From the Potential to the First Hochschild Cohomology Group of a Cluster Tilted Algebra

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Dedicated to José Antonio de la Peña for his 60th birthday

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Abstract

The objective of this paper is to give a concrete interpretation of the dimension of the first Hochschild cohomology space of a cyclically oriented or tame cluster tilted algebra in terms of a numerical invariant arising from the potential.

Keywords Hochschild cohomology · Cluster tilted algebras · Triangulated surfaces

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Introduction

In this paper, we present a concrete computation of the first Hochschild cohomology group of cyclically oriented and tame cluster tilted algebras. Cluster tilted algebras were introduced in [21] and independently in [22] for type $\mathbb{A}$, as an application of the categorification of the cluster algebras of Fomin and Zelevinsky, see [19]. Cyclically oriented cluster tilted algebras were defined in [16] where it is shown to be the largest known class of cluster tilted algebras for which a system of top relations, called minimal relations in [20], is known.

Our motivation is twofold. First, it can be argued that any (co)-homology theory has for object to detect and even compute cycles. In cluster tilted algebras, there are cycles which naturally occur: indeed, any cluster tilted algebra can be represented as the jacobian algebra of a quiver with potential, the latter being a linear combination of cycles in its quiver [29]. We are interested here in the relation between the cycles appearing in the potential and the first Hochschild cohomology group of the given cluster tilted algebra. Our second motivation, more *ad hoc*, comes from the very simple formula given in [10, Theorem 1.2], for a representation-finite cluster tilted algebra, allowing to read the dimension of the first Hochschild cohomology space directly in the ordinary quiver of the algebra. It is natural to ask for which class of (cluster tilted) algebras does this dimension depend only on the quiver. Because representation-finite cluster tilted algebras are cyclically oriented, the latter class is a natural candidate. However, cyclically oriented cluster tilted algebras are generally representation-infinite, they may be tame or wild. For tame, not necessarily cyclically oriented cluster tilted algebras, the first Hochschild cohomology group has been studied in [10]. We therefore investigate this class as well.

We first recall from [21] that an algebra B is called *cluster tilted* if there exists a basic tilting object T in the cluster category $\mathcal{C}$ of a hereditary algebra such that $B = \text{End}_{\mathcal{C}} T$. This is related to the well-known notion of tilted algebras, see, for instance, [12], namely an algebra is *tilted* if there exist a hereditary algebra A and a basic tilting module T_A such that $C = \text{End } T_A$. Indeed, it was shown in [3] that an algebra B is cluster tilted if and only if there exists a tilted algebra C such that B is isomorphic to the trivial extension of C by the $C - C$ -bimodule $E = \text{Ext}_{\mathcal{C}}^2(DC, C)$, we then say that B is the *relation extension* of C . In [4], a certain equivalence relation was defined on the set of cycles occurring in the Keller potential W of B . The number N_W of equivalence classes under this relation is called the *potential invariant* of B . If B is a cyclically oriented cluster tilted algebra, then it follows from [16] that W , and therefore N_W , only depend on the ordinary quiver of B . In contrast, if B is a tame cluster tilted algebra, not necessarily cyclically oriented, then W does not only depend on the quiver of B , but also on its presentation. However, our main result says that if B is a cluster tilted algebra which is cyclically oriented or tame of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$, then the potential invariant N_W equals the dimension of the first Hochschild space of B . Moreover E viewed as a $C - C$ -bimodule has N_W indecomposable summands which are pairwise orthogonal bricks so that N_W equals also the dimension of the endomorphism algebra of the bimodule ${}_C E_C$. We thus state our Theorem.

Theorem A. *Let B be a cyclically oriented cluster tilted algebra or a cluster tilted algebra of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$, having N_W as potential invariant, C a tilted algebra such that B is the relation extension of C by $E = \text{Ext}_{\mathcal{C}}^2(DC, C)$. Then we have*

$$\dim_{\mathbb{k}} \text{HH}^1(B) = N_W = \dim_{\mathbb{k}} \text{End}_{C-C}(E).$$

Moreover, the indecomposable summands of ${}_C E_C$ are pairwise orthogonal bricks.

For a cluster tilted algebra of type $\tilde{A}$, the situation is slightly different: in this case, the dimension of $\mathrm{HH}^1(B)$ equals $N_W + \epsilon$, where $\epsilon = 3$ if B contains a double arrow, $\epsilon = 2$ if it contains a proper bypass, and $\epsilon = 1$ otherwise, see [9, 10] or 3.9 below. Our theorem shows that, if B is a cyclically oriented or tame cluster tilted algebra, then the dimension of $\mathrm{HH}^1(B)$ depends only on the quiver of B . The formula of [10, Theorem 1.2] in the representation-finite case is a special case of our theorem. Furthermore, the last statement proves Conjecture 4.6 of [5] for tilted algebras whose relation extensions are cyclically oriented. Another nice consequence of the theorem is that, in this case, the number of indecomposable summands of the bimodule ${}_C E_C$, and the dimension of $\mathrm{End}_{C-C}(E)$ do not depend on C , but only on the quiver of B .

We also present another point of view. It is known that for gentle algebras, a geometric model allows to compute the Hochschild cohomology group [23, 33]. Of course, not all the cluster-tilted algebras we are dealing with arise from marked surfaces. This is however the case for cluster-tilted algebras of type $\mathbb{D}$, which are representation-finite and hence cyclically oriented, and those of type $\tilde{\mathbb{D}}$, which are tame and not always cyclically oriented. For the notation below, we refer the reader to Section 4. We prove the following Theorem.

Theorem B. *Let B be a cluster tilted algebra of type $\mathbb{D}_n$ or $\tilde{\mathbb{D}}_n$ and (S, T) its geometric realisation. Then*

$$\dim_k \mathrm{HH}^1(B) = |\Delta_{\mathrm{NRel}(p,q)}| + m_p + m_q - m_{p,q}$$

where $m_{p,q}$ is equal to one if and only if there exist a triangle $\Delta \in \Delta_{\mathrm{Rel}(p)} \cap \Delta_{\mathrm{Rel}(q)}$

Recall that the geometric model associated with cluster tilted algebras of type $\mathbb{D}$ and $\tilde{\mathbb{D}}$ involves punctures. One of the features of this theorem is that the first Hochschild cohomology group depends on the triangles incident to punctures.

On the other hand, we also give a geometric description of the notion of admissible cuts of [14], which completes the previous description of [1, 24].

The paper is organised as follows. Section 1 is devoted to preliminaries, Section 2 to cluster tilted algebras, Section 3 to the proof of our Theorem A, and Section 4 to the proof of Theorem B.

1 Preliminaries

1.1 Notation

Let $\mathbb{k}$ be an algebraically closed field. It is well-known that any basic and connected finite dimensional $\mathbb{k}$ -algebra C can be written as $C \simeq \mathbb{k}Q/I$, with Q a finite connected quiver and I an admissible ideal of $\mathbb{k}Q$. The pair (Q, I) is then called a *bound quiver*, and the isomorphism $C \simeq \mathbb{k}Q/I$ is a *presentation* of C , see [12]. We denote by Q_0 the set of points of Q and by Q_1 its set of arrows. Following [18], we sometimes consider an algebra C as a category of which the object class C_0 is Q_0 and the set of morphisms from x to y is $C(x, y) = e_x C e_y$, where e_x, e_y are the primitive idempotents of C associated to x and y respectively. A full subcategory D of C is *convex* if for any $x, y \in D_0$, and any path $x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_t = y$ in the category C , we have $x_i \in D_0$ for all i . An algebra C is *constricted* if for any arrow $\alpha: x \rightarrow y$ we have $\dim_{\mathbb{k}} C(x, y) = 1$.

A *relation* from $x \in Q_0$ to $y \in Q_0$ is a linear combination $\rho = \sum_{i=1}^m \lambda_i w_i$ where each λ_i is a nonzero scalar and the w_i are pairwise different paths in Q of length at least two from x to y . Relations in a bound quiver (Q, I) are generated by top relations (called minimal in

[20]). Let $\mathbb{k}Q^+$ be the two-sided ideal of $\mathbb{k}Q$ consisting of the linear combinations of paths of length at least one, then a relation $\rho \in e_x I e_y$ is called a *top relation* if its residual class in $e_x \left(\frac{I}{\mathbb{k}Q^+ \cdot I + I \cdot \mathbb{k}Q^+} \right) e_y$ is nonzero. We use the word “top” in order to underline that these relations belong to the top of I considered as a $\mathbb{k}Q^+ - \mathbb{k}Q^+$ -bimodule. If $C \simeq \mathbb{k}Q/I$, then the ideal I is always generated by finitely many top relations. A *system of relations* for a bound quiver algebra $C \simeq \mathbb{k}Q/I$ is a subset $\mathcal{R}$ of I consisting of relations such that $\mathcal{R}$ but no proper subset of it generates I as an ideal [17]. A relation $\rho = \sum_{i=1}^m \lambda_i w_i$ is *monomial* if $m = 1$ and *minimal* if, for every nonempty proper subset J of $\{1, \dots, m\}$ we have $\rho = \sum_{i \in J} \lambda_i w_i \notin I$. It is *strongly minimal* if, for any subset J as above and every set of nonzero scalars $\{\lambda'_j | j \in J\}$, we have $\rho = \sum_{j \in J} \lambda'_j w'_j \notin I$. We have the following Lemma.

Lemma [10, (1.2)] *Let $C = \mathbb{k}Q/I$ with I generated by a system of relations $\mathcal{R} = \{\rho_1, \dots, \rho_t\}$. If each ρ_i is a linear combination of paths that do not contain oriented cycles, then $\mathcal{R}$ can be replaced by a system of strongly minimal relations $\mathcal{R}' = \{\rho'_1, \dots, \rho'_t\}$ such that each ρ'_i is a linear combination of the paths appearing in ρ_i .*

Two paths u, v in a quiver Q are *parallel* if they have the same source and the same target, and *antiparallel* if the source (or the target) of u is the target (or the source, respectively) of v . A quiver is *acyclic* if it contains no oriented cycles. Algebras with acyclic quivers are called *triangular*. For more notions or results of representation theory, we refer the reader to [12].

