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On the Radical of a Hecke–Kiselman Algebra

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Abstract
The Hecke-Kiselman algebra of a finite oriented graph Θ over a field K is studied. If Θ is an oriented cycle, it is shown that the algebra is semiprime and its central localization is a finite direct product of matrix algebras over the field of rational functions $K(x)$. More generally, the radical is described in the case of PI-algebras, and it is shown that it comes from an explicitly described congruence on the underlying Hecke-Kiselman monoid. Moreover, the algebra modulo the radical is again a Hecke-Kiselman algebra and it is a finite module over its center.

Keywords Hecke–Kiselman algebra · Monoid · Simple graph · Reduced words · Algebra of matrix type · Noetherian algebra · PI algebra · Jacobson radical

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1 Introduction

Let Θ be a finite simple oriented graph with n vertices $\{1, \dots, n\}$. The Hecke–Kiselman monoid HK_Θ associated to Θ, introduced by Ganyushkin and Mazorchuk in [7], is generated by elements $x_1, \dots, x_n$ subject to the defining relations:

(i) $x_i = x_i^2$, for $1 \leqslant i \leqslant n$,
(ii) $x_i x_j = x_j x_i$ if the vertices i, j are not connected in Θ,
(iii) $x_i x_j x_i = x_j x_i x_j = x_i x_j$, if i, j are connected by an arrow $i \to j$ in Θ.

Thus, HK_Θ is a natural homomorphic image of the corresponding Coxeter monoid, where relations (iii) are replaced by the braid relations $x_i x_j x_i = x_j x_i x_j$. Hence, information

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on the structure and representations of HK_Θ in particular contributes to the understanding of representation theory of the latter. Several combinatorial properties of HK_Θ and their representations were studied in [1, 5, 7, 9, 11]. We continue the study in [13], where the structure of HK_Θ, and of the associated algebra $K[HK_\Theta]$ over a field K, is investigated. The case where Θ is the oriented cycle $x_1 \to x_2 \to \cdots \to x_n \to x_1$, with $n \geqslant 3$, plays a crucial role. Our first main result shows that the associated Hecke-Kiselman algebra, denoted by $K[C_n]$, is semiprime. It is also Noetherian, as shown in [13]. Consequently, since $K[C_n]$ is an algebra of Gelfand-Kirillov dimension one [10], from [16] it follows that $K[C_n]$ is a finite module over its center. Moreover, its classical quotient ring can be completely described.

Theorem 1 *Let $n \geqslant 3$. Then $K[C_n]$ is a semiprime Noetherian PI-algebra. Moreover, its classical quotient ring is isomorphic to $\prod_{i=0}^{n-2} M_{n_i}(K(x))$, where $n_i = \binom{n}{i+1}$, for $i = 0, \ldots, n-2$.*

Recall that the classical quotient ring of a semiprime Goldie PI-algebra is its central localization, see [14], Theorem 1.7.34.

In particular, this result answers a question asked in [13]. Next, we apply it to derive a description of the Jacobson radical $\mathcal{J}(K[HK_\Theta])$ of an arbitrary algebra $K[HK_\Theta]$, provided it satisfies a polynomial identity. The latter condition is equivalent to a simple condition expressed in terms of the graph Θ, [10]. Namely, it is equivalent to saying that Θ does not contain two cyclic subgraphs (i.e. subgraphs which are oriented cycles) connected by an oriented path. We prove that the radical is the ideal determined by an explicitly described congruence ρ on HK_Θ, so that $K[HK_\Theta]/\mathcal{J}(K[HK_\Theta]) \cong K[HK_\theta/\rho]$ is again a Hecke-Kiselman algebra and it has a very transparent structure. For a congruence η on a semigroup S, the kernel of the natural homomorphism $K[S] \longrightarrow K[S/\eta]$ will be denoted by $I(\eta)$. So $K[S/\eta] \cong K[S]/I(\eta)$.

Namely, let ρ be the congruence on HK_Θ generated by all pairs (xy, yx) such that there is an arrow $x \to y$ that is not contained in any cyclic subgraph of Θ. Let Θ' be the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ. Then $HK_{\Theta'} \cong HK_\Theta/\rho$. (If there is no such a pair then we assume that ρ is the trivial congruence.) Then, because of the assumption that $K[HK_\Theta]$ is a PI-algebra, the connected components of Θ' are either singletons or cyclic subgraphs. Recall from [13] that this implies that $K[HK_{\Theta'}]$ is a Noetherian algebra. Indeed, Noetherian algebras $K[HK_\Theta]$ are characterized by the condition: each of the connected components of the graph Θ either is acyclic or it is a cyclic graph of length n for some $n \geqslant 3$. Our second main result reads as follows.

