Degenerate 0-Schur Algebras and Nil-Temperley-Lieb Algebras

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Abstract

In Jensen and Su (J. Pure Appl. Algebra **219**(2), 277–307 2014) constructed 0-Schur algebras, using double flag varieties. The construction leads to a presentation of 0-Schur algebras using quivers with relations and the quiver presentation naturally gives rise to a new class of algebras, which are introduced and studied in this paper. That is, these algebras are defined on the quivers of 0-Schur algebras with relations modified from the defining relations of 0-Schur algebras by a tuple of parameters $\underline{t}$. In particular, when all the entries of $\underline{t}$ are 1, we recover 0-Schur algebras. When all the entries of $\underline{t}$ are zero, we obtain a class of basic algebras, which we call the degenerate 0-Schur algebras. We prove that the degenerate algebras are both associated graded algebras and quotients of 0-Schur algebras. Moreover, we give a geometric interpretation of the degenerate algebras using double flag varieties, in the same spirit as Jensen and Su (J. Pure Appl. Algebra **219**(2), 277–307 2014), and show how the centralizer algebras are related to nil-Hecke and nil-Temperley-Lieb algebras.

Keywords 0-Schur algebras · Quivers · Nil-Hecke algebras · Nil-Temperley-Lieb algebras · Double flag varieties

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1 Introduction

It is well known that the classical Schur algebras are specialisations of q -Schur algebras (see [5] and [6]) at $q = 1$. Analogously, 0-Schur algebras are specialisations of q -Schur algebras at $q = 0$. The 0-Schur algebras have been studied by Donkin [6, §2.2] in terms of 0-Hecke algebras of symmetric groups, by Krob and Thibon [13] in connection with noncommutative symmetric functions, and by Deng and Yang on their presentations and representation types [3, 4].

A new approach towards 0-Schur algebras was investigated by Jensen and Su [11], by considering their monoid structure. Inspired by Beilinson, Lusztig and MacPherson's geometric construction of q -Schur algebras [1] and Reineke's work on a monoid structure of Hall algebras [16], Su defined a generic multiplication in the positive part of q -Schur algebras [19]. The generic multiplication was then generalized by Jensen and Su [11] to give a global geometric construction of 0-Schur algebras. This geometric construction produces a monoid structure, simplifies the multiplication and provides a new approach to studying the structure of 0-Schur algebras. In [12] we gave a construction of indecomposable projective modules and studied homomorphism spaces between projective modules.

The nature of 0-Schur algebras exposed in [11] leads to several interesting related algebras. First of all, we can modify the generating relations of 0-Schur algebras, relying on multiple parameters $\underline{t}$. In particular, when all the parameters are 1, we recover 0-Schur algebras and when all parameters are 0, we obtain a class of basic algebras. We prove that similarly to 0-Schur algebras, these newly defined algebras have reduced paths as basis (see [12]), which are in one-to-one correspondence with $GL(V)$ -orbits in double flag varieties or certain sets of integral matrices. So they have the same dimension as the corresponding 0-Schur algebras and they are proved to be degenerations of 0-Schur algebras. We further show that they are isomorphic to quotients of 0-Schur algebras by naturally defined ideals and to the associated graded algebras of 0-Schur algebras, which notably have a natural structure of filtered algebras. We also investigate relations between their centralizer algebras, Nil-Hecke and Nil-Temperley-Lieb algebras.

The remainder of this paper is organised as follows. In Section 2, we give a brief background on q -Schur and 0-Schur algebras and discuss how to consider 0-Schur algebras as filtered algebras. In Section 3, we prove some preliminary results on a family of idempotent ideals. In Section 4, we construct a class of algebras $D_{\underline{t}}(n, r)$ using quivers and modified relations of 0-Schur algebras and prove an isomorphism theorem between different $D_{\underline{t}}(n, r)$. In Section 5, we first construct the associated graded algebras $DS_0(n, r)$ of 0-Schur algebras and give them a geometric interpretation. We then show that $D_{\underline{t}}(n, r)$ for $\underline{t} = \underline{0}$ is a degenerate 0-Schur algebra and prove our main result that the three algebras, $DS_0(n, r)$, $D_0(n, r)$ and the quotient of $S_0(n, n + r)$ modulo a natural idempotent ideal are isomorphic. In Section 6, we discuss relations between centralizer algebras of the degenerate algebras, nil-Hecke and nil-Temperley-Lieb algebras.

2 Background on the Algebra $S_0(n, r)$

2.1 The Algebra $S_q(n, r)$

We first recall the construction of quantised Schur algebras given by Beilinson-Lusztig-MacPherson [1]. Let k be a field and let V be a k -vector space of dimension r . Let $\mathcal{F} = \mathcal{F}(V)$ be the set of n -step flags

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$

The natural action of $GL(V)$ on the vector space V induces a diagonal action of $GL(V)$ on $\mathcal{F} \times \mathcal{F}$ defined by

$$g(f, f') = (gf, gf'),$$

where $g \in GL(V)$ and $f, f' \in \mathcal{F}$. Denote the orbit of (f, f') by $[f, f']$.

Let $\Xi(n, r)$ be the set of $n \times n$ -matrices $A = (a_{ij})_{i,j}$ with a_{ij} nonnegative integers and $\sum_{1 \leq i, j \leq n} a_{ij} = r$. Then there is a bijection from $\mathcal{F} \times \mathcal{F}/GL(V)$ to $\Xi(n, r)$ sending the orbit of (f, f') to $A = (a_{i,j})_{i,j}$ with

$$a_{ij} = \dim \frac{V_i \cap V'_j}{V_{i-1} \cap V'_j + V_i \cap V'_{j-1}} \quad \text{for } 1 \leq i, j \leq n, \quad (1)$$

where $f = (0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V)$ and $f' = (0 = V'_0 \subseteq V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n = V)$. Denote by e_A the orbit in $\mathcal{F} \times \mathcal{F}$ corresponding to a matrix A . Consider the diagram

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{F} \times \mathcal{F} & \xrightarrow{\Delta} & (\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F}) \\ \downarrow \pi & & \\ \mathcal{F} \times \mathcal{F} & & \end{array}$$

where $\pi(f, g, h) = (f, h)$ and $\Delta(f, g, h) = ((f, g), (g, h))$.

Following [1, Prop. 1.2], for any given $A, B, C \in \Xi(n, r)$, there is a polynomial $g_{A,B,C}(q) \in \mathbb{Z}[q]$ such that for all finite fields k ,

$$g_{A,B,C}(|k|) = \frac{|\pi^{-1}(e_C) \cap \Delta^{-1}(e_A \times e_B)|}{|e_C|},$$

where $|X|$ denotes the cardinality of a set X .

Following a remark by Du [7], the q -Schur algebra studied by Dipper and James in [5] (see also [6]) can now be defined as follows.