1.2 Hochschild cohomology

Let C be an algebra and E a $C - C$ -bimodule which is finite dimensional over $\mathbb{k}$. The *Hochschild complex* is the complex

$$0 \longrightarrow E \xrightarrow{b^1} \text{Hom}_{\mathbb{k}}(C, E) \xrightarrow{b^2} \dots \longrightarrow \text{Hom}_{\mathbb{k}}(C^{\otimes i}, E) \xrightarrow{b^{i+1}} \text{Hom}_{\mathbb{k}}(C^{\otimes i+1}, E) \xrightarrow{b^{i+2}} \dots$$

where $C^{\otimes i}$ is defined inductively by $C^{\otimes 1} = C$ and $C^{\otimes i} = C^{\otimes (i-1)} \otimes_{\mathbb{k}} C$ for $i > 1$. The map $b^1 : E \rightarrow \text{Hom}_{\mathbb{k}}(C, E)$ is defined by $(b^1 x)(c) = cx - xc$ for $x \in E$, $c \in C$, and b^{i+1} is defined by

$$\begin{aligned} (b^{i+1} f)(c_0 \otimes \dots \otimes c_i) &= c_0 f(c_1 \otimes \dots \otimes c_i) \\ &\quad + \sum_{j=1}^i (-1)^j f(c_0 \otimes \dots \otimes c_{j-1} c_j \otimes \dots \otimes c_i) \\ &\quad + (-1)^{i+1} f(c_0 \otimes \dots \otimes c_{i-1}) c_i \end{aligned}$$

for a $\mathbb{k}$ -linear map $f : C^{\otimes i} \rightarrow E$ and elements $c_0, \dots, c_i \in C$.

The i^{th} cohomology group of this complex is the i^{th} *Hochschild cohomology group* of C with coefficients in E , denoted $H^i(C, E)$. If ${}_C E_C = {}_C C_C$, we write $\text{HH}^i(C)$ instead of $H^i(C, C)$. For instance, $\text{HH}^0(C)$ is the centre of the algebra C .

Let $\text{Der}(C, E)$ be the vector space of all *derivations*, that is, $\mathbb{k}$ -linear maps $d : C \rightarrow E$ such that for any $c, c' \in C$ we have

$$d(cc') = cd(c') + d(c)c'.$$

A derivation d is *inner* if there exists $x \in E$ such that $d(c) = cx - xc$ for any $c \in C$. Letting $\text{Inn}(C, E)$ denote the subspace of $\text{Der}(C, E)$ consisting of all inner derivations, we have

$$H^1(C, E) = \frac{\text{Der}(C, E)}{\text{Inn}(C, E)}.$$

Further, given a complete set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$ of C , a derivation $d : C \rightarrow E$ is called *normalised* if $d(e_i) = 0$ for all i . Let $\text{Der}_0(C, E)$ be the subspace of $\text{Der}(C, E)$ consisting of the normalised derivations, and $\text{Inn}_0(C, E) = \text{Der}_0(C, E) \cap \text{Inn}(C, E)$. Then we also have (see [28]).

$$H^1(C, E) = \frac{\text{Der}_0(C, E)}{\text{Inn}_0(C, E)}.$$

When we deal with a derivation, we may always assume implicitly that it is normalised.

We recall the definition of a simply connected algebra, see, for instance [32]. Let (Q, I) be a bound quiver. For an arrow α , we denote its formal inverse by α^{-1} . A walk from x to y in Q is a formal composition $\alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \dots \alpha_t^{\epsilon_t}$ (where $\alpha_i \in Q_1$, $\epsilon_i = \pm 1$ for i such that $1 \leq i \leq t$) starting at x and ending at y . We denote by ε_x the stationary path at x . Next, let $\sim$ be the least equivalence on the set of walks in Q such that:

- (a) If $\alpha : x \rightarrow y$ is an arrow, then $\alpha^{-1}\alpha \sim \varepsilon_x$ and $\alpha\alpha^{-1} \sim \varepsilon_y$.
- (b) If $\rho = \sum_{i=1}^m \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j such that $1 \leq i, j \leq m$.
- (c) If $u \sim v$, then $wuw' \sim wv w'$ whenever these compositions make sense.

Denote by $[u]$ the equivalence class of a walk u . Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes of closed walks starting and ending at x is a group for the operation $[u][v] = [uv]$. This group does not depend on the choice of x and therefore is denoted as $\pi_1(Q, I)$ and called the *fundamental group* of (Q, I) . A triangular algebra C is said to be *simply connected* if, for any presentation $C \simeq \mathbb{k}Q/I$, the group $\pi_1(Q, I)$ is trivial. It is *strongly simply connected* if every full convex subcategory of C is simply connected. This is equivalent to requiring that, for every full convex subcategory D of C , one has $\text{HH}^1(D) = 0$, see [32].

2 Cluster Tilted Algebras

2.1 Bound Quivers

Cluster tilted algebras were originally defined as endomorphism rings of basic tilting objects in the cluster category of a hereditary algebra [21]. We use the following equivalent definition [3]. Let C be a triangular algebra of global dimension at most two. Its trivial extension $\tilde{C} = C \ltimes E$ by the so-called *relation bimodule* $E = \text{Ext}_C^2(DC, C)$ with the natural C -actions, is called the *relation extension* of C . If C is tilted of type Q , then $\tilde{C}$ is called *cluster tilted of type Q* . Assume $C = \mathbb{k}Q/I$ and $\mathcal{R} = \{\rho_1, \dots, \rho_k\}$ is a system of relations for I , then the quiver $\tilde{Q}$ of $\tilde{C}$ is as follows:

- (a) $\tilde{Q}_0 = Q_0$.
- (b) For $x, y \in Q_0$, the set of arrows in $\tilde{Q}$ from x to y equals the set of arrows of Q from x to y , called *old arrows* plus, for each top relation ρ_i from y to x , an additional arrow $\alpha_i : x \rightarrow y$ (called *new arrow*), see [3, Theorem 2.6].

A *potential* on a quiver is a linear combination of oriented cycles in the quiver. The *Keller potential* on $\tilde{Q}$ is the sum

$$W = \sum_{i=1}^k \alpha_i \rho_i$$

where α_i, ρ_i are as in (b), above. Oriented cycles are considered up to cyclic permutation: two potentials are *cyclically equivalent* if their difference lies in the $\mathbb{k}$ -vector space generated by all elements of the form $\beta_1\beta_2\cdots\beta_m - \beta_m\beta_1\cdots\beta_{m-1}$, where $\beta_1\cdots\beta_m$ is an oriented cycle in $\tilde{Q}$. For an arrow β , the *cyclic partial derivative* ∂_β is the $\mathbb{k}$ -linear map defined on an oriented cycle $\beta_1\beta_2\cdots\beta_m$ by

$$\partial_\beta(\beta_1\beta_2\cdots\beta_m) = \sum_{\beta=\beta_i} \beta_i + 1\cdots\beta_m\beta_1\cdots\beta_{i-1}$$

and extended by linearity to W . Thus $\partial_\beta(W)$ is invariant under cyclic permutations. The *Jacobian algebra* $\mathcal{J}(\tilde{Q}, W)$ is the one given by the quiver $\tilde{Q}$ bound by all cyclic partial derivatives of the Keller potential with respect to each arrow of $\tilde{Q}$. If C is a tilted algebra, so that $\tilde{C} = C \ltimes E$ is cluster tilted, then $\tilde{C} \simeq \mathcal{J}(\tilde{Q}, W)$, see, for instance, [29].

Clearly, the Keller potential W depends on the relations, and thus on the presentation of the algebra. For instance, if C is given by the quiver Q

$$1 \xrightarrow{\alpha''} 2 \xrightarrow[\gamma]{\beta} 3$$

and we consider the two-sided ideals $I_1 = \langle \alpha\beta \rangle$ and $I_2 = \langle \alpha\beta - \alpha\gamma \rangle$, then $\mathbb{k}Q/I_1 \simeq \mathbb{k}Q/I_2$. In the first case $\tilde{C}$ is given by the quiver $\tilde{Q}$

$$1 \xrightarrow{\alpha} 2 \xrightarrow[\gamma]{\beta''} 3 \quad \text{with a curved arrow } \eta \text{ from } 3 \text{ to } 1$$

with potential $W_1 = \alpha\beta\eta$, and in the second case by the same quiver $\tilde{Q}$ but with potential $W_2 = \alpha\beta\eta - \alpha\gamma\eta$.

2.2 Sequential walks

Let $C \simeq \mathbb{k}Q/I$ be a tilted algebra. A walk in Q is called *reduced* if it contains no subwalk of one of the forms $\alpha^{-1}\alpha$ or $\alpha\alpha^{-1}$. Let $w = uw'v$ be a reduced walk in Q . We say that the subwalks u, v *point to the same direction* if both u, v or both u^{-1}, v^{-1} are paths in Q . Following [10, 11], a reduced walk $w = uw'v$ with u, v pointing to the same direction is called a *sequential walk* in (Q, I) if the following hold:

- There is a top relation $\rho = \sum_i \lambda_i u_i$ with $u = u_1$ or $u = u_1^{-1}$, there is a top relation $\sigma = \sum_j \mu_j v_j$ with $v = v_1$ or $v = v_1^{-1}$.
- No subpath of w' or $(w')^{-1}$ lies in I nor is equal to some w_k , where $\varsigma = \sum_k v_k w_k$ is a relation which has a point in common with one of the u_i or the v_j (such a w_k is called a *branch* of ς).
- w' itself has no arrows in common with one of the u_i or the v_j

Let $\tilde{C}$ be the relation extension of C . A walk $w = \alpha w' \beta$ is C -sequential if w' consists of old arrows, α, β are new arrows corresponding to old relations $\rho = \sum_i \lambda_i u_i$ and $\sigma = \sum_j \mu_j v_j$ such that, for any i, j , the walk $u_i w' v_j$ is sequential. We need essentially the following result.

Lemma [10, 11] *Let C be a tilted algebra. Then the bound quiver of its relation extension $\tilde{C}$ contains no C -sequential walk.*

2.3 Tame cluster tilted algebras

Because of Lemma 1.1, if $C = \mathbb{k}Q/I$ is a tilted algebra having $B = \tilde{C} = \mathbb{k}\tilde{Q}/\tilde{I}$ as relation extension, then one may choose a system of strongly minimal relations as generating set for $\tilde{I}$. Moreover, if $\rho = \sum \lambda_i w_i$ is a strongly minimal relation lying in $\tilde{I}$ but not in I , then each of the w_i contains exactly one new arrow α_i , because of Lemma 2.2, and each new arrow appears in this way. If this is the case, then we write $\alpha_i | w_i$.

