Monomial Gorenstein algebras and the stably Calabi–Yau property

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Abstract

A celebrated result by Keller–Reiten says that 2-Calabi–Yau tilted algebras are Gorenstein and stably 3-Calabi–Yau. This note shows that the converse holds in the monomial case: a 1-Gorenstein monomial algebra and stably 3-Calabi–Yau is 2-Calabi–Yau tilted, moreover is Jacobian. We study the case of other stably Calabi–Yau Gorenstein monomial algebras.

Keywords 2-Calabi–Yau tilted algebras · Monomial · Jacobian algebras · Gorenstein-projective · Stably Calabi–Yau

Mathematics Subject Classification (2010) 16G20 · 16G50

1 Introduction

Cluster-tilted algebras are inspired by Fomin–Zelevinsky cluster algebras, a class of commutative algebras that was introduced in [15], also they are related to tilted algebras [19]. Whereas tilted algebras are defined as endomorphism algebras of tilting objects over a hereditary algebra, a cluster-tilted algebra is the endomorphism algebra of a cluster-tilting object in a triangulated category called the cluster category of a hereditary algebra. The crucial step to define a cluster algebra, and the way to obtain new cluster-tilted algebras from old is called mutation. The mutation algorithm takes a quiver (or matrix) and transforms it into a new one, and the cluster-tilting mutation process takes a certain finite dimensional algebra and defines a new one that is closely related to the original. All the algebras given by this operation share interesting homological properties and their module categories are very close to each other. Cluster categories and cluster-tilted algebras were introduced in [9, 10, 12].

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The clear interaction between cluster algebras and cluster tilted algebras has motivated hundreds of works. Some of them are devoted to study other ‘cluster like’ categories and their cluster-tilting objects. So is the case of 2-Calabi–Yau tilted algebras which are obtained by replacing the cluster category by a 2-Calabi–Yau triangulated category. The definition of 2-Calabi–Yau tilted algebra appeared explicitly in [8]. The reader can also find important references in [27]. One of the main articles on the theory is the one by Keller–Reiten [21] where it is proved that 2-Calabi–Yau tilted algebras are 1-Iwanaga–Gorenstein and stably 3-Calabi–Yau, i. e. with a 3-Calabi–Yau stable category of Gorenstein projective objects (singularity category).

A large family of 2-Calabi–Yau tilted algebras are Jacobian algebras of a quiver with potential (Q, W) . Jacobian algebras are defined by applying cyclic derivatives to a quiver with potential (a generalization called quiver with hyperpotential can be considered [23]). As before, new quivers with potential can be obtained from old ones by mutation, see [14]. The 2-Calabi–Yau category that allows us to obtain Jacobian algebras as endomorphism algebras of cluster-tilting objects is called the generalized cluster category $\mathcal{C}_{(Q,W)}$ and was introduced by Amiot [1] building on previous works by Keller and Ginzburg.

On the other hand, in recent years several authors studied Gorenstein monomial algebras and their singularity categories [13, 25]. These articles show that combinatorics behind Gorenstein projective stable categories is nice when we are dealing with 1-Iwanaga–Gorenstein monomial algebras.

As mentioned above, we know that Jacobian algebras are 2-Calabi–Yau tilted, and also that 2-Calabi–Yau tilted algebras are 1-Iwanaga–Gorenstein and stably 3-Calabi–Yau. But are these conditions equivalent?

In the case of monomial algebras, this article uses the combinatorics of Gorenstein projective stable categories to show that the answer is affirmative.

The main result of the article is the following:

Theorem 1 (Theorem 3.5) *Let k be an algebraically closed field of characteristic zero and $\Lambda = kQ/I$ a monomial k -algebra. The following are equivalent:*

- (1) Λ is 2-Calabi–Yau tilted,
- (2) Λ is a 1-Iwanaga–Gorenstein algebra and stably 3-Calabi–Yau,
- (3) Λ is Jacobian.

With the consideration of hyperpotentials instead of potentials, the condition $\text{char } k = 0$ can be removed.

Jacobian algebras include a well known class of gentle 2-Calabi–Yau tilted algebras whose quivers with potential can be obtained from triangulations of unpunctured surfaces, see [3]. This family includes the cluster-tilted algebras of type $\mathbb{A}$ and $\tilde{\mathbb{A}}$. A related family of gentle algebras, that also have a geometric realization via unpunctured surfaces, are the m -cluster-tilted algebras of types $\mathbb{A}$ and $\tilde{\mathbb{A}}$, see [5, 18, 28]. This family is a subfamily of gentle algebras defined by a bound quiver that can be realized as a dual quiver obtained from $(m+2)$ -angulations as well. In Section 4 we consider a family of algebras arising from surfaces that generalize the class of m -cluster tilted algebras of types $\mathbb{A}$ and $\tilde{\mathbb{A}}$. The main result of this section characterizes their Gorenstein projective modules.

Theorem 2 (Theorem 4.5) *Let $\Lambda_{\mathcal{T}} = kQ_{\mathcal{T}}/I_{\mathcal{T}}$ be an algebra arising from a $(m+2)$ -angulation and N a $\Lambda_{\mathcal{T}}$ -module. Then N is Gorenstein projective if and only if*

$\Omega^{m+1}\tau N \simeq N$ over $\text{mod } \Lambda\tau$. In particular, this holds for m -cluster tilted algebras of type $\mathbb{A}$ and $\tilde{\mathbb{A}}$.

2 Preliminaries

Throughout these notes, we consider k -algebras where k is an algebraically closed field of characteristic zero (if the contrary is not stated). Let $Q = (Q_0, Q_1)$ be a finite quiver, where Q_0 is the set of vertices and Q_1 the set of arrows. Let $s, t: Q_1 \rightarrow Q_0$ be the functions that indicate the source and the target of each arrow, respectively. We will only consider *finite dimensional basic k -algebras*. Every finite dimensional basic k -algebra Λ is isomorphic to a quotient kQ/I , where I is an admissible ideal. The pair (Q, I) is called a *bound quiver* [4, Chapter III].

Let Ω be the usual syzygy operator, τ the Auslander–Reiten (AR) translation, and $D = \text{Hom}_k(-, k)$. Denote by $\mathcal{D}^b(\mathbb{A}_m)$ the derived category of a hereditary algebra of Dynkin type $\mathbb{A}_m$.

Definition 2.1 A k -algebra Λ is *Iwanaga–Gorenstein* if $\text{inj.dim } \Lambda = \text{proj.dim } D(\Lambda^{op}) \leq d$ for some non-negative integer d . In this case we say that Λ is d -Iwanaga–Gorenstein.