Definition 2.1 [1] The quantised Schur algebra $S_q(n, r)$ is the free $\mathbb{Z}[q]$ -module with basis $\{e_A \mid A \in \Xi(n, r)\}$ and multiplication given by

$$e_A \cdot e_B = \sum_{C \in \Xi(n, r)} g_{A,B,C}(q) e_C, \quad \text{for all } A, B \in \Xi(n, r).$$

For an $n \times n$ -matrix $A = (a_{ij})$, define the row and column vectors of A by

$$\text{ro}(A) = \left(\sum_{j=1}^n a_{1j}, \dots, \sum_{j=1}^n a_{nj} \right) \quad \text{and} \quad \text{co}(A) = \left(\sum_{i=1}^n a_{i1}, \dots, \sum_{i=1}^n a_{in} \right).$$

Note that if $g_{A,B,C}(q) \neq 0$, then

$$\text{ro}(A) = \text{ro}(C), \quad \text{co}(A) = \text{ro}(B) \quad \text{and} \quad \text{co}(B) = \text{co}(C). \quad (2)$$

2.2 Definition of $S_0(n, r)$

In [11], Jensen and Su defined a generic multiplication of orbits in $\mathcal{F} \times \mathcal{F} = \mathcal{F}(V) \times \mathcal{F}(V)$ by

$$e_A \cdot e_B = \begin{cases} e_C & \text{if } \Delta^{-1}(e_A \times e_B) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where V is defined over an algebraically closed field k and e_C is the unique open orbit in $\pi \Delta^{-1}(e_A \times e_B)$. This defines an associative $\mathbb{Z}$ -algebra $G(n, r)$ with basis $\Xi(n, r)$ and in fact $G(n, r)$ is isomorphic to the 0-Schur algebra (Theorem 7.2.1 in [11])

$$S_0(n, r) = S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Z},$$

where $\mathbb{Z}$ is viewed as the $\mathbb{Z}[q]$ -module $\mathbb{Z}[q]/(q)$. As the multiplication in $G(n, r)$ is much simplified (e.g. the multiplication of two orbits is either 0 or again an orbit), in the rest of this paper, we will take $G(n, r)$ as the 0-Schur algebra $S_0(n, r)$.

Remark 2.2 We will see below (Lemma 2.3, 2.4 and 2.7) that the multiplication can be described combinatorially, independent of the choice of k .

Let $\Lambda(n, r)$ be the set of compositions of r into n parts. For each $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda(n, r)$, let $\text{diag}(\lambda)$ denote the diagonal matrix $\text{diag}(\lambda_1, \dots, \lambda_n)$ and write $k_\lambda = e_{\text{diag}(\lambda)}$. By definition, for each $A \in \Xi(n, r)$,

$$k_\lambda \cdot e_A = \begin{cases} e_A, & \text{if } \lambda = \text{ro}(A); \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad e_A \cdot k_\lambda = \begin{cases} e_A, & \text{if } \lambda = \text{co}(A); \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Thus, $\sum_{\lambda \in \Lambda(n, r)} k_\lambda$ is the identity of $S_0(n, r)$.

Let E_{ij} be the elementary $n \times n$ matrix with a single nonzero entry 1 at (i, j) . Denote by $e_{i, \lambda}$ (resp. $f_{j, \lambda}$) the basis element of $S_0(n, r)$ corresponding to the matrix that has column vector λ and the only nonzero off diagonal entry is 1 at $(i, i+1)$ (resp. $(j+1, j)$).

2.3 The Fundamental Multiplication Rules

Note that $S_0(n, r)$ is generated by $e_{i, \lambda}$, $f_{i, \lambda}$ and k_λ , where $1 \leq i \leq n-1$ and $\lambda \in \Lambda(n, r)$ (see Lemma 6.9 in [11]). Let

$$e_i = \sum_{\lambda \in \Lambda(n, r)} e_{i, \lambda} \quad \text{and} \quad f_i = \sum_{\lambda \in \Lambda(n, r)} f_{i, \lambda}.$$

Note that for any given orbit e_A , by the definition of the multiplication,

$$e_i e_A = e_{i, \text{co}(A)} e_A.$$

This says that only one term remains in the product and the same holds for $e_A e_i$, $e_A f_i$ and $f_i e_A$. The following are the fundamental multiplication rules in $S_0(n, r)$, which describe the action of generators on basis elements.

Lemma 2.3 [11, Lemma 6.11] *Let $e_A \in S_0(n, r)$ with $\text{ro}(A) = \lambda$.*

- (1) *If $\lambda_{i+1} > 0$, then $e_i e_A = e_X$, where $X = A + E_{i, p} - E_{i+1, p}$ with $p = \max\{j \mid a_{i+1, j} > 0\}$.*
- (2) *If $\lambda_i > 0$, then $f_i e_A = e_Y$, where $Y = A - E_{i, p} + E_{i+1, p}$ with $p = \min\{j \mid a_{i, j} > 0\}$.*

Symmetrically, there are the following formulas.

Lemma 2.4 [12, Lemma 2.2] *Let $e_A \in S_0(n, r)$ with $\text{co}(A) = \mu$.*

- (1) *If $\mu_{i+1} > 0$, then $e_A f_i = e_X$, where $X = A + E_{p, i} - E_{p, i+1}$ with $p = \max\{j \mid a_{j, i+1} > 0\}$.*
- (2) *If $\mu_i > 0$, then $e_A e_i = e_Y$, where $Y = A - E_{p, i} + E_{p, i+1}$ with $p = \min\{j \mid a_{j, i} > 0\}$.*

2.4 Presenting $S_0(n, r)$ by Quiver with Relations

Let

$$\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0) \in \mathbb{Z}^n,$$

where the only nonzero entries 1 and -1 are at the i th and $(i + 1)$ th positions, respectively. We define a quiver $\Sigma(n, r)$ with vertices corresponding to $\lambda \in \Lambda(n, r)$ and arrows

$$\begin{array}{ccc} & \xrightarrow{f_{i, \lambda + \alpha_i}} & \\ k_{\lambda + \alpha_i} & & k_{\lambda} \\ & \xleftarrow{e_{i, \lambda}} & \end{array}$$

and let $\mathcal{J}$ be the ideal of the path algebra $\mathbb{Z}\Sigma(n, r)$ generated by the binomial relations

$$\begin{aligned} P_{ij, \lambda} &= k_{\mu} P_{ij} k_{\lambda}, \\ N_{ij, \lambda} &= k_{\mu} N_{ij} k_{\lambda}, \\ C_{ij, \lambda} &= k_{\lambda + \alpha_i - \alpha_j} C_{ij} k_{\lambda}, \end{aligned}$$

where

$$\begin{aligned} P_{ij} &= \begin{cases} e_i^2 e_j - e_i e_j e_i & \text{for } i = j - 1, \\ -e_i e_j e_i + e_j e_i^2 & \text{for } i = j + 1, \\ e_i e_j - e_j e_i & \text{otherwise,} \end{cases} \quad \text{and } \mu = \begin{cases} \lambda + 2\alpha_i + \alpha_j & \text{if } i = j \pm 1 \\ \lambda + \alpha_i + \alpha_j & \text{otherwise;} \end{cases} \\ N_{ij} &= \begin{cases} -f_i f_j f_i + f_j f_i^2 & \text{for } i = j - 1, \\ f_i^2 f_j - f_i f_j f_i & \text{for } i = j + 1, \\ f_i f_j - f_j f_i & \text{otherwise,} \end{cases} \quad \text{and } \mu = \begin{cases} \lambda - 2\alpha_i - \alpha_j & \text{if } i = j \pm 1 \\ \lambda - \alpha_i - \alpha_j & \text{otherwise;} \end{cases} \end{aligned}$$

and

$$C_{ij} = e_i f_j - f_j e_i - \delta_{ij} \left(\sum_{\mu: \mu_{i+1}=0} k_{\mu} - \sum_{\mu: \mu_i=0} k_{\mu} \right).$$

Note that the ideal $\mathcal{J}$ is not admissible, as the relations C_{ij} involve idempotents.

The quotient algebra $\mathbb{Z}\Sigma(n, r)/\mathcal{J}$ is isomorphic to $S_0(n, r)$, by an isomorphism that maps the arrows $e_{i, \lambda}$, $f_{i, \lambda}$ and vertices k_{λ} to the corresponding elements in $S_0(n, r)$ (Theorem 7.1.2 in [11]). The relations $P_{ij, \lambda}$, $N_{ij, \lambda}$ are called the *Serre relations* of $S_0(n, r)$. The relations $C_{ij, \lambda}$ are commutative relations if $i \neq j$ or $\lambda_i \lambda_{i+1} \neq 0$, idempotent relations if exactly one of λ_i , λ_{i+1} is zero, and empty relations otherwise.