This brings us to our next definition. We define a relation $\approx_Q$ on the set $\tilde{Q}_1 \setminus Q_1$ of new arrows by setting $\alpha \approx_Q \beta$ if α, β are equal or else there exists a strongly minimal relation $\rho = \sum \lambda_i w_i$ in $\tilde{I}$ and indices i, j such that $\alpha | w_i$ and $\beta | w_j$. We next let $\sim_Q$ be the least equivalence relation containing $\approx_Q$, that is, its transitive closure. The relation invariant $N_{B,C}$ of $B = \tilde{C}$ with respect to C is the number of equivalence classes of new arrows with respect to $\sim_Q$.

Assume now that B is tame. Because of [21], the tame representation-infinite cluster tilted algebras are just those of euclidean type, and the representation-finite are those of Dynkin type. We have the following Theorem.

Theorem *Let C be a tilted algebra, and B its relation extension. Then:*

- (a) [5, (6.3)], [7, (5.7)] *There exists a short exact sequence of vector spaces*

$$0 \longrightarrow H^1(B, E) \longrightarrow HH^1(B) \longrightarrow HH^1(C) \longrightarrow 0.$$

- (b) [10, 5.5] $H^1(B, E) \simeq H^1(C, E) \oplus \text{End}_{C-C}(E)$, where the endomorphisms of E are the $C - C$ -bimodule endomorphisms.
- (c) [10, (1.1)] *If B is tame, then $H^1(B, E) = \mathbb{k}^{N_{B,C}}$.*
- (d) [10, (3.2)(3.8)] *If B is of type $\tilde{\mathbb{A}}$ and $\mathcal{R}$ is a system of relations for C , then $N_{B,C} = |\mathcal{R}|$ and does not depend on the choice of C .*
- (e) [10, (4.3)(5.6)(5.7)] *If B is of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$, then one can assume that C is constricted and then $H^1(C, E) = 0$ and $H^1(B, E) = \text{End}_{C-C}(E) = \mathbb{k}^{N_{B,C}}$.*

In particular, in the latter case, E is, as $C - C$ -bimodule, the direct sum of $N_{B,C}$ pairwise orthogonal bricks.

2.4 Cyclically Oriented Cluster Tilted Algebras

Let Q be a quiver. A chordless cycle in Q is the full subquiver generated by a set of points $\{x_1, x_2, \dots, x_t\}$ which is topologically a cycle, that is, the edges between the x_i 's are precisely those of the form $x_i - x_{i+1}$ (with $x_{t+1} = x_1$), see [15]. The quiver Q is cyclically oriented if each chordless cycle in Q is an oriented cycle, see [16]. A cluster tilted algebra is cyclically oriented if it has a cyclically oriented quiver.

In particular, cyclically oriented cluster tilted algebras contain no multiple arrows in their quivers. For instance, as shown in [20], the representation-finite cluster tilted algebras are cyclically oriented.

We now give an example of a family of cyclically oriented cluster tilted algebras which can be representation-finite, tame or wild.

Example Consider the algebra C_t given by the quiver

with $t \geq 1$ bound by the relation $\sum_{i=1}^t \alpha_i \beta_i = 0$. If $t \leq 2$, then C_t is a representation-finite tilted algebra. If $t = 3$, then it is tame concealed of type $\mathbb{D}_4$, while if $t > 3$, then C_t is wild concealed of type

Thus, C_t is tilted for any t . Its relation extension $\tilde{C}_t$ is given by the quiver

If $t \geq 3$, it is cluster concealed, tame for $t = 3$, wild if $t > 3$. It has Keller potential

$$W = \sum_{i=1}^t \gamma(\alpha_i \beta_i).$$

It is easily seen to be cyclically oriented.

The set of relations described in Section 2.1 for cluster tilted algebras is usually not a system of top relations, see [16]. The problem of finding a system of top relations for cluster tilted algebras is still open in general. It is only solved for representation-finite cluster tilted algebras [20], for cluster tilted algebras of type $\tilde{A}$, see [2], and for cyclically oriented cluster tilted algebras [16].

Let Q be a cyclically oriented quiver. A subset of Q_1 consisting of exactly one arrow from each chordless oriented cycle of Q is called an *admissible cut* of Q . If Q is equipped with a potential W , then the *algebra of the cut* is the quotient of the jacobian algebra $\mathcal{J}(Q, W)$ which is obtained by deleting the arrows of the cut, see [14]. Finally a path γ in Q which is antiparallel to an arrow ξ is called a *shortest path* if the full subquiver generated by the oriented cycle $\xi\gamma$ is chordless.

Theorem [16] *Let B be a cyclically oriented cluster tilted algebra, then:*

- (a) (4.2) *The arrows of Q occurring in a chordless cycle are in bijection with elements of a system of top relations for any presentation of B . Let ξ be such an arrow and $\gamma_1, \dots, \gamma_t$ be the shortest paths antiparallel to ξ . Then the corresponding relation is of the form $\sum_{i=1}^t a_i \gamma_i$ where the a_i are nonzero scalars. Moreover the subquiver of Q restricted to the points involved in the γ_i looks as follows:*

In particular, the paths $\gamma_i \xi$ share only the endpoints

- (b) (3.4) *Any chordless cycle is of the form $\xi\gamma$ where ξ is an arrow and γ a shortest path antiparallel to ξ .*
- (c) (4.7)(4.8) *Assume C has global dimension two. Then $B \simeq \tilde{C}$ if and only if C is the quotient of B by an admissible cut. Moreover, any such C is strongly simply connected.*

We warn the reader that in (c) above, the algebra of the cut C is not necessarily tilted: as pointed out in [14, 16] it may be iterated tilted of global dimension 2.

It follows from (a) above that, if ξ is a new arrow and ρ_ξ is the relation corresponding to it, then the Keller potential is

$$W = \sum_{\xi} \xi \rho_{\xi}.$$

Because of (b), it is the sum of all chordless cycles. Thus, it is completely determined by the quiver.

Finally, it follows from the last statement of (c) above that an admissible cut of a cyclically oriented cluster tilted algebra contains no pair of parallel paths of the form (a, p) ,

where α is an arrow and p is a path, see [16, 3.9]. We recall that such a pair (a, p) is called a *bypass*. It is a *proper bypass* if the length of p is at least two.

2.5 Direct decomposition of the potential

Let (Q, W) be a quiver with potential. Following [4], we define an equivalence relation $\sim_W$ between the oriented cycles which appear as summands of the potential as follows. If γ, γ' are two summands of W , we set $\gamma \approx_W \gamma'$ in case there exists an arrow which is common to γ and γ' . Then $\sim_W$ is defined to be the least equivalence relation containing $\approx_W$. Thus, $\sim_W$ is the transitive closure of $\approx_W$. The relation $\sim_W$ is called the *potential equivalence* and the number N_W of equivalence classes of cycles in the potential under $\sim_W$ is called the *potential invariant* of (Q, W) .

A sum decomposition of the potential

$$W = W' + W''$$

is called *direct* if, whenever γ' is a cycle in W' , γ'' a cycle in W'' , then we have $\gamma' \not\sim_W \gamma''$. In this case, we write $W = W' \oplus W''$. Thus, N_W equals the number of indecomposable direct summands of the potential.

The motivation for introducing these concepts is their relation with the direct sum decompositions of the $C - C$ -bimodule $E = \text{Ext}_C^2(DC, C)$ when C is a tilted algebra.

Theorem [4] *Let C be a tilted algebra, $E = \text{Ext}_C^2(DC, C)$ and $B = C \ltimes E$. Further, let W be the Keller potential of B , then:*

- (a) (1.2.2) *Assume $W = W' \oplus W''$ and denote by E', E'' the $C - C$ -bimodules generated by the classes of new arrows appearing in a cycle of W', W'' respectively. Then $E = E' \oplus E''$ as $C - C$ -bimodules.*
- (b) (1.3.1) *Conversely, if B is cyclically oriented or of type $\tilde{\mathbb{A}}$ and $E = E' \oplus E''$ as $C - C$ -bimodules, then there is a decomposition $W = W' \oplus W''$ of W such that E', E'' are the $C - C$ -bimodules generated by the classes of arrows belonging to W', W'' respectively.*

As an easy consequence of (a), there is an injection from the set of equivalence classes of cycles occurring in W into the set of indecomposable direct summands of ${}_C E_C$.

Actually, it is easy to see that this injection is a bijection. Indeed, assume $E = \bigoplus_{i=1}^s E_i$, where the E_i are indecomposable C - C -bimodules. For each i , with $1 \leq i \leq s$, $\text{top} E_i \neq 0$, hence E_i contains a new arrow α_i in its support. Denoting by ρ_i the relation on C corresponding to α_i , the cycle $w_i = \alpha_i \rho_i$ is a summand of the Keller potential, and the equivalence class of w_i maps into E_i under the injection. This indeed follows from the concrete description of this injection, see [4]. In particular, the number s of indecomposable direct summands of E as a C - C -bimodule is equal to the potential invariant N_W .

2.6 A Lower Bound

As a consequence of Theorem 2.3 and the remarks following it, we deduce a lower bound for the dimension of the first Hochschild cohomology group of a cluster tilted algebra.

Proposition *Let C be a tilted algebra, and B its relation extension by $E = \text{Ext}_C^2(DC, C)$. Then we have*

- (a) $\dim_{\mathbb{k}} \text{HH}^1(B) \geq \dim_{\mathbb{k}} \text{HH}^1(C) + N_W$.
- (b) *If B is not hereditary, then $\dim_{\mathbb{k}} \text{HH}^1(B) \neq 0$.*
- (c) *We have $\dim_{\mathbb{k}} \text{HH}^1(B) = \dim_{\mathbb{k}} \text{HH}^1(C) + N_W$ if and only if:*
 - i) $H^1(C, E) = 0$
 - ii) E decomposes as a direct sum of pairwise orthogonal bricks.

Proof Because of Theorem 2.3 (a) and (b), we have $\dim_{\mathbb{k}} \text{HH}^1(B) = \dim_{\mathbb{k}} \text{HH}^1(C) + \dim_{\mathbb{k}} H^1(C, E) + \dim_{\mathbb{k}} \text{End}(E)$.

Now, let $E = \bigoplus_{i=1}^{N_W} E_i$ be a direct sum decomposition of the $C - C$ -bimodule E into indecomposable summands. Because the identity morphisms on each E_i induce N_W linearly independent endomorphisms of E , we have $\dim_{\mathbb{k}} \text{End}_{C-C}(E) \geq N_W$ and equality holds if and only if E is the direct sum of pairwise orthogonal bricks, thus $\dim_{\mathbb{k}} \text{HH}^1(B) \geq \dim_{\mathbb{k}} \text{HH}^1(C) + N_W$ and equality holds if and only if $H^1(C, E) = 0$ and $\dim_{\mathbb{k}} \text{End}_{C-C}(E) = N_W$, that is, if and only if E is the direct sum of pairwise orthogonal bricks. This proves (a) and (c). Finally, if B is not hereditary, then its Keller potential W contains at least a nonzero summand, so that $N_W \neq 0$ which implies $\text{HH}^1(B) \neq 0$, thus proving (b). $\square$

It follows from the results of [10] that if B is a tame cluster tilted algebra, then the two conditions of (c) are satisfied, see [10] (5.6), (5.8) and (5.9). As we shall see, they are also satisfied if B is cyclically oriented. So, for both of these classes, the equality (c) between the dimensions holds.