Remark 2.2 For a d -Iwanaga–Gorenstein algebra Λ , a Λ -module M is Gorenstein-projective if and only if M is a d -th syzygy, see [6, Proposition 6.20]. In this case each Λ -module either has infinite projective dimension or has projective dimension at most d . The stable subcategory of Gorenstein-projective modules, also called singularity category, $\underline{\text{GP}}(\Lambda)$ is triangulated, with shift given by the formal inverse of Ω .

An algebra $\Lambda = kQ/I$ is *monomial* if all the generators of I are paths $\alpha_1 \dots \alpha_t$. Denote by F the minimal set of paths generating I . Following [25, Section 6.2], when kQ/I is monomial and 1-Iwanaga–Gorenstein the elements of F are very particular: F is a disjoint union $\bigsqcup_{i=1}^m F_i$ where the subset F_i consists of all sub-paths over a cyclic path $c_i = x_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n_i}} x_1$ of certain common length. The cycles c_i are considered up to cyclic equivalence, and without arrow repetition. Denote by $\mathcal{C}(\Lambda)$ the set of cycles $c_1, \dots, c_m$ that contain the paths in $F_1, \dots, F_m$ respectively, by n_i the length of the cycle c_i and by r_i the (common) length of paths in F_i . We refer to the generators in F as *zero-relations*. By [25, Lemma 6.12], two different cycles c_i, c_j in $\mathcal{C}(\Lambda)$ do not share arrows.

Example 2.3 Let Q be the quiver with ideal $I = \langle \alpha\beta\gamma, \beta\gamma\delta, \gamma\delta\alpha, \delta\alpha\beta, \lambda^4 \rangle$

$$\begin{array}{ccccc} 1 & \xrightarrow{\alpha} & 2 & & \\ \delta \uparrow & & \downarrow \beta & & \\ 4 & \xleftarrow{\gamma} & 3 & \xrightarrow{\lambda} & 3 \end{array}$$

Then $F = F_1 \sqcup F_2$ with $F_1 = \{\alpha\beta\gamma, \beta\gamma\delta, \gamma\delta\alpha, \delta\alpha\beta\}$, $F_2 = \{\lambda^4\}$. We have $c_1 = \alpha\beta\gamma\delta$ and $c_2 = \lambda$, so $n_1 = 4$, $n_2 = 1$, $r_1 = 3$ and $r_2 = 4$.

Lemma 2.4 [25, Theorem 6.16] Let $\Lambda = kQ/I$ be a 1-Iwanaga–Gorenstein monomial algebra. The singularity category $\underline{\text{GP}}(\Lambda)$ is equivalent to the coproduct $\bigsqcup_{c_i \in \mathcal{C}(\Lambda)} \mathcal{D}^b(\mathbb{A}_{r_i-1})/\tau^{n_i}$.

2.1 Gentle algebras

We recall the definition of gentle algebra. By [17] it is known that gentle algebras are Iwanaga–Gorenstein.

Definition 2.5 A k -algebra $\Lambda = kQ/I$ is *gentle* if

- (g1) For each $x_0 \in Q_0$ there are at most two arrows with source x_0 , and at most two arrows with target x_0 ,
- (g2) the ideal I is generated by paths of length 2,
- (g3) for each $\beta \in Q_1$ there is at most one arrow $\alpha \in Q_1$ and at most one arrow γ such that $\alpha\beta \in I$ and $\beta\gamma \in I$, and
- (g4) for each $\beta \in Q_1$ there is at most one arrow α and at most one arrow γ such that $\alpha\beta \notin I$ and $\beta\gamma \notin I$.

An algebra $\Lambda = kQ/I$, where I is generated by paths and (Q, I) satisfies the two conditions (g1) and (g4), is called a *string algebra*. Thus every gentle algebra is a string algebra. A *string* in Λ is by definition a reduced walk w in Q avoiding the zero-relations, thus w is a sequence $x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_{n+1}$ where the x_i are vertices of Q and each α_i is an arrow between the vertices x_i and x_{i+1} in either direction such that there is no $\xrightarrow{\beta} \xleftarrow{\beta}$, no $\xleftarrow{\beta} \xrightarrow{\beta}$, and no $\xleftarrow{\beta_1} \cdots \xleftarrow{\beta_r}$ or $\xrightarrow{\beta_1} \cdots \xrightarrow{\beta_r}$ with $\beta_1 \cdots \beta_r \in I$. If the first and the last vertex of w coincide, then the string is cyclic. A *band* is a cyclic string b such that each power b^n is a cyclic string but b is not a power of some string. The classification of indecomposable modules over a string algebra $\Lambda = kQ/I$ is given by Butler–Ringel [11] in terms of strings and bands in (Q, I) . Each string w defines an indecomposable module $M(w)$, called a *string module*, and each band b defines a family of indecomposable modules $M(b, \lambda, n)$, called *band modules*, with parameters $\lambda \in k$ and $n \in \mathbb{N}$.

Definition 2.6 Let $\Lambda = kQ/I$ be a gentle algebra.

- (a) A cycle $x_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} x_n \xrightarrow{\alpha_n} x_1$ is *saturated* if $\alpha_i \alpha_{i+1} \in I$, for $i \in \mathbb{Z}/n\mathbb{Z}$. In particular, a *saturated loop* is an arrow δ such that $s(\delta) = t(\delta)$ and $\delta^2 \in I$.
- (b) An arrow β is *gentle* if there is no arrow α such that $\alpha\beta \in I$.
- (c) A path $\alpha_1 \cdots \alpha_n$ is formed by *consecutive relations* if $\alpha_i \alpha_{i+1} \in I$ for $1 \leq i < n$.
- (d) A path $\alpha_1 \cdots \alpha_n$ is *critical* if it is formed by consecutive relations and α_1 is a gentle arrow.

When there is no gentle arrow, we set $n(\Lambda) = 0$. When there is a gentle arrow, let $n(\Lambda)$ be the maximal length computed over all critical paths. This number is bounded since Q is finite.

Theorem 2.7 [17] Let $\Lambda = kQ/I$ be a gentle algebra with $n(\Lambda)$ the maximum length over all critical paths. Then $\text{inj.dim} \Lambda = n(\Lambda) = \text{proj.dim} D(\Lambda^{op})$ if $n(\Lambda) > 0$, and $\text{inj.dim} \Lambda = \text{proj.dim} D(\Lambda^{op}) \leq 1$ if $n(\Lambda) = 0$. In particular, Λ is Iwanaga–Gorenstein.