2.5 Bases of Reduced Paths and $S_0(n, r)$ as a Filtered Algebra

We say that a path in $\Sigma(n, r)$ is reduced if it is not equal to a path of shorter length in $\Sigma(n, r)$ modulo $\mathcal{J}$. Equivalence classes of reduced paths form a multiplicative basis for $S_0(n, r)$, which coincides with the basis e_A , $A \in \Xi(n, r)$. There are in general many paths equal to e_A , modulo $\mathcal{J}$, but by the fundamental multiplication rules and the presentation of $S_0(n, r)$, the numbers of occurrences of e_i and f_i in any reduced path are

$$E_i(e_A) = \sum_{l \leq i < m} a_{l, m} \quad \text{and} \quad F_i(e_A) = \sum_{m \leq i < l} a_{l, m},$$

respectively.

Lemma 2.5 *For any $1 \leq i \leq n - 1$, we have*

$$E_i(e_A) + E_i(e_B) - E_i(e_A \cdot e_B) = F_i(e_A) + F_i(e_B) - F_i(e_A \cdot e_B) \geq 0.$$

Proof The generating relations P_{ij} and N_{ij} do not change the length of a path. The length of a path either remains unchanged or decreases by 2, when we apply the relation $C_{ii,\lambda}$. It decreases if and only if $\lambda_i = 0$ or $\lambda_{i+1} = 0$ (but not both), as in these cases the relation $C_{ii,\lambda}$ becomes

$$k_\lambda f_i e_i k_\lambda = k_\lambda \text{ if } \lambda_i = 0 \text{ and } \lambda_{i+1} \neq 0;$$

or

$$k_\lambda e_i f_i k_\lambda = k_\lambda \text{ if } \lambda_{i+1} = 0 \text{ and } \lambda_i \neq 0.$$

In this situation, the numbers of the occurrences of e_i and f_i both decrease by 1. Therefore

$$E_i(e_A) + E_i(e_B) - E_i(e_A \cdot e_B) = F_i(e_A) + F_i(e_B) - F_i(e_A \cdot e_B) \geq 0,$$

as stated. $\square$

We define vectors $E(e_A)$ and $F(e_A)$ by

$$E(e_A) = (E_1(e_A), E_2(e_A), \dots, E_{n-1}(e_A)) \text{ and } F(e_A) = (F_1(e_A), F_2(e_A), \dots, F_{n-1}(e_A)).$$

Comparing tuples $(E(e_A), F(e_A))$ component-wise gives the following.

Corollary 2.6 $S_0(n, r)$ is a filtered algebra with degree function $e_A \mapsto (E(e_A), F(e_A))$.

For $j \geq i$, let

$$e(i, j) = e_i \cdot e_{i+1} \cdot \dots \cdot e_j \text{ and } f(j, i) = f_j \cdot \dots \cdot f_{i+1} \cdot f_i.$$

We give two explicit descriptions of reduced paths equal to a basis element e_A , analogous to the monomial and PBW bases of $S_q(n, r)$, respectively.

Lemma 2.7 Let $e_A \in S_0(n, r)$ with $A = (a_{i,j})_{i,j}$. Then we can write e_A as reduced paths as follows.

$$\begin{aligned} (1) \quad e_A &= \left(\prod_{s=n-1}^1 \prod_{l=1}^s e_l^{\sum_{1 \leq p \leq l} a_{p,s+1}} \right) \cdot \left(\prod_{s=1}^{n-1} \prod_{l=n-1}^s f_l^{\sum_{l < p \leq n} a_{p,s}} \right) \cdot k_{\text{co}(A)}; \\ (2) \quad e_A &= \left(\prod_{j=n}^2 \prod_{i=j-1}^1 e(i, j-1)^{a_{ij}} \right) \cdot \left(\prod_{j=1}^{n-1} \prod_{i=n}^{j+1} f(i-1, j)^{a_{ij}} \right) \cdot k_{\text{co}(A)}. \end{aligned}$$

Proof The formulas follow by repeatedly applying the fundamental multiplication rules in Lemma 2.3. In both (1) and (2), the numbers of occurrences of e_i and f_i are $E_i(e_A)$ and $F_i(e_A)$. So the paths are reduced. $\square$

We explain by an example on how to achieve the above formulas.

Example 2.8 Let $A = (a_{ij})$, where

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Note that $e_B = k_{\text{co}(A)}$. Using the fundamental multiplication rules, we have the following.

$$\begin{aligned} (1) \quad B &= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{f_2^6} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{f_1^8} \begin{pmatrix} 0 & 0 & 0 \\ 8 & 1 & 0 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{f_2^5} \\ &\begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{e_1} \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{e_2^6} \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 6 \\ 5 & 6 & 0 \end{pmatrix} \xrightarrow{e_1^2} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} = A. \end{aligned}$$

That is,

$$\begin{aligned} e_A &= e_1^2 \cdot e_2^6 \cdot e_1 \cdot f_2^5 \cdot f_1^8 \cdot f_2^6 \cdot e_B \\ &= e_1^{a_{13}} \cdot e_2^{(a_{13}+a_{23})} \cdot e_1^{a_{12}} \cdot f_2^{a_{31}} \cdot f_1^{(a_{21}+a_{31})} \cdot f_2^{a_{32}} \cdot e_B. \end{aligned}$$

$$\begin{aligned} (2) \quad B &= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{f_2^6} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 6 \end{pmatrix} \xrightarrow{(f_2 f_1)^5} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{f_1^3} \\ &\begin{pmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{e_1} \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{(e_1 e_2)^2} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \\ 5 & 6 & 4 \end{pmatrix} \xrightarrow{e_2^4} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} = A. \end{aligned}$$

That is,

$$\begin{aligned} e_A &= e_2^4 \cdot (e_1 e_2)^2 \cdot e_1 \cdot f_1^3 \cdot (f_2 f_1)^5 \cdot f_2^6 \cdot e_B \\ &= e(2, 2)^4 \cdot e(1, 2)^2 \cdot e(1, 1) \cdot f(1, 1)^3 \cdot f(2, 1)^5 \cdot f(2, 2)^6 \cdot e_B \\ &= e(2, 2)^{a_{23}} \cdot e(1, 2)^{a_{13}} \cdot e(1, 1)^{a_{12}} \cdot f(1, 1)^{a_{21}} \cdot f(2, 1)^{a_{31}} \cdot f(2, 2)^{a_{32}} \cdot e_B. \end{aligned}$$

In both cases, the lower triangular parts/upper triangular parts are created column-wise. The main differences are, for instance in the lower triangular parts, in (2) each step creates an entry a_{ij} at a time, starting from the lowest entry and then moving upwards, while in (1) it goes downwards and in each step we apply the maximal times of f_i so that afterwards we have an exactly right entry in row i .

3 Idempotents and Idempotent Ideals of $S_0(n, r)$

The set of compositions $\Lambda(n, r)$ (i.e. the vertices in the quiver $\Sigma(n, r)$) can be drawn on an $(n - 1)$ -simplex, where compositions with zero entries lie on the boundary. We call an idempotent k_λ *boundary* if λ lies on the boundary of the simplex, and *interior* otherwise. In this section, we are interested in the ideal generated by all boundary idempotents. We obtain a dimension formula for the quotient algebra, which turns out to be useful when we consider degenerations of 0-Schur algebras in Section 5.

Let $I_i(n, r)$ be the ideal generated by the idempotents k_λ with the number of nonzero entries in λ less than or equal to i . That is, $I_i(n, r)$ is generated by idempotents corresponding to $(i - 1)$ -faces of the simplex. There are in general several $(i - 1)$ -faces in $\Sigma(n, r)$, but as the following lemma shows, any one of them contains enough idempotents to generate the whole of $I_i(n, r)$.