Theorem 2 *Assume that Θ is a finite oriented graph such that $K[HK_\Theta]$ is a PI-algebra. Let Θ' be the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ and let ρ be the congruence on HK_Θ defined above. Then*

1. *the Jacobson radical $\mathcal{J}(K[HK_\Theta])$ of $K[HK_\Theta]$ is equal to the ideal $I(\rho)$ determined by ρ,*
2. *$K[HK_\Theta]/\mathcal{J}(K[HK_\Theta]) \cong K[HK_{\Theta'}]$ and it is the tensor product of algebras $K[HK_{\Theta_i}]$ of the connected components $\Theta_1, \ldots, \Theta_m$ of Θ', each being isomorphic to $K \oplus K$ or to the algebra $K[C_j]$, for some $j \geqslant 3$,*
3. *$K[HK_{\Theta'}]$ is a finitely generated module over its center.*

Recall that the Jacobson radical of a finitely generated PI-algebra R is nilpotent, see [15], Theorem 6.3.39. However, for $R = K[HK_\Theta]$ this can also be derived from our proof.

This result opens a perspective for developing representation theory of such monoids HK_Θ, which was one of the motivations in [7].

2 Some background

A Gröbner basis for C_n has been found in [11], by applying the diamond lemma, see [2]. Consequently, the elements of C_n can be treated as words in the free monoid $F = \langle x_1, \ldots, x_n \rangle$ that are reduced in terms of certain rewriting system in F. Let $|w|_i$ denote the degree of a word w (treated as an element of C_n) in the generator x_i. If $i, j \in \{1, \ldots, n\}$ then $x_i \cdots x_j$ denotes the product of all consecutive generators from x_i up to x_j if $i < j$, or down to x_j, if $i > j$.

Theorem 3 *Let $\Theta = C_n$ for some $n \geqslant 3$. Let S be the system of reductions in F consisting of all pairs of the form*

(1) $(x_i x_i, x_i)$ *for all* $i \in \{1, \ldots, n\}$,
(2) $(x_j x_i, x_i x_j)$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $1 < j - i < n - 1$,
(3) $(x_n(x_1 \cdots x_i)x_j, x_j x_n(x_1 \cdots x_i))$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $i + 1 < j < n - 1$,
(4) $(x_i u x_i, x_i u)$ *for all* $i \in \{1, \ldots, n\}$ *and* $1 \neq u \in F$ *such that* $|u|_i = |u|_{i-1} = 0$. *Here, we write* $i - 1 = n$ *if* $i = 1$,
(5) $(x_i v x_i, v x_i)$ *for all* $i \in \{1, \ldots, n\}$ *and* $1 \neq v \in F$ *such that* $|v|_i = |v|_{i+1} = 0$. *Here we write* $i + 1 = 1$ *if* $i = n$.

Then the set $\{w - v \mid$ *for* $(w, v) \in S\}$ *is a Gröbner basis of the algebra* $K[C_n]$.

It follows that an element $w \in F$ is a reduced word if and only if w has no factors that are leading terms of the reductions (1) - (5) listed above. This reduction system is compatible with the degree-lexicographical ordering on the free monoid F defined by $x_1 < x_2 < \cdots < x_n$. We will use this result from [11] several times without further comment.

Our approach heavily depends on the results of [13]. In particular, a very transparent description of the reduced forms of almost all elements of C_n has been found in [13], Theorem 2.1. Namely, for $i = 0, 1, \ldots, n-2$, the set $\tilde{M}_i$ of reduced forms of elements of C_n that have a factor of the form $x_n q_i = x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1}$ can be described as follows

$$\tilde{M}_i = \{a(x_n q_i)^k b \in C_n : a \in A_i, b \in B_i, k \geqslant 1\}, \tag{1}$$

