Polynomial Quantization and Overalgebra

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Abstract We construct polynomial quantization (a variant of quantization in the spirit of Berezin) on para-Hermitian symmetric spaces. For that we use two approaches: (a) using a reproducing function, (b) using an “overgroup”. Also we show that the multiplication of symbols is an action of an overalgebra.

Keywords Symplectic manifolds · Symbol calculus · Quantization · Berezin transform

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Let G/H be a para-Hermitian symmetric space. We can consider that G/H is a manifold in the Lie algebra $\mathfrak{g}$ of G . Quantization on G/H in the spirit of Berezin (see papers [1] and [2] of Berezin himself) has been constructed in [6]. Polynomial quantization (the most algebraic variant of quantization) has been given in [9], with explicit formulae for rank one spaces. Here an initial algebra operators is the algebra of operators in a maximal degenerate series representation of the universal enveloping algebra $\text{Env}(\mathfrak{g})$ of $\mathfrak{g}$. In this paper we give both a modification of [9] (using a reproducing function) and a new approach (using the notion of an “overgroup”). In the latter case covariant and contravariant symbols and the Berezin transform appear quite naturally and transparently, so that polynomial quantization turns out to be included in representation theory. Moreover, we show that the multiplication of symbols is exactly an action of an overalgebra.

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1 Para-Hermitian Symmetric Spaces

Let G/H be a *semisimple symmetric space*. Here G is a connected semisimple Lie group with an involutive automorphism $\sigma \neq 1$, and H is an open subgroup of G^σ , the subgroup of fixed points of σ . We consider that groups act on their homogeneous spaces *from the right*, so that G/H consists of right cosets Hg .

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and of H respectively. Let $B_{\mathfrak{g}}$ be the Killing form of G . There is a decomposition of $\mathfrak{g}$ into direct sums of $+1$, -1 -eigenspaces of the involution σ :

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}.$$

The subspace $\mathfrak{q}$ is invariant with respect to H in the adjoint representation Ad . It can be identified with the tangent space to G/H at the point $x^0 = He$.

The dimension of Cartan subspaces of $\mathfrak{q}$ (maximal Abelian subalgebras in $\mathfrak{q}$ consisting of semisimple elements) is called the rank of G/H .

Now let G/H be a *symplectic* manifold. Then $\mathfrak{h}$ has a non-trivial center $Z(\mathfrak{h})$. For simplicity we assume that G/H is an orbit $\text{Ad } G \cdot Z_0$ of an element $Z_0 \in \mathfrak{g}$. In particular, then $Z_0 \in Z(\mathfrak{h})$.

Further, we can also assume that G is *simple*. Such spaces G/H are divided into 4 classes (see [3, 4]):

- (a) Hermitian symmetric spaces;
- (b) semi-Kählerian symmetric spaces;
- (c) para-Hermitian symmetric spaces;
- (d) complexifications of spaces of class (a).

Spaces of class (a) are Riemannian, of other three classes are pseudo-Riemannian (not Riemannian).

We focus on spaces of class (c). Here the center $Z(\mathfrak{h})$ is one-dimensional, so that $Z(\mathfrak{h}) = \mathbb{R}Z_0$, and Z_0 can be normalized so that the operator $I = (\text{ad } Z_0)_{\mathfrak{q}}$ on $\mathfrak{q}$ has eigenvalues ± 1 . A symplectic structure on G/H is defined by the bilinear form $\omega(X, Y) = B_{\mathfrak{g}}(X, IY)$ on $\mathfrak{q}$.

The ± 1 -eigenspaces $\mathfrak{q}^{\pm} \subset \mathfrak{q}$ of I are Lagrangian, H -invariant, and irreducible. They are Abelian subalgebras of $\mathfrak{g}$. So $\mathfrak{g}$ becomes a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{q}^- + \mathfrak{h} + \mathfrak{q}^+,$$

with commutation relations $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{q}^-] \subset \mathfrak{q}^-$, $[\mathfrak{h}, \mathfrak{q}^+] \subset \mathfrak{q}^+$. The pair $(\mathfrak{q}^+, \mathfrak{q}^-)$ is a Jordan pair with multiplication $\{XYZ\} = (1/2)[[X, Y], Z]$, see [5]. Let $\varkappa$ be the genus of this Jordan pair.

Set $Q^{\pm} = \exp \mathfrak{q}^{\pm}$. The subgroups $P^{\pm} = HQ^{\pm} = Q^{\pm}H$ are maximal parabolic subgroups of G . One has the following decompositions:

$$G = \overline{Q^+ H Q^-} \tag{1.1}$$

$$= \overline{Q^- H Q^+}, \tag{1.2}$$

where bar means closure and the sets under the bar are open and dense in G . Let us call (1.1) and (1.2) the *Gauss decomposition* and (allowing some slang) the *anti-Gauss decomposition* respectively. Decompositions (1.1), (1.2) mean that almost any element $g \in G$ can be decomposed as (the Gauss decomposition):

$$g = \exp \eta \cdot h \cdot \exp \xi, \tag{1.3}$$

or (the anti-Gauss decomposition):

$$g = \exp \xi \cdot h \cdot \exp \eta, \quad (1.4)$$

where $h \in H$, $\xi \in \mathfrak{q}^-$, $\eta \in \mathfrak{q}^+$, all three factors in (1.3) and (1.4) are defined uniquely. We also use the Gauss decomposition (1.3) in a little different form:

$$g = \exp \eta \cdot \exp \xi \cdot h, \quad (1.5)$$

where η and h are the same as in (1.3), and ξ is obtained from ξ in (1.3) by $\text{Ad } h$.

Decompositions (1.3) and (1.4) generate actions of G on $\mathfrak{q}^-$ and $\mathfrak{q}^+$ respectively, namely, $\xi \mapsto \tilde{\xi} = \xi \bullet g$ and $\eta \mapsto \hat{\eta} = \eta \circ g$:

$$\exp \xi \cdot g = \exp Y \cdot \tilde{h} \cdot \exp \tilde{\xi}, \quad (1.6)$$

$$\exp \eta \cdot g = \exp X \cdot \hat{h} \cdot \exp \hat{\eta}, \quad (1.7)$$

where $X \in \mathfrak{q}^-$, $Y \in \mathfrak{q}^+$. These actions are defined on open and dense sets depending on g . Therefore, G acts on $\mathfrak{q}^- \times \mathfrak{q}^+ : (\xi, \eta) \mapsto (\tilde{\xi}, \hat{\eta})$. The stabilizer of the point $(0, 0) \in \mathfrak{q}^- \times \mathfrak{q}^+$ is $P^+ \cap P^- = H$, so that we get an embedding

$$\mathfrak{q}^- \times \mathfrak{q}^+ \hookrightarrow G/H. \quad (1.8)$$

It is defined on an open and dense set, its image is also an open and dense set. Therefore, we can consider $(\xi, \eta) \in \mathfrak{q}^- \times \mathfrak{q}^+$ as coordinates on G/H , let us call them *horospherical coordinates*.