Lemma 3.1 *The ideal $I_i(n, r)$ is generated by all idempotents k_λ lying in any chosen $(i - 1)$ -face in $\Sigma(n, r)$.*

Proof For any $\lambda = (\dots, \lambda_{i-1}, 0, \lambda_{i+1}, \dots) \in \Lambda(n, r)$,

$$k_\lambda = f_i^{\lambda_{i+1}} k_\mu e_i^{\lambda_{i+1}},$$

where $\mu = (\dots, \lambda_{i-1}, \lambda_{i+1}, 0, \dots)$ with the other entries equal to those of λ . Therefore k_λ is contained in the ideal generated by k_μ . Similarly, k_μ is contained in the ideal generated by k_λ , and so these two ideals are equal. This shows that $I_i(n, r)$ is generated by all idempotents k_λ lying in any chosen $(i - 1)$ -face in $\Sigma(n, r)$. $\square$

Denote by $I(n, r)$ the ideal of $S_0(n, r)$ generated by all the boundary idempotents. If $r \geq n$, then $I(n, r) = I_{n-1}(n, r)$. If $r < n$, then $I(n, r) = S_0(n, r)$.

Lemma 3.2 *The ideal $I(n, r)$ has a basis consisting of e_A , where $A \in \Xi(n, r)$ is a matrix with at least one diagonal entry equal to 0.*

Proof Denote by S the subspace of $S_0(n, r)$ spanned by e_A with at least one diagonal entry of A equal to 0. We first show that S is an ideal. It is enough to prove that for any generator x and any $e_D \in S$, both xe_D and e_Dx are contained in S . We only prove that $xe_D \in S$ for $x = e_j$ and $D = (d_{ij})_{ij}$ with $d_{ii} = 0$, as the other case can be done similarly. Let $e_X = e_j e_D$ and $X = (x_{ij})_{ij}$. By Lemma 2.3, multiplying e_j with e_D from the left only affects two entries in the $(j+1)$ th-row and j th-row in D , respectively. We have either

$$x_{ii} = d_{ii} = 0 \text{ or } x_{ii} = 1.$$

The latter case occurs only if $d_{i+1,i+1} = 0$ (and $j = i+1$), but then

$$x_{i+1,i+1} = 0$$

as well. In either case, $xe_D = e_X \in S$, as required. So S is an ideal in $S_0(n, r)$.

As $I(n, r)$ is generated by idempotents k_λ with $\lambda_i = 0$ for some i , we have $k_\lambda \in S$ and therefore $I(n, r) \subseteq S$, since S is an ideal.

It remains to show that $S \subseteq I(n, r)$. Suppose that $e_A \in S$ has the diagonal entry $a_{ii} = 0$, for some $1 \leq i \leq n$. We will show that $e_A \in I(n, r)$. Let B be the matrix with

$$b_{ss} = \begin{cases} \sum_{s \leq l \leq i} a_{ls} & \text{if } s < i, \\ 0 & \text{if } s = i, \\ \sum_{i \leq l \leq s} a_{ls} & \text{if } s > i \end{cases}$$

on the diagonal and

$$b_{st} = \begin{cases} 0 & \text{if } i \geq s > t, \\ 0 & \text{if } i \leq s < t, \\ a_{st} & \text{otherwise} \end{cases}$$

off the diagonal. The i 'th row in B is zero, and so we have $e_B = k_\lambda e_B \in I(n, r)$, where $\lambda = \text{ro}(B)$. To complete the proof we will construct elements x and y such that $xye_B = e_A$, which implies that $e_A \in I(n, r)$. In fact, y is a product of generators e_j for $j > i$ and x is a product of f_j for $j \leq i$. Explicitly, for instance, we have

$$y = \prod_{s=n-1}^i \prod_{l=i}^s e_l^{\sum_{i \leq p \leq l} a_{p,s+1}} = \underbrace{(e_i^{a_{in}} e_{i+1}^{a_{in}+a_{i+1,n}} \cdots e_{n-1}^{\sum_{i \leq p \leq n-1} a_{pn}})}_{s=n-1} \cdots \underbrace{e_i^{a_{i,i+1}}}_{s=i}$$

and a similar formula for x . Multiplying e_B with x and y produces the correct entries at (s, t) in the two zero regions $i \geq s > t$ and $i \leq s < t$ from the diagonal entries in B . So

$$xye_B = e_A,$$

as required. $\square$

We give an example to illustrate the construction in the proof above.

Example 3.3 Let $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$, where the diagonal entry $a_{33} = 0$. Then $B = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix}$, $x = f_2^2 f_1^2 f_2$ and $y = e_3$. We have $e_A = xye_B$.

Lemma 3.2 shows that $I(n, r)$ is a summand of $S_0(n, r)$ as $\mathbb{Z}$ -modules, and so $S_0(n, r)/I(n, r)$ is a free $\mathbb{Z}$ -module. We have the following formulas which will be used Section 5.

Corollary 3.4

$$(1) \quad \text{rank } I(n, r) = \sum_{s=1}^n \binom{n}{s} \binom{n^2 + r - n - 1}{r + s - n}.$$

$$(2) \quad \text{rank } S_0(n, r)/I(n, r) = \binom{n^2 + r - n - 1}{r - n}.$$

Note that in part (1), each term in the sum counts the matrices with exactly s zero diagonal entries.

4 Modified Algebras of $S_0(n, r)$

In the remainder of this paper, we will work with algebras defined over a field $\mathbb{F}$. Note that the 0-Schur algebra $S_0(n, r)$ is a free $\mathbb{Z}$ -module. When we extend the ground ring $\mathbb{Z}$ to a field $\mathbb{F}$,

$$S_0(n, r) \otimes_{\mathbb{Z}} \mathbb{F},$$

the results in Sections 2 and 3 still hold, where replace rank by dimension in Corollary 3.4. By abuse of notation, we continue to denote the $\mathbb{F}$ -algebra by $S_0(n, r)$.

Modifying the generating relations in Section 2.4, we construct a family of modified algebras $D_{\underline{t}}(n, r)$ of $S_0(n, r)$ and prove when two modified algebras are isomorphic. Further properties will be discussed in the next section, in relation to degenerations of 0-Schur algebras and their geometric interpretation.

Let $B(n, r) \subseteq \Lambda(n, r)$ be the set of elements corresponding to boundary idempotents. That is, this is the set of λ such that there is some $\lambda_i = 0$. Let

$$\underline{t} = (t_i)_{1 \leq i \leq n-1}$$

with each entry of $\underline{t}$ in $\mathbb{F}$. Denote by $\mathcal{J}(\underline{t})$ the ideal of $\mathbb{F}\Sigma(n, r)$ generated by $P_{ij, \lambda}$, $N_{ij, \lambda}$ and $C_{ij, \lambda}(\underline{t})$, where $P_{ij, \lambda}$ and $N_{ij, \lambda}$ are defined as in Section 2.4 and

$$C_{ij, \lambda}(\underline{t}) = k_{\lambda + \alpha_i - \alpha_j} C_{ij}(\underline{t}) k_{\lambda}, \quad (5)$$

where

$$C_{ij}(\underline{t}) = e_i f_j - f_j e_i - \delta_{ij} \cdot t_i \cdot \left(\sum_{\mu: \mu_{i+1}=0} k_{\mu} - \sum_{\mu: \mu_i=0} k_{\mu} \right).$$

Note that if $\lambda_i = \lambda_{i+1} = 0$, then $C_{ij, \lambda}(\underline{t})$ is an empty relation.

Definition 4.1 Let $D_{\underline{t}}(n, r) := \mathbb{F}\Sigma(n, r)/\mathcal{J}(\underline{t})$.

