Correction to: Automorphisms of the Doubles of Purely Non-Abelian Finite Groups

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The original version of the article unfortunately contained a mistake. Lemma 3.10, Lemma 4.4, and Theorem 9.4 are incorrect.

We let all notation be as in [3]. The following is a correction to [3, Lemma 3.10], which improperly states an equivalence.

Lemma 3.10 *Let G, H be groups; $u: \mathbb{k}^G \rightarrow \mathbb{k}^H$ a morphism of coalgebras, and $v: \mathbb{k}G \rightarrow \mathbb{k}H$ an algebra morphism. Then the following are equivalent.*

(1) u, v satisfy

$$v(x) \rightarrow u(e_g) = u(e_{xgx^{-1}}) \quad (1)$$

for all $x, g \in G$

(2) $(u^*v)(g)g^{-1} \in C_{\mathbb{k}G}(\text{Im}(u^*))$ for all $g \in G$.

(3) $xu^*(h)x^{-1} = u^*(v(x)hv(x)^{-1})$ for all $x \in G$ and $h \in H$.

Any of these imply that u^*v is a normal algebra endomorphism of $\mathbb{k}G$.

Proof The equivalence stated here was in fact correctly proven in [3, Lemma 3.10] by simply rewriting the equation. The error there is in the subsequent assertion that the condition $(u^*v)(g)g^{-1} \in C_{\mathbb{k}G}(\text{Im}(u^*))$ for all $g \in G$ is equivalent to normality of u^*v . A counterexample is simple. Let $G = H$, v the trivial morphism of Hopf algebras, and u the identity morphism. Then u^*v is trivial and so normal, but does not satisfy $(u^*v)(g)g^{-1} = g^{-1} \in C_G(\text{Im}(u^*)) = Z(G)$ for all $g \in G$ unless G is abelian.

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So let u, v be as in the statement. By definition u^*v is normal if and only if $(u^*v)(g^x) = (u^*v)(g)^x$ for all $g, x \in G$. As in [5, Lemma 3.2] (see also [4, Section 3]) u^*v is normal if and only if $(Su^*v) * \text{id}$ is an algebra morphism, if and only if $((Su^*v) * \text{id}) \curlywedge (u^*v)$, if and only if $(u^*v)(g)g^{-1} \in C_{\mathbb{k}G}(\text{Im}(u^*v))$ for all $g \in G$. Since we always have $C_{\mathbb{k}G}(\text{Im}(u^*)) \subseteq C_{\mathbb{k}G}(\text{Im}(u^*v))$, it follows that Eq. 1 implies that u^*v is normal, as desired. $\square$

Example 1 This rules out $(p, u, r, v) = (0, 1, 0, 0)$ as defining an element of $\text{End}(\mathcal{D}(G))$ whenever G is non-abelian, as the corresponding map $e_g \# x \mapsto e_g \# 1$ is easily seen to be an algebra morphism if and only if G is abelian.

However, $(0, 0, 0, 1)$, as in [3, Example 7.2], remains a well-defined element of $\text{End}(\mathcal{D}(G))$ for all G .

This error affects a few other results, but thankfully does so in a fairly nominal fashion. To see this, we must extend [3, Lemma 4.4] as follows.

Lemma 4.4 *Let G, H be groups (not necessarily finite). Let $v: G \rightarrow H$ and $w: H \rightarrow G$ be group homomorphisms. Suppose that $Z(H) \text{Im}(v) = H$ and $Z(G) \text{Im}(w) = G$. Then the following all hold:*

- (1) $C_G(\text{Im}(w)) = Z(G)$;
- (2) $C_H(\text{Im}(v)) = Z(H)$;
- (3) $v(Z(G)) \subseteq Z(H)$;
- (4) $w(Z(H)) \subseteq Z(G)$;
- (5) $C_H(\text{Im}(vw)) = Z(H)$;
- (6) $C_G(\text{Im}(wv)) = Z(G)$;
- (7) *The following are equivalent:*
 - (a) $wv \in \text{End}(G)$ is normal;
 - (b) $vw \in \text{End}(H)$ is normal;
 - (c) $w(h)^g = w(h^{v(g)})$ for all $g \in G$ and $h \in H$;
 - (d) $v(g)^h = v(g^{w(h)})$ for all $g \in G$ and $h \in H$.

When any of these hold we also have $\ker(v) \subseteq Z(G)$ and $\ker(w) \subseteq Z(H)$.

Proof Items 1 to 4 were proven in [3, Lemma 4.4]. The equivalence of Items 7a and 7b was also stated, but there is a minor omission in the proof thereof we will correct.

By Item 3 and the assumption $G = Z(G) \text{Im}(w)$ we have

$$C_H(\text{Im}(v)) = C_H(v(Z(G) \text{Im}(w))) = C_H(v(Z(G)) \text{Im}(vw)) = C_H(\text{Im}(vw)).$$

Thus Item 2 implies Item 5. Similarly, Item 1 and 4 and the assumption $H = Z(H) \text{Im}(v)$ implies Item 6.