Let us write explicit formula for embedding (1.8). We use a redecomposition “anti-Gauss” to “Gauss”. We take $\xi \in \mathfrak{q}^-$, $\eta \in \mathfrak{q}^+$ and decompose the anti-Gauss product $\exp \xi \cdot \exp(-\eta)$ according to formula (1.5) (the “Gauss”):

$$\exp \xi \cdot \exp(-\eta) = \exp Y \cdot \exp X \cdot h, \quad (1.9)$$

where $X \in \mathfrak{q}^-$, $Y \in \mathfrak{q}^+$. The obtained element $h \in H$ depends on ξ and η only, denote it by $h(\xi, \eta)$. Using (1.9), let us form the following element $g \in G$:

$$g = \exp Y \exp \xi = \exp X \cdot h \cdot \exp \eta. \quad (1.10)$$

Then the pair ξ, η goes just to the point $x = x^0 g$ where g is defined by (1.10).

Under the action of the group G the element $h(\xi, \eta)$ is transformed as follows:

$$h(\tilde{\xi}, \hat{\eta}) = \tilde{h}^{-1} \cdot h(\xi, \eta) \cdot \hat{h}, \quad (1.11)$$

where $\tilde{h}$ and $\hat{h}$ are taken from (1.6) and (1.7) respectively.

Determinant

$$\det \text{Ad } h(\xi, \eta)^{-1} \Big|_{\mathfrak{q}^+}$$

is a polynomial in ξ, η . Moreover, see [5], it is the power $N(\xi, \eta)^{\mathcal{Z}}$ of an irreducible polynomial $N(\xi, \eta)$ of degree r in ξ and η separately.

In horospherical coordinates the G -invariant measure on G/H is:

$$dx = dx(\xi, \eta) = |N(\xi, \eta)|^{-\mathcal{Z}} d\xi d\eta, \quad (1.12)$$

where $d\xi$ and $d\eta$ are Euclidean measures on $\mathfrak{q}^-$ and $\mathfrak{q}^+$ respectively.

2 Maximal Degenerate Series Representations

In this Section we introduce two series of representations induced by characters of maximal parabolic subgroups $P^\pm$ of G (maximal degenerate series representations).

Let $\lambda \in \mathbb{C}$. We take the following character ω_λ of H :

$$\omega_\lambda(h) = |\det(\text{Ad}h)|_{\mathfrak{q}^+}|^{-\lambda/\varkappa}$$

and then we extend this character to the subgroups $P^\pm$, setting it equal to 1 on $Q^\pm$.

We consider induced representations $\pi_\lambda^\pm$ of G :

$$\pi_\lambda^\pm = \text{Ind} \left(G, P^\mp, \omega_{\mp\lambda} \right).$$

They act on the space $\mathcal{D}_\lambda^\pm(G)$ of functions $f \in C^\infty(G)$ having the uniformity property:

$$f(pg) = \omega_{\mp\lambda}(p)f(g), \quad p \in P^\mp,$$

by translations from the right:

$$(\pi_\lambda^\pm(g)f)(s) = f(sg).$$

Realize them in the *non-compact picture*: we restrict functions from $\mathcal{D}_\lambda^\pm(G)$ to the subgroups $Q^\pm$ and identify them (as manifolds) with $\mathfrak{q}^\pm$, we obtain

$$(\pi_\lambda^-(g)f)(\xi) = \omega_\lambda(\tilde{h})f(\tilde{\xi}), \quad (\pi_\lambda^+(g)f)(\eta) = \omega_\lambda(\hat{h}^{-1})f(\hat{\eta}),$$

where $\tilde{\xi}, \tilde{h}, \hat{\eta}, \hat{h}$ are taken from decompositions (1.6), (1.7).

Let us write intertwining operators. Introduce operators $A_\lambda^\mp$ by:

$$(A_\lambda^\pm \varphi)(\eta) = \int_{\mathfrak{q}^\pm} |N(\xi, \eta)|^{-\lambda-\varkappa} \varphi(\xi) d\xi,$$

The operator $A_\lambda^\pm$ intertwines $\pi_\lambda^\pm$ with $\pi_{-\lambda-\varkappa}^\mp$. Their composition is a scalar operator:

$$A_\lambda^\pm A_{-\lambda-\varkappa}^\mp = c(\lambda)^{-1} \cdot \text{id}, \quad (2.1)$$

where lower or upper indexes are taken, $c(\lambda)$ is a meromorphic function of λ , invariant with respect to the change $\lambda \mapsto -\lambda - \varkappa$.

The representation $\pi_\lambda^\pm$ of the universal enveloping algebra $\text{Env}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ (we preserve the same symbols) is given by some differential operators. In particular, for $L \in \mathfrak{g}$ these operators have the first order. On the product $\varphi\psi$ of functions they act as follows:

$$\pi_\lambda^\pm(L)(\varphi\psi) = (\pi_\lambda^\pm(L)\varphi) \cdot \psi + \varphi \cdot (\pi_0^\pm(L)\psi). \quad (2.2)$$

Let us introduce the following bilinear form on functions on $\mathfrak{q}^\pm$:

$$\langle\langle f, h \rangle\rangle = \int f(\xi) h(\xi) d\xi = \int f(\eta) h(\eta) d\eta. \quad (2.3)$$

It is invariant with respect to the pair $(\pi_{-\lambda-\varkappa}^\pm, \pi_\lambda^\pm)$:

$$\langle\langle \pi_{-\lambda-\varkappa}^\pm(g^{-1})f, h \rangle\rangle = \langle\langle f, \pi_\lambda^\pm(g)h \rangle\rangle. \quad (2.4)$$

The principal anti-automorphism $X \mapsto X^\vee$ in $\text{Env}(\mathfrak{g})$ corresponds to the map $g \mapsto g^{-1}$ in the group G : if $X = L_1 L_2 \dots L_k$, where $L_i \in \mathfrak{g}$, then

$$X^\vee = (-1)^k L_k \dots L_2 L_1.$$

It is an anti-involution: $(XY)^\vee = Y^\vee X^\vee$. Formula (2.4) gives:

$$\langle\langle \pi_{-\lambda-\varkappa}^\pm(X^\vee)f, h \rangle\rangle = \langle\langle f, \pi_\lambda^\pm(X)h \rangle\rangle, \quad X \in \text{Env}(\mathfrak{g}). \quad (2.5)$$