Remark 4.2 Note that for $i \neq j$, $C_{ij}(\underline{t}) = e_i f_j - f_j e_i$, which gives the following commutative diagram in $D_{\underline{t}}(n, r)$

$$\begin{array}{ccc} k_{\lambda} & \xrightarrow{e_i} & k_{\lambda+\alpha_i} \\ f_j \downarrow & & \downarrow f_j \\ k_{\lambda-\alpha_j} & \xrightarrow{e_i} & k_{\lambda+\alpha_i-\alpha_j}. \end{array}$$

When $i = j$ and either $\lambda_i \neq 0$ or $\lambda_{i+1} \neq 0$, we have the following three cases

$$C_{ii,\lambda}(\underline{t}) = \begin{cases} (e_i f_i - f_i e_i)k_{\lambda}, & \text{if } \lambda_i \lambda_{i+1} \neq 0; \\ (e_i f_i - t_i k_{\lambda})k_{\lambda}, & \text{if } \lambda_{i+1} = 0; \\ (t_i k_{\lambda} - f_i e_i)k_{\lambda}, & \text{if } \lambda_i = 0. \end{cases} \quad (6)$$

For the first case, we have the commutative relation at k_{λ} as follows

$$k_{\lambda-\alpha_i} \xleftarrow{e_i} k_{\lambda} \xrightarrow{e_i} k_{\lambda+\alpha_i} \\ f_i \downarrow \quad \quad \quad f_i \downarrow$$

For the second and the third case, λ is boundary and the relations are not admissible if $t_i \neq 0$.

When all the $t_i = 1$, we recover the 0-Schur algebra $S_0(n, r)$. When all the $t_i = 0$, $D_{\underline{t}}(n, r)$ is a basic algebra, which we call the degenerate 0-Schur algebra.

Define the map

$$\Phi_{\underline{t}, \underline{s}} : D_{\underline{t}}(n, r) \longrightarrow D_{\underline{s}}(n, r)$$

given by

$$k_{\lambda} \mapsto k_{\lambda}, \quad f_{i,\lambda} \mapsto f_{i,\lambda} \text{ and } e_{i,\lambda} \mapsto \frac{t_i}{s_i} e_{i,\lambda},$$

if $t_i s_i \neq 0$ and $e_{i,\lambda} \mapsto e_{i,\lambda}$ otherwise.

For $\underline{a} = (a_1, a_2, \dots, a_{n-1}) \in \mathbb{F}^{n-1}$, we write $\underline{t} = \underline{a} \underline{s}$ if $t_i = a_i s_i$ for all $1 \leq i \leq n-1$. We have the following proposition.

Proposition 4.3 *The map $\Phi_{\underline{t}, \underline{s}}$ defined above is an isomorphism of algebras if and only if the sets $\{i \mid t_i \neq 0\}$ and $\{i \mid s_i \neq 0\}$ coincide.*

Proof If the sets $\{i \mid t_i \neq 0\}$ and $\{i \mid s_i \neq 0\}$ coincide, it can be easily verified that the maps $\Phi_{\underline{s}, \underline{t}}$ and $\Phi_{\underline{t}, \underline{s}}$ are well-defined and that they are mutually inverse. Therefore $\Phi_{\underline{t}, \underline{s}}$ is an isomorphism.

On the other hand, if the two sets do not coincide, then either $\Phi_{\underline{t}, \underline{s}}$ has a non-trivial kernel or is not a homomorphism of algebras. So it is not an isomorphism. $\square$

Corollary 4.4 *Suppose that $s_i = 0$ if $t_i = 0$ and $s_i = 1$ if $t_i \neq 0$. Then*

$$D_{\underline{t}}(n, r) \cong D_{\underline{s}}(n, r).$$

In particular, if $t_i \neq 0$ for all $1 \leq i \leq n-1$, then

$$D_{\underline{t}}(n, r) \cong S_0(n, r).$$

5 The Degenerations $D_{\underline{t}}S_0(n, r)$ of $S_0(n, r)$

5.1 A New Algebra Defined on Double Flag Varieties

Let $D_{\underline{t}}S_0(n, r)$ be the $\mathbb{F}$ -space with basis $\{e_A \mid A \in \Xi(n, r)\}$. Define a multiplication on $D_{\underline{t}}S_0(n, r)$ as follows,

$$e_A \star_{\underline{t}} e_B = \underline{t}^{E(e_A)+E(e_B)-E(e_A \cdot e_B)} e_A \cdot e_B,$$

where $\underline{t}^{E(e_X)}$ is the product of $t_i^{E_i(e_X)}$ and we use the convention that $t_i^0 = 1$ when $t_i = 0$. Following Lemma 3.2, let $I(n, r) \subseteq D_{\underline{t}}S_0(n, r)$ be the subspace with basis e_A , where A has a zero on the diagonal.

Lemma 5.1 $D_{\underline{t}}S_0(n, r)$ with the product $\star_{\underline{t}}$ is an associative $\mathbb{F}$ -algebra. Moreover, $I(n, r)$ is a two-sided ideal.

Proof We need only show that the product $\star_{\underline{t}}$ is associative. We have

$$\begin{aligned} (e_A \star_{\underline{t}} e_B) \star_{\underline{t}} e_C &= \underline{t}^{E(e_A)+E(e_B)-E(e_A e_B)} (e_A e_B) \star_{\underline{t}} e_C \\ &= \underline{t}^{E(e_A)+E(e_B)-E(e_A e_B)+E(e_A e_B)+E(e_C)-E(e_A e_B e_C)} e_A e_B e_C \\ &= \underline{t}^{E(e_A)+E(e_B)+E(e_C)-E(e_A e_B e_C)} e_A e_B e_C \end{aligned}$$

Similarly,

$$e_A \star_{\underline{t}} (e_B \star_{\underline{t}} e_C) = \underline{t}^{E(e_A)+E(e_B)+E(e_C)-E(e_A e_B e_C)} e_A e_B e_C$$

and so the product is associative.

As $e_A \star_{\underline{t}} e_B$ and the multiplication $e_A e_B$ in $S_0(n, r)$ only differ by a scalar, that $I(n, r)$ is a two sided ideal in $D_{\underline{t}}S_0(n, r)$ follows from Lemma 3.2. $\square$

For the special case where $t_i = 0$ for all i , we write $\star$ for the product $\star_{\underline{t}}$, and we write $DS_0(n, r)$ for $D_{\underline{t}}S_0(n, r)$.

Example 5.2 Let $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$e_A e_B = e_C, \text{ but } e_A \star e_B = 0.$$

By Lemma 3.2, the e_C in Example 5.2 is contained in the ideal $I(2, 5)$. This indicate that there should be a link between $DS_0(n, r)$ and the quotient algebra $S_0(n, r)/I(n, r)$. We will explore the link and their relation to $D_0(n, r)$ in the remaining part of this section.

Lemma 5.3 The algebra $D_{\underline{t}}S_0(n, r)$ is generated by $e_{i,\lambda}$, $f_{i,\lambda}$ and k_λ , where $\lambda \in \Lambda(n, r)$ and $1 \leq i \leq n-1$.

Proof Recall that Lemma 2.7 expresses each basis element e_A in $S_0(n, r)$ as reduced paths. Observe that the corresponding products in $D_{\underline{t}}S_0(n, r)$ determined by the reduced paths are not zero, as the products satisfy

$$E(e_A) + E(e_B) = E(e_A \cdot e_B).$$

So the lemma follows. $\square$

Recall the algebra

$$D_{\underline{l}}(n, r) = \mathbb{F}\Sigma(n, r)/\mathcal{J}(t)$$

defined in Section 4. Up to isomorphism (see Corollary 4.4), the difference between $D_{\underline{l}}(n, r)$ and $S_0(n, r)$ is that some of the defining relations of $S_0(n, r)$ involving idempotents are replaced by zero relations. Recall also for $j \geq i$,

$$e(i, j) = e_i \cdot e_{i+1} \cdot \dots \cdot e_j.$$

Lemma 5.4 *Let $i < j$ and $a, b \geq 0$ such that $j - i > a + b$. Then*

$$e(i, j) \cdot e(i + a, j - b) = e(i + a, j - b) \cdot e(i, j)$$

in $D_{\underline{l}}(n, r)$.