where A_i, B_i are certain well defined sets. Moreover, if $\tilde{M} = \bigcup_{i=0}^{n-2} \tilde{M}_i$ then the set $C_n \setminus \tilde{M}$ is finite and each $M_i = \tilde{M}_i^0$ ($\tilde{M}_i$ with zero adjoined) is isomorphic to a semigroup of matrix type $\mathcal{M}^0(S_i, A_i, B_i; P_i)$, where S_i denotes the cyclic semigroup generated by $s_i = x_n q_i$, P_i is a matrix of size $B_i \times A_i$ with coefficients in $\langle x_n q_i \rangle \cup \{\theta\}$, where $\langle s_i \rangle$ is the monoid generated by s_i. Recall that, if S is a semigroup, A, B are nonempty sets and $P = (p_{ba})$ is a $B \times A$ - matrix with entries in S^0, then the semigroup of matrix type $\mathcal{M}^0(S, A, B; P)$ over S is the set of all triples (s, a, b), where $s \in S, a \in A, b \in B$, with the zero element θ, with operation $(s, a, b)(s', a', b') = (s p_{ba'} s', a, b')$ if $p_{ba'} \in S$ and θ otherwise. So, $\mathcal{M}^0(S, A, B; P)$ is an order in the completely 0-simple semigroup $\mathcal{M}^0(G, A, B; P)$ over a cyclic infinite group, in the sense of [6]. Moreover $\mathcal{M}^0(K[S], A, B; P)$ denotes the corresponding algebra of matrix type. It is defined as the contracted semigroup algebra

$K_0[\mathcal{M}^0(S, A, B; P)]$ and (if A, B are finite) it can be interpreted as the set of all $A \times B$ - matrices over $K[S]$ with operation $\alpha\beta = \alpha \circ P \circ \beta$, where $\circ$ stands for the standard matrix product. For basic results on semigroups and algebras of matrix type we refer to [12], Chapter 5.They play a fundamental role in representation theory of semigroup algebras.

It is shown in [13] that $|A_i| = |B_i|$ and P_i is not a zero divisor in the matrix ring $M_{n_i}(K[s_i])$. Therefore, P_i is invertible as a matrix in $M_{n_i}(K(s_i))$ and hence the algebra of matrix type $\mathcal{M}^0(K(s_i), A_i, B_i; P_i) \cong M_{n_i}(K(s_i))$; this isomorphism is accomplished via the map $x \mapsto x \circ P$. Moreover, the latter is a central localization of the prime algebra $K_0[M_i] \cong \mathcal{M}^0(K[S_i], A_i, B_i; P_i)$, where S_i is the cyclic semigroup generated by s_i.

Lemma 1 *$K_0[M_i]$ is a prime algebra. Moreover, it does not have nonzero finite dimensional ideals.*

Proof The first assertion was proved in [13], Theorem 5.8. Suppose that J is a nonzero ideal. Then there exist $v, w \in M_i$ such that $vJw \neq 0$. Hence, the matrix type structure of $K_0[M_i]$ implies easily that there exist $v', w' \in M_i$ such that $0 \neq v'vJww' \subseteq K[x_n q_i]$. Then, clearly, $J \cap K[x_n q_i]$ is infinite dimensional. □

We start with calculating the size of the set A_i, for every $i = 0, \ldots, n-2$ and $n \geqslant 3$.

Proposition 4 *For any $i \in \{0, \ldots, n-2\}$ and $n \geqslant 3$ we have $|A_i| = \binom{n}{i+1}$.*

Proof For $i = n-2$ the assertion follows from Lemma 2.5 in [13], so next we assume that $i \leqslant n-3$.
From the description of the set A_i from Theorem 2.1 in [13] it is clear that every element w of A_i is exactly of one of the forms

1. $w = (x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})$ where $i+1 \geqslant s \geqslant 1, s+1 < k_{s+1} < \cdots < k_{i+1} \leqslant n-1$ and $s \geqslant k_s$; for $s = i+1$ we assume that $w = (x_{k_{i+1}} \cdots x_{i+1})$ with $i+1 \geqslant k_{i+1}$;
2. $w = (x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})$ where $i+1 \geqslant s \geqslant 1$, $s < k_s < \cdots < k_{i+1} \leqslant n-1$;
3. $w = 1$.