We rename the kernel of intertwining operators:

$$\Phi_\lambda(\xi, \eta) = |N(\xi, \eta)|^\lambda = \omega_\lambda(h(\xi, \eta)). \quad (2.6)$$

Formula (1.11) gives

$$\Phi_{\lambda}(\tilde{\xi}, \widehat{\eta}) = \Phi_{\lambda}(\xi, \eta) \cdot \omega_{\lambda}(\tilde{h})^{-1} \cdot \omega_{\lambda}(\widehat{h}),$$

which can be interpreted as an invariance property of the function $\Phi_{\lambda}(\xi, \eta)$:

$$\left[\pi_{\lambda}^{-}(g) \otimes \pi_{\lambda}^{+}(g) \right] \Phi_{\lambda}(\xi, \eta) = \Phi_{\lambda}(\xi, \eta).$$

This formula can be rewritten as

$$\left(\pi_{\lambda}^{-}(g^{-1}) \otimes 1 \right) \Phi_{\lambda}(\xi, \eta) = \left(1 \otimes \pi_{\lambda}^{+}(g) \right) \Phi_{\lambda}(\xi, \eta). \quad (2.7)$$

For elements L of the Lie algebra $\mathfrak{g}$, formula (2.7) gives:

$$- \left(\pi_{\lambda}^{-}(L) \otimes 1 \right) \Phi_{\lambda}(\xi, \eta) = \left(1 \otimes \pi_{\lambda}^{+}(L) \right) \Phi_{\lambda}(\xi, \eta). \quad (2.8)$$

3 Symbols and Transforms

In this Section we apply to a para-Hermitian symmetric space G/H the scheme of quantization in the spirit of Berezin offered in [6]. We consider the most algebraic version of the quantization, we call it the *polynomial quantization*. For an initial algebra of operators we take here the algebra of operators $\pi_{\lambda}^{-}(\text{Env}(\mathfrak{g}))$, $\lambda \in \mathbb{C}$. The role of the Fock space is played by a space of functions $\varphi(\xi)$, $\xi \in \mathfrak{q}^{-}$, so that our operators act in functions $\varphi(\xi)$. We introduce covariant and contravariant symbols of operators, the Berezin transform etc.

As a (an analog of) *supercomplete system* we take the function (2.6). Introduce *covariant and contravariant symbols* of operators $D = \pi_{\lambda}^{-}(X)$, where $X \in \text{Env}(\mathfrak{g})$. We start from formula (2.1), it is reproducing formula:

$$f(\xi) = c(\lambda) \int \Phi_{\lambda}(\xi, v) \Phi_{-\lambda-\varkappa}(u, v) f(u) du dv. \quad (3.1)$$

Theorem 3.1 *The function $Df(\xi)$ is expressed in terms of $f(\xi)$ by one of two following formulas:*

$$(Df)(\xi) = c(\lambda) \int \left((\pi_{\lambda}^{-}(X) \otimes 1) \Phi_{\lambda} \right) (\xi, v) \Phi_{-\lambda-\varkappa}(u, v) f(u) du dv, \quad (3.2)$$

$$(Df)(\xi) = c(\lambda) \int \Phi_{\lambda}(\xi, v) \left((1 \otimes \pi_{-\lambda-\varkappa}^{+}(X)) \Phi_{-\lambda-\varkappa} \right) (u, v) f(u) du dv. \quad (3.3)$$

Proof Let us apply the operator D to both hand sides of (3.1) (as functions of ξ) we at once get (3.2).

Now let us write formula (3.1) replacing f by $\pi_{\lambda}^{-}(g)f$. We obtain

$$(\pi_{\lambda}^{-}(g)f)(\xi) = c(\lambda) \int \Phi_{\lambda}(\xi, v) dv \int \Phi_{-\lambda-\varkappa}(u, v) (\pi_{\lambda}^{-}(g)f)(u) du. \quad (3.4)$$

By (2.4) the inner integral here becomes as follows:

$$\int \left(\left(\pi_{-\lambda-\varkappa}^{-}(g^{-1}) \otimes 1 \right) \Phi_{-\lambda-\varkappa}(u, v) \right) \cdot f(u) du,$$

then we use (2.7) and turn this integral into:

$$\int \left((1 \otimes \pi_{-\lambda-\varkappa}^{+}(g)) \Phi_{-\lambda-\varkappa}(u, v) \right) \cdot f(u) du,$$

hence formula (3.4) becomes as follows:

$$(\pi_{\lambda}^{-}(g)f)(\xi) = c(\lambda) \int \Phi_{\lambda}(\xi, v) \left((1 \otimes \pi_{-\lambda-\varkappa}^{+}(g)) \Phi_{-\lambda-\varkappa}(u, v) \right) \cdot f(u) du dv,$$

Let us pass here from the group G to the algebra $\text{Env}(\mathfrak{g})$: we change g by X and obtain just (3.3). $\square$

Comparing (3.2), (3.3) with (3.1), we introduce the following two functions:

$$F(\xi, \eta) = \frac{1}{\Phi_{\lambda}(\xi, \eta)} (\pi_{\lambda}^{-}(X) \otimes 1) \Phi_{\lambda}(\xi, \eta), \quad (3.5)$$

$$F^{\natural}(\xi, \eta) = \frac{1}{\Phi_{-\lambda-\varkappa}(\xi, \eta)} (1 \otimes \pi_{-\lambda-\varkappa}^{+}(X)) \Phi_{-\lambda-\varkappa}(\xi, \eta). \quad (3.6)$$

Then we write

$$(Df)(\xi) = c(\lambda) \int F(\xi, v) \Phi_{\lambda}(\xi, v) \Phi_{-\lambda-\varkappa}(u, v) f(u) du dv,$$

$$(Df)(\xi) = c(\lambda) \int F^{\natural}(u, v) \Phi_{\lambda}(\xi, v) \Phi_{-\lambda-\varkappa}(u, v) f(u) du dv.$$

Therefore, remembering invariant measure dx on G/H , see (1.12), we can write (3.2) and (3.3) as follows:

$$(Df)(\xi) = c(\lambda) \int F(\xi, v) \frac{\Phi_{\lambda}(\xi, v)}{\Phi_{\lambda}(u, v)} f(u) dx(u, v). \quad (3.7)$$

$$(Df)(\xi) = c(\lambda) \int F^{\natural}(u, v) \frac{\Phi_{\lambda}(\xi, v)}{\Phi_{\lambda}(u, v)} f(u) dx(u, v). \quad (3.8)$$

Thus, to the operator D two functions F and $F^{\natural}$ correspond: $D \longrightarrow F$ and $D \longrightarrow F^{\natural}$. Let us call them *covariant and contravariant symbols* of the operator D , respectively. We shall denote them also $\text{co}_{\lambda}D$ and $\text{contra}_{\lambda}D$. The following theorem converts these correspondences.