Let $x_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} x_n \xrightarrow{\alpha_n} x_1$ be a saturated cycle. Let u_i be the maximal path avoiding zero-relations starting at the vertex x_i and let v_i be the maximal path avoiding zero-relations ending at x_i . Then the indecomposable projective module $P(x_i)$ and the

Fig. 1 Local situation for a saturated cycle

indecomposable injective module $I(x_i)$ are string modules given by $P(x_i) = M(u_i^{-1}\alpha_i u_{i+1})$ and $I(x_i) = M(v_{i-1}\alpha_{i-1}v_i^{-1})$, see Fig. 1.

Theorem 2.8 [20, Theorem 2.5] *Let $\Lambda = kQ/I$ be a gentle algebra. Let $x_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} x_n \xrightarrow{\alpha_n} x_1$ be a saturated cycle. The string module $M(u_i)$, where u_i is the string starting at x_i , is Gorenstein-projective. Moreover, all indecomposable modules in $\underline{\text{GP}}(\Lambda)$ are obtained in such manner.*

3 Monomial 2-Calabi–Yau Tilted Algebras Are Jacobian

A triangulated k -category $\mathcal{C}$, Hom-finite with split idempotents, is d -Calabi–Yau if there is a bifunctorial isomorphism $\text{Hom}_{\mathcal{C}}(X, Y) \simeq D\text{Hom}_{\mathcal{C}}(Y, X[d])$ for all $X, Y \in \mathcal{C}$. This means that the d -th power of [1] is a Serre functor for the category, and this is to say that there is a functorial isomorphism $\tau^{-1}[d-1] = \text{Id}$.

Definition 3.1 Let $\mathcal{C}$ be a 2-Calabi–Yau category

- (1) An object T is *cluster-tilting* if it is basic and $\text{add}T = \{X \in \mathcal{C} : \text{Hom}_{\mathcal{C}}(X, T[1]) = 0\}$.
- (2) The endomorphism algebra $\text{End}_{\mathcal{C}}(T)$ of a cluster-tilting object is called a *2-Calabi–Yau tilted algebra*.

Examples of 2-Calabi–Yau tilted algebras are the *cluster-tilted* algebras defined in [10]. A more general family is obtained from generalized cluster categories defined by Amiot, see [1]. *Quivers with potential* were introduced in [14]. A potential W is a (possibly infinite) linear combination of cycles in Q , up to cyclic equivalence. Given an arrow α and a cycle $c = \alpha_1 \dots \alpha_l$, the cyclic derivative $\partial_{\alpha}(c)$ is defined by

$$\partial_{\alpha}(\alpha_1 \dots \alpha_l) = \sum_{k=1}^l \delta_{\alpha\alpha_k} \alpha_{k+1} \dots \alpha_l \alpha_1 \dots \alpha_{k-1},$$

where $\delta_{\alpha\alpha_k}$ is the Kronecker delta. This derivative can be applied to a sum of cycles extending by linearity. Notice that a cycle $\alpha_1 \dots \alpha_l$ may have arrow repetitions. Let $R\langle\langle Q \rangle\rangle$ be the complete path algebra consisting of all (possibly infinite) linear combinations of paths in Q . Let (Q, W) be a quiver with potential, the *Jacobian algebra* is defined to be $\text{Jac}(Q, W) = R\langle\langle Q \rangle\rangle / \langle\partial_{\alpha} W, \alpha \in Q_1\rangle$. When one considers the algebra $R\langle\langle Q \rangle\rangle / \langle\partial_{\alpha} W, \alpha \in Q_1\rangle$ and the result is finite dimensional then it is a Jacobian algebra $\text{Jac}(Q, W)$ and there is no need to consider completions and ideal closures.

Amiot [1, Sec. 3] showed that Jacobian algebras are 2-CY tilted whenever they are finite dimensional by defining a generalized cluster category $\mathcal{C}_{(Q,W)}$. In [2, Question 2.20], the author asked whether all 2-Calabi–Yau tilted algebras are Jacobian, and warned that the answer might be negative in a general setting. The first part of this note is devoted to this question, provided the algebra is monomial.

Now we recall some main properties of 2-Calabi–Yau tilted algebras. The first part of the next theorem is a celebrated result by Keller–Reiten, the second part is a characterization of Gorenstein projective modules that can be computed over the module category.

Theorem 3.2 [16, 21] *Let Λ be a 2-Calabi–Yau tilted algebra,*

- (1) *Λ is 1-Iwanaga–Gorenstein and stably 3-Calabi–Yau.*
- (2) *A Λ -module M is Gorenstein projective if and only if $\Omega^2 \tau M \simeq M$ in $\underline{\text{mod}} \Lambda$.*

Let $\Lambda = kQ/I$ be a monomial 2-Calabi–Yau tilted algebra. Since it is a monomial 1-Iwanaga–Gorenstein algebra by the result above, the singularity category $\text{GP}(\Lambda)$ is described in Lemma 2.4. Using this, we can gather information on cycles and relations in (Q, I) . Remember that the zero-relations of length r_i in F lie in a cycle c_i of length n_i .

Lemma 3.3 *Let $\Lambda = kQ/I$ be a monomial 1-Iwanaga–Gorenstein algebra and stably 3-Calabi–Yau, and $F = \coprod F_i$ the corresponding set of zero-relations. Then $r_i = b_i n_i - 1$, where $b_i \in \mathbb{Z}^+$ is such that $r_i > 0$.*

Proof We want to know for which parameters $x, n \in \mathbb{Z}^+$ we have that an orbit category $\mathcal{D}^b(\mathbb{A}_x)/\tau^n$ is 3-Calabi–Yau. Recall that in the orbit category [1] and τ are induced from $\mathcal{D}^b(\mathbb{A}_x)$.

So, at least, we need the functor $\tau^{-1}[2]$ to send objects of a certain orbit to objects in the same orbit. One can easily compute that $\tau^{-1}[2]X = \tau^{-x-2}X$ over $\mathcal{D}^b(\mathbb{A}_x)$. Hence $\tau^{x+2}X$ has to be $(\tau^n)^b X = \text{Id}X = X$ in the orbit category, so $x = nb - 2$. If we go back to the notation in Lemma 2.4, the claim follows. $\square$

With the last lemma we have restricted the possible lengths of cycles and zero-relations in kQ/I , so now there is a constraint for the elements in F . In the following we show that a finite dimensional algebra described by a bound quiver (Q, I) satisfying this constraint has to be 2-Calabi–Yau tilted.

