_j n_j \sigma_{j,j} \quad \text{and} \quad c(x, y) := \sum_{\substack{i, j \in J \\ \text{with } i \leq j}} x_i y_j \sigma_{i,j}$$

defines an abelian 3-cocycle such that the trace map of Eq. 1.2 sends it to the given quadratic form q . Here x_j (resp. y_j, z_j) refers to coordinates with values $x_j \in \mathbb{Z}$ for $j \in J_1$ resp. $x_j \in \{0, 1, 2, \dots, n_j - 1\}$ for $j \in J_2$. The map $q \mapsto (h, c)$ is linear, so it provides a group homomorphism $\text{Quad}(G, M) \rightarrow Z_{ab}^3(G, M)$, which makes Diagram (1.3) commute.

Proof of Theorem 7.1 (Step 1) Given the format of our input abelian group G , we can use Lemma/Construction 4.5 to set up an optimal admissible presentation. Let us quickly walk through the relevant steps of the construction, adapted to our setting: Define

$$F_1 := \bigoplus_{j \in J_2} \mathbb{Z} \quad \text{and} \quad F_0 := \bigoplus_{j \in J_1 \cup J_2} \mathbb{Z}$$

(with J_1 and J_2 as in Eq. 7.1) and this sets up a resolution

$$0 \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\pi} G \longrightarrow 0.$$

Write $(e_j)_{j \in J}$ for the standard basis of F_0 , i.e. the j -th summand $\mathbb{Z}$ is spanned by e_j (so that $e_j = \pi e_j$ in terms of the elements in the statement of the theorem). As in Lemma 4.5, define $B(x, y) = b(\pi x, \pi y)$ and $Q(x) = q(\pi x)$ and then

$$\sigma_{i,j} := C(e_i, e_j) = \begin{cases} B(e_i, e_j) & \text{if } i < j \\ Q(e_i) & \text{if } i = j \\ 0 & \text{if } i > j \end{cases} \quad (7.3)$$

describes a $\mathbb{Z}$ -bilinear form on F_0 (this is the same as in the construction given loc. cit.). As guaranteed by the quoted lemma, (F_0, π, C) is an optimal admissible presentation. (Step 2) Each element of G has a unique presentation as

$$g = \sum_{j \in J} g_j \pi(e_j) \quad \text{with} \quad g_j \in \{0, 1, \dots, n_j - 1\} \text{ if } j \in J_2$$

and $g_j \in \mathbb{Z}$ if $j \in J_1$. Sending this g to the vector

$$\tilde{g} := \sum_{j \in J} g_j e_j \in F_0$$

pins down an admissible lift in the sense of Definition 5.1 (including $\tilde{0} = 0$). With respect to the basis $(e_j)_{j \in J}$ we can write $C(-, -)$ as

$$C(x, y) = \sum_{i,j \in J} x_i y_j C(e_i, e_j) = \sum_{i \leq j} x_i y_j \sigma_{i,j}. \quad (7.4)$$

We can write the admissible lift coordinate-wise for any vector $x \in G$ as

$$(\tilde{x})_j = \begin{cases} x_j & \text{for } j \in J_1 \\ x_j - n_j \left\lfloor \frac{x_j}{n_j} \right\rfloor & \text{for } j \in J_2. \end{cases} \quad (7.5)$$

In particular,

$$\begin{aligned} L(x, y)_j &= \widetilde{(x+y)}_j - (\tilde{x})_j - (\tilde{y})_j \\ &= \begin{cases} 0 & \text{for } j \in J_1 \\ -n_j \left(\left\lfloor \frac{x_j+y_j}{n_j} \right\rfloor - \left\lfloor \frac{x_j}{n_j} \right\rfloor - \left\lfloor \frac{y_j}{n_j} \right\rfloor \right) & \text{for } j \in J_2. \end{cases} \end{aligned}$$

Now assume we are given $x, y, z \in F_0$ such that the coordinates satisfy the bound $x_j \in \{0, 1, \dots, n_j - 1\}$ for all $j \in J_2$ (this will simplify the formulas). Demand the same for y_j resp. z_j . There is no condition if $j \in J_1$. Invoke Theorem 6.2 to obtain that the pair (h, c) with

$$h(x, y, z) := -C(\tilde{x}, L(y, z)) \quad \text{and} \quad c(x, y) := C(\tilde{x}, \tilde{y})$$

is an abelian 3-cocycle mapping to q under the trace map. Next, let us unravel these expressions. Expand h using Eq. 7.4 to

$$h(x, y, z) = - \sum_{i \leq j} x_i L(y, z)_j \sigma_{i,j} \quad (7.6)$$

$$= \sum_{i \leq j \text{ with } j \in J_2} x_i \left(\left\lfloor \frac{y_j + z_j}{n_j} \right\rfloor - \left\lfloor \frac{y_j}{n_j} \right\rfloor - \left\lfloor \frac{z_j}{n_j} \right\rfloor \right) n_j \sigma_{i,j}. \quad (7.7)$$

Since $x_j, y_j \in \{0, 1, \dots, n_j - 1\}$ for every $j \in J_2$, we may rewrite the expression for h as

$$h(x, y, z) = \sum_{i \leq j \text{ with } j \in J_2} x_i \begin{cases} 0 & \text{if } y_j + z_j < n_j \\ 1 & \text{if } y_j + z_j \geq n_j \end{cases} n_j \sigma_{i,j} \quad (7.8)$$

and this simplifies to

$$= \sum_{\substack{i \leq j \\ \text{with } j \in J_2 \text{ and } y_j + z_j \geq n_j}} x_i n_j \sigma_{i,j}.$$

Finally, if $i \neq j$, we have by Eq. 7.3 and the bilinearity of B that

$$n_j \sigma_{i,j} = n_j B(e_i, e_j) = B(e_i, n_j e_j) = b(\pi e_i, \pi(n_j e_j)) = 0 \quad (7.9)$$

since $n_j e_j \in \ker(\pi)$. Hence,

$$h(x, y, z) = \sum_{\substack{j \in J_2 \\ \text{with } y_j + z_j \geq n_j}} x_j n_j \sigma_{j,j}.$$

This finishes the proof. $\square$

Example 7.2 Note that along the way, we have found some other possibly useful presentations of the 3-cocycle. For example, Eq. 7.7 expresses the associator for arbitrary representatives/lifts $x_i, y_i, z_i \in \mathbb{Z}$.