The equivalence of Item 7a and 7c, and of Item 7b and 7d, now follows from Lemma 3.10 and its proof. We can then fill in the omission from the proof given in [3, Lemma 4.4] of the equivalence of Item 7a and 7b by using Item 5 and 6 as follows.

Suppose that Item 7a holds. For any $h \in H$, we may write $h = bv(g)$ for some $b \in Z(H)$ and $g \in G$. Then

$$(vw)(h)h^{-1} = (vw)(bv(g))(bv(g))^{-1} = ((vw)(b)b^{-1}) \cdot v((wv)(g)g^{-1}).$$

From Item 3 and 4 we have $(vw)(b)b^{-1} \in Z(H)$. By normality of wv and Item 5 we have $(wv)(g)g^{-1} \in C_G(\text{Im}(wv)) = Z(G)$. Therefore $(vw)(h)h^{-1} \in Z(H) \subseteq C_H(\text{Im}(vw))$ (indeed, equality holds by Item 5) for all $h \in H$, which implies vw is normal, as desired. The other direction of the equivalence follows by symmetry. The statement about $\ker(v)$ follows from $(wv)(g)g^{-1} \in C_G(\text{Im}(wv)) = Z(G)$, and similarly for $\ker(w)$.

This completes the proof. $\square$

The extended equivalence guarantees that all arguments and statements concerning isomorphisms $\mathcal{D}(G) \rightarrow \mathcal{D}(H)$ (which forces $G \cong H$ by [3, Theorem 3.4]) carry through essentially unaltered, as then $w = u^*$ and v satisfy the above lemma by [3, Lemma 3.3 and Theorem 3.4]. Thus the error is only particularly relevant to results about morphisms which are not necessarily isomorphisms.

The principle such relevance is in the proof of [3, Theorem 9.4]. The proof of this result needs the corrected Lemma 3.10 to correctly establish that centralizers are mapped appropriately. There is an additional issue where the proof explicitly requires $|\text{class}(s)| = |\text{class}(u^*(s))|$, but does not provide adequate justification or assumptions to ensure this. As such we fully state and prove the corrected version here.

Theorem 9.4 Fix groups G, H and a homomorphism $\psi = (p, u, r, v) \in \text{Hom}(\mathcal{D}(G), \mathcal{D}(H))$. Suppose that all of the following hold:

- (1) ψ is flippable;
- (2) $Z(G) \text{Im}(u^*) = G$;
- (3) $\ker(u^*)$ contains no non-trivial commutators.

Then the induced map $\text{Rep}(\mathcal{D}(H)) \rightarrow \text{Rep}(\mathcal{D}(G))$ is given by

$$(s, \chi) \mapsto (u^*(s)P(\chi), r^*(s) * \chi \circ v).$$

Proof Let assumptions and notation be as in the statement. By [3, Lemma 4.2] the assumption that ψ is flippable means we have $A \subseteq Z(G)$, $B \subseteq Z(H)$, and that the components are all morphisms of Hopf algebras. Moreover, flippability of ψ by definition means $(p^*, v^*, r^*, u^*) \in \text{Hom}(\mathcal{D}(H), \mathcal{D}(G))$, so by [3, Corollary 2.3] and Lemma 3.10 we have

$$v(g)^h = v(g^{u^*(h)}) \quad (2)$$

for all $g \in G$ and $h \in H$.

For the remainder of the proof we fix an arbitrary irreducible representation (s, χ) of $\mathcal{D}(H)$.

By [3, Lemma 9.2] (s, χ) is sent to a module supported at $u^*(s)P(\chi)$. To obtain the desired result, we need only show that the character of (s, χ) under ψ is the same as the character for $(u^*(s)P(\chi), r^*(s) * (\chi \circ v))$.

First we must justify that $\chi \circ v$ is a well-defined character of

$$C_G(P(\chi)u^*(s)) = C_G(u^*(s)),$$

or equivalently that v restricts to a group homomorphism $C_G(u^*(s)) \rightarrow C_H(s)$ for any $s \in H$. By Lemma 3.10 we have $v(u^*(s))s^{-1} \in C_G(\text{Im}(u^*))$. The assumption $Z(G) \text{Im}(u^*) = G$ implies $C_G(\text{Im}(u^*)) = Z(G)$. Therefore $C_H(v(u^*(s))) = C_H(s)$. Using this, the assumption that $Z(G) \text{Im}(u^*) = G$, and taking $g = u^*(s)$ in Eq. 2 we obtain the desired claim.

Next, the assumption $Z(G) \operatorname{Im}(u^*) = G$ additionally implies $\operatorname{class}(u^*(s)) = u^*(\operatorname{class}(s))$ for all $s \in H$. Given this equality we then have $|\operatorname{class}(s)| = |\operatorname{class}(u^*(s))|$ for all $s \in H$ if and only if u^* is one-to-one on conjugacy classes. This in turn is equivalent to $\ker(u^*)$ containing no non-trivial commutators. So by assumptions $|\operatorname{class}(s)| = |\operatorname{class}(u^*(s))|$.