Choose $1 \leqslant s \leqslant i+1$ and $0 \leqslant i \leqslant n-3$. Then the elements w from Case 1. are in a bijection with strictly increasing sequences $(k_s, \ldots, k_{i+1})$ of natural numbers such that $1 \leqslant k_s \leqslant s < s+2 \leqslant k_{s+1} < \cdots < k_{i+1} \leqslant n-1$. It is easy to see that there exist exactly $s\binom{n-s-2}{i-s+1}$ sequences of the above form. Similarly, elements w of the form as in Case 2. are in a bijection with strictly increasing sequences $(k_s, \ldots, k_{i+1})$ of natural numbers such that $s+1 \leqslant k_s < \cdots < k_{i+1} \leqslant n-1$. There are exactly $\binom{n-s-1}{i-s+2}$ such sequences.

It follows that

$$|A_i| = 1 + \sum_{s=1}^{i+1} \left(\binom{n-s-1}{i-s+2} + s\binom{n-s-2}{i-s+1} \right).$$

Thus, it is enough to prove that $1 + \sum_{s=1}^{i+1}(\binom{n-s-1}{i-s+2} + s\binom{n-s-2}{i-s+1}) = \binom{n}{i+1}$ for $n \geqslant 3$ and $0 \leqslant i \leqslant n-3$.

Moreover, if $i = n - 3$, then by a direct calculation we get that

$$1 + \sum_{s=1}^{n-2} \left(\binom{n-s-1}{n-s-1} + s \binom{n-s-2}{n-s-2} \right) = \binom{n}{n-2},$$

as desired.

It is easy to check that

$$1 + \sum_{s=1}^{i+1} \left(\binom{n-s-1}{i-s+2} + s \binom{n-s-2}{i-s+1} \right) = \sum_{k=0}^{i+1} (i+2-k) \binom{n-i-3+k}{k}.$$

Indeed, substituting $k = i + 1 - s$ in the sum in the left hand side, we get that this sum is equal to

$$\begin{aligned} 1 + \sum_{k=0}^{i} \binom{n-i-2+k}{k+1} + \sum_{k=0}^{i} (i+1-k) \binom{n-i-3+k}{k} &= \\ = 1 + \sum_{k=1}^{i+1} \binom{n-i-3+k}{k} + \sum_{k=0}^{i} (i+1-k) \binom{n-i-3+k}{k} &= \\ = \sum_{k=0}^{i+1} \binom{n-i-3+k}{k} + \sum_{k=0}^{i+1} (i+1-k) \binom{n-i-3+k}{k} &= \\ = \sum_{k=0}^{i+1} (i+2-k) \binom{n-i-3+k}{k}&, \end{aligned}$$

as claimed.

We proceed by induction on n to prove that

$$\sum_{k=0}^{i+1} (i+2-k) \binom{n-i-3+k}{k} = \binom{n}{i+1}.$$

For $i = 0$ and arbitrary $n \geqslant 3$ we have $1 + \binom{n-2}{1} + \binom{n-3}{0} = \binom{n}{1}$ and the assertion follows. If $n = 3$, then we have $0 \leqslant i \leqslant 0$, so the proposition holds.

Assume now that the equality is true for some n and every $i \leqslant n - 3$. Consider the sum

$$\sum_{k=0}^{i+1} (i+2-k) \binom{(n+1)-i-3+k}{k}$$

for $n - 2 > i > 0$. Using $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$ if $k \geqslant 1$ and $\binom{m+1}{0} = \binom{m}{0}$ we get

$$\begin{aligned} &\sum_{k=0}^{i+1} (i+2-k) \binom{(n+1)-i-3+k}{k} = \\ &= \sum_{k=0}^{i+1} (i+2-k) \binom{n-i-3+k}{k} + \sum_{k=1}^{i+1} (i+2-k) \binom{n-i-3+k}{k-1}. \end{aligned}$$

From the induction hypothesis it follows that the first sum is equal to $\binom{n}{i+1}$. Substituting $m = k-1$ and $j = i-1$ we get

$$\sum_{k=1}^{i+1}(i+2-k)\binom{n-i-3+k}{k-1} = \sum_{m=0}^{j+1}(j+2-m)\binom{n-j-3+m}{m}.$$

From the induction hypothesis it follows that the above sum is equal to $\binom{n}{i}$. Now, using $\binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1}$ we get

$$\sum_{k=0}^{i+1}(i+2-k)\binom{(n+1)-i-3+k}{k} = \binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1}$$

and the assertion follows. □

3 Main results

We will identify, without further comments, elements of the monoid C_n with words in free monoid F that are reduced with respect to the system S described in Theorem 3.