2.7 Potential and arrow equivalences

We now prove that the potential invariant N_W equals the relation invariant $N_{B,C}$ defined in Section 2.3.

Lemma *Let C be a tilted algebra, and B its relation extension. If W is the Keller potential of B , then $N_W = N_{B,C}$. If moreover B is tame, then N_W equals the number of indecomposable summands of $E = \text{Ext}_C^2(DC, C)$.*

Proof Because of Lemma 2.2, each cycle γ of W contains exactly one new arrow α_γ . The correspondence $\gamma \leftrightarrow \alpha_\gamma$ is actually bijective, because each new arrow lies on a cycle of W . We first claim that $\gamma \approx_W \sigma$ implies $\alpha_\gamma \approx_Q \alpha_\sigma$. Indeed, the hypothesis says that the cycles γ and σ share an arrow, say α . If $\alpha = \alpha_\gamma$, then, because of Lemma 2.2 we also have $\alpha = \alpha_\sigma$. If $\alpha \neq \alpha_\gamma$ then, for the same reason $\alpha \neq \alpha_\sigma$. Then the cyclic derivatives of the cycles containing α yield a minimal relation $\rho = \sum a_i w_i$ and two indices i, j such that $\alpha_\gamma | w_i$, $\alpha_\sigma | w_j$. Because of Lemma 1.1 there exists a strongly minimal relation having the same property. Therefore $\alpha_\gamma \approx_Q \alpha_\sigma$, as required.

Conversely, assume now $\alpha_\gamma \approx_Q \alpha_\sigma$. Then there exists a strongly minimal relation $\rho = \sum a_i w_i$ from x to y , say, and indices i, j such that $\alpha_\gamma | w_i$ and $\alpha_\sigma | w_j$. This implies the existence of an arrow $\eta : y \rightarrow x$ (actually in C) so that the summand of W corresponding to ρ is actually $\eta\rho = a_i(\eta w_i) + a_j(\eta w_j) + \sum_{k \neq i, j} a_k(\eta w_k)$. Therefore $\gamma = \eta w_i$ and $\sigma = \eta w_j$ and these two cycles γ, σ share the arrow η . Hence $\gamma \approx_W \sigma$.

As a consequence, if γ, σ are two cycles in W , then we have $\gamma \sim_W \sigma$ if and only if $\alpha_\gamma \sim_Q \alpha_\sigma$. Therefore $N_W = N_{B,C}$.

If B is tame, then it follows from Theorem 2.3(e) that $N_{B,C}$ equals the number of indecomposable summands of E , hence the last statement. $\square$

3 Proof of Theorem A

3.1 The cyclically oriented case

Let C be a tilted algebra and $E = \text{Ext}_C^2(DC, C)$ be such that $B = C \ltimes E$ is cyclically oriented. Because of Theorem 2.3(a) there exists a short exact sequence of vector spaces

$$0 \longrightarrow H^1(B, E) \longrightarrow HH^1(B) \longrightarrow HH^1(C) \longrightarrow 0.$$

This gives $HH^1(B) \simeq HH^1(C) \oplus H^1(B, E)$ as vector spaces. Because of Theorem 2.4 above, C is strongly simply connected, so $HH^1(C) = 0$.

Moreover, [7, 4.8], see also [10], gives $H^1(B, E) \cong H^1(C, E) \oplus \text{End}_{C-C}(E)$, so that $HH^1(B) \cong H^1(C, E) \oplus \text{End}_{C-C}(E)$. We start by proving that $H^1(C, E) = 0$. This will imply that $HH^1(B) \cong \text{End}_{C-C}(E)$. We shall complete the proof of Theorem A by proving that the indecomposable summands of E are pairwise orthogonal bricks, in Sections 3.3 and 3.4, respectively.

Lemma *Let C be a tilted algebra such that $B = C \ltimes E$ is cyclically oriented. Then $H^1(C, E) = 0$.*

Proof It suffices to show that $\text{Der}_0(C, E) = 0$. Let thus $\delta \in \text{Der}_0(C, E)$ be nonzero. Then, there exists an old arrow α from i to j , say, such that $\delta(\alpha) \neq 0$. We show that this leads to a contradiction.

Because δ is normalised, we have

$$\delta(\alpha) = \delta(e_i \alpha e_j) = e_i \delta(\alpha) e_j \in e_i E e_j.$$

Then there exist old paths $u : i \rightsquigarrow x$ and $v : y \rightsquigarrow j$ as well as a new arrow $\beta : x \rightarrow y$ such that $\delta(\alpha) = u\beta v$. Indeed, u and v consist solely of old arrows, because $E^2 = 0$. If both u, v are trivial paths, then $\alpha\beta^{-1}$ is a chordless cycle in the quiver of B , contradicting the fact that B is cyclically oriented. Hence, u or v is nontrivial. On the other hand, u and v do not intersect each other, otherwise the old arrow α would be a bypass of a path in C , and this is impossible because C is strongly simply connected, see Theorem 2.4(c). We then have a closed walk $u\beta v\alpha^{-1}$ in B with u, v old paths and β a new arrow. Without loss of generality we may assume that this closed walk is of minimal length among all the closed walks of the form $u'\beta'v'\alpha^{-1}$ in B with u', v' old paths and β' a new arrow.

Because β is a new arrow, there exists an old relation $\rho = \sum_i \lambda_i w_i$ corresponding to it. We claim that no w_i has a point in common with the paths u and v except their endpoints x and y respectively. For, assume this is the case and let $z \neq x$ be, for instance, a point common to w_1 and u . Denoting respectively by w'_1, w''_1 and by u', u'' the subpaths of w_1 and u ending and starting in z ,

we get two paths $w_1'w_1'' (= w)$ and $w_1'u_1''$ which are both antiparallel to the new arrow β . This is impossible, because two paths antiparallel to a new arrow can only intersect in their endpoints, see Theorem 2.4(a). The situation is exactly similar if w_1 intersects the path v . This establishes our claim.

Because $\beta v \alpha^{-1} u \beta$ is not a C -sequential walk, there is a subpath of $v \alpha^{-1} u$ which is a branch of a relation on C . We claim that there is exactly one such subpath and it lies either on u or on v . Assume that there are two such subpaths, one lying on u , and the other lying on v , and let the corresponding new arrows be denoted respectively by $\gamma' : k' \rightarrow l'$ and $\gamma : k \rightarrow l$. We prove that this implies the existence of a C -sequential walk

Let $\rho' = \sum_i \lambda_i u_i$ and $\rho = \sum_j \mu_j v_j$ be the relations in C defining the new arrows γ' and γ respectively. If some u_i cuts some v_j , then α would be a bypass in C , a contradiction. If some u_i intersects the path u between l' and i , then we have an oriented cycle in C , an impossibility. Similarly, no v_j intersects the subpath of v from k to j . We then have a C -sequential walk $\gamma'(u')^{-1} \alpha (v')^{-1} \gamma$, as required. Because the situation is similar in case γ and γ' are both antiparallel to u or both antiparallel to v , this proves our claim.

Assume that this subpath lies on v and γ is the corresponding new arrow. Then there exist subpaths v' and v'' of v such that $v' \gamma^{-1} v''$ is parallel to v .

We claim that the cycle $\beta v' \gamma^{-1} v'' \alpha^{-1} u$ is chordless, which will give a contradiction to the fact that B is cyclically oriented. Indeed, if this is not the case, then there exists a chord $\eta : a \rightarrow b$ with a, b points on the cycle $\beta v' \gamma^{-1} v'' \alpha^{-1} u$.

We have several possibilities.

1. Both a, b lie on u . Assume that η is parallel to a subpath of u . Then η cannot be a new arrow for, otherwise, there exists a relation in C from b to a , and, in this case, the subpath of u parallel to η together with one branch of the relation constitutes an oriented cycle entirely consisting of old arrows which contradicts the triangularity of C . Then η is an old arrow. But then, it is a bypass to a subpath of u , which contradicts Theorem 2.4(c).

Assume now that η is antiparallel to a subpath of u . Then η cannot be an old arrow, because C is triangular. Hence, η is a new arrow. Consequently, there exists a subpath u_1 of u such that we have a C -sequential walk $\eta u_1^{-1} \alpha (v'')^{-1} \gamma$ in B as before, a contradiction.

2. The proof is entirely similar if a, b lie on v', v'' .
3. Assume a lies on u and b lies on v . If η is an old arrow, then α would be a bypass of a path of the form $u_1 \eta v_1$, with u_1, v_1 subpaths of u, v respectively. Because this path consists only of old arrows, this yields a contradiction to Theorem 2.4(c). Therefore η is a new arrow. But we may replace the cycle $u \beta v \alpha^{-1}$ by the shorter one $u_1 \eta v_1 \alpha^{-1}$, a contradiction to the minimality (again here u_1 or v_1 is nontrivial for, otherwise, we have a double arrow).
4. Assume a lies on v and b lies on u . If η is a new arrow, then there is a relation in C from b to a hence a path w in C from b to a . But then α is a bypass to $u_1 w v_1$, where u_1, v_1 are subpaths of u, v respectively. Because $u_1 w v_1$ consists only of old arrows, this is a path in C , hence we get a contradiction to Theorem 2.4(c). Consequently η is an old arrow. We thus have a cycle of the form $u_1 \eta^{-1} v_1 \alpha^{-1}$ with u_1, v_1 subpaths of u, v respectively, and the cycle consisting entirely of old arrows. Take a cycle of minimal length of the form $u'_1 \eta'^{-1} v'_1 \alpha^{-1}$ with u'_1, v'_1 subpaths of u, v respectively and the cycle consisting entirely of old arrows. Notice that u'_1 or v'_1 is nontrivial, because otherwise the cycle $u'_1 \eta'^{-1} v'_1 \alpha^{-1}$ would reduce to an oriented cycle $(\eta')^{-1} \alpha^{-1}$ in C , a contradiction to triangularity. But then the cycle $u'_1 \eta'^{-1} v'_1 \alpha^{-1}$ is chordless but not oriented. This contradiction completes the proof. $\square$

3.2 Indecomposable summands of ${}_C E_C$

The previous lemma implies that, if B is cyclically oriented, then $\text{HH}^1(B) = \text{End}_{C-C}(E)$. We thus turn to the computation of the latter.