Example 7.3 [14] A lively description how one attaches an abelian 3-cocycle to a quadratic form is also given by Kapustin and Saulina in [14, §3.2] (again in the situation with G finite). For readers familiar with their paper, let us note that $\vec{A} \odot \vec{B}$ (in their notation) corresponds to our $\widetilde{x + y}$. They obtain the formula

$$h(x, y, z) = \sum_i n_i x_i \left\lfloor \frac{y_i + z_i}{n_i} \right\rfloor \sigma_{i,i}$$

at the end of [14, §3]. This is Eq. 7.8, again with the summands $i \neq j$ removed by the same argument as in Eq. 7.9.

8 Abelian 3-Cocycle Formulas in Exponential Format

Aside from Quinn's formula, a lot of literature prefers to explicitly spell out the abelian 3-cocycle in terms of exponential functions when $M := \mathbb{C}^\times$. Let us also provide this.

Theorem 8.1 (Exponential format 3-cocycles) *Suppose*

$$G = \bigoplus_{k \in J} \mathbb{Z}/n_k \mathbb{Z} \quad (8.1)$$

for $n_k \geq 1$ and J some totally ordered index set. Write $(e_k)_{k \in J}$ for the generator 1 of the k -th summand. Then there is a bijection between the following three sets:

(1) All possible choices of values

- $p^{(k)} \in \{0, 1, \dots, \gcd(n_k^2, 2n_k) - 1\}$ for every $k \in J$,
- $q^{(k,l)} \in \{0, 1, \dots, \gcd(n_k, n_l) - 1\}$ for all $k < l$ with $k, l \in J$.

(2) All quadratic forms $q \in \text{Quad}(G, \mathbb{C}^\times)$, uniquely described by the following properties

$$\begin{aligned} q(e_k) &= \exp\left(\frac{2\pi i}{\gcd(n_k^2, 2n_k)} p^{(k)}\right), \\ b(e_k, e_l) &= \exp\left(\frac{2\pi i}{\gcd(n_k, n_l)} q^{(k,l)}\right) \quad (\text{for } k < l), \end{aligned}$$

where b is the polarization of q (and further we necessarily then have $b(e_k, e_l) = b(e_l, e_k)$ for $k > l$ and $b(e_k, e_k) = 2q(e_k)$ as well).

- (3) All abelian 3-cocycles $(h, c) \in H_{ab}^3(G, \mathbb{C}^\times)$, uniquely pinned down by the cocycle representative

$$c(x, y) = \prod_{k < l} \exp \left(\frac{2\pi i q^{(k,l)}}{\gcd(n_k, n_l)} x_k y_l \right) \cdot \prod_k \exp \left(\frac{2\pi i p^{(k)}}{\gcd(2n_k, n_k^2)} x_k y_k \right), \quad (8.2)$$

and

$$h(x, y, z) = \prod_k \exp \left(\frac{2\pi i p^{(k)}}{\gcd(2n_k, n_k^2)} (x_k ([y_k]_{n_k} + [z_k]_{n_k} - [y_k + z_k]_{n_k})) \right),$$

where x_k (resp. y_k, z_k) denotes the coordinates of vectors $x, y, z \in G$ according to Eq. 8.1. Here $[-]_{n_k}$ refers to the remainder of division by n_k , expressed as an element in $\{0, 1, \dots, n_k - 1\}$.

Really, $\text{Quad}(G, \mathbb{C}^\times)$ and $H_{ab}^3(G, \mathbb{C}^\times)$ are abelian groups and the above bijections are abelian group isomorphisms, given in terms of the parameters $p^{(k)}, q^{(k,l)}$ by elementwise addition in the quotient groups (i.e. $\mathbb{Z}/(n_k^2, 2n_k)$ for $p^{(k)}$ etc.).

The map $q \mapsto (h, c)$ is linear, so it provides a group homomorphism $\text{Quad}(G, M) \rightarrow Z_{ab}^3(G, M)$, which makes Diagram (1.3) commute.

Before we prove this, let us first establish an explicit parametrization of the quadratic forms.

We apologize that the following repeats part of the statement of the above theorem, but we prefer to be clear about what we prove here amidst a lot of notation.

Lemma 8.2 Suppose

$$G = \bigoplus_{k \in J} \mathbb{Z}/n_k \mathbb{Z}$$

for $n_k \geq 1$ and J some totally ordered index set. Then all elements of $\text{Quad}(G, \mathbb{C}^\times)$ are in bijection to all possible choices

- (1) $p^{(k)} \in \{0, 1, \dots, \gcd(n_k^2, 2n_k) - 1\}$ for every $k \in J$;
- (2) $q^{(k,l)} \in \{0, 1, \dots, \gcd(n_k, n_l) - 1\}$ for all $k < l$ with $k, l \in J$.

Once these choices are made, the corresponding quadratic form and its polarization satisfy

$$q(e_k) = \exp \left(\frac{2\pi i}{\gcd(n_k^2, 2n_k)} p^{(k)} \right),$$

$$b(e_k, e_l) = \exp \left(\frac{2\pi i}{\gcd(n_k, n_l)} q^{(k,l)} \right) \quad (\text{for } k < l)$$

(and then necessarily $b(e_k, e_l) = b(e_l, e_k)$ for $k > l$ and $b(e_k, e_k) = 2q(e_k)$ as well).

Proof (Step 0) It suffices to prove this for J finite, since we can write G as the colimit over all finite subsets $J_0 \subset J$, and correspondingly the subset of parameters $p^{(k)}, q^{(k,l)}$ with $k, l \in J_0$. This is compatible under inclusion of finite subsets of J .