Our first main aim is to show that $K[C_n]$ is semiprime. To prove this, we strengthen some of the results from [13].

Consider the automorphism σ of C_n given by $\sigma(x_i) = x_{i+1}$ for $i = 1, \ldots, n$, where we agree that $x_{n+1} = x_1$. The natural extension to an automorphism of $K[C_n]$ also will be denoted by σ. For basic properties of this automorphism we refer to Section 3 in [13].

We have an ideal chain in C_n

$$\emptyset = I_{n-2} \subseteq I_{n-3} \subseteq I_{n-4} \subseteq \cdots \subseteq I_0 \subseteq I_{-1} \tag{2}$$

where $I_i = \{w \in C_n : C_n w C_n \cap \langle x_n q_i \rangle = \emptyset\}$ for $i = 0, \ldots, n-2$, and

$$I_{-1} = I_0 \cup C_n x_n q_0 C_n.$$

In particular, using Corollary 3.17 in [13] we obtain that $\sigma(I_k) = I_k$ for $k = 0, \ldots, n-3$. The key structural result obtained in [13] reads as follows.

Proposition 1 *Consider the ideal chain* (2). *Then*

1. *for* $i = 0, \ldots, n-2$, *the semigroups of matrix type* $M_i = \mathcal{M}^0(S_i, A_i, B_i; P_i)$, *satisfy* $M_i \subseteq I_{i-1}/I_i$,
2. *for* $i = 1, \ldots, n-2$, *the sets* $(I_{i-1}/I_i) \setminus M_i$ *are finite;*
3. $I_{-1}/I_0 = M_0$;
4. $\tilde{M}_{n-2} = M_{n-2} \setminus \{\theta\}$ *is an ideal in* C_n;
5. C_n/I_{-1} *is a finite semigroup.*

The following observation can be deduced from the results and methods of [13].

Lemma 2 M_i *is a right ideal in* C_n/I_i *for every* $i = 0, 1, \ldots, n-2$.

Proof Let $a(x_n q_i)^k b \in \tilde{M}_i$ and take any generator $x_r \in C_n$. Assume that the element $a(x_n q_i)^k b x_r$ is not in $\tilde{M}_i$. We claim that then $a(x_n q_i)^k b x_r \in I_i$. Let b' be the reduced form of $b x_r$. If $b' = x_j \bar{b}$ for some word $\bar{b}$, where $j \leqslant i+1$, then using reduction (4) from Theorem 3 we get that $a(x_n q_i)^k b'$ can be reduced to $a(x_n q_i)^k \bar{b}$. Therefore we can assume that a prefix of b' is equal to x_j, for some $j > i+1$. If $j < n$, then it can be calculated that $a(x_n q_i)^k b x_r$ can be rewritten as a word with a factor of the form $x_{j-1}\cdots x_{i+2}x_n x_1\cdots x_{i+1}x_{n-1}\cdots x_j$ and this element is in I_i by Lemma 3.8 in [13]. Let us now consider the case when x_n is a prefix of b'. As we assume that $a(x_n q_i)^k b' \notin \tilde{M}_i$, this word can be rewritten in C_n as an element without the factor $x_n q_i$. From Theorem 3 it is easy to see that to obtain a word without such a factor one has to use a reduction of type (5). Therefore $a(x_n q_i)^k b'$ can be written as a word with a prefix of the form $a(x_n q_i)^k x_n v x_j$, where $|x_n v|_j = |x_n v|_{j+1} = 0$. Moreover, for $j \leqslant i$ or $j = n-1$ the generator x_{j+1} occurs in $x_n q_i x_n$ after x_j, thus the reduction of x_j of type (5) is not possible in this case. Therefore $n-1 > j \geqslant i+1$. It follows (see Lemma 2.3 in [13]) that such a prefix is of the form $a(x_n q_i)^k x_n x_1\cdots x_j$. Therefore this element has a factor $x_n x_1\cdots x_i x_{n-1}\cdots x_{i+1}x_n x_1\cdots x_j$ for some $n-1 > j \geqslant i+1$. It can be checked (using the reductions from Theorem 3) that the latter word can be rewritten as an element with the factor $x_{n-1}\cdots x_{j+1}x_n x_1\cdots x_j$, which is in $I_{j-1}\subseteq I_i$, by Lemma 3.8 in [13]. The assertion follows. □

The following lemma provides a crucial step in the proof of Theorem 1. By $\mathcal{P}(K[C_n])$ we denote the prime radical of $K[C_n]$.