Theorem 3.2 *The operator D is recovered by its covariant and contravariant symbols by means of equalities (3.7) and (3.8). Therefore, the maps co_{λ} and contra_{λ} are one-to-one.*

For generic λ , the space of symbols of both types is the space $S(G/H)$ of all *polynomials* on G/H .

Establish a connection between co- and contravariant symbols. Comparing (3.6) with (3.5) and using (2.7), we obtain

$$\text{contra}_{\lambda} \left(\pi_{\lambda}^{-}(X) \right) = \text{co}_{-\lambda-\varkappa} \left(\pi_{-\lambda-\varkappa}^{-}(X^{\vee}) \right). \quad (3.9)$$

Formula (2.5) says that the operator conjugate to an operator $D = \pi_{\lambda}^{-}(X)$ with respect to the form (2.3) is $D^{*} = \pi_{-\lambda-\varkappa}^{-}(X^{\vee})$, so that (3.9) means

$$\text{contra}_{\lambda}(D) = \text{co}_{-\lambda-\varkappa}(D^{*}).$$

Since ξ, η are horospherical coordinates on G/H , symbols (co- and contra-) become functions on G/H and, moreover, *polynomials* on $G/H \subset \mathfrak{g}$. It is why we call this variant of quantization the *polynomial quantization*. For generic λ , the space of symbols is the space of all *polynomials* on G/H .

In particular, the symbol of the identity operator is the function on G/H equal to 1 identically. If X belongs to the Lie algebra $\mathfrak{g}$ itself, then the symbol of the operator $\pi_\lambda^-(X)$ is a linear function $B_g(X, x)$ of $x \in G/H \subset \mathfrak{g}$ up to a factor depending on λ .

The multiplication of operators gives rise to the multiplication of covariant symbols, denote it by $*$. Namely, let F_1 and F_2 be covariant symbols of operators D_1 and D_2 respectively. Then the covariant symbol $F_1 * F_2$ of the product $D_1 D_2$ is

$$(F_1 * F_2)(\xi, \eta) = \frac{1}{\Phi_\lambda(\xi, \eta)} (D_1 \otimes 1) (\Phi_\lambda(\xi, \eta) F_2(\xi, \eta)) . \quad (3.10)$$

Putting in (3.7) $D = D_1$, $F = F_1$ and $f(u) = \Phi_\lambda(u, \eta) F_2(u, \eta)$, we get

$$(F_1 * F_2)(\xi, \eta) = \int_{G/H} F_1(\xi, v) F_2(u, \eta) \mathcal{B}_\lambda(\xi, \eta; u, v) dx(u, v),$$

where

$$\mathcal{B}_\lambda(\xi, \eta; u, v) = c(\lambda) \frac{\Phi_\lambda(\xi, v) \Phi_\lambda(u, \eta)}{\Phi_\lambda(\xi, \eta) \Phi_\lambda(u, v)} .$$

Let us call this function $\mathcal{B}_\lambda$ the *Berezin kernel*. It can be regarded as a function $\mathcal{B}_\lambda(x, y)$ on $G/H \times G/H$. It is invariant with respect to G :

$$\mathcal{B}_\lambda(\text{Ad } g \cdot x, \text{Ad } g \cdot y) = \mathcal{B}_\lambda(x, y).$$

Thus, the space of covariant symbols is an associative algebra with 1.

In particular, let us write formulas for multiplication by symbols V corresponding to elements L of the Lie algebra $\mathfrak{g}$.

Theorem 3.3 *We have (the point means pointwise multiplication)*

$$V * F = V \cdot F + (\pi_0^-(L) \otimes 1) F \quad (3.11)$$

$$F * V = V \cdot F - (1 \otimes \pi_0^+(L)) F \quad (3.12)$$

Proof To prove (3.10), we take in formula (3) $D_1 = \pi_\lambda^-(L)$ and $F_2 = F$, then we differentiate by (2.2), as a result we get (3.10). Now by (3.5) we have

$$F * V = \frac{1}{\Phi_\lambda} (D \otimes 1) (\pi_\lambda^-(L) \otimes 1) \Phi_\lambda,$$

then by (2.8) we can change here the latter operator by the operator $\{-(1 \otimes \pi_\lambda^+(L))\}$ and then transpose it with $D \otimes 1$ since they act on different variables. We obtain

$$F * V = -\frac{1}{\Phi_\lambda} (1 \otimes \pi_\lambda^+(L)) (\Phi_\lambda F), \quad (3.13)$$

then we differentiate by (2.2) and use (2.8) again. It gives (3.11). $\square$

The multiplication of contravariant symbols is obtained from the multiplication of these functions as covariant symbols by the permutation of multipliers and by change $\lambda \rightarrow -\lambda - \varkappa$. Corresponding kernel is get from the Berezin kernel $\mathcal{B}_\lambda$ by the same change.

Theorem 3.4 *Correspondences co_λ and $contra_\lambda$ are equivariant with respect to $\mathfrak{g}$. Namely, let $L \in \mathfrak{g}$. If F and $F^\natural$ are symbols of an operator $D = \pi_\lambda^-(X)$, then the operator $D^L = \pi_\lambda^-(ad L \cdot X)$ has symbols $U(L)F$ and $U(L)F^\natural$:*

$$co_\lambda(D^L) = U(L) co_\lambda(D), \quad contra_\lambda(D^L) = U(L) contra_\lambda(D). \quad (3.14)$$

Proof We have $\text{co}_\lambda(D^L) = V * F - F * V$. By Theorem 3.3 this symbol is equal to $(\pi_0^-(L) \otimes 1 + 1 \otimes \pi_0^+(L))F = U(L)F$. The second formula in (3.13) is reduced to the first one by means of (3.9), we use the relation $[L, X]^\vee = [L, X^\vee]$. $\square$

Thus, we have two maps: co_λ and contra_λ which connect polynomials on G/H and operators acting on functions $f(\xi)$.

The composition $\mathcal{B}_\lambda = \text{co}_\lambda \circ \text{contra}_\lambda$ maps the contravariant symbol of an operator D to its covariant symbol. Let us call $\mathcal{B}_\lambda$ the *Berezin transform*. The kernel of this transform is just the Berezin kernel.

If a polynomial F on G/H is the covariant symbol of an operator $D = \pi_\lambda^-(X)$, $X \in \text{Env}(\mathfrak{g})$, and is the contravariant symbol of an operator A simultaneously, then $A = \pi_{-\lambda-\varkappa}^-(X^\vee)$. So, the composition $\mathcal{O} = \text{contra}_\lambda \circ \text{co}_\lambda$ is

$$\mathcal{O} : \pi_\lambda^-(X) \mapsto \pi_{-\lambda-\varkappa}^-(X^\vee).$$

Such a map was absent in Berezin's theory for Hermitian symmetric spaces.