Proof It suffices to prove that $e(i, j) \cdot e_l = e_l \cdot e(i, j)$ for all $l = i, \dots, j$. In fact, due to the commutativity relation $e_m e_{m'} = e_{m'} e_m$ when $|m - m'| > 1$, we need only prove $e(i, j) \cdot e_l = e_l \cdot e(i, j)$ for $(i, j) = (l - 1, l)$, $(l - 1, l + 1)$, $(l, l + 1)$. All the three cases follow from the relations $P_{mm'}$. $\square$

Proposition 5.5 $\dim_{\mathbb{F}} D_{\underline{l}}(n, r) = \dim_{\mathbb{F}} S_0(n, r)$.

Proof Observe that in both algebras, a complete list of representatives of nonzero paths is a basis. In particular, the paths of the form in Lemma 2.7 (2), which are all reduced paths, form a basis for $S_0(n, r)$.

As the relations P_{ij} and N_{ij} are the same in both algebra, if two nonzero paths are equal in $D_{\underline{l}}(n, r)$, they are equal in $S_0(n, r)$. Further, any reduced path that is nonzero in $S_0(n, r)$ is also nonzero in $D_{\underline{l}}(n, r)$. Indeed, take a nonzero reduced path $\rho = \dots e_{i,\lambda} \dots f_{j,\mu} \dots$ in $\Sigma(n, r)$. Now suppose that $0 = \rho \in D_{\underline{l}}(n, r)$. This implies that using relations $P_{ij,\lambda}$, $N_{ij,\lambda}$,

$$\rho = \dots e_{i,\lambda-\alpha_i} f_{i,\lambda} k_{\lambda} \dots \text{ or } \dots k_{\lambda} f_{i,\lambda+\alpha_i} e_{i,\lambda} \dots \quad (\dagger)$$

with λ at the boundary. So the new expression $(\dagger)$ also holds in $S_0(n, r)$. By the relations $C_{ij,\lambda}$,

$$e_{i,\lambda-\alpha_i+\alpha_{i+1}} f_{i,\lambda} k_{\lambda} = k_{\lambda} f_{i,\lambda+\alpha_i-\alpha_{i+1}} e_{i,\lambda} = k_{\lambda},$$

which contradicts the minimality of the number of arrows in ρ . So ρ is a non-zero path in $D_{\underline{l}}(n, r)$. Therefore

$$\dim_{\mathbb{F}} D_{\underline{l}}(n, r) \geq \dim_{\mathbb{F}} S_0(n, r). \quad (7)$$

Next we claim that any reduced path ρ in $D_{\underline{l}}(n, r)$ is equal to a path of the form (2) in Lemma 2.7 and thus by the inequality (7), the dimensions of the two algebras are the same. We proceed with the proof of the claim by induction on the length of ρ . When the length of ρ is at most 1, then it already has the required form. Assume that the length is larger than 1, and that $\rho = \rho \cdot k_{\lambda}$. We have

$$\rho = \rho' e_i k_{\lambda} \text{ or } \rho' f_i k_{\lambda}$$

where ρ' is a path of length one less than ρ . By the induction hypothesis, we may write ρ' of the form (2) in Lemma 2.7,

$$\rho' = E F k_{\lambda'}$$

where $E = e(i_s, j_s) \cdot \dots \cdot e(i_1, j_1)$ and $F = f(i'_1, j'_1) \cdot \dots \cdot f(i'_t, j'_t)$. We first consider the case $\rho = \rho' e_i k_{\lambda}$. Since ρ is reduced, we have

$$\rho = E F e_i k_{\lambda} = E e_i F k_{\lambda}.$$

If $j_m \neq i - 1$ for all m , the relations P_{ij} give us the required form

$$Ee_l Fk_\lambda = e(i_s, j_s) \cdot \dots \cdot e(i_l, j_l) \cdot e(i, i) \cdot \dots \cdot e(i_1, j_1) Fk_\lambda,$$

where l is the smallest integer such that $j_l > i$.

Let m be the smallest integer with $j_m = i - 1$. We have

$$Ee_i = e(i_s, j_s) \cdot \dots \cdot e(i_m, j_m + 1) \cdots e(i_1, j_1).$$

By repeatedly applying Lemma 5.4 and relations P_{ij} , we can move $e(i_m, j_m + 1)$ to the appropriate position.

The case $\rho = \rho' f_i k_\lambda$ is similar, and so we skip the details. In either case, the path ρ is equal to a path of the form (2) in Lemma 2.7, as claimed. $\square$

We can now prove the first main result of this paper.

Theorem 5.6 *There is an isomorphism of $\mathbb{F}$ -algebras*

$$D_{\underline{t}}(n, r) \cong D_{\underline{t}}S_0(n, r),$$

where $k_\lambda \mapsto k_\lambda$, $e_{i,\lambda} \mapsto e_{i,\lambda}$ and $f_{i,\lambda} \mapsto f_{i,\lambda}$.

Proof As the defining relations of $D_{\underline{t}}(n, r)$ are satisfied in $D_{\underline{t}}S_0(n, r)$, the map is well-defined. By Lemma 5.3, the map is an epimorphism. Note that by definition,

$$\dim D_{\underline{t}}S_0(n, r) = \dim S_0(n, r)$$

and thus by Proposition 5.5,

$$\dim D_{\underline{t}}S_0(n, r) = \dim D_{\underline{t}}(n, r).$$

Thus the map is an isomorphism, as claimed. $\square$

Our second main result realises the algebra $DS_0(n, r)$ as a quotient of the algebra $D_{\underline{t}}S_0(n, r + n)$.

Theorem 5.7 *There is an isomorphism of $\mathbb{F}$ -algebras*

$$D_{\underline{t}}S_0(n, r + n)/I(n, r + n) \cong DS_0(n, r).$$

Proof We embed $\Sigma(n, r)$ into the interior of $\Sigma(n, r + n)$ via $\phi : k_\lambda \mapsto k_\mu$ and embed the arrows correspondingly, where $\mu = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_n + 1) \in \Lambda(n, r + n)$. By abuse of notation, we also denote the ideal of $D_{\underline{t}}(n, r + n)$ generated by the boundary idempotents by $I(n, r)$. Observe that the relations $\phi(P_{ij,\lambda})$, $\phi(N_{ij,\lambda})$ and $\phi(C_{ij,\lambda}(\underline{0}))$ are zero in $D_{\underline{t}}(n, r + n)/I_{\underline{t}}(n, r + n)$. So we have a surjective map

$$D_{\underline{0}}(n, r) \twoheadrightarrow D_{\underline{t}}(n, r + n)/I(n, r + n).$$

Further, following the isomorphism in Theorem 5.6, which maps the ideal $I(n + r)$ of $D_{\underline{t}}(n, r + n)$ into the ideal $I(n, n + r)$ of $D_{\underline{t}}S_0(n, n + r)$, we have a surjective map

$$D_{\underline{t}}(n, r + n)/I(n, r + n) \twoheadrightarrow D_{\underline{t}}S_0(n, r + n)/I(n, r + n).$$

By Corollary 3.4,

$$\dim S_0(n, r + n)/I(n, r + n) = \binom{n^2 + (n + r) - n - 1}{(n + r) - n} = \binom{n^2 + r - 1}{r},$$

which is the dimension of $S_0(n, r)$. Therefore,

$$\begin{aligned}\dim D_{\underline{t}}S_0(n, r+n)/I(n, r+n) &= \dim S_0(n, r+n)/I(n, r+n) \\ &= \dim S_0(n, r) \\ &= \dim DS_0(n, r).\end{aligned}$$

Applying the isomorphism in Theorem 5.6 for $\underline{t} = \underline{0}$, we have surjective maps

$$DS_0(n, r) \twoheadrightarrow D_{\underline{t}}(n, r+n)/I(n, r+n) \twoheadrightarrow D_{\underline{t}}S_0(n, r+n)/I(n, r+n).$$

with the dimensions of two end terms the same. Therefore we have the isomorphism as claimed. $\square$

The construction of the isomorphism in the previous theorem gives us the following inverse system of algebra surjections (similar to the analogous inverse system for q -Schur algebras when q is invertible [1]).