Lemma 3 *Assume that J is a finite dimensional ideal of $K[C_n]$. Then $J = 0$. In particular, the left annihilator $A = \{\alpha \in K[C_n] : \alpha K[M] = 0\}$ of $K[M]$ in $K[C_n]$ is zero. Moreover, $K[C_n]$ is a semiprime algebra.*

Proof Suppose that $J \neq 0$ is a finite dimensional ideal of $K[C_n]$. First, we claim that a nonzero element $\alpha \in J$ can be chosen so that for every $i = 1,\ldots,n$ we have $wx_i = w$ for all $w \in supp(\alpha)$ or $\alpha x_i = 0$.

Let $0 \neq \alpha \in J$ be such that $|supp(\alpha)|$ is minimal possible. Let $supp(\alpha) = \{v_1,\ldots,v_k\}$. Since J is finite dimensional, the set Z consisting of all such k-tuples is finite.

Let $\mathcal{R}$ denote the Green's relation on the monoid C_n, see [3]. Consider the $\mathcal{R}$-order $\leq_{\mathcal{R}}$ on C_n; in other words, we write $w \leq_{\mathcal{R}} v$ if $wC_n \subseteq vC_n$. Then define a relation $\preceq$ on C_n^k by: $(u_1,\ldots,u_k) \preceq (w_1,\ldots,w_k)$ if $u_i \leq_{\mathcal{R}} w_i$ for every $i = 1,\ldots,k$.

Now, by the choice of α, for every $x \in C_n$ we have that either $\alpha x = 0$ or $supp(\alpha x) = \{v_1 x,\ldots,v_k x\}$ and in the latter case $(v_1 x,\ldots,v_k x) \preceq (v_1,\ldots,v_k)$. Since the set Z introduced above is finite, we may further choose an element α for which the k-tuple $(v_1,\ldots,v_k)$ is minimal possible with respect to $\preceq$. Then $v_i \mathcal{R} v_i x$ for every i. Since the monoid C_n is $\mathcal{J}$-trivial by [4], Theorem 4.5.3, it follows that for every j we either have $wx_j = w$ for every $w \in supp(\alpha)$ or $\alpha x_j = 0$, as claimed.

Next, assume that $\beta \in K[C_n]$ is a nonzero element such that $wx_1 = w$ holds in C_n for every $w \in supp(\beta)$. Then $|w|_1 > 0$ for every such w. Write $w = w_0 x_1 w_1$, for some reduced words w_0, w_1 such that $|w_1|_1 = 0$. We claim that then $|w_1|_n = 0$. Indeed, if $w_1 = ux_n v$ with $|v|_n = 0$, then $wx_1 = w_0 x_1 u x_n v x_1$ and then the only possible reduction that allows to decrease the length of this word (needed in order to get $wx_1 = w$ in C_n) is of the form $x_1 z x_1 \to z x_1$, where z is a prefix of $ux_n v$ containing ux_n. But then we do not get $wx_1 = w$ in C_n because x_1 appears after the last occurrence of x_n in the reduced form of wx_1), a contradiction. So $|w_1|_n = 0$, as claimed.

Assume first that $|w|_n > 0$. Write $w = sx_n t x_1 w_1$, for some reduced words s, t (so $w_0 = sx_n t$) such that $|t|_n = 0$. Then also $|t|_1 = 0$ because w is reduced. Hence, either $wx_n = sx_n t x_1 w_1 x_n$ is a reduced word with $|wx_n|_n \geqslant 2$ (if $|tw_1|_{n-1} > 0$) or $wx_n = w$ in C_n and the reduced form of $wx_n = w$ does not end with generator x_n (if $|tw_1|_{n-1} = 0$).

Next, consider the case when $|w|_n = 0$. It is clear that in this case wx_n is a reduced word, and $|wx_n|_n = 1$. Together with the previous paragraph of the proof this implies that $wx_n \neq w'x_n$ in C_n for all $w, w' \in supp(\beta)$ with $w \neq w'$.

We have proved that the hypotheses on β imply that $\beta x_n \neq 0$.