4 Polynomial Quantization and the Overgroup

As an overgroup for G we take the direct product $\tilde{G} = G \times G$. It contains G as the diagonal $\{(g, g), g \in G\}$. First we describe a series of representations $\tilde{R}_\lambda$ of $\tilde{G}$.

Let $\tilde{P}$ be a parabolic subgroup $\tilde{P}$ consisting of pairs (zh, hn) , $z \in Q^-, h \in H, n \in Q^+$. Let $\tilde{\omega}_\lambda$ be a character of $\tilde{P}$ equal to $\omega_\lambda(h)$ at these pairs. The representation of $\tilde{G}$ induced by the character $\tilde{\omega}_\lambda$ of the subgroup $\tilde{P}$ is denoted $\tilde{R}_\lambda$.

Let us give some realizations of representations $\tilde{R}_\lambda$.

Denote by $\mathcal{C}$ (a "cone") the manifold of "double" cosets

$$y = s_1^{-1} Q^- Q^+ s_2, \quad s_1, s_2 \in G.$$

The group $\tilde{G}$ acts on $\mathcal{C}$ as follows:

$$y \mapsto g_1^{-1} y g_2, \quad g_1, g_2 \in G. \quad (4.1)$$

Denote by $\mathcal{D}_\lambda(\mathcal{C})$ the space of functions f on $\mathcal{C}$ of class C^∞ satisfying the following homogeneity condition

$$f(s_1^{-1} h Q^- Q^+ s_2) = \omega_\lambda(h) f(s_1^{-1} Q^- Q^+ s_2). \quad (4.2)$$

The representation $\tilde{R}_\lambda$ acts on $\mathcal{D}_\lambda(\mathcal{C})$ by

$$(\tilde{R}_\lambda(g_1, g_2) f)(y) = f(g_1^{-1} y g_2), \quad g_1, g_2 \in \tilde{G}.$$

Let us take in $\mathcal{C}$ two sections: "hyperbolic" section $\mathcal{X}$ and "parabolic" section Γ .

The manifold $\mathcal{X} \subset \mathcal{C}$ consists of cosets

$$x = s^{-1} Q^- Q^+ s, \quad s \in G.$$

The group G acts on $\mathcal{X}$ by $x \mapsto g^{-1} x g$. The stabilizer of the initial point $x^0 = Q^- Q^+$ is H , so that $\mathcal{X}$ can be identified with G/H .

The manifold $\Gamma \subset \mathcal{C}$ consists of cosets

$$\gamma = \exp(-\eta) Q^- Q^+ \exp \xi, \quad \xi \in \mathfrak{q}^-, \eta \in \mathfrak{q}^+. \quad (4.3)$$

This manifold can be identified with $\mathfrak{q}^- \times \mathfrak{q}^+$. We can embed $\Gamma \hookrightarrow \mathcal{X}$ repeating the embedding $\mathfrak{q}^- \times \mathfrak{q}^+ \hookrightarrow G/H$, see (1.8)–(1.10). Namely, let a point $x = s^{-1} Q^- Q^+ s$, $s \in G$, has

horospherical coordinates ξ, η . By (1.9) we find the element $h(\xi, \eta)$ and then by (1.10) we obtain an element g . We rename it by s , so that

$$s = \exp Y \cdot \exp \xi = \exp X \cdot h_0 \cdot \exp \eta, \quad (4.4)$$

where $X \in \mathfrak{q}^-, Y \in \mathfrak{q}^+$. Therefore

$$x = \exp(-\eta) \cdot h_0^{-1} Q^- Q^+ \exp \xi, \quad h_0 = h(\xi, \eta). \quad (4.5)$$

Thus, the embedding above assigns to a point $\gamma \in \Gamma$, given by (4.3), the point $x \in \mathcal{X}$, given by (4.5).

The representation $\tilde{R}_\lambda$ can be realized in functions on these manifolds $\mathcal{X}$ and Γ .

First consider $\mathcal{X}$. A point $x = s^{-1} Q^- Q^+ s$ in $\mathcal{X}$ under action (4.1) goes to the point $g_1^{-1} x g_2 = g_1^{-1} s^{-1} Q^- Q^+ s g_2$ in $\mathcal{C}$. Take the element $sg_2(sg_1)^{-1}$, i. e. the element $sg_2 g_1^{-1} s^{-1}$, and decompose it "by Gauss":

$$sg_2 g_1^{-1} s^{-1} = \exp(-Y^*) \cdot \exp X^* \cdot h^*, \quad X^* \in \mathfrak{q}^-, Y^* \in \mathfrak{q}^+. \quad (4.6)$$

Here the element $h^* \in H$ depends on the point x only and does not depend on its representative s . Let us form an element $s^* \in G$:

$$s^* = \exp Y^* \cdot sg_2 = \exp X^* \cdot h^* sg_1. \quad (4.7)$$

It gives the point $x^* = (s^*)^{-1} Q^- Q^+ s^*$ in $\mathcal{X}$. By (4.7) we have

$$x^* = g_1^{-1} s^{-1} (h^*)^{-1} Q^- Q^+ sg_2.$$

Therefore,

$$f(x^*) = \omega_\lambda((h^*)^{-1}) f(g_1^{-1} x g_2),$$

so that $\tilde{R}_\lambda$ acts in functions on $\mathcal{X} = G/H$ as follows:

$$(\tilde{R}_\lambda(g_1, g_2)f)(x) = \omega_\lambda(h^*) f(x^*). \quad (4.8)$$

Theorem 4.1 *In horospherical coordinates ξ, η on $\mathcal{X}$ the representation $\tilde{R}_\lambda$ is*

$$(\tilde{R}_\lambda(g_1, g_2)f)(\xi, \eta) = \frac{\Phi_\lambda(\xi \bullet g_2, \eta \circ g_1)}{\Phi_\lambda(\xi, \eta)} \omega_\lambda(\tilde{h}_2) \omega_\lambda(\hat{h}_1^{-1}) f(\xi \bullet g_2, \eta \circ g_1), \quad (4.9)$$

where $\tilde{h}_2$ and $\hat{h}_1$ are taken from decompositions (1.6) and (1.7) with $g = g_2$ and $g = g_1$ respectively.