(Step 1) So assume J finite. Let $q \in \text{Quad}(G, \mathbb{C}^\times)$ be arbitrary. We first claim that $q(e_k)$ must be a $\gcd(n_k^2, 2n_k)$ -torsion element in $\mathbb{C}^\times$, i.e.

$$q(e_k) = \exp\left(\frac{2\pi i}{\gcd(n_k^2, 2n_k)} p^{(k)}\right) \quad (8.3)$$

for some uniquely determined $p^{(k)} \in \{0, 1, \dots, \gcd(n_k^2, 2n_k) - 1\}$. The proof for this goes as follows: The generator e_k spans a subgroup $\iota: \mathbb{Z}/n_k\mathbb{Z} \hookrightarrow G$, so we can pull the quadratic form back to this subgroup.

$$\text{Quad}(G, \mathbb{C}^\times) \xrightarrow{\iota^*} \text{Quad}(\mathbb{Z}/n_k\mathbb{Z}, \mathbb{C}^\times) \cong \text{Hom}(\mathbb{Z}/(2n_k, n_k^2)\mathbb{Z}, \mathbb{C}^\times) \quad (8.4)$$

The last isomorphism, the determination of quadratic forms on $\mathbb{Z}/n_k\mathbb{Z}$, goes back to Whitehead. We explain the argument: One can further pull back along $\mathbb{Z} \rightarrow \mathbb{Z}/n_k\mathbb{Z}$ and it is easy to see that any quadratic form on $\mathbb{Z}$ must have the shape

$$q'(x) := x^2 m \quad (8.5)$$

for some $m \in \mathbb{C}^\times$ (e.g., use Example 2.3). Here and for the rest of this sub-argument, we stick to the additive notation. One then only needs to check for which m such a q' descends to a quadratic form on the quotient $\mathbb{Z}/n_k\mathbb{Z}$, which will then give (8.4). We claim that this holds whenever m is a $\gcd(2n_k, n_k^2)$ -torsion element in $\mathbb{C}^\times$.

Necessity: Suppose it descends. Then $0 = q'(0) = q'(n_k) = n_k^2 q'(1)$, again by Example 2.3. Hence, we must have $n_k^2 m = 0$ in $\mathbb{C}^\times$. Moreover, if q' descends to $\mathbb{Z}/n_k\mathbb{Z}$, so does the polarization $b': G \otimes_{\mathbb{Z}} G \rightarrow \mathbb{C}^\times$. Hence,

$$b'(x, y) = 2xym,$$

satisfies $b'(x + Nn_k, y) \equiv b'(x, y) \pmod{n_k}$ for all N . This forces $2n_k m = 0$. Thus, m must be both $2n_k$ - and n_k^2 -torsion, i.e. $\gcd(2n_k, n_k^2)$ -torsion in $\mathbb{C}^\times$. Conversely, suppose it is. Then

$$\begin{aligned} q'(x + Nn_k) &= (x + Nn_k)^2 m \\ &= x^2 m + 2n_k Nxm + N^2 n_k^2 m \equiv x^2 m \end{aligned}$$

since the second and third summand vanish because of the torsion assumption.

This proves the subclaim. That is: the valid choices for $m \in \mathbb{C}^\times$ in Eq. 8.5 are precisely the $\gcd(2n_k, n_k^2)$ -torsion elements. However, these are precisely such as described in Eq. 8.3. Next, by bilinearity $nb(e_k, e_l) = 1$ once $\gcd(n_k, n_l) \mid n$ (because e_k is n_k -torsion and e_l is n_l -torsion), so we may write

$$b(e_k, e_l) = \exp\left(\frac{2\pi i}{\gcd(n_k, n_l)} q^{(k,l)}\right)$$

for some uniquely determined $q^{(k,l)} \in \{0, 1, \dots, \gcd(n_k, n_l) - 1\}$.

(Step 2) We have now seen that $q(e_k)$ and $b(e_k, e_l)$ can only be of the described forms. This defines a set-theoretic map

$$\text{Quad}(G, \mathbb{C}^\times) \longrightarrow \left\{ \begin{array}{l} \text{parameter values } p^{(k)}, q^{(k,l)} \\ \text{(in the ranges described)} \end{array} \right\}. \quad (8.6)$$

It is also clear that this map is injective. It remains to check that conversely any choice of parameters defines a quadratic form on G . Since we already have an injective map, it suffices to count elements on the right side. For any abelian groups A, B we have

$$\text{Quad}(A \oplus B, \mathbb{C}^\times) \cong \text{Quad}(A, \mathbb{C}^\times) \oplus \text{Quad}(B, \mathbb{C}^\times) \oplus \text{Hom}(A \otimes_{\mathbb{Z}} B, \mathbb{C}^\times).$$

To see this, note that

$$\text{Quad}(G, \mathbb{C}^\times) = \text{Hom}(\Gamma G, \mathbb{C}^\times)$$

for Whitehead's universal quadratic functor Γ and the latter is a quadratic functor, i.e.

$$\Gamma(A \oplus B) \cong \Gamma(A) \oplus \Gamma(B) \oplus (A \otimes B).$$

For details, we refer to [1, Chapter I, §4] (or the classic [19]). Hence,

$$\begin{aligned} \text{Quad}\left(\bigoplus_k \mathbb{Z}/n_k \mathbb{Z}, \mathbb{C}^\times\right) &\cong \bigoplus_k \text{Quad}(\mathbb{Z}/n_k \mathbb{Z}, \mathbb{C}^\times) \\ &\quad \oplus \bigoplus_{k < l} \text{Hom}(\mathbb{Z}/\gcd(n_k, n_l) \mathbb{Z}, \mathbb{C}^\times) \end{aligned}$$

and this unravels to

$$\cong \bigoplus_k \mathbb{Z}/(2n_k, n_k^2) \mathbb{Z} \oplus \bigoplus_{k < l} \mathbb{Z}/\gcd(n_k, n_l) \mathbb{Z}.$$

For the first type of summands we have again used the isomorphism of Eq. 8.4, and for the second type of summand note that any linear map of a torsion group to $\mathbb{C}^\times$ must have its image in the torsion of $\mathbb{C}^\times$ and thus certainly in $U(1)$, and $\text{Hom}(-, U(1))$ is just the Pontryagin dual, which (non-canonically) can be identified with the input abelian group. Without spelling out the actual count, it is clear that this set has the same cardinality as our set of parameter values on the right side in Eq. 8.6. This finishes the proof. $\square$

Now the proof of Theorem 8.1 can be done in a similar fashion to the one we used to obtain Quinn's formula.