Because B is cyclically oriented, its Keller potential W is the sum of all chordless cycles in the quiver of B . Two chordless cycles γ', γ'' are equivalent if and only if there exists a sequence of chordless cycles $\gamma' = \gamma_1, \gamma_2 \dots \gamma_t = \gamma''$ such that for each i , the cycles γ_i and γ_{i+1} share an arrow. Because of the last statement of Theorem 2.4(a), γ_i and γ_{i+1} cannot share more than one arrow, so they share exactly one.

Lemma *The number of indecomposable summands of E as a $C - C$ -bimodule equals the potential invariant N_W .*

Proof Write $W = W_1 \oplus W_2 \oplus \cdots \oplus W_s$ where the W_i are the indecomposable summands of W . Each W_i is the sum of equivalent chordless cycles and no cycle which is a summand of W_i is equivalent to a cycle which is a summand of W_j for $j \neq i$. Therefore the number s of summands of W equals the number of equivalence classes of cycles, which is precisely the potential invariant N_W . Because, as pointed out at the end of Section 2.5, there is a bijection between the indecomposable summands of the potential and those of ${}_C E_C$, we infer the statement. $\square$

We have proven in Lemma 2.7 that the same statement holds true for tame cluster tilted algebras.

3.3 Endomorphisms of E

Let thus $E_1, \dots, E_{N_W}$ denote the indecomposable summands of ${}_C E_C$. In order to prove that $\mathrm{HH}^1(B) \cong \mathrm{End}_{C-C}(E)$ is N_W -dimensional, we need to prove that the E_i are pairwise orthogonal bricks in the category of $C-C$ -bimodules. We start with the following Lemma.

Lemma *With the above notation, for every nonzero $\delta \in \mathrm{Hom}_{C-C}(E_i, E)$ and every new arrow α in E_i , we have*

$$\delta(\alpha) = \lambda_\alpha \alpha$$

for some scalar $\lambda_\alpha \in \mathbb{k}$

Proof Denote by $\{\alpha_1, \dots, \alpha_t\}$ the set of new arrows and by $\{\rho_1, \dots, \rho_t\}$ the corresponding relations in C , so that the Keller potential is

$$W = \sum_{i=1}^t \alpha_i \rho_i$$

We may assume that the equivalence class of the chordless cycle $\alpha_1 \rho_1$ contains as new arrows $\alpha_1, \dots, \alpha_r$ with $r \leq t$. For δ as in the statement and any i with $1 \leq i \leq r$ we have

$$\delta(\alpha_i) = \sum_j \lambda_{ij} u_{ij} \alpha_j v_{ij}$$

where the λ_{ij} are scalars and u_{ij}, v_{ij} are paths such that α_i and $u_{ij} \alpha_j v_{ij}$ are parallel. Note that the absence of double arrows in B implies that, for each j , the path u_{ij} or the path v_{ij} is nontrivial. Moreover, the fact that $E^2 = 0$ implies that u_{ij}, v_{ij} are paths in C .

We claim that $\lambda_{ij} = 0$ when $i \neq j$ and this implies that $\delta(\alpha_i) = \lambda_{ii} \alpha_i$ (for, otherwise, the nontriviality of u_{ij} or v_{ij} would imply a contradiction to the triangularity of C). We may assume without loss of generality that $i = 1$ so that $j \neq 1$. The paths u_{1j}, v_{1j} do not intersect for, otherwise, we have an oriented cycle in C , a contradiction. We consider the cycle $u_{1j} \alpha_j v_{1j} \alpha_1^{-1}$. This is a cycle which we may assume of minimal length among all cycles of the form $u' \alpha_j v' \alpha_1^{-1}$ with u', v' paths in C . If it is chordless, then we are done because it is not oriented. Therefore we may assume that it has a chord $\beta : a \rightarrow b$. We study the different possibilities for a and b . For ease of notation, we set $u = u_{1j}, v = v_{1j}$.

1. Both a and b lie on u . Assume that $\beta : a \rightarrow b$ is an old arrow. If β is parallel to u then it is a bypass of a subpath of u , a contradiction to Theorem 2.4(c). If β is antiparallel to u , then it generates with the subpath of u from b to a an oriented cycle in C , a

contradiction to its triangularity. Therefore β is a new arrow. If $\beta : a \rightarrow b$ is parallel to u , then it corresponds to a relation from b to a and so generates an oriented cycle in C , a contradiction to triangularity. But then β is antiparallel to u and β together with α_1 yield a C -sequential walk in B : indeed, if $\rho = \sum_i \lambda_i w_i$ is the relation corresponding to β and one of the w_i intersects the subpath of u from the source of α_1 to b , then we have an oriented cycle in C , an impossibility. Thus in all cases we reach a contradiction.

2. The situation is exactly similar if β is an arrow between two points of v .
3. Assume a lies on v and b lies on u . If $\beta : a \rightarrow b$ was a new arrow and $\rho = \sum_i \lambda_i w_i$ is the relation corresponding to it, then no w_i can intersect the subpath of u from the source of α_1 to b nor the subpath of v from a to the target of α_1 , for, if this is the case, then we have an oriented cycle in C . Consequently β would form with α_1 (or α_j) a C -sequential walk in B . Therefore it is an old arrow. But then this is a C -sequential walk of the form $\alpha_1 v_1^{-1} \beta u_1^{-1} \alpha_1$ with u_1, v_1 subpaths of u, v respectively. This shows that there is no such chord β .
4. The only possibility left is that a lies on u and b lies on v . If $\beta : a \rightarrow b$ is an old arrow, then there is an oriented cycle in C consisting of β , a subpath of v , a branch of the relation ρ_1 corresponding to α_1 and a subpath of u . This contradiction implies that β is a new arrow. But then we have a cycle of the form $u' \beta v' \alpha_1^{-1}$ with u', v' paths in C and β, α_1 new arrows, a contradiction to the assumed minimality of the cycle $u \alpha_j v \alpha_1^{-1}$. $\square$

This proof resembles that of Lemma 3.1. The essential difference is that in Lemma 3.1 the arrow α is old while the arrow β is new, but here both of the arrows α_1 and α_j are new.

3.4 Pairwise orthogonal bricks

We now prove that the indecomposable summands of ${}_C E_C$ are pairwise indecomposable bricks.

Lemma *With the above notation, we have*

$$\dim_{\mathbb{k}} \operatorname{Hom}_{C-C}(E_i, E_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Proof Assume first that $i \neq j$. Then $\operatorname{Hom}_{C-C}(E_i, E_j) = 0$ follows directly from Lemma 3.3. Therefore we just need to prove that for each i with $1 \leq i \leq N_W$ we have

$$\operatorname{End}_{C-C}(E_i) \simeq \mathbb{k}.$$

As $C-C$ -bimodule, E_i is generated by those new arrows $\alpha_1, \dots, \alpha_r$ which occur in the chordless cycles in the equivalence class corresponding to E_i . It follows from Lemma 3.3 that for every $\delta \in \operatorname{End}_{C-C}(E_i)$, and every j with $1 \leq j \leq r$ we have

$$\delta(\alpha_j) = \lambda_j \alpha_j$$

for some scalar λ_j . We claim that $\lambda_j = \lambda_1$. This will establish the statement.

The new arrow α_1 belongs to a cycle in the potential. Because there are no loops in the quiver of B , each arrow appearing in the potential is antiparallel to a relation. Therefore

there exists a relation in B involving the arrow α_1 . Because of Lemma 1.1, we may assume this relation to be strongly minimal, that is a relation of the form

$$\rho = \sum_{l=1}^r \mu_l (w_l \alpha_l w'_l)$$

where w_l, w'_l are old paths, the μ_l are nonzero scalars and, for every proper subset $J \subset \{1, \dots, r\}$ and every set of nonzero scalar μ'_l with $l \in J$, we have

$$\rho = \sum_{l \in J} \mu'_l (w_l \alpha_l w'_l) \neq 0.$$

Applying δ to the relation ρ yields

$$0 = \delta(\rho) = \sum_{l=1}^r \mu_l \lambda_l (w_l \alpha_l w'_l)$$

subtracting from this expression $\lambda_1 \rho$ we get

$$\sum_{l \neq 1} \mu_l (\lambda_l - \lambda_1) (w_l \alpha_l w'_l) = 0.$$

The strong minimality of ρ and the fact that $\mu_l \neq 0$ for every l imply that $\lambda_l = \lambda_1$ for every l . $\square$

The previous two lemmata can be interpreted as saying that every derivation of B is diagonalisable.

3.5 Proof of the main Theorem in the cyclically oriented case

For the benefit of the reader we repeat the statement of the theorem in the cyclically oriented case.

Theorem *Let B be a cyclically oriented cluster tilted algebra having N_W as potential invariant, C a tilted algebra such that B is the relation extension of C and $E = \text{Ext}_C^2(DC, C)$. Then we have*

$$\dim_{\mathbb{k}} \text{HH}^1(B) = N_W = \dim_{\mathbb{k}} \text{End}_{C-C}(E).$$

Moreover, the indecomposable summands of ${}_C E_C$ are pairwise orthogonal bricks.

Proof The proof was outlined at the beginning of Section 3. It follows from Lemma 3.1 that $\text{H}^1(C, E) = 0$. On the other hand, Lemmata 3.3 and 3.4 show that the indecomposable summands of ${}_C E_C$ are pairwise orthogonal bricks. The fact that their number equals the potential invariant follows from Lemma 3.2. $\square$

3.6 The representation-finite case

Let Q be the quiver of a representation-finite cluster tilted algebra. An arrow in Q is called an *inner arrow* if it belongs to two chordless cycles. We deduce from our main result above the following corollary, which is [10, Theorem 1.2].