Proof Let a point $x = s^{-1} Q^- Q^+ s$, $s \in G$, has horospherical coordinates ξ, η . By (4.4) and (1.6), (1.7) we have

$$sg_2 = \exp Y \cdot \exp \xi \cdot g_2 = \exp Y \cdot \exp Y_2 \cdot \tilde{h}_2 \cdot \exp \tilde{\xi}_2,$$

$$sg_1 = \exp X \cdot h_0 \cdot \exp \eta \cdot g_1 = \exp X \cdot h_0 \cdot \exp X_1 \cdot \hat{h}_1 \cdot \exp \hat{\eta}_1.$$

where $\tilde{\xi}_2 = \xi \bullet g_2$, $\hat{\eta}_1 = \eta \circ g_1$. Hence

$$s^* = \exp Y^* \cdot sg_2 = \exp Y_3 \cdot \tilde{h}_2 \cdot \exp \tilde{\xi}_2, \quad (4.10)$$

$$s^* = \exp X^* \cdot sg_1 = \exp X_3 \cdot h^* \cdot h_0 \cdot \hat{h}_1 \cdot \exp \hat{\eta}_1. \quad (4.11)$$

Therefore, using (4.10) and (4.11), we obtain

$$\begin{aligned} x^* &= (s^*)^{-1} Q^- Q^+ s^* \\ &= \exp \hat{\eta}_1 \cdot (h^* h_0 \hat{h}_1)^{-1} \cdot Q^- Q^+ \cdot \tilde{h}_2 \cdot \exp \tilde{\xi}_2 \\ &= \exp \hat{\eta}_1 \cdot (h^* h_0 \hat{h}_1)^{-1} \cdot \tilde{h}_2 \cdot Q^- Q^+ \cdot \exp \tilde{\xi}_2. \end{aligned}$$

By homogeneity condition (4.2) we have

$$f(x^*) = f(\exp \widehat{\eta}_1 \cdot Q^- Q^+ \cdot \exp \widetilde{\xi}_2) \cdot \omega_\lambda \left((h^* h_0 \widehat{h}_1)^{-1} \widetilde{h}_2 \right) \quad (4.12)$$

On the other hand, by (4.5) we can write the point x^* in the following form:

$$x^* = \exp \widehat{\eta}_1 \cdot (h_0^*)^{-1} Q^- Q^+ \cdot \exp \widetilde{\xi}_2,$$

where $h_0^* = h(\widetilde{\xi}_2, \widehat{\eta}_1)$. Whence again by homogeneity condition (4.2) we obtain

$$f(x^*) = f(\exp \widehat{\eta}_1 \cdot Q^- Q^+ \cdot \exp \widetilde{\xi}_2) \cdot \omega_\lambda \left((h_0^*)^{-1} \right). \quad (4.13)$$

Comparing (4.12) and (4.13) we get

$$\omega_\lambda \left(\widehat{h}_1^{-1} h_0^{-1} (h^*)^{-1} \widetilde{h}_2 \right) = \omega_\lambda \left((h_0^*)^{-1} \right),$$

whence

$$\omega_\lambda(h^*) = \frac{\omega_\lambda(h_0^*)}{\omega_\lambda(h_0)} \omega_\lambda(\widehat{h}_1^{-1}) \omega_\lambda(\widetilde{h}_2).$$

Substitute it to (4.8) and remember (2.4) and (3.1), as result we obtain (4.9). $\square$

Theorem 4.2 *In horospherical coordinates ξ, η on Γ the representation $\widetilde{R}_\lambda$ is*

$$(\widetilde{R}_\lambda(g_1, g_2) f)(\xi, \eta) = \omega_\lambda(\widetilde{h}_2) \omega_\lambda(\widehat{h}_1^{-1}) f(\xi \bullet g_2, \eta \circ g_1).$$

It shows that $\widetilde{R}_\lambda$ is equivalent to a tensor product:

$$\widetilde{R}_\lambda(g_1, g_2) = \pi_\lambda^-(g_2) \otimes \pi_\lambda^+(g_1).$$

The group $\widetilde{G}$ contains three subgroups isomorphic to G . The first one is the diagonal consisting of pairs (g, g) , $g \in G$. The restriction of the representation $\widetilde{R}_\lambda$ to this subgroup is the representation U by translations on G/H :

$$(\widetilde{R}_\lambda(g, g) f)(x) = (U(g)f)(x) = f(g^{-1}xg).$$

Indeed, (4.6) and (4.7) with $g_1 = g_2 = g$ give $h^* = e$ and $s^* = sg$.

Two other subgroups G_1 and G_2 consist of pairs (g, e) and (e, g) , where $g \in G$.

By virtue of Theorem 4.1, the restriction of the representation $\widetilde{R}_\lambda$ to the subgroup G_2 is given by

$$\begin{aligned} (\widetilde{R}_\lambda(e, g) f)(\xi, \eta) &= \frac{\Phi_\lambda(\widetilde{\xi}, \eta)}{\Phi_\lambda(\xi, \eta)} \omega_\lambda(\widetilde{h}) f(\widetilde{\xi}, \eta) \\ &= \frac{1}{\Phi_\lambda(\xi, \eta)} (\pi_\lambda^-(g) \otimes 1) \left[f(\xi, \eta) \Phi_\lambda(\xi, \eta) \right]. \end{aligned} \quad (4.14)$$

Similarly, the restriction of the representation $\widetilde{R}_\lambda$ to the subgroup G_1 is given by

$$(\widetilde{R}_\lambda(g, e) f)(\xi, \eta) = \frac{1}{\Phi_\lambda(\xi, \eta)} (1 \otimes \pi_\lambda^+(g)) \left[f(\xi, \eta) \Phi_\lambda(\xi, \eta) \right]. \quad (4.15)$$

Let us go from the group G to the universal enveloping algebra $\text{Env}(\mathfrak{g})$. Then from (4.14) and (4.15) for $X \in \text{Env}(\mathfrak{g})$ we obtain

$$\begin{aligned} (\widetilde{R}_\lambda(0, X) f)(\xi, \eta) &= \frac{1}{\Phi_\lambda(\xi, \eta)} (\pi_\lambda^-(X) \otimes 1) \left[f(\xi, \eta) \Phi_\lambda(\xi, \eta) \right], \\ (\widetilde{R}_\lambda(X, 0) f)(\xi, \eta) &= \frac{1}{\Phi_\lambda(\xi, \eta)} (1 \otimes \pi_\lambda^+(X)) \left[f(\xi, \eta) \Phi_\lambda(\xi, \eta) \right]. \end{aligned}$$

Let us take as f the function f_0 equal to the 1 identically. Then we have

$$(\tilde{R}_\lambda(0, X)f_0)(\xi, \eta) = \frac{1}{\Phi_\lambda(\xi, \eta)}(\pi_\lambda^-(X) \otimes 1)\Phi_\lambda(\xi, \eta), \quad (4.16)$$

$$(\tilde{R}_{-\lambda-\varkappa}(X, 0)f_0)(\xi, \eta) = \frac{1}{\Phi_{-\lambda-\varkappa}(\xi, \eta)}(1 \otimes \pi_{-\lambda-\varkappa}^+(X))\Phi_{-\lambda-\varkappa}(\xi, \eta). \quad (4.17)$$

Right hand sides of formulae (4.16) and (4.17) are just covariant and contravariant symbols of operator $D = \pi_\lambda^-(X)$, see Section 3.