Proof of Theorem 8.1 We follow the same proof as for Theorem 7.1, so let us just describe how certain details need to be changed. Firstly, we are now in the special case $M := \mathbb{C}^\times$. Using our parametrization of quadratic forms of Lemma 8.2, we may rewrite (7.3) in the concrete shape

$$\sigma_{k,l} = C(e_k, e_l) = \begin{cases} \exp\left(\frac{2\pi i}{\gcd(n_k, n_l)} q^{(k,l)}\right) & \text{if } k < l \\ \exp\left(\frac{2\pi i}{\gcd(2n_k, n_k^2)} p^{(k)}\right) & \text{if } k = l \\ 0 & \text{if } k > l. \end{cases} \quad (8.7)$$

Write $[x]_n$ for the remainder in $\{0, 1, \dots, n-1\}$ of $x \in \mathbb{Z}$ under division by n . Rewrite the admissible lifting in Eq. 7.5 in the shape

$$(\tilde{x})_j = [x]_{n_j}.$$

This is an admissible lift in the sense of Definition 5.1 (and only optically different from the choice in the proof of Quinn's formula). We compute

$$L(x, y)_j = [x_j + y_j]_{n_j} - [x_j]_{n_j} - [y_j]_{n_j}$$

and obtain

$$h(x, y, z) = \prod_{k \leq l} \sigma_{k,l}^{x_k ([y_l]_{n_l} + [z_l]_{n_l} - [y_l + z_l]_{n_l})}$$

$$c(x, y) = \prod_{k \leq l} \sigma_{k,l}^{x_k y_l}$$

because of the multiplicative notation in $\mathbb{C}^\times$. We readily read off (8.2) for $c(x, y)$, just by plugging in the values of $\sigma_{k,l}$ as provided by Eq. 8.7. Similarly,

$$h(x, y, z) = \left(\prod_{k < l} \exp \left(\frac{2\pi i q^{(k,l)}}{\gcd(n_k, n_l)} (x_k ([y_l]_{n_l} + [z_l]_{n_l} - [y_l + z_l]_{n_l})) \right) \right) \cdot \left(\prod_k \exp \left(\frac{2\pi i p^{(k)}}{\gcd(2n_k, n_k^2)} (x_k ([y_k]_{n_k} + [z_k]_{n_k} - [y_k + z_k]_{n_k})) \right) \right).$$

As in Eq. 7.9, for $k < l$ the expression $[y_l]_{n_l} + [z_l]_{n_l} - [y_l + z_l]_{n_l}$ is a multiple of n_l , so the entire input to the exponential function lies in $2\pi i\mathbb{Z}$. Thus,

$$h(x, y, z) = \prod_k \exp \left(\frac{2\pi i p^{(k)}}{\gcd(2n_k, n_k^2)} (x_k ([y_k]_{n_k} + [z_k]_{n_k} - [y_k + z_k]_{n_k})) \right),$$

which is what we had claimed. $\square$

9 Constructing Associators and Braidings

9.1 Recollections

As we had explained in the introduction, this note applies to both (a) pointed braided fusion categories over a field k , as well as (b) braided categorical groups.

The reason for this is that both types of braided monoidal categories can be classified in terms of (a slight generalization of) pre-metric groups as in [4, 13].

Definition 9.1 [13, §3] A *quadratic triple* is a triple (G, M, q) , where G, M are abelian groups and $q \in \text{Quad}(G, M)$ a quadratic form. A morphism $(G, M, q) \rightarrow (G', M', q')$ is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ q \downarrow & & \downarrow q' \\ M & \xrightarrow{g} & M' \end{array} \quad (9.1)$$

where f, g are group homomorphisms. Write Quad for the category of quadratic triples. A *symmetric triple* is a quadratic triple such that the polarization of the quadratic form q vanishes. Write Quad_{sym} for the full subcategory of symmetric triples.

Write $\mathcal{BCG}$ for the 1-category whose objects are braided categorical groups and whose morphisms are the equivalence classes of braided monoidal functors.⁴

Theorem 9.2 (Joyal–Street [13, §3]) *There is an equivalence of 1-categories*

$$\begin{aligned} T : \mathcal{BCG} &\longrightarrow \mathcal{Quad} \\ (\mathbb{C}, \otimes) &\longmapsto (\pi_0(\mathbb{C}, \otimes), \pi_1(\mathbb{C}, \otimes), q), \end{aligned} \quad (9.2)$$

where q is defined as follows: For any object $X \in \mathbb{C}$ the self-braiding $s_{X,X} : X \otimes X \xrightarrow{\sim} X \otimes X$ induces an automorphism of the tensor unit, namely

$$s_{X,X} \otimes X^{-1} \otimes X^{-1} : 1_{\mathbb{C}} \xrightarrow{\sim} 1_{\mathbb{C}},$$

and $q([X]) := (s_{X,X} \otimes X^{-1} \otimes X^{-1}) \in \text{Aut}(1_{\mathbb{C}}) \cong M$ extends to a well-defined quadratic form on $\pi_0(\mathbb{C}, \otimes)$.