Corollary *Let B be a representation-finite cluster tilted algebra and Q its quiver. Then the dimension of $\mathrm{HH}^1(B)$ equals the number of chordless cycles in Q minus the number of inner arrows in Q .*

Proof In the representation-finite case, relations are monomial or binomial relations. Two chordless cycles γ' and γ'' are equivalent if and only if there is a sequence of chordless cycles $\gamma' = \gamma_1, \gamma_2, \dots, \gamma_t = \gamma''$ such that for every i , the cycle γ_i shares exactly one arrow with γ_{i+1} . That is, γ_i is connected to γ_{i+1} by an inner arrow. Therefore, the total number of equivalence classes equals the number of chordless cycles minus the number of inner arrows. $\square$

3.7 Partial relation extensions

Let C be a tilted algebra and $E = \mathrm{Ext}_C^2(DC, C)$ be such that $B = C \ltimes E$ is cyclically oriented. Assume that E splits as a direct sum of $C - C$ -bimodules as $E = E' \oplus E''$. Then the trivial extension $B' = C \ltimes E'$ is called a *partial relation extension*, see [4]. Because of Theorem 2.5 above, corresponding to the decomposition $E = E' \oplus E''$ is a decomposition $W = W' \oplus W''$ of the potential W such that E', E'' are respectively the $C - C$ -bimodules generated by the classes of new arrows in W', W'' . The bound quiver of B' is described in [4, 2.2.2] and it is shown in [6] that there is a short exact sequence of vector spaces of the form

$$0 \rightarrow H^1(B', E) \rightarrow \mathrm{HH}^1(B') \rightarrow \mathrm{HH}^1(C) \rightarrow 0$$

Further, [6] Remark (3.5) says that

$$H^1(B', E') = H^1(C, E') \oplus \mathrm{End}_{C-C}(E')$$

Corollary *With the above notation, let $N_{W'}$ denote the number of equivalence classes of cycles in W' , then $\dim_{\mathbb{k}} \mathrm{HH}^1(B') = N_{W'} = \dim_{\mathbb{k}} \mathrm{End}_{C-C}(E')$*

Proof Because C is strongly simply connected, see Theorem 2.4(c), we have $\mathrm{HH}^1(C) = 0$. Therefore, as vector spaces, we have

$$\mathrm{HH}^1(B') \simeq H^1(B', E') \simeq H^1(C, E') \oplus \mathrm{End}_{C-C}(E')$$

Because $H^1(C, E') \oplus H^1(C, E'') = H^1(C, E' \oplus E'') = H^1(C, E) = 0$, due to Lemma 3.1, we have $H^1(C, E') = 0$. On the other hand, E' is a direct summand of E . Because the latter is isomorphic, as $C - C$ -bimodule, to the direct sum of pairwise orthogonal bricks, so is E' . Therefore the dimension of $\mathrm{End}_{C-C}(E')$ equals the number of indecomposable summands of E' which, because of Section 2.5 and the comments above, is equal to $N_{W'}$. $\square$

We observe that, if B is a tame cluster tilted algebra, then the same arguments prove that

$$\dim_{\mathbb{k}} \mathrm{HH}^1(B') = N_{W'} + \dim_{\mathbb{k}} \mathrm{HH}^1(C).$$

3.8 Relation with the fundamental group

For the properties of the fundamental group of a bound quiver, we refer, for instance, to [32].

Corollary Let $B = \mathbb{k}\tilde{Q}/\tilde{I}$ be a cyclically oriented cluster tilted algebra. Then

- (a) The fundamental group $\pi_1(\tilde{Q}, \tilde{I})$ does not depend on the presentation of B ,
- (b) We have $\mathrm{HH}^1(B) \cong \mathrm{Hom}(\pi_1(\tilde{Q}, \tilde{I}), \mathbb{k}^+)$.

Proof (a) This follows from the fact that, up to scalars, the presentation of B is determined by its quiver, see [16, (4.2)].

- (b) In Lemma 3.3 we established that any derivation is diagonalisable. The conclusion then follows from [25, Corollary 3]. $\square$

3.9 The tame case

Let C be a tilted algebra of euclidean type, $E = \mathrm{Ext}_C^2(DC, C)$ and $B = \tilde{C}$ have Keller potential W . It follows from Lemma 2.7 that $N_W = N_{B,C}$ and that this common value, which we denote by N , is the number of indecomposable summands of ${}_C E_C$. The following Theorem, of which part (a) was already proven in [10] and [9] gives the dimension of the first Hochschild cohomology space of B .

Theorem Let B be a cluster tilted algebra, and C a tilted algebra such that $B = \tilde{C}$, then

- (a) If B is of type $\tilde{\mathbb{A}}$, then $\dim_{\mathbb{k}} \mathrm{HH}^1(B) = N + \varepsilon$ where

$$\varepsilon = \begin{cases} 3 & \text{if } C \text{ is a double arrow (and then so is } B), \\ 2 & \text{if } C \text{ is either a hereditary bypass, (and then so is } B), \text{ or contains a double} \\ & \text{arrow extended or coextended by just one branch.} \\ 1 & \text{otherwise.} \end{cases}$$

- (b) If B is of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$, then $\dim_{\mathbb{k}} \mathrm{HH}^1(B) = N$.

Proof (a) Let B be of type $\tilde{\mathbb{A}}$. Because of Theorem 2.3 we have $\mathrm{HH}^1(B) \simeq \mathrm{HH}^1(C) \oplus \mathbb{k}^N$. The proof reduces to the calculation of $\epsilon = \dim_{\mathbb{k}} \mathrm{HH}^1(C)$.

Assume first that C is constricted, it follows from the proof of [9, Proposition 5.1] that $\dim_{\mathbb{k}} \mathrm{HH}^1(C) = 1$. On the other hand, if C is not constricted, then C contains either a double arrow or a hereditary bypass, that is, B is of one of the two forms of [10, Lemma 4.3] or their duals. If B is hereditary, then it coincides with C , and there are two cases to consider: either C is a double arrow, then it is well-known that $\dim_{\mathbb{k}} \mathrm{HH}^1(C) = 3$, or C is a hereditary bypass, and then one can easily compute (use for instance Happel's long exact sequence [28]) that $\dim_{\mathbb{k}} \mathrm{HH}^1(C) = 2$. On the other hand, if B is not hereditary, the result follows from [10, Lemmata 3.3 and 4.5].

- (b) Assume now that B is of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$. Then C is a tilted algebra of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$. Because of [13], C is simply connected. Because of [8], see also [31], we have $\mathrm{HH}^1(C) = 0$. Therefore, Theorem 2.3(c) yields in this case $\mathrm{HH}^1(B) \simeq \mathbb{k}^N$. $\square$

3.10 Invariants are invariant

The following obvious corollary arises from the fact that $\dim_{\mathbb{k}} \mathrm{HH}^1(B)$ depends only on B .

Corollary Let B be a tame cluster tilted algebra, C_1, C_2 be tilted algebras such that $B \simeq \tilde{C}_1 \simeq \tilde{C}_2$. For each $i = 1, 2$, let $E_i = \text{Ext}_{C_i}^2(DC_i, C_i)$ and W_i the Keller potential arising from the relations of C_i . Then

- (a) $N_{B, C_1} = N_{B, C_2}$.
- (b) $N_{W_1} = N_{W_2}$ and does not depend on the presentation.
- (c) The bimodules E_1 and E_2 have the same number of indecomposable summands.
- (d) $\dim_{\mathbb{k}} \text{End}_{C_1-C_1}(E_1) = \dim_{\mathbb{k}} \text{End}_{C_2-C_2}(E_2)$.

3.11 Examples

- (a) Consider the tilted algebra C given by the quiver

bound by $\rho\theta = 0$, $\rho\gamma = 0$, $\alpha\beta + \gamma\delta = 0$, and $\mu\lambda = 0$. It is representation-infinite of (euclidean) type $\tilde{\mathbb{E}}_8$ and its relation extension $\tilde{C}$ is given by the quiver

with Keller potential

$$W = \varepsilon\alpha\beta + \varepsilon\delta\gamma + \xi\rho\gamma + \eta\rho\theta + \sigma\mu\lambda$$

It is easily seen to be cyclically oriented. Further, there are two classes of cycles, namely $\mathcal{S}_1 = \{\varepsilon\alpha\beta, \varepsilon\delta\gamma, \xi\rho\gamma, \eta\rho\theta\}$ and $\mathcal{S}_2 = \{\sigma\mu\lambda\}$. These may be represented in the following diagram, where the bottom line represents the common arrows to the corresponding cycles.

The connected components of the diagram are precisely the equivalence classes. Accordingly, the bimodule ${}_C E_C$ has two indecomposable summands $E = E_1 \oplus E_2$,

with E_i corresponding to $\mathcal{S}_i$. Then $E_2 = C\sigma C$ is a simple module while E_1 is 11-dimensional with top corresponding to the new arrows ε, ξ and η .

Applying our Theorem we get we get $\dim_{\mathbb{k}} \mathrm{HH}^1(\tilde{C}) = 2$.

- (b) Consider the tilted algebra C given by the quiver

bound by $\varepsilon\gamma = 0, \delta\alpha\beta + \delta\gamma = 0$. It is representation-infinite of euclidean type $\tilde{\mathbb{D}}_4$ and its relation extension $\tilde{C}$ is given by the quiver

with Keller potential $W = \lambda\delta\alpha\beta + \lambda\delta\gamma + \mu\varepsilon\gamma$. It is not cyclically oriented, because it contains the nonoriented chordless cycle given by the parallel paths $(\gamma, \alpha\beta)$.

In this case, the same diagram as in example (a) shows that we have just one equivalence class

so that $\dim_{\mathbb{k}} \mathrm{HH}^1(\tilde{C}) = 1$.

In contrast to the cyclically oriented case, the two cycles $\lambda\delta\alpha\beta$ and $\lambda\delta\gamma$ have more than one arrow in common.

4 Proof of Theorem B

In this section, we give a geometric version of the main Theorem for cluster tilted algebras of types $\mathbb{D}_n$ and $\tilde{\mathbb{D}}_n$, therefore we shall work with the geometric model of those algebras, namely the triangulation of a disc with n marked points in the boundary and at most two marked points in the interior.

4.1 Unreduced Potential for $\mathbb{D}_n$ and $\tilde{\mathbb{D}}_n$

A bordered surface with marked points, or simply, surface is a pair (S, M) where S is a compact oriented Riemann surface with (possibly empty) boundary and M is a non empty finite set of points on S , called *marked points*, such that M has at least one point in each connected component of the boundary of S . The marked points that lie in the interior of S are called *punctures*. For technical reasons, we require that (S, M) is not a sphere with 1, 2 or 3 punctures; a monogon with 0 or 1 puncture; or a bigon or a triangle without punctures.

A *triangulation* T of (S, M) is any maximal collection of non-crossing *arcs*, which are isotopy classes of curves whose endpoints are marked points and which are not isotopic to a point or a *boundary segment*, that is any segment connecting two marked points in the same boundary component without passing through a third marked point. The arcs of a triangulation T cut the surface (S, M) into ideal triangles. Triangles that have only two distinct sides are called *self-folded triangles*. A triangle is *internal* if all its sides are arcs of a triangulation. We refer to the tuple (S, M, T) as a *triangulated surface*. The *valency* $\text{val}_T(x)$ of a puncture x is the number of arcs in T incident to x , where each loop at x is counted twice.