Let us change the position of arguments in $\tilde{R}_\lambda$, then we have a new representation $\hat{R}_\lambda$ of $\tilde{G}$, namely, $\hat{R}_\lambda(g_1, g_2) = \tilde{R}_\lambda(g_2, g_1)$. Using the realization of $\hat{R}_\lambda$ on the section Γ , we see that the tensor product $A_\lambda \otimes B_\lambda$ intertwines the representation $\hat{R}_\lambda$ with the representation $\hat{R}_{-\lambda-\varkappa}$. Passing from Γ to $\mathcal{X}$ and replacing λ by $-\lambda - \varkappa$, we obtain that the operator $c(\lambda)A_{-\lambda-\varkappa} \otimes B_{-\lambda-\varkappa}$ intertwines the representation $\hat{R}_{-\lambda-\varkappa}$ with the representation $\tilde{R}_\lambda$ and transfers contravariant symbols to covariant ones. It has the kernel $\mathcal{B}_\lambda(\xi, \eta; u, v)$, i. e. it is precisely the Berezin transform.

Theorem 4.3 *Let V be a covariant symbol of the first order (corresponding to an element L of the Lie algebra $\mathfrak{g}$). The multiplication of covariant symbols by V is the action of the overalgebra $\tilde{\mathfrak{g}}$ on the space of covariant symbols;*

$$V * F = \tilde{R}_\lambda(0, L)F, \quad F * V = -\tilde{R}_\lambda(L, 0)F. \quad (4.18)$$

Proof Formula (3) with $D_1 = \pi_\lambda^-(L)$ and $F_2 = F$ gives exactly the first formula in (4.18). The second formula is just (3.12). $\square$

This theorem adjoins to themes of [7, 8, 10] and gives a new point of view on the multiplication of symbols.

5 Example: A Hyperboloid of One Sheet

Here we determine *explicitly* an action of the overalgebra on the space of covariant symbols.

The group G is $\mathrm{SL}(2, \mathbb{R})$, the subgroup H consists of diagonal matrices, the space G/H is a hyperboloid of one sheet in $\mathbb{R}^3$. The overgroup $\tilde{G} = G \times G$ contains three subgroups $G^d, G_1 \amalg G_2$ isomorphic to G , see Section 4. Let $\mathfrak{g}$ be the Lie algebra of G . Then the Lie algebras of $\tilde{G}$ and G^d, G_1, G_2 are $\tilde{\mathfrak{g}} = \mathfrak{g} + \mathfrak{g}$ and $\mathfrak{g}^d, \mathfrak{g}_1, \mathfrak{g}_2$, respectively.

In order to write an action of the overalgebra $\tilde{\mathfrak{g}}$, it is sufficient to take some subspace complementary to $\mathfrak{g}^d$. Now we take the subalgebra $\mathfrak{g}_2$. It consists of pairs $(0, X)$, where $X \in \mathfrak{g}$.

In this example we take a few greater store of representations: besides of parameter λ there is a discrete parameter $\nu = 0, 1$. We shall use the notation

$$t^{\lambda, \nu} = |t|^\lambda \mathrm{sgn}^\nu t.$$

The group $G = \mathrm{SL}(2, \mathbb{R})$ consists of real matrices of the second order with unit determinant:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

For $\lambda \in \mathbb{C}$, $\nu = 0, 1$, denote by $\mathcal{D}_{\lambda, \nu}(\mathbb{R})$ the space of functions f in $C^\infty(\mathbb{R})$ such that the function $\widehat{f}(t) = t^{\lambda, \nu} f(1/t)$ belongs to $C^\infty(\mathbb{R})$ too. The representation $\pi_{\lambda, \nu}$ of the group G acts on $\mathcal{D}_{\lambda, \nu}(\mathbb{R})$ by (we consider that G acts from the right):

$$(\pi_{\lambda, \nu}(g)f)(t) = f\left(\frac{\alpha t + \gamma}{\beta t + \delta}\right)(\beta t + \delta)^{\lambda, \nu}.$$

Any irreducible finite-dimensional representation ρ_k of the group G is labeled by the number k (the *highest weight*) such that $2k \in \mathbb{N} = \{0, 1, 2, \dots\}$. It acts on the space V_k of polynomials $\varphi(t)$ in t of degree $\leq 2k$ (so that $\dim V_k = 2k + 1$) by

$$(\rho_k(g)\varphi)(t) = \varphi\left(\frac{\alpha t + \gamma}{\beta t + \delta}\right)(\beta t + \delta)^{2k}.$$

The Lie algebra $\mathfrak{g}$ of the group G consists of real matrices of the second order with zero trace. A basis in $\mathfrak{g}$ consists of matrices:

$$L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \quad (5.1)$$

The commutation relations are:

$$[L_+, L_-] = -2L_1, \quad [L_+, L_1] = -L_+, \quad [L_1, L_-] = -L_-. \quad (5.2)$$

Operators corresponding to elements of $\mathfrak{g}$ and $\text{Env}(\mathfrak{g})$ in representations $\pi_{\lambda, \nu}$ etc. do not depend on ν , so we do not write ν in indexes. For basis elements (5.1) we have

$$\pi_\lambda(L_-) = \frac{d}{dt}, \quad \pi_\lambda(L_1) = t \frac{d}{dt} - \frac{\lambda}{2}, \quad \pi_\lambda(L_+) = t^2 \frac{d}{dt} - \lambda t.$$

Replacing here λ by $2k$, we obtain formulas for ρ_k .