The idea is as follows: Firstly, one picks a skeleton of the category, using that every braided monoidal category is braided monoidal equivalent to any of its skeleta (with a suitably defined braided monoidal structure on the skeleton). On this skeleton, the associator

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

and braiding

$$s_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

are automorphisms of objects, because in a skeleton any two mutually isomorphic objects must be the same object. Thus, $a_{X,Y,Z} \in M$ and $s_{X,Y} \in M$ for any objects X, Y, Z . The pentagon and hexagon axioms of a braided monoidal structure then become equivalent to the axioms of an abelian 3-cocycle (h, c) (namely Eq. 2.1 for the pentagon, and Eqs. A, A' for the hexagons).

This means that up to braided monoidal equivalence it suffices to work with the following explicit skeletal model in the situation $G := \pi_0(\mathbb{C}, \otimes)$, $M := \pi_1(\mathbb{C}, \otimes) = \text{Aut}_{\mathbb{C}}(1_{\mathbb{C}})$ and how h, c enter is explained below:

Definition 9.3 (Skeletal model) Suppose we are given abelian groups G, M and an abelian 3-cocycle $(h, c) \in Z_{ab}^3(G, M)$. Let $\mathcal{T} := \mathcal{T}(G, M, (h, c))$ denote the following braided categorical group:

- (1) The objects are the elements in G .
- (2) The automorphisms of an object $X \in G$ are $\text{Aut}(X) := M$, and their composition is the addition of M .
- (3) There are no morphisms except for automorphisms (so no other composition of morphisms needs to be defined).
- (4) The monoidal structure is

$$(X \xrightarrow{f} X) \otimes_{\mathcal{T}} (X' \xrightarrow{f'} X') := (X + X' \xrightarrow{f+f'} X + X'),$$

where addition is just addition in G (on objects) resp. in M (for f, f').

- (5) The associator

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \quad (9.3)$$

⁴In [13] the category $\mathcal{BCG}$ is defined without identifying morphisms which only differ by equivalence. This leads to the statement of the classification [13, Theorem 3.3] to sound more convoluted.

is the automorphism defined by $h(X, Y, Z) \in M$. The associativity of G settles that the objects on either side are the same.

(6) The braiding

$$s_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

is the automorphism given by $c(X, Y) \in M$.

One then checks that changing the abelian 3-cocycle by an abelian 3-coboundary amounts to the structure rendering the identity functor $\text{id} : \mathcal{T} \rightarrow \mathcal{T}$ a braided monoidal self-equivalence, i.e. only the cohomology class of $[(h, c)] \in H_{ab}^3(G, M)$ is well-defined when regarding $\mathcal{T}$ as an object in $\mathcal{BCG}$. Finally, the latter identifies uniquely with a quadratic form $q \in \text{Quad}(G, M)$ via the Eilenberg–Mac Lane isomorphism, Eq. 1.2 (or Section 11). For details regarding the proof of Theorem 9.2 we refer to [13, §3] (or the slightly different treatment in [12]).

Let $\mathcal{BCG}_{\text{sym}} \subset \mathcal{BCG}$ denote the full subcategory of those braided categorical groups which are symmetric monoidal (the braided monoidal equivalences then automatically become symmetric monoidal equivalences). These are also known as *Picard groupoids*.

Theorem 9.4 (Sính [18]) *The equivalence of Eq. 9.2 restricts to an equivalence of 1-categories*

$$T : \mathcal{BCG}_{\text{sym}} \longrightarrow \text{Quad}_{\text{sym}}.$$

Proof This was originally proven in Sính’s thesis [18], and this essentially started the entire subject from scratch. In particular, the formulation as a special case of the Joyal–Street equivalence is anachronistic. Sính defined her own functor and the right side was different (but equivalent to Quad_{sym}). We explain how the above arises as a special case within the Joyal–Street classification: If the braiding is symmetric, we have

$$s_{Y,X} \circ s_{X,Y} = \text{id}_{X \otimes Y},$$

and in terms of the abelian 3-cocycle (h, c) built from the braiding and associator this amounts to $c(y, x) + c(x, y) = 0$. Among all abelian 3-cocycles, this additional constraint isolates the symmetric 3-cocycles, $H_{\text{sym}}^3(G, M) \subseteq H_{ab}^3(G, M)$, but there is a commutative diagram, going back to Whitehead, whose top row is exact (see [3, Lemma 4.10])

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(G/2G, M) & \hookrightarrow & \text{Quad}(G, M) & \xrightarrow{q \mapsto b} & \text{Hom}(G \otimes G, M) \\ & & \uparrow \cong & & \uparrow \cong & & \\ 0 & \longrightarrow & H_{\text{sym}}^3(G, M) & \hookrightarrow & H_{ab}^3(G, M), & & \end{array}$$

where “ $q \mapsto b$ ” refers to the map sending a quadratic form to its polarization. In particular, $H_{\text{sym}}^3(G, M)$ identifies precisely with those quadratic forms whose polarization vanishes, which was the defining property for Quad_{sym} on the right side and proves the claim. The embedding $\text{Hom}(G/2G, M) \hookrightarrow \text{Quad}(G, M)$ is based on the observation that any linear map $G/2G \rightarrow M$ satisfies the axioms of a quadratic form (see [3, Lemma 4.10] for details). $\square$

Now return to the concept of a quadratic triple as in Definition 9.1. Let k be a field. In the special case where G is finite, $M := k^\times$ and g in Diagram (9.1) is constrained to be

the identity map, the definition transforms into the concept of a pre-metric group. We will mildly generalize this and drop the finiteness assumption.

Definition 9.5 For us, a *big fusion category* is a semisimple rigid k -linear monoidal category $(\mathcal{C}, \otimes)$ with finite-dimensional Hom-spaces such that the monoidal unit $1_{\mathcal{C}}$ is simple. It is *pointed* if all simple objects are invertible.

An ordinary (non-big) fusion category is a big fusion category such that there are only finitely many isomorphism classes in $\mathcal{C}$; this is the standard definition. The definition of a semisimple category includes that every object decomposes as a *finite* direct sum of simple objects. Nonetheless, this would have to hold even if we only required that every object is a direct sum of simples: If $X \simeq \bigoplus_{i \in I} S_i$ with each S_i simple (or merely non-zero), the assumption $\dim_k \text{End}(X) < \infty$ already forces the index set I to be finite.