Now given $g \in \operatorname{class}(P(\chi)u^*(s)) = P(\chi)\operatorname{class}(u^*(s))$ we let $y_g \in \operatorname{class}(s)$ be such that $P(\chi)u^*(y_g) = g$. Such a value of y_g exists and is unique by the preceding remarks. Fixing an element $y'_g \in H$ such that $s^{y'_g} = y_g$, we define $g' = u^*(y'_g)$. By definition, $(P(\chi)u^*(s))^{g'} = g$.

Now let ξ be the character of the image of (s, χ) under ψ , and let β be the character of $(u^*(s)P(\chi), r^*(s) * (\chi \circ v))$. By [3, Lemma 9.1],

$$\beta(e_g \# h) = r^*(s)(h)\chi(v(t)) \quad (3)$$

whenever $g \in \operatorname{class}(P(\chi)u^*(s)) = P(\chi)\operatorname{class}(u^*(s))$ and $h = t^{g'}$ for some $t \in C_G(P(\chi)u^*(s)) = C_G(u^*(s))$, with g' defined as above; and $\beta(e_g \# h) = 0$ otherwise.

On the other hand, by [3, equation (9.6)] we have

$$\xi(e_g \# h) = r^*(s)(h)\chi(\gamma) \quad (4)$$

whenever $g \in \operatorname{class}(P(\chi)u^*(s))$, and $v(h) = \gamma^{y'_g}$ for some $\gamma \in C_H(s)$, with y'_g defined as above; and $\xi(e_g \# h) = 0$ otherwise.

So suppose $\beta(e_g \# h) \neq 0$. We then have $v(h) = v(t^{g'}) = v(t)^{(vu^*)(y'_g)}$. We have previously noted that $(vu^*)(y'_g)(y'_g)^{-1} \in C_H(\operatorname{Im}(v))$, from which it follows that $v(h) = v(t)^{y'_g}$. We therefore conclude that $\xi(e_g \# h) = \beta(e_g \# h)$ whenever $\beta(e_g \# h) \neq 0$.

Conversely, if $\xi(e_g \# h) \neq 0$ then $v(h) = \gamma^{y'_g}$ implies $(u^*v)(h) = u^*(\gamma)^{g'}$. From Lemma 3.10 and the assumption that $Z(G) \operatorname{Im}(u^*) = G$ we conclude that $(u^*v)(h) = d^{-1}h$ for some $d \in C_G(\operatorname{Im}(u^*)) = Z(G)$. Therefore $h = (du^*(\gamma))^{g'}$ and $du^*(\gamma) \in C_G(u^*(s))$. We conclude that $\beta(e_g \# h) \neq 0$, and so from the preceding argument we must have that $\beta(e_g \# h) = \xi(e_g \# h)$ for all $g, h \in G$.

This completes the proof. $\square$

As was previously mentioned, Lemma 3.10 is used here to properly justify that v restricts to a group homomorphism $C_G(u^*(s)) \rightarrow C_H(s)$. The original statement and proof also omitted the assumption that $\ker(u^*)$ contains no non-trivial commutator. We can see the necessity of this by simply comparing dimensions:

$$\begin{aligned} \dim(s, \chi) &= |\operatorname{class}(s)| \chi(1); \\ \dim(u^*(s)P(\chi), r^*(s) * (\chi \circ v)) &= |\operatorname{class}(u^*(s))| \chi(1). \end{aligned}$$

The proof given in [3] explicitly relied upon $|\operatorname{class}(s)| = |\operatorname{class}(u^*(s))|$, but did not provide logically sound justification for why this would hold without the explicit assumption on $\ker(u^*)$. Observe that from [3, Theorem 3.12] we have $\ker(u^*) \cap H' = 1$ is always true for isomorphisms, so Theorem 9.4 is applicable to all isomorphisms.

Example 2 While it is known that derived subgroups of finite groups can contain elements which are not commutators [2], it is perhaps not particularly well-known if it is possible for the derived subgroup to contain a subgroup consisting entirely of such elements (other than the identity) which is normal in the parent group. Any such subgroup would necessarily be central. A search with GAP [1] shows that the smallest such example has order 128. In

particular, $G = \text{SmallGroup}(128, 731)$ has an element of order 2 in $Z(G) \subset G'$ which is also the unique element of G' which is not a commutator.

Subsequently, while $\ker(u^*) \cap H' = 1$ implies $\ker(u^*)$ contains no non-trivial commutator, the converse need not hold.

The remaining consequences of the error in [3, Lemma 3.10] are as follows.

- The statement of [5, Lemma 9.2] must replace the statement " u^*v is normal" with Eq. 1, or any equivalent form from Lemma 3.10.
- Similar changes must be made in the statement of [5, Corollary 9.3].

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