Lemma 3.4 *Let $\Lambda = kQ/I$ be a 1-Iwanaga–Gorenstein monomial algebra where $F = \coprod_{i \in [1,m]} F_i$ is such that $r_i = b_i n_i - 1$, $b_i \in \mathbb{Z}^+$ and $r_i > 0$. Then the zero-relations arise from a potential $W = \sum_{i \in [1,m]} c_i^{b_i}$.*

Proof Since two different cycles in $\mathcal{C}(\Lambda)$ do not share arrows, each $\alpha \in Q_1$ is either in one of the cycles or is not present in $\mathcal{C}(\Lambda)$. If α is not in any cycle then the associated partial derivation adds zero to the ideal I . If α is in the cycle $c_j \in \mathcal{C}(\Lambda)$ of length n_j (with no repetition), then $\partial_\alpha(\sum_i c_i^{b_i}) = \partial_\alpha(c_j^{b_j})$ and this is an element of kQ given by $b_j u$ where u is a subpath of length $r_j = n_j b_j - 1$ in the cycle $c_j^{b_j}$ without the arrow α . Our general hypothesis is that k is a field of characteristic zero, so these paths $b_j u$ can be replaced by

u as generators of I . Doing this for all $\alpha \in Q_1$ we cover exactly all the elements in F . Therefore Λ is a Jacobian algebra hence it is 2-Calabi–Yau tilted. $\square$

The two lemmas above sum up in the following theorem.

Theorem 3.5 *Let k be an algebraically closed field of characteristic zero and $\Lambda = kQ/I$ a monomial k -algebra. The following are equivalent:*

- (1) Λ is 2-Calabi–Yau tilted,
- (2) Λ is a 1-Iwanaga–Gorenstein algebra and stably 3-Calabi–Yau,
- (3) Λ is Jacobian.

Proof (1 $\Rightarrow$ 2) Let Λ be monomial 2-Calabi–Yau tilted. By Theorem 3.2 (1) Λ is a 1-Iwanaga–Gorenstein algebra with stably 3-Calabi–Yau singularity category.

(2 $\Rightarrow$ 3) By Lemmas 3.3 and 3.4, a 1-Iwanaga–Gorenstein monomial algebra $\Lambda = kQ/I$ with a 3-Calabi–Yau singularity category is such that the ideal I arises from the cyclic derivatives of a potential, and this potential can be easily defined from I .

(3 $\Rightarrow$ 1) It follows from [1, Theorem 3.6] that Λ is 2-Calabi–Yau tilted in the sense of Keller–Reiten. $\square$

The previous result is written with the restriction: $b_j \in \mathbb{Z}^+$ is non-zero over k . To guarantee this we say k is of characteristic zero.

To solve the problem presented by integration-differentiation of potentials over fields of positive characteristic, an alternative is proposed: using *hyperpotentials*. A hyperpotential on a quiver Q is a collection of elements $(\rho_\alpha)_{\alpha \in Q_1}$ over the complete algebra $R\langle\langle Q \rangle\rangle$ such that: for $\alpha: i \rightarrow j$, ρ_α is a (possibly infinite) linear combination of paths $j \rightsquigarrow i$, and $\sum_{\alpha \in Q_1} [\alpha, \rho_\alpha] = 0$. The *Jacobian algebra of a hyperpotential* is the quotient $R\langle\langle Q \rangle\rangle / \langle \rho_\alpha \rangle$ (see [23, Proposition 1]). In view of this, if we admit hyperpotentials, the last result extends to algebraically closed fields of positive characteristic.

Remark 3.6 Let $\Lambda = kQ/I$ a monomial k -algebra, where k is a field of arbitrary characteristic. The following are equivalent:

- (1) Λ is 2-Calabi–Yau tilted,
- (2) Λ is a 1-Iwanaga–Gorenstein algebra and stably 3-Calabi–Yau,
- (3) Λ is a Jacobian algebra of a hyperpotential.

Example 3.7 The algebra kQ/I in Example 2.3 is Jacobian. The potential is $W = \alpha\beta\gamma\delta + \lambda^5$.

3.1 Gentle case

A particular case of monomial algebras is gentle algebras, see Definition 2.6. In particular gentle Jacobian algebras were studied in [3, Section 2]. There it is proved that a Jacobian algebra arising from a triangulation is gentle and 1-Iwanaga–Gorenstein [3, Theorem 2.7], and a gentle algebra kQ/I , where Q has no loops, such that relations are restricted to saturated 3-cycles arises from a surface triangulation.

Now we apply the results in Section 3 but adding the gentle hypothesis.

Fig. 2 Gentle bound quiver (Q, I) , Example 3.9

Proposition 3.8 *Let kQ/I be a gentle 2-Calabi–Yau tilted algebra, then every relation lies on a saturated 3-cycle or a saturated loop.*

Proof A gentle algebra is monomial, and by Theorem 3.2 (1) we can apply Lemma 3.3. Then we have $r_i = b_i n_i - 1 = 2$, so $b_i n_i = 3$. We have only two options: $b_i = 1$ and $n_i = 3$, or $b_i = 3$ and $n_i = 1$. In the first case the relations arise from a potential term of length three $\alpha\beta\gamma$ and we obtain a saturated 3-cycle, and in the second case the potential term is δ^3 so we get a loop such that $\delta^2 \in I$. $\square$

We can easily build gentle 2-Calabi–Yau tilted algebras. We only need to be careful so we do not create a cycle that is not in I .

Example 3.9 Let Q be the quiver in Fig. 2, and consider the potential $W = \delta_1^3 + \sum_{i=1}^3 \alpha_i \beta_i \gamma_i$. Then $\text{Jac}(Q, W)$ is a gentle 2-Calabi–Yau tilted algebra.

4 Other Monomial Algebras and their Singularity Categories

In this section we study $\underline{\text{GP}}$ for a family of gentle algebras and some examples of non-gentle m -cluster tilted algebras. We will obtain an analogous property to Theorem 3.2 (2) by explicit computation. This property is deeply related to the CY dimension of $\underline{\text{GP}}$.¹

An algebra arising from an $(m+2)$ -angulation $\mathcal{T}$ is defined as $kQ_{\mathcal{T}}/I_{\mathcal{T}}$ where the bound quiver $(Q_{\mathcal{T}}, I_{\mathcal{T}})$ is obtained from the $(m+2)$ -angulation. The set of vertices $(Q_{\mathcal{T}})_0$ is indexed by the internal arcs in $\mathcal{T}$. For any two vertices i, j , there is an arrow $i \rightarrow j$ when the corresponding m -diagonals i and j share a vertex, they are edges of the same $(m+2)$ -gon and i follows j clockwise. Given consecutive arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$, then $\alpha\beta \in I_{\mathcal{T}}$ if and only if i, j and k are edges in the same $(m+2)$ -gon.