Corollary 5.8 *There is a sequence of surjective algebra homomorphisms*

$$DS_0(n, r) \leftarrow DS_0(n, r+n) \leftarrow \cdots \leftarrow DS_0(n, r+kn) \leftarrow \cdots$$

Proof If $t_i = 0$ for all i , then the previous theorem gives us the surjective composition

$$DS_0(n, r+kn) \rightarrow DS_0(n, r+kn)/I(n, r+kn) \cong DS_0(n, r+(k-1)n)$$

for any $k \geq 0$. $\square$

Remark 5.9 (1) Assume that $\underline{t}$ and $\underline{s}$ are given with the property that $s_i = 0$ if $t_i = 0$. Theorem 5.6 and Corollary 4.4 imply that $D_{\underline{s}}S_0(n, r)$ is a degeneration of $D_{\underline{t}}S_0(n, r)$. In particular, $D_{\underline{t}}S_0(n, r)$ is a degeneration of $S_0(n, r)$ and $DS_0(n, r)$ is a degeneration of $D_{\underline{t}}S_0(n, r)$ for all $\underline{t}$.

(2) These degenerations can be understood as passing from a filtered algebra to its associated graded algebra. In particular, $DS_0(n, r)$ is the graded algebra associated to the filtered algebra structure on $S_0(n, r)$ given in Corollary 2.6.

We end this section by giving a more geometric interpretation of the multiplication in $DS_0(n, r)$ using double flag varieties. Suppose that

$$e_A = [f, g], \quad e_B = [g, h], \quad \text{and} \quad e_C = [f', h']$$

with

$$e_A \cdot e_B = e_C$$

in $S_0(n, r)$. Recall that V is an r -dimensional vector space defined over a field k . We view an n -step flag in V as a representation of a linear quiver A_n , where vertex 1 is a source and vertex n is a sink. Associated to such a representation f , we denote the dimension vector by $\underline{\dim} f$.

Lemma 5.10 $\underline{\dim} f' \cap h' \geq \underline{\dim} f \cap g + \underline{\dim} g \cap h - \underline{\dim} g$. Consequently,

$$\underline{\dim} f' \cap h' = \underline{\dim} f \cap g + \underline{\dim} g \cap h - \underline{\dim} g$$

if and only if

$$E(e_A) + E(e_B) = E(e_A \cdot e_B).$$

Proof Note that

$E(e_A) = \underline{\dim} f - \underline{\dim} f \cap g$, $E(e_B) = \underline{\dim} g - \underline{\dim} h \cap g$, $E(e_A e_B) = \underline{\dim} f' - \underline{\dim} f' \cap h'$ and $f \cong f'$ as representations. Now the lemma follows from Lemma 2.5. $\square$

Therefore the product $\star$ can be defined as below.

Lemma 5.11

$$[f, g] \star [g, h] = \begin{cases} [f', h'] & \text{if } \underline{\dim} f' \cap h' = \underline{\dim} f \cap g + \underline{\dim} g \cap h - \underline{\dim} g, \\ 0 & \text{otherwise.} \end{cases}$$

6 Centralizer Algebras of $D_t(n, r)$

In this section, we discuss relations between the 0-Hecke algebra $H_0(r)$, the nil-Hecke algebra $NH_0(r)$, the nil-Temperley-Lieb algebra $NTL(r)$ and the degenerate 0-Schur algebra $DS_0(r, r)$. Throughout this section, we let $\alpha = (1, \dots, 1)$ be the composition in $\Lambda(r, r)$ with all the entries equal to 1.

6.1 The Algebras $H_0(r)$, $NH_0(r)$ and $DS_0(r, r)$

We first recall some definitions and key facts. We then show that the associated graded algebras of $H_0(r)$ and $NH_0(r)$, with the filtration induced by the degree function defined in Section 2.5, are isomorphic to a subalgebra of $k_\alpha DS_0(r, r)k_\alpha$.

Definition 6.1 [2, 8] The 0-Hecke algebra, denoted by $H_0(r)$, is the $\mathbb{F}$ -algebra generated by $T_1, T_2, \dots, T_{r-1}$ with defining relations

$$\begin{cases} T_i^2 = T_i, & \text{for } 1 \leq i \leq r-1; \\ T_i T_j = T_j T_i, & \text{for } 1 \leq i, j \leq r-1 \text{ with } |i-j| > 1; \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq r-2. \end{cases} \quad (8)$$

Theorem 6.2 [11, Theorem 10.4] The algebras $H_0(r)$ and $k_\alpha S_0(r, r)k_\alpha$ are isomorphic via the map $T_i \mapsto k_\alpha f_i e_i k_\alpha$.

Definition 6.3 [10, 17] The nil-Hecke algebra, denoted by $NH_0(r)$, is the unital $\mathbb{F}$ -algebra generated by $T_1, T_2, \dots, T_{r-1}$ with defining relations

$$\begin{cases} T_i^2 = 0, & \text{for } 1 \leq i \leq r-1; \\ T_i T_j = T_j T_i, & \text{for } 1 \leq i, j \leq r-1 \text{ with } |i-j| > 1; \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i \leq r-2. \end{cases} \quad (9)$$

Assume that $w = s_{i_1} \cdots s_{i_t} = s_{j_1} \cdots s_{j_t}$ are reduced expressions in the symmetric group S_r on r letters, where s_i is the transposition $(i, i+1)$. As reduced expressions of w can be obtained from each other by using the braid relations only (see [14, 20]), we have

$$T_{i_1} \cdots T_{i_t} = T_{j_1} \cdots T_{j_t}$$

in both $H_0(r)$ and $NH_0(r)$, and thus the element

$$T_w = T_{i_1} \cdots T_{i_t}$$

is well-defined in both algebras. One can also deduce that $\{T_w \mid w \in S_r\}$ is a basis for both algebras. Further,

$$\text{in } H_0(r), T_i T_w = \begin{cases} T_{s_i w}, & \text{if } \ell(s_i w) = \ell(w) + 1; \\ T_w, & \text{otherwise,} \end{cases} \quad (10)$$

and

$$\text{in } NH_0(r), T_{w_1} T_{w_2} = \begin{cases} T_{w_1 w_2}, & \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2); \\ 0, & \text{otherwise,} \end{cases} \quad (11)$$

where $\ell : W \rightarrow \mathbb{N} \cup \{0\}$ is the length function of elements in S_r .

By Theorem 6.2, the filtration of the 0-Schur algebra $S_0(r, r)$ discussed in Section 2.5 induces a filtration on $H_0(r)$. When the length equations in Eqs. 10 or 11 hold, the multiplications of T_w and $T_{w'}$ in $H_0(r)$ and $NH_0(r)$ are the same, so we also have a filtration on $NH_0(r)$. Further, together with the fact that $DS_0(r, r)$ is the associated graded algebra of $S_0(r, r)$, this implies the following.