If X is a simple object in a pointed big fusion category, it is invertible and thus X^{-1} is also simple; and if X, Y are both simple, then $X \otimes Y$ can only have a single simple direct summand. Thus, $X \otimes Y$ must be simple, too.

Definition 9.6 If $(\mathcal{C}, \otimes)$ is a big pointed braided fusion category, let $\mathcal{C}_{\text{simp}}$ be the full subcategory of simple objects, and keep only the isomorphisms as morphisms. Thus, $\mathcal{C}_{\text{simp}}$ is a groupoid. Moreover, $\mathcal{C}_{\text{simp}}$ is braided monoidal by restricting the braided monoidal structure to the subcategory.

Definition 9.7 ([4, Definition 2.38], [5, §8.4]) Fix a field k . A *big pre-metric group* is a pair (G, q) , where G is an abelian group and $q \in \text{Quad}(G, k^\times)$ a quadratic form. A morphism $(G, q) \rightarrow (G', q')$ is a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ & \searrow q & \swarrow q' \\ & k^\times & \end{array}$$

where f is a group homomorphism. Write $\mathcal{PM}_k$ for the category of big pre-metric groups. A *symmetric big pre-metric group* is a pair (G, q) such that the polarization of the quadratic form vanishes. Write $\mathcal{PM}_{k, \text{sym}}$ for the corresponding full subcategory. The original definition without the word “big” refers to the same, except that we demand G to be a finite abelian group.

Write $\mathcal{PB}_k$ for the 1-category whose objects are pointed braided k -linear big fusion categories and whose morphisms are the equivalence classes of k -linear braided monoidal functors. Write $\mathcal{PS}_k \subset \mathcal{PB}_k$ for the full subcategory of those big fusion categories which are symmetric monoidal.

Theorem 9.8 [4, Proposition 2.41] *Let k be an algebraically closed field. There is an equivalence of 1-categories*

$$\begin{aligned} E : \mathcal{PB}_k &\longrightarrow \mathcal{PM}_k \\ (\mathcal{C}, \otimes) &\longmapsto (\pi_0(\mathcal{C}, \otimes), q), \end{aligned}$$

where q is defined as follows: For any simple object $X \in \mathcal{C}$ the self-braiding $s_{X,X} : X \otimes X \xrightarrow{\sim} X \otimes X$ induces an automorphism of the tensor unit, namely

$$s_{X,X} \otimes X^{-1} \otimes X^{-1} : 1_{\mathcal{C}} \xrightarrow{\sim} 1_{\mathcal{C}},$$

and $q([X]) := (s_{X,X} \otimes X^{-1} \otimes X^{-1}) \in \text{Aut}(1_{\mathcal{C}}) \cong k^{\times}$ extends to a well-defined quadratic form on $\pi_0(\mathcal{C}, \otimes)$.

Proof This result essentially reduces to the classification of Joyal–Street of braided categorical groups. The main differences are as follows: (a) Since $1_{\mathcal{C}}$ is simple by assumption, $\text{End}(1_{\mathcal{C}})$ must be a division algebra over k by Schur’s Lemma. As k is algebraically closed, we must have $\text{End}(1_{\mathcal{C}}) = k$ and thus $\text{Aut}(1_{\mathcal{C}}) = k^{\times}$. The category $\mathcal{C}_{\text{simp}}$ thus is a braided categorical group with $\pi_1(\mathcal{C}_{\text{simp}}) = k^{\times}$. (b) Since the braided monoidal functors between big fusion categories are assumed k -linear, they correspond to morphisms of quadratic triples as in (9.1), but must be the identity on $k^{\times}$. $\square$

See also [5, Theorem 8.4.12] for a direct proof which differs in certain parts. In analogy to Theorem 9.4 one obtains the following.

Theorem 9.9 ([4, Example 2.45]) *The equivalence of Eq. 9.2 restricts to an equivalence of 1-categories*

$$T : \mathcal{PS}_k \longrightarrow \mathcal{PM}_{k,\text{sym}}.$$

9.2 Structure Results

Now, let us draw some conclusions from these equivalences. The following is certainly known in the setting of fusion categories.

Theorem 9.10 *Let $(\mathcal{C}, \otimes)$ be*

- (a) *a k -linear pointed braided fusion category with k algebraically closed, or*
- (b) *a braided categorical group.*

Write $\pi_i := \pi_i(\mathcal{C}_{\text{simp}}, \otimes)$ in situation (a), and $\pi_i := \pi_i(\mathcal{C}, \otimes)$ in situation (b). Then the following statements are equivalent:

- (1) *$(\mathcal{C}, \otimes)$ is braided monoidal equivalent to a simultaneously skeletal and strictly associative braided monoidal category.*
- (2) *The abelian 3-cocycle of $(\mathcal{C}, \otimes)$ admits a cocycle representative (h, c) such that*

$$c : \pi_0 \times \pi_0 \longrightarrow \pi_1$$

is bilinear and h vanishes.

- (3) *The abelian 3-cocycle of $(\mathcal{C}, \otimes)$ admits a cocycle representative (h, c) such that*

$$h : \pi_0 \times \pi_0 \times \pi_0 \longrightarrow \pi_1 \quad \text{and} \quad c : \pi_0 \times \pi_0 \longrightarrow \pi_1$$

are trilinear resp. bilinear in π_0 .

- (4) *There exists a bilinear form S on π_0 such that*

$$q(x) = S(x, x)$$

holds for the quadratic form attached to (h, c) under the Eilenberg–Mac Lane isomorphism of Eq. 11.1.

The equivalence (1) $\Leftrightarrow$ (4) in the case of fusion categories is found in [5, Exercise 8.4.11].