Now, we apply this observation to the element α. Because of the choice of α, we get that if $\alpha x_1 = \alpha$ then $\alpha x_n = \alpha$. Using the automorphism σ (and noting that $\sigma(\alpha)$, as an element of the finite dimensional ideal $\sigma(J)$ of $K[C_n]$, inherits the hypotheses on α) we get that $\sigma(\alpha)x_1 = \sigma(\alpha)$, so that $\sigma(\alpha)x_n = \sigma(\alpha)$, by the above argument applied to $\sigma(\alpha)$ in place of α. Thus, $\alpha x_{n-1} = \alpha$, by applying α^{-1}. Repeating this argument several times, we then get $\alpha x_j = \alpha$ for every j. A similar argument shows that if $\alpha x_k \neq 0$ for some k, then $\alpha x_j \neq 0$ for every j. However, $\alpha = \alpha x_n x_1 x_2 \cdots x_{n-1} \in J \cap K[\tilde{M}_{n-2}]$, a finite dimensional ideal of $K[\tilde{M}_{n-2}]$, because $x_n x_1 \cdots x_{n-1} \in \tilde{M}_{n-2} \subseteq M$ and $\tilde{M}_{n-2}$ is an ideal of C_n. Therefore, Lemma 1 implies that $\alpha = 0$. This contradiction shows that we may assume that $\alpha x_j = 0$ for every j.

Let $w \in supp(\alpha)$ be maximal with respect to the order $\leq_{\mathcal{R}}$. If x_j is the last letter of the (reduced form of the) word w then $w = wx_j = w'x_j$ in C_n, for some $w' \in supp(\alpha)$. This implies that $w \leq_{\mathcal{R}} w'$, so by the choice of w we get $w = w'$, a contradiction. Therefore $J = 0$.

By Theorem 5.9 in [13], $\mathcal{P}(K[C_n]) \cap K[\tilde{M}] = 0$ because $\tilde{M} = \bigcup_{i=0}^{n-2} \tilde{M}_i$ and every $K[M_i]$ is prime. So, we know that $A \cap K[\tilde{M}] = 0$ and therefore A and $\mathcal{P}(K[C_n])$ are finite dimensional, because $C_n \setminus \tilde{M}$ is finite. Hence, the assertion follows. □

We are now in a position to prove Theorem 1.

Proof In view of Lemma 3, from Theorem 5.9 in [13] we know that $K[C_n]$ is a Noetherian semiprime PI-algebra.

For any fixed $i = 0, \dots, n-2$, let J_i be a maximal among all ideals of $K[C_n]$ intersecting $K[x_n q_i]$ trivially and such that $K[I_i] \subseteq J_i$. Then J_i is a prime ideal. By Corollary 10.16 in [8], $GK\dim(R) = clK\dim(R)$ (the Gelfand-Kirillov and the classical Krull dimensions) for every finitely generated Noetherian PI-algebra R. Since $GK\dim(K[C_n]) = 1$, it follows that J_i is a minimal prime ideal of $K[C_n]$. Clearly, the image J_i' of J_i in $K[C_n]/K[I_i]$ is a prime ideal. M_i is a right ideal in C_n/I_i by Lemma 3, and thus it is a two-sided ideal because C_n/I_i is endowed with a natural involution which preserves M_i, by Corollary 3.12 and Lemma 3.18 in [13]. Since $K[M_i]$ is a prime algebra, it follows that the classical quotient rings of $K[M_i]$ and $K[C_n]/J_i$ are equal. Moreover, as explained in the introduction, the classical ring of quotients of $K[M_i]$ is naturally isomorphic to $M_{n_i}(K(x))$, where $n_i = |A_i|$ for $i = 0, \dots, n-2$. Therefore, $J = \bigcap_{i=0}^{n-2} J_i$ is a semiprime ideal of $K[C_n]$ such that $J \cap K[M] = 0$ (by the definition of the ideals J_i). Since $C_n \setminus M$ is finite, J is finite dimensional, whence $J = 0$ by Lemma 3. We obtain that the quotient ring Q of $K[C_n]$ satisfies $Q \cong \prod_{i=0}^{n-2} M_{n_i}(K(x))$, $i = 0, \dots, n-2$. In view of Proposition 4, this completes the proof. □

Our second main result describes the radical of a Hecke–Kiselman algebra $K[HK_\Theta]$, as well as the algebra modulo the radical, in the case of PI-algebras. So, assume that Θ is a finite oriented graph such that $K[HK_\Theta]$ is a PI-algebra. This is equivalent to saying that Θ does not contain two cyclic subgraphs (i.e. subgraphs which are cycles) connected by an oriented path, [10]. Let ρ be the congruence on HK_Θ generated by all pairs (xy, yx) such that there is an arrow $x \to y$ that is not contained in any cyclic subgraph of Θ. (If there is no such a pair then we assume that ρ is the trivial congruence.) Let Θ' be the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ. Then $HK_{\Theta'} \cong HK_\Theta/\rho$. Then the connected components of Θ' are either singletons or cyclic subgraphs.