Proposition 6.4 *The associated graded algebras of $H_0(r)$ and $NH_0(r)$ are isomorphic to the algebra $k_\alpha DS_0(r, r)k_\alpha$.*

Define a degree function d for a $\mathbb{Z}$ -grading on $k_\alpha DS_0(r, r)k_\alpha$ given by

$$d(w) := d(T_w) = \sum_i E_i(e_A),$$

where A is the permutation matrix corresponding to w . Then via the isomorphism in Theorem 6.2, we can reformulate the multiplication in $k_\alpha DS_0(r, r)k_\alpha$ as follows

$$T_{w_1} T_{w_2} = \begin{cases} T_{w_1 w_2}, & \text{if } d(w_1 w_2) = d(w_1) + d(w_2); \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

Note that in $k_\alpha DS_0(r, r)k_\alpha$, some T_w can not be expressed as a product in the T_i .

Example 6.5 Consider the product $T_1 T_2 T_1$, which has length 3 and so is non-zero in the nil-Hecke algebra. The corresponding element $T_{s_1 s_2 s_1}$ has degree 2, therefore the product $T_1 T_2 T_1$ is zero in $k_\alpha DS_0(r, r)k_\alpha$.

By inspecting the entries above the diagonal in the matrix A , we can reinterpret $d(e_A)$ in terms of permutations as

$$d(w) = \sum_{w(j) < j} (j - w(j)).$$

It is shown in [15], that any permutation can be written as a product of transpositions, such that the degrees add up. In other words, the corresponding product in $k_\alpha DS_0(r, r)k_\alpha$ is non-zero. Hence the set of elements $T_{(i,j)}$ corresponding to transpositions (i, j) generates the algebra $k_\alpha DS_0(r, r)k_\alpha$.

6.2 Nil-Temperley-Lieb Algebras

The nil-Temperley-Lieb algebra $NTL(r)$ is the quotient algebra of $NH_0(r)$ modulo the ideal generated by $T_i T_{i+1} T_i$ for $1 \leq i \leq r - 2$ (see e.g. [9]).

Lemma 6.6 *The algebra $NTL(r)$ has a basis consisting of T_w , where w does not contain a subword of the form $s_i s_j s_i$ in any of its reduced expressions, for any i and j with $|i - j| = 1$.*

Proof The lemma follows from the fact that $\{T_w | w \in S_r\}$ is a basis of $NH_0(r)$ and the definition of $NTL(r)$. $\square$

Note that the dimension of $NTL(r)$ is the Catalan number $\frac{1}{r+1} \binom{2r}{r}$ and there is a useful combinatorial parametrisation of the elements in a basis of $NTL(r)$, using Dyck words. Pictorially, Dyck words can be described using peak pictures in a triangle with r dots on each edge (cf [18]). For instance, when $r = 3$, the five peak pictures are as follows.

Let

$$x_i = k_\alpha f_i e_i k_\alpha \in k_\alpha DS_0(n, r) k_\alpha.$$

Denote by $\underline{DS}(r, \alpha)$ the subalgebra of $k_\alpha DS_0(r, r) k_\alpha$ generated by x_i for $1 \leq i \leq r - 1$. When $r = 3$, the algebra $\underline{DS}(r, \alpha)$ is five dimensional, the orbit basis elements are determined by the following matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Similar to T_w ,

$$x_w = x_{i_1} \dots x_{i_l}$$

is well-defined, where $s_{i_1} \dots s_{i_l}$ is a reduced expression of w . By direct computation following the fundamental multiplication rules and the definition of $DS_0(n, r)$, we have the following lemma.

Lemma 6.7 *The elements x_i for $1 \leq i \leq r - 1$ satisfy the generating relations of $NTL(r)$. Consequently, there is an epimorphism*

$$NTL(r) \longrightarrow \underline{DS}(r, \alpha), \quad T_i \mapsto x_i, \quad \forall i.$$

Any matrix that determines a nonzero orbit in $\underline{DS}(r, \alpha)$ is a permutation matrix. We call a nonzero entry that is below the diagonal a *peak entry*. For a peak entry (i, j) , which implies that $j < i$, we call the entries (j, j) and (i, i) the *feet of the peak*. For the five matrices above, which determine the orbit basis elements in $\underline{DS}(r, \alpha)$, we connect the peaks, feet and diagonal 1-entry, using zig-zag lines as follows.

In this way, we obtain a well defined one-to-one correspondence between the basis elements of $N\mathcal{T}\mathcal{L}(3)$ and $\underline{DS}(3, \alpha)$. In fact, such a one-to-one correspondence exists for all r .

Theorem 6.8 *The two algebras $N\mathcal{T}\mathcal{L}(r)$ and $\underline{DS}(r, \alpha)$ are isomorphic.*

Proof First note the nonzero orbits e_A in $\underline{DS}(r, \alpha)$ form a basis and each orbit e_A is uniquely determined by the matrix A . By Lemma 6.7, there is a surjective homomorphism from $N\mathcal{T}\mathcal{L}(r)$ to $\underline{DS}(r, \alpha)$. So it suffices to show that there exists a nonzero element generated by the x_i s such that the corresponding matrix produces the peak picture.

First identify the triangle, in which we draw the peak pictures, with the lower triangular part of an $r \times r$ matrix. The peaks give us peak entries, $(i_1, j_1), \dots, (i_s, j_s)$ where for any l ,

$$j_l < i_l < i_{l+1} \text{ and } j_l < j_{l+1}.$$

Let

$$x^{(l)} = x_{i_l-1} \dots x_{j_l+1} x_{j_l} \text{ and } x = x^{(1)} \dots x^{(s)}.$$

By the fundamental multiplication rules, the multiplication of x_{i+1} with $x_i \dots x_{j_s} k_\alpha$ moves a 1 on or above the diagonal further up in the same column and a 1 on or below the diagonal further down in the same column and thus

$$E(x_{i+1}) + E(x_i \dots x_{j_s} k_\alpha) = E(x_{i+1} x_i \dots x_{j_s} k_\alpha).$$

That is, the equality in the definition of $\star$ holds. Therefore,

$$x^{(s)} k_\alpha \neq 0.$$

Similarly,

$$x^{(l)} (x^{(l+1)} \dots x^{(s)} k_\alpha) \neq 0$$

and when compared with $x^{(l+1)} \dots x^{(s)} k_\alpha$, it has a new peak at (i_l, j_l) . Thus we have found a nonzero element $x = x k_\alpha$ such that the corresponding matrix gives us the required peak picture. $\square$

Example 6.9 In this example, we demonstrate the process of obtaining the matrix

$$\begin{array}{ccc} 0 & 0 & 1 \\ | & & \\ 1 & \text{---} & 0 & 0 \\ | & & \\ 0 & 1 & \text{---} & 0 \end{array}$$

which as indicated has two peaks, using the construction in the proof of the theorem. The two peak entries are $(2, 1)$ and $(3, 2)$. By definition,

$$x^{(2)} = x_2 \text{ and } x^{(1)} = x_1.$$

Then $x^{(2)} k_\alpha$ and $x^{(1)} x^{(2)} k_\alpha$ are the orbit basis elements corresponding to the matrices, respectively,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We have the following special version of Lemma 5.11.

Lemma 6.10 *Use the same notation Lemma 5.11. Assume that all the flags are isomorphic to $\bigoplus_{i=1}^n P_i$ and $[f, g] = x_i$ for some i . Then the following is true in $\underline{DS}(r, \alpha)$*

$$[f, g] \star [g, h] = \begin{cases} [f', h'] & \text{if } \dim f' \cap h' = \dim g \cap h - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Note that as a representation, $g/(f \cap g)$ is isomorphic to the simple representation of the linear quiver A_n at vertex i , since $[f, g] = x_i$. Now the lemma follows from Lemma 5.10 and 5.11. $\square$

Remark 6.11 In the light of Lemma 6.10, Theorem 6.8 gives a geometric realisation of nil-Temperley-Lieb algebras, via double flag varieties.

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