Example 4.1 Draw an annulus as a strip Σ and remove a disk from its interior. Add two marked points in this boundary component and other marked points in the boundary of the strip, see Fig. 3. Given a 4-angulation of this marked surface, we can obtain a bound quiver $(Q_{\mathcal{T}}, I_{\mathcal{T}})$.

In particular, when the surface is a disk (resp. an annulus) the gentle algebra obtained is m -cluster tilted of type $\mathbb{A}$ (resp. $\hat{\mathbb{A}}$) in the sense of Thomas [28], as it was studied in several works [3, 5, 12, 18, 26, 29].

¹This discussion is the main topic of the Appendix. The author is deeply grateful with the Appendix authors for their answer on this subject.

Fig. 3 4-angulation of the sphere with 3 three disks removed, and bound quiver (Q_T, I_T)

The following properties were observed in [26, Rem. 2.18] and [18, Sec. 7] in the case of the disc and annulus, but it is easy to see that they hold when the algebra arises from an $(m + 2)$ -angulation of a general surface.

Proposition 4.2 *Let (Q_T, I_T) be a bound quiver arising from an $(m + 2)$ -angulation.*

- (1) $\Lambda_T = kQ_T/I_T$ is a gentle algebra.
- (2) The only possible saturated cycles in (Q_T, I_T) are $(m + 2)$ -cycles.
- (3) There can be at most $m - 1$ consecutive zero-relations not lying in a saturated cycle.

Immediately, we have the following observation.

Lemma 4.3 *Let $\Lambda_T = kQ_T/I_T$ be an algebra arising from a $(m + 2)$ -angulation. Then, Λ_T is m -Iwanaga–Gorenstein.*

Proof The case $m = 1$ follows from Theorem 3.2 (1), and also from [3, Lemma 2.6]. Let $m \geq 2$. Since Λ_T is gentle, we can apply Theorem 2.7. First assume that there is no gentle arrow in (Q_T, I_T) , then $n(\Lambda_T) = 0$, so d is zero or one and $d \leq m$. The statement follows.

Now, assume there are gentle arrows in (Q_T, I_T) , and let α_1 be one of them. It follows that α_1 is not part of a saturated cycle. Let $\alpha_1 \dots \alpha_r$ be a critical path. Since α_1 is not part of a saturated cycle, then none of the arrows α_i for $1 \leq i \leq r$ is part of a saturated cycle. By Proposition 4.2 (3), the maximal number of consecutive zero-relations outside of a saturated cycle is $m - 1$. Therefore, $r \leq m$, and by Theorem 2.7, Λ_T is Gorenstein of dimension $d \leq m$. $\square$

Most of the arguments in the following lemma can be found also in [20, Section 4].

Lemma 4.4 *Let $\Lambda = kQ/I$ be a gentle d -Iwanaga–Gorenstein algebra, $d \geq 1$. Let $x \in Q_0$, and let N be an indecomposable direct summand of $\text{rad}P(x)$. Then,*

- (a) $N \in \underline{\text{GP}}(\Lambda)$, or
- (b) $\text{proj.dim}N \leq d - 1$.

Proof If N is projective, we are in case (b). Let N be non projective. Let $P(x) = M(u^{-1}\alpha^{-1}\beta w)$ be the indecomposable projective and $N = M(u)$ so that $S(t(\alpha)) = \text{top}M(u)$. We study the cases:

- (i) α is part of a saturated cycle $x_1 \rightarrow \cdots \rightarrow x_i \xrightarrow{\alpha} x_{i+1} \cdots \rightarrow x_1$.
 - (ii) α is not part of a saturated cycle.
- (i) Let $x_i \xrightarrow{\alpha} x_{i+1}$, then $M(u)$ is a direct summand of $\text{rad}P(x_i)$. By Theorem 2.8, $M(u) \in \underline{\text{GP}}(\Lambda)$.
 - (ii) Since $N = M(u)$ is not projective, there exists an arrow δ_1 such that $\alpha\delta_1 \in I$. The arrow δ_1 is not part of a saturated cycle, since then α would be part of the saturated cycle. Let $P(t(\alpha)) = M(c^{-1}\delta_1^{-1}u)$, then there is an exact sequence

$$0 \rightarrow M(c) \rightarrow P(t(\alpha)) \rightarrow M(u) \rightarrow 0. \quad (1)$$

If the string module $M(c)$ is not projective, then it satisfies the same conditions as $M(u)$, so we can construct a new exact sequence

$$0 \rightarrow M(c_1) \rightarrow P(t(\delta_1)) \rightarrow M(c) \rightarrow 0. \quad (2)$$

Recursively, we obtain a path $\alpha\delta_1 \cdots \delta_n$ such that each quadratic factor belongs to I . This process has to finish after a finite number of steps, being the direct summand $M(c_n)$ of $P(t(\delta_{n-1}))$ a projective module. If there were not finitely many steps and $M(c_n)$ was not projective, we would find new arrows $\delta_{n+1}, \dots$ and form a path $\alpha\delta_1 \cdots \delta_n \cdots$ such that each quadratic factor is in I . The quiver Q is finite, so the only way to construct an infinite path $\alpha\delta_1 \cdots \delta_n \cdots$ is by reaching a saturated cycle. By the gentleness, if one of the arrows δ_i is in a saturated cycle, then all $\alpha, \delta_1, \dots, \delta_n$ are in the saturated cycle, contradicting the condition imposed on α . Therefore the procedure to find the short exact sequences stops. Splicing the short exact sequences we get a projective resolution for $M(u)$, which is finite, so $\text{proj.dim}M(u) < \infty$. By Remark 2.2, we have $\text{proj.dim}M(u) \leq d$. Now, we can also express $M(u)$ as $M(u) = \Omega M(\beta w)$. If we had $\text{proj.dim}M(u) = d$, then we would have $\text{proj.dim}M(\beta w) = d + 1$ and this is impossible by Remark 2.2. Thus, $\text{proj.dim}M(u) \leq d - 1$. $\square$

To complete the previous lemma, observe that if Λ is selfinjective (that is, Λ is 0-Iwanaga–Gorenstein) then every indecomposable module is Gorenstein-projective.