Proof We discuss the braided categorical group case. Suppose (G, M, q) is the quadratic triple attached to $(\mathcal{C}, \otimes)$ by Theorem 9.2 (1 $\Rightarrow$ 2) Suppose $(\mathcal{C}, \otimes)$ is braided monoidal equivalent to a skeletal and strictly associative braided monoidal category. Being skeletal, it is of the form of a skeletal model $\mathcal{T}(G, M, (h, c))$, Definition 9.3, where (h, c) is an abelian 3-cocycle. Since the model is strictly associative, the maps

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

of Eq. 9.3 are identity maps in this model, i.e. $h(x, y, z) = 0$ holds for all x, y, z . Now Eqs. A and A' imply that c is $\mathbb{Z}$ -bilinear. (2 $\Rightarrow$ 3) Trivial. (3 $\Rightarrow$ 4) Define $S(x, y) := c(x, y)$. This is bilinear by assumption. Moreover,

$$S(x, x) = c(x, x) = q(x)$$

since the Eilenberg–Mac Lane isomorphism maps (h, c) to $x \mapsto c(x, x)$. (4 $\Rightarrow$ 1) We can apply Lemma 4.1 (2), so (G, id_G, S) is an optimal admissible presentation with $F_1 = 0$. According to Theorem 6.2 it follows that (h, c) with

$$h(x, y, z) := -S(\tilde{x}, L(y, z)), \quad c(x, y) := S(\tilde{x}, \tilde{y})$$

is an abelian 3-cocycle representative for the quadratic form q for any admissible lifting $(-)$. Since $F_0 = G$, we can just take the identity as a lifting. In particular, $L(x, y) = (x + y) - \tilde{x} - \tilde{y} = 0$, so $h(x, y, z) = 0$ for all $x, y, z \in G$. It follows that the skeletal model $\mathcal{T}(G, M, (0, c))$ is braided monoidal equivalent to $(\mathcal{C}, \otimes)$, and this is skeletal, and thanks to $h = 0$ the associators are trivial. In the big fusion category case, instead work with the triple $(G, k^\times, q)$ of Theorem 9.8, and replace the skeletal model $\mathcal{T}(G, M, (h, c))$ by its natural k -linear analogue. $\square$

In the symmetric monoidal situation the equivalent properties of Theorem 9.10 are always satisfied.

Theorem 9.11 (Johnson–Osorno [11]) *Let $(\mathcal{C}, \otimes)$ be*

- *a k -linear pointed symmetric fusion category with k algebraically closed, or*
- *a Picard groupoid.*

Then $(\mathcal{C}, \otimes)$ is symmetric monoidal equivalent to a simultaneously skeletal and strictly associative symmetric monoidal category. More precisely, all characterizations in Theorem 9.10 are always met in this setting.

This was originally observed in the context of Picard groupoids [11, Theorem 2.2] with a different method.

Proof We apply the equivalence of categories in the symmetric setting, Theorem 9.4, so $(\mathcal{C}, \otimes)$ corresponds to a symmetric triple (G, M, q) in $Quad_{sym}$, or $(G, k^\times, q)$ in the fusion category setting. Being in $Quad_{sym}$, we know that the polarization of q is trivial, i.e.

$$0 = b(x, y) = q(x + y) - q(x) - q(y),$$

showing that q is a linear map (the complete deduction of this is as we had done it around (3.7)). As the above correspondence of Theorem 9.4 is just a special case of the Joyal–Street equivalence of Theorem 9.2, we can also use Theorem 9.10 in this setting. The conclusion

(4) $\Rightarrow$ (1) shows that our claim is proven if we can show that q comes from a bilinear form. Define

$$S(x, y) = \sum_i x_i y_i q(\gamma_i) \quad (9.4)$$

with respect to the coordinates $x_i, y_i \in \mathbb{F}_2$ in any chosen basis $(\gamma_i)_{i \in I}$ of $G/2G$ as an $\mathbb{F}_2$ -vector space. Since $q(-\gamma_i) = q(\gamma_i)$ holds for any quadratic form, the linearity of q yields $2q(\gamma_i) = 0$, so the scalar multiplication with elements from $\mathbb{F}_2$ in Eq. 9.4 is well-defined and S is indeed bilinear. Then $S(x, x) = \sum x_i^2 q(\gamma_i) \equiv \sum x_i q(\gamma_i) = q(x)$, proving the claim. The remaining characterizations follow since by Theorem 9.10 they are all equivalent. $\square$

As a further refinement of the characterization in Theorem 9.10 relating the possibility to trivialize associators to the linearity of the braiding, we can show the following.

Theorem 9.12 *Suppose $(\mathcal{C}, \otimes)$ is a braided categorical group such that $\pi_1(\mathcal{C}, \otimes)$ is a divisible group. Then $(\mathcal{C}, \otimes)$ is braided monoidal equivalent to a skeletal model as in Definition 9.3 such that*

$$h(x, y, z) = c(x, y) + c(x, z) - c(x, y + z)$$

holds for the abelian 3-cocycle. That is: For any objects X, Y, Z we have

$$a_{X,Y,Z} = s_{X,Y} + s_{X,Z} - s_{X,Y \otimes Z} \quad \text{and} \quad a_{Z,X,Y} = s_{X \otimes Y,Z} - s_{X,Z} - s_{Y,Z}. \quad (9.5)$$

We call this a normal form skeletal model.

We do not know whether this is also true for general $\pi_1(\mathcal{C}, \otimes)$.