An alternative representation of an arbitrary triangulated surface consists in gluing together a finite collection of *puzzle pieces* along selected arcs of them, in such a way that some arcs of those puzzle pieces correspond to boundary segments of S , see [27]. This puzzle piece decomposition plays an important rôle in this section, as it is the basis for the construction of the quiver and potential of a triangulated surface. It is easy to see that given any triangulated surface (S, M, T) , its puzzle piece decomposition might have seven different puzzle pieces (boundary segments are coloured in grey and non-self-folded triangles are coloured in blue), see Fig. 1.

Following [27], given a triangulation T of a surface (S, M) , the unreduced adjacency quiver $\widehat{Q}_T$ is built by gluing blocks corresponding to each kind of puzzle pieces, these identifications are allowed only in points of type $\bullet$ which correspond to arcs which are part of a non-self-folded triangle in each puzzle piece. These blocks are depicted in Fig. 2.

Moreover, by [30], the unreduced potential $\widehat{W}_T$ associated to triangulation T is also defined by the blocks of type II, IV, V, and the cycles surrounding punctures. More precisely, $\widehat{W}_T$ is defined by the 3-cycles arising from the non-self-folded triangles in each puzzle pieces of type II, IV and V, the 3-cycles arising from the self-folded triangles in the puzzle pieces of type IV and V, the 3-cycle arising from the two self-folded sides in the puzzle piece of type V and the cycles surrounding punctures.

For now on, we focus in the geometric realisation of the cluster tilted algebra of type $\mathbb{D}_n$ (or $\widetilde{\mathbb{D}}_n$), namely a disc D with n marked points in the boundary and exactly one marked point p (or exactly two marked points p and q , respectively) in the interior of D , denote by M the set of the marked points of D . We say that the tuple (D, M, T) is a *triangulated punctured disc*, where T is a triangulation of (D, M) .

Fig. 1 Puzzle pieces in a triangulated surface

Fig. 2 Quivers blocks

Before giving an explicit formula for the unreduced potential $\widehat{W}_T$ of a triangulation T of a punctured disc (D, M) , we fix some notation. For each non-self-folded triangle Δ in a puzzle piece of type II, IV or V of the puzzle piece decomposition of (D, M, T) , we denote by C_Δ the 3-cycle arising from Δ . Let x be a puncture in (D, M, T) , then C_x is either

- (i) the 3-cycle arising from the self-folded triangle enclosing x in the puzzle pieces of type IV or V if $\text{val}_T(x) = 1$ or
- (ii) the cycle surrounding the puncture x otherwise.

Finally, if (S, M) is a twice punctured disc, let p and q be the two punctures in (S, M) , then $C_{p,q}$ is either

- (i) the 3-cycle arising from the two self-folded sides in the puzzle piece of type V if p and q are enclosed in a puzzle piece of type V or
- (ii) zero otherwise.

The unreduced potential $\widehat{W}_T$ of a triangulation T of a once punctured disc (D, M) is written as follows:

$$\widehat{W}_T = \sum_{\Delta} C_\Delta + C_p,$$

where the sum runs over all non-self-folded triangles in the puzzle pieces of type II or IV of the puzzle piece decomposition of (S, M, T) and p is the puncture of (D, M) . The unreduced potential $\widehat{W}_T$ of a triangulation T of an twice punctured disc (D, M) is written as follows:

$$\widehat{W}_T = \sum_{\Delta} C_\Delta + C_p + C_q + C_{pq},$$

where the sum runs over all non-self-folded triangles in the puzzle pieces of type II, IV or V of the puzzle piece decomposition of (S, M, T) and p and q are the punctures in (D, M) .

In order to obtain the reduced quiver with potential (Q_T, W_T) of the triangulation T of a punctured disc (D, M) , one needs an algebraic procedure to delete 2-cycles from the (non-necessarily 2-acyclic) quiver with potential $(\widehat{Q}_T, \widehat{W}_T)$. Such algebraic procedure is provided by the Derksen-Weyman-Zelevinsky Splitting Theorem, see [26]. Recall that the unreduced quiver with potential $(\widehat{Q}_T, \widehat{W}_T)$ is already reduced if and only if $\text{val}_T(x) \neq 2$ for every puncture x , otherwise the reduction affects the 2-cycle C_x surrounding x and the 3-cycles that share an arrow with the 2-cycle. Observe that a puncture x of valency 2 could be involved in three different types of configurations depicted on Fig. 3.

We give a geometric interpretation of the equivalence relation $\sim_W$ in cycles in the reduced potential in terms of internal triangles related to a puncture. Notice that not every cycle of a potential arises from triangles and not every internal triangle gives a cycle in the

reduced potential. In this section, we work with two quivers with potential coming from the same surface, the reduced one and the unreduced one.

Definition Let (S, M, T) be a triangulated surface.

- (i) Suppose x is a puncture in (S, M, T) . An internal triangles Δ is *related to the puncture* x if one of the following conditions holds:
 - (a) a side of Δ is incident to the puncture x ;
 - (b) the triangle Δ is the non-self-folded triangle of a puzzle piece of type IV or V such that x is enclosed in this puzzle piece.
- (ii) Suppose x and y are punctures in (S, M, T) . An internal triangle Δ is *related to the punctures* x or y if one of the following conditions holds:
 - (a) a side of Δ is incident to the punctures x or y ;
 - (b) the triangle Δ is the non-self-folded triangle of a puzzle piece of type IV such that x or y is enclosed in this puzzle piece.
 - (c) the triangle Δ is the non-self-folded triangle of a puzzle piece of type V such that x and y are enclosed in this puzzle piece.

We set $\Delta \approx \Delta'$ in case there exists a puncture x such that Δ and Δ' are related to it. Then $\sim$ is defined to be the least equivalence relation containing $\approx$. Thus, $\sim$ is the transitive closure of $\approx$.

Denote by $\overline{\Delta_{\text{Rel}(x)}}$ the set of internal triangles related to the puncture x and by $\Delta_{\text{Rel}(x)}$ the subset of non-self-folded internal triangles related to the puncture x . Denote by $\overline{\Delta_{\text{Rel}(x,y)}}$ the set of internal triangles related to x or y and by $\Delta_{\text{Rel}(x,y)}$ the subset of non-self-folded internal triangles related to x or y .

Remark (a) By construction of $\widehat{Q}_T$, the 3-cycles C_Δ and $C_{\Delta'}$ do not share arrows, for any pair of non-self-folded triangles Δ and Δ' .

- (b) Denote by $\Delta_{\text{NRel}(x,y)}$ the subset of non-self-folded internal triangles neither related to x nor to y , that is $\Delta_{\text{NRel}(x,y)} = \Delta_{\text{NRel}(x)} \cup \Delta_{\text{NRel}(y)}$. The subset of non-self-folded internal triangles is a disjoint union of $\Delta_{\text{Rel}(x,y)} \sqcup \Delta_{\text{NRel}(x,y)}$.

Fig. 3 Configurations of punctures with valency 2

- (c) If $\text{val}_T(x) = 1$, then $|\overline{\Delta_{\text{Rel}(x)}}| \leq 2$ and the equality holds if and only if the triangulation has a triangle of type IV or V.
- (d) By definition, if Δ is the non-self-folded triangle of a puzzle piece of type of type II, IV or V and $\Delta \in \Delta_{\text{Rel}(x)}$, then C_Δ is potential equivalent to C_x in the unreduced potential.

Lemma *Let (D, M, T) be a triangulated punctured disc, Δ and Δ' two non-self-folded internal triangles, and C_Δ and $C_{\Delta'}$ the 3-cycle associated to each triangle. Then Δ and Δ' are related if and only if C_Δ and $C_{\Delta'}$ are potential-equivalent in the unreduced potential.*

Proof First of all, observe that if a triangle Δ of type V is part of the decomposition of puzzle pieces induced by T , then the equivalence class $[C_\Delta]$ is the set of the four 3-cycles of the block V, therefore there is no other triangle Δ' such that $\Delta \sim \Delta'$. Assume that the decomposition of puzzle pieces induced by T has no non-self-folded triangle being part of a puzzle piece of type V.

By definition $\Delta \approx \Delta'$ if and only if $C_\Delta \approx_{\widehat{W}_T} C_x \approx_{\widehat{W}_T} C_{\Delta'}$ for some puncture x , then it is clear that if $\Delta' \sim \Delta$, then $C_\Delta \sim_{\widehat{W}_T} C_{\Delta'}$.

Now, suppose C_Δ and $C_{\Delta'}$ are potential-equivalent. Let $C_\Delta = C_0 \approx_{\widehat{W}_T} C_1 \approx_{\widehat{W}_T} \dots \approx_{\widehat{W}_T} C_{k-1} \approx_{\widehat{W}_T} C_k = C_{\Delta'}$ be a sequence of cycles related by a common arrow such that $C_i \neq C_j$ if and only if $i \neq j$. Since no pair of 3-cycles associated to triangles of type II, IV or V shares arrows, the cycles C_1 and C_{k-1} are C_p or C_q , therefore Δ and Δ' are related to a puncture. If $C_1 = C_{k-1}$, then both triangles Δ and Δ' are related to the same puncture, and by definition $\Delta \sim_{\widehat{W}_T} \Delta'$.

Now suppose $C_1 \neq C_{k-1}$, without loss of generality suppose $C_1 = C_p$ and $C_{k-1} = C_q$. We claim there is a non-self-folded internal triangle Δ'' related to p and q . By the minimality conditions over the sequences of cycles and because no pair of 3-cycles associated to non-self-folded internal triangles shares arrow, then the sequences are either $C_\Delta = C_0 \approx_{\widehat{W}_T} C_p \approx_{\widehat{W}_T} C_q \approx_{\widehat{W}_T} C_k = C_{\Delta'}$ or $C_\Delta = C_0 \approx_{\widehat{W}_T} C_p \approx_{\widehat{W}_T} C_{\Delta''} \approx_{\widehat{W}_T} C_q \approx_{\widehat{W}_T} C_k = C_{\Delta'}$ where Δ'' is a non-self-folded internal triangle.