Theorem 4.5 *Let $\Lambda_{\mathcal{T}} = kQ_{\mathcal{T}}/I_{\mathcal{T}}$ be an algebra arising from a $(m+2)$ -angulation and let N be a $\Lambda_{\mathcal{T}}$ -module. Then N is Gorenstein projective if and only if $\Omega^{m+1}\tau N \simeq N$ in $\underline{\text{mod}}\Lambda_{\mathcal{T}}$. In particular, this holds for m -cluster tilted algebras of type $\mathbb{A}$ and $\tilde{\mathbb{A}}$.*

Proof Let M be an indecomposable module in $\underline{\text{GP}}(\Lambda_{\mathcal{T}})$. By Theorem 2.8, $M = M(u_i)$ where u_i is the maximal non-zero path starting at x_i .

$$\begin{array}{ccccc}
 \begin{array}{c} \downarrow \\ v_i \\ \downarrow \end{array} & & \begin{array}{c} \downarrow \\ v_{i+1} \\ \downarrow \end{array} & & \begin{array}{c} \downarrow \\ v_{i+2} \\ \downarrow \end{array} \\
 x_i & \xrightarrow{\alpha_i} & x_{i+1} & \xrightarrow{\alpha_{i+1}} & x_{i+2} \\
 \begin{array}{c} \downarrow \\ u_i \\ \downarrow \end{array} & & \begin{array}{c} \downarrow \\ u_{i+1} \\ \downarrow \end{array} & & \begin{array}{c} \downarrow \\ u_{i+2} \\ \downarrow \end{array}
 \end{array}$$

We compute a minimal projective presentation of $M(u_i)$.

$$\begin{array}{ccccc} M(u_{i+1}^{-1}\alpha_{i+1}u_{i+2}) & \xrightarrow{p_1} & M(u_i^{-1}\alpha_i u_{i+1}) & \longrightarrow & M(u_i) \longrightarrow 0. \\ \nearrow & & \nwarrow & & \\ M(u_{i+2}) & & M(u_{i+1}) & & \end{array}$$

Observe that $\Omega^t M(u_i) = M(u_{i+t})$, where t is an integer considered modulo $m+2$. Applying the Nakayama functor we get

$$\begin{array}{ccccccc} 0 \longrightarrow & M(v_{i+1}) \hookrightarrow & M(v_i \alpha_i v_{i+1}^{-1}) & \xrightarrow{vp_1} & M(v_{i-1} \alpha_{i-1} v_i^{-1}) & \twoheadrightarrow & M(v_{i-1}). \\ & & \searrow & & \nearrow & & \\ & & M(v_i) & & & & \end{array}$$

Then $\tau M(u_i) = \ker vp_1 = M(v_{i+1})$. Let $S(y_{i+1}) = \text{top} M(v_{i+1})$. Let $P(y_{i+1}) = M(w_{i+1}^{-1}v_{i+1}\alpha_{i+1}u_{i+2})$ be the projective cover of $M(v_{i+1})$.

$$\begin{array}{ccccc} & & y_{i+1} & & \\ & & \downarrow \{u_{i+1}\} & & \\ x_i & \xrightarrow{\alpha_i} & x_{i+1} & \xrightarrow{\alpha_{i+1}} & x_{i+2} \\ & & & & \downarrow \{u_{i+2}\} \end{array}$$

Therefore, $\Omega M(v_{i+1}) = M(w'_{i+1}) \oplus M(u_{i+2})$, where $M(w'_{i+1})$ is the maximal submodule of $M(w_{i+1})$. The syzygy functor is additive, so

$$\Omega^{m+1} \tau M(u_i) = \Omega^{m+1} M(v_{i+1}) = \Omega^m M(w'_{i+1}) \oplus \Omega^m M(u_{i+2}).$$

Since $\Omega^t M(u_i) = M(u_{i+t})$, we have

$$\Omega^m M(w'_{i+1}) \oplus \Omega^m M(u_{i+2}) = \Omega^m M(w'_{i+1}) \oplus M(u_i).$$

Now, we only need to prove that $\Omega^m M(w'_{i+1}) = 0$. Observe that $M(w'_{i+1})$ is a direct summand of $\text{rad} P(y_{i+1})$.

We know by Lemma 4.3 that $\Lambda_{\mathcal{T}}$ is Gorenstein of dimension $d \leq m$. By Lemma 4.4 one of the following holds:

- (1) $\text{proj.dim} M(w'_{i+1}) \leq m-1$, or
- (2) $M(w'_{i+1}) \in \underline{\text{GP}}(\Lambda_{\mathcal{T}})$.

If (1) holds, then $\Omega^m M(w'_{i+1}) = 0$ and we are done.

We assume (2) holds, so $M(w'_{i+1}) \in \underline{\text{GP}}(\Lambda_{\mathcal{T}})$. We prove that this leads to a contradiction. Let z_{i+1} be the vertex such that $\text{top} M(w'_{i+1}) = S(z_{i+1})$. By the description in Theorem 2.8, the vertex z_{i+1} is the target of an arrow γ in a saturated $(m+2)$ -cycle and $\gamma w'_{i+1} \neq 0$.

(2a) If the arrow γ is $y_{i+1} \xrightarrow{\gamma} z_{i+1}$, as appears in the figure below, then there is an arrow a_j in the saturated $(m+2)$ -cycle, such that $a_j \gamma \in I_{\mathcal{T}}$. Then, $a_j v_{i+1} \neq 0$,

contradicting that $I(x_{i+1}) = M(v_i \alpha_i v_{i+1}^{-1})$ is the indecomposable injective associated to x_i . This is absurd.

$$\begin{array}{ccccc} & \downarrow a_j & & & \\ y_{i+1} & \xrightarrow{\gamma} & z_{i+1} & \xrightarrow{a_{j+2}} & \\ \downarrow \wr v_{i+1} & & \downarrow \wr w'_{i+1} & & \end{array}$$

(2b) If the arrow γ in a saturated cycle is such that $s(\gamma) \neq y_{i+1}$, as we see in the figure below, there is an arrow b_{j+2} following the saturated cycle such that $\gamma b_{j+2} \in I_{\mathcal{T}}$. Thus, we have $\gamma w'_{i+1} \neq 0$ and by gentleness, $a_{j+1} b_{j+2} \notin I_{\mathcal{T}}$.

$$\begin{array}{ccccc} & \downarrow a_j & & \downarrow \gamma & \\ y_{i+1} & \xrightarrow{a_{j+1}} & z_{i+1} & \xrightarrow{b_{j+2}} & \\ \downarrow \wr v_{i+1} & & \downarrow \wr w'_{i+1} & & \end{array}$$

But recall that $M(w'_{i+1})$ is a submodule of $\text{rad} P(y_{i+1})$, so b_{j+2} has to be the first arrow in the string w'_{i+1} . This contradicts that $M(w'_{i+1})$ is a submodule of $\text{rad} P(y_{i+1})$. To sum up, $M(w'_{i+1})$ is not a trivial Gorenstein projective module and just (1) holds. $\square$

As a corollary, we obtain the next result that generalizes the properties known for cluster-tilted algebras: Theorem 3.2 (1) and (2).