Now, we are in a position to prove Theorem 2.

Proof Suppose that a vertex $x \in V(\Theta)$ is a source vertex. In other words, there is an arrow $x \to y$ for some $y \in V(\Theta)$ but there are no arrows of the form $z \to x$. For any $w \in HK_\Theta$ consider the element $\beta = (xy - yx)w(xy - yx) \in K[HK_\Theta]$. Since x is a source vertex, we know that $xvx = xv$ in HK_Θ for every $v \in HK_\Theta$. Hence $xwxy = xwy, xwyx = xwy$. Similarly, $xywxy = xywy$ and $xywyx = xywy$. Therefore $\beta = 0$. It follows that $xy - xy \in \mathcal{P}(K[HK_\Theta])$.

If x is a sink, that is there is an arrow $z \to x$ for some $z \in V(\Theta)$ but there are no arrows of the form $x \to y$ in the graph Θ, a symmetric argument shows that $xz - zx \in \mathcal{P}(K[HK_\Theta])$ for all z such that $z \to x$ in Θ. Let ρ_1 be the congruence generated by all pairs (xy, yx) such that x or y is either source or sink and there is an arrow $x \to y$ that is not contained in any cyclic subgraph of Θ. Equivalently, we may consider the graph Γ_1 obtained by erasing in Θ all such arrows $x \to y$ and $z \to x$ as above. Then $K[HK_{\Gamma_1}] \cong K[HK_\Theta]/I(\rho_1)$. We have shown that $I(\rho_1) \subseteq \mathcal{P}(K[HK_\Theta])$. Repeating this argument finitely many times we easily get that $I(\rho) \subseteq \mathcal{P}(K[HK_\Theta])$ (and our argument shows that $I(\rho)$ is nilpotent, because Θ is finite).

Since we know that $\mathcal{J}(K[HK_\Theta]) = \mathcal{P}(K[HK_\Theta])$, to prove the first assertion of the theorem it is now enough to check that $K[HK_{\Theta'}]$ is semiprime. $HK_{\Theta'}$ is the direct product of all HK_{Θ_i}, where Θ_i, $i = 1, \ldots, m$, are the connected components of Θ' . From [10] we know that each HK_{Θ_i} is either a band with two elements (if Θ_i has only one vertex) or it is isomorphic to C_k for some $k \geqslant 3$. In the former case $K[HK_{\Theta_i}] \cong K \oplus K$, in the latter $K[HK_{\Theta_i}]$ is a semiprime PI-algebra (by Theorem 1) of Gelfand-Kirillov dimension one [10], and hence it is a finitely generated module over its center [16]. It follows easily that $K[HK_\Theta]$ is a finitely generated module over its center.

Let Q_i be the classical ring of quotients of $K[HK_{\Theta_i}]$. If $HK_{\Theta_i} = C_{m_i}$ for some m_i then we know that Q_i is a central localization of the form described in Theorem 1. Clearly, $HK_{\Theta'}$ is the direct product $\prod_{i=1}^m HK_{\Theta_i}$. Then in the localization $Q = Q_1 \otimes \cdots \otimes Q_m$ of $K[HK_{\Theta'}] \cong \bigotimes_{i=1}^m K[HK_{\Theta_i}]$ each of the factors is isomorphic to $K \oplus K$ or to $\prod_{j=0}^{m_i-2} M_{r_j}(K(x))$, where $r_j = \binom{m_i}{j+1}$. Therefore, the tensor product is semiprime. Hence $K[HK_{\Theta'}]$ is semiprime, because Q is its central localization. It is now clear that $K[HK_{\Theta'}] \cong K[HK_\Theta]/\mathcal{P}(K[HK_\Theta])$. The result follows. □

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