Proof By Theorem 4.6 since $M := \pi_1(\mathcal{C}, \otimes)$ is divisible, an optimal admissible presentation (F_0, π, C) exists. Let $\widetilde{(-)}$ be any admissible lift for this presentation. Then by Theorem 6.2 we find an abelian 3-cocycle representative (h, c) such that

$$\begin{aligned} h(x, y, z) &= -C(\widetilde{x}, L(y, z)) = C(\widetilde{x}, \widetilde{y}) + C(\widetilde{x}, \widetilde{z}) - C(\widetilde{x}, \widetilde{(y + z)}) \\ &= c(x, y) + c(x, z) - c(x, y + z), \end{aligned}$$

which is precisely our claim. Combine it with Theorem 9.2 and Definition 9.3 to relate it to the concrete associator and braiding. This yields $a_{X,Y,Z} = s_{X,Y} + s_{X,Z} - s_{X,Y \otimes Z}$. For the second identity,

$$\begin{aligned} c(x + y, z) - c(x, z) - c(y, z) &= C(\widetilde{(x + y)}, \widetilde{z}) - C(\widetilde{x}, \widetilde{z}) - C(\widetilde{y}, \widetilde{z}) = C(L(x, y), \widetilde{z}) \\ &= b(\pi L(x, y), \pi \widetilde{z}) - C(\widetilde{z}, L(x, y)) = h(z, x, y) \end{aligned}$$

since $\pi L(y, z) = 0$ holds by construction. $\square$

Analogously, we obtain:

Theorem 9.13 *Suppose k is an algebraically closed field. Let $(\mathcal{C}, \otimes)$ be a k -linear pointed braided big fusion category. Then $(\mathcal{C}, \otimes)$ is braided monoidal equivalent to a skeletal big fusion category such that*

$$a_{X,Y,Z} = \frac{s_{X,Y} \cdot s_{X,Z}}{s_{X,Y \otimes Z}} \quad \text{and} \quad a_{Z,X,Y} = \frac{s_{X \otimes Y,Z}}{s_{X,Z} \cdot s_{Y,Z}}$$

hold for all simple objects X, Y, Z .

Proof The only difference to the proof for Theorem 9.12 is that we need to show that $\pi_1(\mathbb{C}, \otimes) = k^\times$ is divisible, but this just amounts to the existence of n -th roots $\sqrt[n]{x}$ for any $n \geq 1$ and $x \in k^\times$, which holds since k is algebraically closed. $\square$

10 Classification of Monoidal Structures on Graded Vector Spaces

Let G be a finite abelian group and k a field. Write Vect_k^G for the category of G -graded finite-dimensional k -vector spaces.

The previous results can be used to achieve a classification of all

- monoidal structures on Vect_k^G up to monoidal equivalence,
- braided monoidal structures on Vect_k^G up to braided monoidal equivalence,
- symmetric monoidal structures on Vect_k^G up to symmetric monoidal equivalence.

Bulacu, Caenepeel and Torrecillas have settled this classification for $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in [2]. They found 8 braided monoidal structures if k does not contain a primitive 4-th root of unity and 24 more otherwise. We recover this in Theorem 8.1, as the set of choices for the parameters is

- (1) $p^{(0)}, p^{(1)} \in \{0, 1, 2, 3\}$,
- (2) $q^{(0,1)} \in \{0, 1\}$,

giving a total of $4 \cdot 4 \cdot 2 = 32$ choices. The theorem also gives the respective braiding and associator and it is easy to distinguish the two families depending on whether they output a primitive 4-th root of unity or not. The paper [8] by Huang, Liu and Ye generalized this to the situation $G = \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$ for arbitrary n_1, n_2 , see [8, Theorem 4.2] (and note that the set $A_{a,b,d}$ loc. cit. can be empty, as is also stressed there).

Most recently, the general case (for G any finite group) has been settled in [9, 10]. The paper [10, §3] uses a different method than we use in the present paper and organizes the answer in a different fashion: they first classify the possible associators in terms of normalized group 3-cocycles and then, when classifying braidings, they study the possible ways to enrich such a cocycle to an abelian 3-cocycle in a second step. Our paper would classify the braided monoidal structures by enumerating the quadratic forms, so it approaches the matter from a different direction.

11 Proof of the Eilenberg-Mac Lane Isomorphism

As we are so close now, let us quickly sketch how one can reprove the original Eilenberg-Mac Lane isomorphism along these lines. Our method does not add anything new to checking injectivity though.

Theorem 11.1 (Eilenberg–Mac Lane) *Let G, M be abelian groups. The trace*

$$\begin{aligned} H_{ab}^3(G, M) &\longrightarrow \text{Quad}(G, M) \\ (h, c) &\longmapsto (q(x) := c(x, x)) \end{aligned} \quad (11.1)$$

is an isomorphism of abelian groups.

Proof (Well-defined) First of all, given an abelian 3-cocycle (h, c) , we need to check that $q(x) := c(x, x)$ is indeed a quadratic form. A complete argument with all details is spelled out in [3, Lemma 3.9]. Next, we have to check that cohomologous cocycles have the same trace. However, by (2.4) the diagonal terms $c(x, x)$ of an abelian 3-coboundary vanish. Thus, the map is a well-defined group homomorphism. *(Surjective)* If G is finitely generated abelian, the generalized form of Quinn's formula, Theorem 7.1, gives an explicit preimage for any quadratic form q . An arbitrary G is the (filtering) colimit of its finitely generated subgroups G' , partially ordered by inclusion, so

$$\varinjlim_{G' \subseteq G} H_{ab}^3(G', M) \longrightarrow \varinjlim_{G' \subseteq G} \text{Quad}(G', M)$$

is also surjective in the limit. *(Injective)* We do not know a way to improve on the argument of Joyal and Street in [12, Theorem 12]. The idea is to check injectivity for $G := \mathbb{Z}$ and $G := \mathbb{Z}/n_j\mathbb{Z}$ individually, and then that both sides of the map are quadratic functors in G , showing injectivity for arbitrary finitely generated abelian groups G , and then use a colimit argument as before. Surely one can run this also explicitly on the level of cocycle computations, but since it is just about proving triviality, it is not clear what knowing an explicit coboundary formula would then be useful for, afterwards. So, we think the approach of Joyal and Street is just right. $\square$

12 Afterword

The following question appears very natural to me, but I have not been able to determine the answer (perhaps embarrassingly).

Problem 2 Suppose G is a torsion-free abelian group and M an arbitrary abelian group. Is any quadratic form $q : G \rightarrow M$ of the shape $x \mapsto S(x, x)$ for S some bilinear form?

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