Corollary 4.6 *Let Λ be a m -cluster tilted algebra of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Then*

- (1) Λ is m -Iwanaga–Gorenstein, and
- (2) $N \in \text{GP}(\Lambda)$ if and only if $\Omega^{m+1} \tau N \cong N$.

Proof Part (1) follows from Lemma 4.3. Part (2) follows from Theorem 4.5. $\square$

From a more general point of view, m -cluster tilted algebras are $(m+1)$ -Calabi–Yau tilted algebras. It is known, and expected, that Corollary 4.6 does not hold in general for d -Calabi–Yau tilted algebras by the next reasons:

- (1) In [21, Section 5.3] there is an example (due to Iyama) of a d -Calabi–Yau tilted algebra that is *not* Iwanaga–Gorenstein.
- (2) Moreover, a recent preprint [24] shows that all finite dimensional k -algebras are d -Calabi–Yau tilted (for all $d > 2$).

Still, there are results in this direction due to Keller and Reiten [22, Section 4.6], and Beligiannis [7, Theorem 6.4], showing that a d -Calabi–Yau tilted algebra is Iwanaga–Gorenstein under certain conditions. In both cases, it is required that $\text{add} T$ is corigid to some degree u , meaning that $\text{Hom}_{\mathcal{C}}(T, T[-t]) = 0$ for all $1 \leq t \leq u$. Over m -cluster tilting categories of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$, the full subcategory defined by a cluster tilting object $\text{add} T$ is $(m+1)$ -cluster tilting. In the next example we show that Corollary 4.6 is independent of these

Fig. 4 A 2-cluster tilting object T which is not corigid

results, by giving an example of a non-corigid cluster-tilting object in the 2-cluster category of type $\mathbb{A}_4$.

Example 4.7 Let $\mathcal{C}_Q^2$ be the 2-cluster category, where Q is of type $\mathbb{A}_4$ and let T be as in Fig. 4. The object $\tilde{T}$ is not corigid since $\text{Hom}(T_3, T_1[-1]) \simeq \text{Hom}(T_3[1], T_1) \neq 0$. By Corollary 4.6 the 2-cluster tilted algebra $\text{End}(T)$ is 2-Iwanaga–Gorenstein. In fact, in this example the algebra, given by the quiver below bound by $\beta\alpha = 0$, is of global dimension two.

$$1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4$$

In the next example we see that there are m -cluster-tilted algebras not arising from $(m + 2)$ -angulations, defined by non-corigid cluster tilting objects, for which the conclusion of Corollary 4.6 holds.

Example 4.8 Let $\mathcal{C}_Q^2$ be the 2-cluster category of type $\mathbb{D}_6$, and $T = \bigoplus_{i=1}^6 T_i$ the 2-cluster tilting object in Fig. 5. The algebra $\Lambda = \text{End}_{\mathcal{C}_Q^2}(T)$ in Example 4.8 is given by the quiver in Fig. 6 and the ideal $I = \langle \lambda\alpha, \alpha\beta\gamma, \beta\gamma\delta, \delta\lambda \rangle$. We see that Λ is 2-Iwanaga–Gorenstein and has infinite global dimension. The indecomposable modules in $\underline{\text{GP}}(\Lambda)$ are $3, 6, \frac{5}{4}, \frac{2}{1}$, exactly those such that $\Omega^3 \tau N = N$.

As a final remark, we may ask the next question:

Question 4.9 m -cluster tilted algebras are still a special subclass of $(m + 1)$ -Calabi–Yau tilted algebras. Are they always Iwanaga–Gorenstein? Does Corollary 4.6 always hold for these algebras?

Fig. 5 A 2-cluster tilting object T such that $\text{Hom}(T_5, T_3[-1]) = \text{Hom}(T_5[1], T_3) \neq 0$, so T is not corigid

Fig. 6 $\Lambda = kQ/I$, Example 4.8

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Appendix A: Auslander–Reiten translations for Gorenstein algebras by Sondre Kvamme and Matthew Pressland

Let Λ be an Iwanaga–Gorenstein algebra. Since $\underline{GP}(\Lambda)$ is a functorially finite subcategory of $\underline{\text{mod}}\Lambda$, it has Auslander–Reiten sequences [30], inducing an Auslander–Reiten translation $\tau_{\underline{GP}}$ on $\underline{GP}(\Lambda)$, typically different from the Auslander–Reiten translation τ_{Λ} on $\underline{\text{mod}}\Lambda$. The goal of this appendix is to relate the objects $\tau_{\Lambda}M$ and $\tau_{\underline{GP}}M$ of $\underline{\text{mod}}\Lambda$ when M is Gorenstein projective. Indeed, we will show that these objects eventually coincide after repeated application of the syzygy functor. While this fact may not be surprising to experts, and can be deduced quickly from results already in the literature (most easily from [31, Thm. 3.7]), we give here a very direct proof, which even exhibits a natural isomorphism of functors. Moreover, we explain how this result provides a new perspective on results of Garcia Elsener (Corollary 4.6 of the present paper) and Garcia Elsener–Schiffler [32, Thm. 1] characterising Gorenstein projective modules over certain Calabi–Yau tilted algebras Λ , by relating these characterisations directly to a Calabi–Yau property of $\underline{GP}(\Lambda)$.

We denote by $\Omega: \underline{\text{mod}}\Lambda \rightarrow \underline{\text{mod}}\Lambda$ the syzygy functor, taking a module to the kernel of a projective cover, and by $\Sigma: \underline{\text{mod}}\Lambda \rightarrow \underline{\text{mod}}\Lambda$ its left adjoint, taking a module to the cokernel of a right $\text{proj}\Lambda$ -approximation. The restrictions of these functors to $\underline{GP}(\Lambda)$ are mutually inverse, the restriction of Σ being the suspension functor on this triangulated category.