Suppose $C_\Delta = C_0 \approx_{\widehat{W}_T} C_p \approx_{\widehat{W}_T} C_q \approx_{\widehat{W}_T} C_k = C_{\Delta'}$, then C_p and C_q share an arrow, and as a consequence there is a non-self-folded internal triangle Δ'' with two sides incident to p and q , as we claim. Finally if $C_\Delta = C_0 \approx_{\widehat{W}_T} C_p \approx_{\widehat{W}_T} C_{\Delta''} \approx_{\widehat{W}_T} C_q \approx_{\widehat{W}_T} C_k = C_{\Delta'}$, and by definition Δ'' is related to p and q . $\square$

4.2 Reduced potential and proof of Theorem B

Until now, we have been working with the equivalence classes of cycles of the unreduced potential $\widehat{W}_T$, it is easy to see that the number of equivalence classes of cycles in the unreduced potential is less than or equal to the number of equivalence classes of cycles in the reduced potential. This subsection is devoted to study unreduced potentials and state and prove Theorem B.

Definition Let (D, M, T) be a triangulated punctured disc and x a puncture in (D, M) .

We define the *coefficient* m_x of x as follows

$$m_x = \begin{cases} 1 & \text{if } \text{val}_T(x) \geq 3 \text{ or } |\overline{\Delta_{\text{Rel}(x)}}| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

The following Lemma shows that the coefficient of a puncture is related to the number of equivalence classes of cycles in the reduced potential. In the following Lemma, we use right-equivalence of quivers with potentials in the sense of [26].

Lemma *Let (D, M, T) be a triangulated punctured disc, x a puncture in (D, M) . Suppose $\widehat{W}_T = \widehat{W}'_T + \widehat{W}''_T$, where $\widehat{W}'_T$ is the potential that does not involve cycles with at least one arrow of C_x and $\widehat{W}''_T$ is the potential that involves cycles with at least one arrow in C_x . Denote by W''_T the potential which replaces $\widehat{W}''_T$ in the reduction process. Then $m_x = 1$ if and only if W''_T is not zero.*

Proof Recall that C_x is by definition the cycle surrounding the puncture x , therefore $C_x = 0$ in $\widehat{W}_T$ if and only if $\text{val}_T(x) = 1$ and x in the puncture of a digon of type IIIa or IIIb, and in this case $m_x = 0$. Suppose $C_x \neq 0$.

Suppose $m_x = 0$, by definition $\text{val}_T(x) \leq 2$ and $|\overline{\Delta_{\text{Rel}(x)}}| \leq 1$, then either x is in a configuration IIIa or IIIb, and in this case $C_x = 0$ or x is in a configuration of type B, see Fig. 3, and one of the triangles is not an internal triangle, and as a consequence $\widehat{W}''_T$ is eliminated in the reduction process.

Now suppose $C_x \neq 0$ and W''_T is zero, then $\text{val}_T(x) = 2$, in this case x is involved in a configuration of type B, and $|\overline{\Delta_{\text{Rel}(x)}}| \leq 1$, then by definition the coefficient m_x of x is zero. $\square$

Theorem *Let B be a cluster tilted algebra of type $\mathbb{D}_n$ or $\widetilde{\mathbb{D}}_n$ and (D, M, T) its geometric realisation. Then*

$$\dim_k \text{HH}^1(B) = |\overline{\Delta_{\text{NRel}(p,q)}}| + m_p + m_q - m_{p,q}$$

where $m_{p,q}$ is equal to one if and only if there exists a triangle $\Delta \in \Delta_{\text{Rel}(p)} \cap \Delta_{\text{Rel}(q)}$.

Proof Let N_{W_T} be the number of indecomposable direct summands of the potential W_T and $N_{\widehat{W}_T}$ be the number of indecomposable direct summands of the unreduced potential $\widehat{W}_T$. We claim that $N_{W_T} = |\overline{\Delta_{\text{NRel}(p,q)}}| + m_p + m_q - m_{p,q}$.

By Remark 4.1,

$$\widehat{W}_T = \sum_{\Delta \in \Delta_{\text{NRel}(p,q)}} C_\Delta + \widehat{W}_{T_{p,q}}$$

where the sum runs over all non-self-folded internal triangles in the set $\Delta_{\text{NRel}(p,q)}$ and $\widehat{W}_{T_{p,q}} = \sum_{\Delta \in \Delta_{\text{Rel}(p,q)}} C_\Delta + C_p + C_q + C_{pq}$.

Observe that each triangle $\Delta \in \Delta_{\text{NRel}(p,q)}$ induces an equivalence class of cycles $[C_\Delta]_{\widehat{W}_T}$ in $\widehat{W}_T$ of cardinality one, otherwise there exists a cycle C in $\widehat{W}_T$ such that $C_\Delta \approx_{W_T} C$. Since no pair of 3-cycles Δ and Δ' of type II, IV or V shares arrows, then C is one of the following: C_p , C_q or C_{pq} , and as a consequence Δ is related to a puncture, which is a contradiction. Moreover, observe that in case one needs to apply a reduction to $\widehat{W}_T$, no element of the sum $\sum_{\Delta \in \Delta_{\text{NRel}(p,q)}} C_\Delta$ is affected. Therefore, $N_{W_T} = |\overline{\Delta_{\text{NRel}(p,q)}}| + N'_{W_T}$,

where N'_{W_T} is the number of indecomposable direct summands in the reduced part of $\widehat{W}_{T_{p,q}}$. Denote by $W_{T_{p,q}}$ the reduced part of $\widehat{W}_{T_{p,q}}$.

It is clear that if (D, M, T) is a triangulated once punctured disc, then $N'_{W_T} = m_p + m_q - m_{p,q} = m_p$.

Suppose (D, M, T) is a triangulated twice punctured disc. Observe that if $C_{p,q} \neq 0$, then $\widehat{W}_{T_{p,q}}$ is already reduced and moreover $\widehat{W}_{T_{p,q}}$ is the sum of the four 3-cycles in the block of type V, therefore $N'_{\widehat{W}_T} = 1$ and by definition of the coefficient of a puncture $m_p + m_q - m_{p,q} = 1$, as we claim.

Suppose $C_{p,q} = 0$, in this case $\widehat{W}_T$ is not necessarily reduced. If $\widehat{W}_{T_{p,q}}$ is already reduced, it is clear that $N'_{\widehat{W}_T} = m_p + m_q - m_{p,q}$, because the equivalence classes of $[C_x]_{\widehat{W}_T}$ and $[C_\Delta]_{\widehat{W}_T}$ in $\widehat{W}_T$ are equal for every $\Delta \in \Delta_{\text{Rel}(x)}$ and $x = p, q$.

Now suppose $\widehat{W}_{T_{p,q}} \neq W''_{T_{p,q}}$, then p or q are punctures of valency 2. Let x be a puncture involved in a configuration of type A, B or C, see Figure 3. If x is involved in a configuration of type C, then $\Delta_{\text{Rel}(p)} \cap \Delta_{\text{Rel}(q)} \neq \emptyset$ and $\widehat{W}_{T_{p,q}}$ is replaced by two cycles that share an arrow, then by Lemma 4.2 $N'_T = 1 = m_p + m_q - m_{p,q}$.

If x is involved in a configuration of type A (or B, respectively), then the reduction process affects the 2-cycle C_x and the cycles C_Δ where $\Delta \in \Delta_{\text{Rel}(x)}$, then $\sum_{\Delta \in \Delta_{\text{Rel}(x)}} C_\Delta + C_x$ is replaced by a 4-cycle C'_x in the reduction process (or by zero, respectively), then by Lemma 4.2 $m_x = 1$ (or $m_x = 0$, respectively). Moreover, if x is in a configuration of type A, then $C'_x \sim_{\widehat{W}_T} C_\Delta$ implies that $C_x \sim_{\widehat{W}_T} C_\Delta$ for every non-self-folded internal triangle $\Delta \in \Delta_{\text{Rel}(p,q)}$, that implies $N'_T = 1 = m_p + m_q - m_{p,q}$. $\square$

4.3 Geometric Interpretation of Admissible Cuts

We give a geometric interpretation of an important definition in this work: *admissible cuts of quivers*. It is known that in the unpunctured case the quiver with potential of any triangulation admits cuts yielding algebras of global dimension at most 2, see [24]. In the punctured case, some triangulations do not admit cuts, and even when they do, the global dimension of the corresponding algebra may exceed 2. In [1] there is a combinatorial characterisation of each of these two situations for ideal valency ≥ 3 -triangulations. In this work, we show an explicit construction of admissible cuts of quivers of Dynkin type $\mathbb{D}_n$ and $\mathbb{D}_n$ yielding algebras of global dimension at most 2 arising for ideal valency 1-triangulations.

Let (S, M, T) be a unpunctured triangulated surface. Following [24], an *admissible cut of the unpunctured triangulated surface* (S, M, T) is defined by choosing exactly one angle of each internal triangle of T , this gives rise to a partial triangulation T' of (S, M) , that is a collection of non-crossing arcs. This new geometric object induces an admissible cut of the quiver Q_T , that consists of the arrows opposite to each chosen angle of each internal triangle.

Now, let (S, M, T) be a triangulated surface such that each puncture is incident to exactly one arc. An *admissible cut* P of (S, M, T) is a set of angles, one for each internal non-self-folded triangle. In a similar way, choosing an angle in a internal non-self-folded triangle of a puzzle piece of type IV or V implies to consider the opposite arrow as well, but in this case, this selection implies to choose as well the closest angle of the self-folded internal triangle, and as a consequence the two arrows facing each angle are elements of the admissible cut of Q_T , as depicted in Fig. 4.

Proposition *Let (S, M, T) be a triangulated punctured disc such that each puncture is incident to exactly one arc and P an admissible cut of (S, M, T) . Then the algebra C_P induced by P is an admissible cut of B_T . Moreover, C_P is of global dimension at most two.*

Fig. 4 Cuts of Blocks

Proof Let (S, M, T) be a triangulated punctured disc such that each puncture is incident to exactly one arc and P an admissible cut of (S, M, T) . As mentioned before, the unreduced quiver with potential $(\hat{Q}_T, \hat{W}_T)$ is already reduced, and moreover, in this case, any oriented cycle in Q_T is fully contained in a block of type II, IV or V, and therefore also each term of the potential W_T . To prove that C_P is an admissible cut it is enough to observe that any chordless oriented cycle in each block is cut just once, see Fig. 4.

Because the quiver Q_T is built by gluing blocks corresponding to each kind of internal triangles and these identifications are allowed only in vertices of type $\bullet$, by definition the quiver Q_P is also built by gluing the cut blocks, and the induced relations are either monomial relations or binomial relations fully contained in each cut block, therefore there are no envelopping relations, and as a consequence the global dimension of C_P is at most two. $\square$

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