Lemma A.1 *The functor $\Sigma^d: \underline{\text{mod}}\Lambda \rightarrow \underline{\text{mod}}\Lambda$ has essential image in $\underline{GP}(\Lambda)$.*

Proof Since Λ is d -Iwanaga–Gorenstein, we have $\Omega^d \Sigma^d M \in \underline{GP}(\Lambda)$ for any Λ -module M . Then $\Sigma^d \Omega^d \Sigma^d M \in \underline{GP}(\Lambda)$ since Σ preserves Gorenstein projectivity. It follows from the triangular identities for the unit and counit of the adjoint pair (Σ^d, Ω^d) that $\Sigma^d M$ is a summand of $\Sigma^d \Omega^d \Sigma^d M$, and hence is itself Gorenstein projective. $\square$

Theorem A.2 *Let Λ be a finite-dimensional Iwanaga–Gorenstein algebra of Gorenstein dimension at most d . Then there is a natural isomorphism*

$$\Omega^d \tau_\Lambda \simeq \Omega^d \tau_{\text{GP}}$$

of endofunctors of $\underline{GP}(\Lambda)$.

Proof Using the Yoneda embedding, it suffices to show that there is a natural isomorphism

$$\underline{\text{Hom}}_\Lambda(-, \Omega^d \tau_{\text{GP}} X) \cong \underline{\text{Hom}}_\Lambda(-, \Omega^d \tau_\Lambda X)$$

of functors on $\text{mod } \Lambda$, for any $X \in \text{GP}(\Lambda)$. By adjunction and the Auslander–Reiten formula for $\text{GP}(\Lambda)$, we obtain natural isomorphisms

$$\underline{\text{Hom}}_\Lambda(-, \Omega^d \tau_{\text{GP}} X) \cong \underline{\text{Hom}}_\Lambda(\Sigma^d -, \tau_{\text{GP}} X) \cong \text{DExt}_\Lambda^1(X, \Sigma^d -),$$

The validity of the second isomorphism depends on Lemma A.1, showing that Σ^d takes values in $\underline{GP}(\Lambda)$. Alternatively, using the Auslander–Reiten formula in $\text{mod } \Lambda$, we get

$$\underline{\text{Hom}}_\Lambda(-, \Omega^d \tau_\Lambda X) \cong \underline{\text{Hom}}_\Lambda(\Sigma^d -, \tau_\Lambda X) \cong \text{DExt}_\Lambda^1(X, \Sigma^d -).$$

The two right-hand sides coincide, completing the proof. $\square$

Theorem A.3 ([32, Thm. 1]) *Let Λ be a 2-Calabi–Yau tilted algebra. Then a Λ -module M is Gorenstein projective if and only if $M \cong \Omega^2 \tau_\Lambda M$ in $\text{mod } \Lambda$.*

Proof By a result of Keller and Reiten [21, Prop. 2.1], a 2-Calabi–Yau tilted algebra is Iwanaga–Gorenstein of Gorenstein dimension at most 1. Thus $\Omega^2 N \in \underline{GP}(\Lambda)$ for any $N \in \text{mod } \Lambda$, and the ‘if’ part of the theorem follows immediately.

For the ‘only if’ direction, Keller and Reiten also prove [21, Thm. 3.3] that $\underline{GP}(\Lambda)$ is a 3-Calabi–Yau category, meaning that Σ^3 is a Serre functor. On the other hand, the Auslander–Reiten formula states that $\Sigma \tau_{\text{GP}}$ is always a Serre functor on $\underline{GP}(\Lambda)$. Since Serre functors are unique up to natural isomorphism, the 3-Calabi–Yau property is equivalent to there being a natural isomorphism $\tau_{\text{GP}} \simeq \Sigma^2$. Now by Theorem A.2, using again the Iwanaga–Gorenstein property of Λ , we have

$$\Omega^2 \tau_\Lambda M \cong \Omega^2 \tau_{\text{GP}} M \cong \Omega^2 \Sigma^2 M \cong M.$$

$\square$

We close by observing that characterisations of Gorenstein projective Λ -modules as in Theorem A.3 are closely related to Calabi–Yau properties of $\underline{GP}(\Lambda)$ in wider generality.

Proposition A.4 *Let Λ be Iwanaga–Gorenstein of Gorenstein dimension at most m . Then $\underline{GP}(\Lambda)$ is $(m+1)$ -Calabi–Yau if and only if there is a natural isomorphism $\Omega^m \tau_\Lambda \simeq \text{Id}_{\underline{GP}(\Lambda)}$. In this case, a Λ -module M is Gorenstein projective if and only if $M \cong \Omega^m \tau_\Lambda M$ in $\text{mod } \Lambda$.*

Proof As in the proof of Theorem A.2, $\underline{GP}(\Lambda)$ is $(m+1)$ -Calabi–Yau if and only if there is a natural isomorphism $\tau_{\text{GP}} \simeq \Sigma^m$ on $\underline{GP}(\Lambda)$. Since Σ^m has quasi-inverse Ω^m on this category, this is equivalent to asking that $\Omega^m \tau_{\text{GP}} \simeq \text{Id}_{\underline{GP}(\Lambda)}$. But Theorem A.2 tells us that $\Omega^m \tau_{\text{GP}} \simeq \Omega^m \tau_\Lambda$ in this setting, and the first part of the statement follows. The second part is then proved as in Theorem A.3, replacing 2 by m . $\square$

Note that the isomorphisms constructed in the proof of Theorem A.3 arise from a natural isomorphism in this way, using the 3-Calabi–Yau property of $\underline{GP}(\Lambda)$ [21, Thm. 3.3]. We also deduce that a similar Calabi–Yau property holds for certain gentle m -cluster-tilted algebras.

Corollary A.5 *Let Λ be an m -cluster-tilted algebra of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Then $\underline{GP}(\Lambda)$ is $(m + 1)$ -Calabi–Yau.*

Proof By Corollary 4.6, Λ is Iwanaga–Gorenstein of Gorenstein dimension at most m , and $M \cong \Omega^m \tau_\Lambda M$ for each $M \in \text{GP}(\Lambda)$. Since Λ is a gentle algebra, meaning that $\underline{GP}(\Lambda)$ is semi-simple [33], there must be a collection of such isomorphisms that is natural in M —for example, by first choosing such an isomorphism for each M in a set of representatives for the isoclasses of indecomposable objects, and then extending to arbitrary objects by choosing direct sum decompositions. (We use here that the ground field k is algebraically closed, so that each indecomposable has endomorphism ring k .) The result then follows by Proposition A.4. $\square$

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