Let us realize the space $\mathbb{R}^4$ of vectors $x = (x_0, x_1, x_2, x_3)$ as the space of real 2×2 matrices:

$$x = \frac{1}{2} \begin{pmatrix} x_0 - x_3 & -x_1 + x_2 \\ x_1 + x_2 & x_0 + x_3 \end{pmatrix}.$$

The overgroup $\tilde{G}$ acts as follows:

$$x \mapsto g_1^{-1} x g_2, \quad (g_1, g_2) \in \tilde{G}.$$

Let $\mathcal{C}$ be the cone $\det x = 0$, $x \neq 0$. For $\lambda \in \mathbb{C}$, $\nu = 0, 1$, let $\mathcal{D}_{\lambda, \nu}(\mathcal{C})$ denote the space of C^∞ functions f on the cone $\mathcal{C}$ homogeneous of degree λ and parity ν :

$$f(tx) = t^{\lambda, \nu} f(x), \quad t \in \mathbb{R}^*.$$

Let $\tilde{R}_{\lambda, \nu}$ be the representation of $\tilde{G}$ by translations on the space $\mathcal{D}_{\lambda, \nu}(\mathcal{C})$ (in fact, it is a representation of the group $\text{SO}_0(2, 2)$ associated with a cone, $\tilde{G}$ covers $\text{SO}_0(2, 2)$ with multiplicity 2):

$$(\tilde{R}_{\lambda, \nu}(g_1, g_2)f)(x) = f(g_1^{-1} x g_2).$$

The section $\mathcal{X}$ of $\mathcal{C}$ by plane $(\text{tr } x) = 1$ can be identified with a hyperboloid of one sheet $-x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$. Restrictions of functions in $\mathcal{D}_{\lambda, \nu}(\mathcal{C})$ to $\mathcal{X}$ form a space $\mathcal{D}_{\lambda, \nu}(\mathcal{X})$ of functions on $\mathcal{X}$. It is contained in $C^\infty(\mathcal{X})$ and contains $\mathcal{D}(\mathcal{X})$. In the realization on $\mathcal{X}$ the representation $\tilde{R}_{\lambda, \nu}$ is:

$$(R_{\lambda, \nu}(g_1, g_2)f)(x) = f\left(\frac{g_1^{-1} x g_2}{\text{tr}(g_1^{-1} x g_2)}\right) \left\{ \text{tr}(g_1^{-1} x g_2) \right\}^{\lambda, \nu}, \quad x \in \mathcal{X}.$$

The section $\mathcal{X}$ is invariant with respect to the action $x \mapsto g^{-1} x g$ of $G^d = G$, it is just the space G/H . The restriction of $\tilde{R}_{\lambda, \nu}$ to $G^d = G$ is the quasiregular representation U of G

on $\mathcal{X}$. It preserves the space $S(\mathcal{X})$ of polynomials on $\mathcal{X}$ and decomposes in the direct sum: $U = \rho_0 + \rho_1 + \rho_2 + \dots$ with the corresponding decomposition $S(\mathcal{X}) = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \dots$

Introduce on $\mathcal{X}$ horospherical coordinates ξ, η :

$$x = \frac{1}{N} \begin{pmatrix} -\eta\xi & -\eta \\ \xi & 1 \end{pmatrix}, \quad N = N(\xi, \eta) = 1 - \xi\eta.$$

In these coordinates, to basis elements (5.1) the following operators correspond:

$$\begin{aligned} \tilde{R}_\lambda(0, L_-) &= \frac{\partial}{\partial \xi} - \lambda \frac{\eta}{N}, \\ \tilde{R}_\lambda(0, L_1) &= \xi \frac{\partial}{\partial \xi} - \lambda \frac{\xi\eta + 1}{2N}, \\ \tilde{R}_\lambda(0, L_+) &= \xi^2 \frac{\partial}{\partial \xi} - \lambda \frac{\xi}{N}. \end{aligned} \quad (5.3)$$

Similarly to Sections 3 and 4, we set

$$\Phi_{\lambda, \nu}(\xi, \eta) = N(\xi, \eta)^{\lambda, \nu}.$$

Covariant symbols of operators $\pi_\lambda(X)$, $X \in \text{Env}(\mathfrak{g})$, we define by (3.5):

$$F(\xi, \eta) = \frac{1}{\Phi_{\lambda, \nu}(\xi, \eta)} (\pi_\lambda(X) \otimes 1) \Phi_{\lambda, \nu}(\xi, \eta)$$

In particular, covariant symbols for basis elements (5.1) are multiplied by $(-\lambda)/2$ polynomials $x_1 - x_2, x_3, x_1 + x_2$, relatively.

For $k \in \mathbb{N}$, we define the Poisson kernel $P_k(x; t)$ as follows. Denote

$$B(x; t) = B(\xi, \eta; t) = \frac{(t - \xi)(1 - \eta t)}{N}, \quad (5.4)$$

then

$$P_k(x; t) = B(x; t)^k. \quad (5.5)$$

This kernel is a fixed vector in the tensor product $U \otimes \rho_k$:

$$(U(g) \otimes \rho_k(g) P_k)(x; t) = P_k(x; t), \quad g \in G.$$

Therefore, $P_k(x; t)$ is a generating function for polynomials in $\mathcal{H}_k$.

Let us introduce the following differential operators $S_k(X)$, $k \in \mathbb{N}$, and $E(X)$ in variable t , linearly depending on $X \in \mathfrak{g}$, for basic elements (5.1) they are

$$\begin{aligned} E(L_-) &= 1, \quad S_k(L_-) = \frac{d^2}{dt^2}, \\ E(L_1) &= t, \quad S_k(L_1) = t \frac{d^2}{dt^2} - (2k + 1) \frac{d}{dt}, \\ E(L_+) &= t^2, \quad S_k(L_+) = t^2 \frac{d^2}{dt^2} - 2(2k + 1)t \frac{d}{dt} + (2k + 1)(2k + 2). \end{aligned}$$

The following commutation relations hold

$$\begin{aligned} S_k([X, Y]) &= \rho_k(X) S_k(Y) - S_k(Y) \rho_{k+1}(X), \\ E([X, Y]) &= \rho_k(X) E(Y) - E(Y) \rho_{k-1}(X). \end{aligned}$$

Then, let us introduce the following coefficients $\alpha_k, \beta_k, \gamma_k$:

$$\begin{aligned}\alpha_k &= \frac{\lambda - k}{(2k + 2)(2k + 1)}, \\ \beta_k &= -\frac{1}{2}, \\ \gamma_k &= -\frac{(\lambda + k + 1)k}{2(2k + 1)}.\end{aligned}$$

Theorem 5.1 *Let $X \in \mathfrak{g}$. The operator $\tilde{R}_\lambda(0, X)$ acts on the Poisson kernel $P_k(x; t)$ as follows:*

$$\tilde{R}_\lambda(0, X) P_k = \alpha_k \cdot S_k(X) P_{k+1} + \beta_k \cdot \rho_k(X) P_k + \gamma_k \cdot E(X) P_{k-1}, \quad (5.6)$$

in the left hand side the operator acts on a function of ξ, η , and in the left hand side the operators act on functions of t .

Proof First we take $X = L_-$. Keeping in mind (5.4), (5.5), we find:

$$\left(\frac{\partial}{\partial \xi} - \lambda \frac{\eta}{N} \right) B^k = -k B^{k-1} \cdot \frac{1 - \eta t}{N} + (-\lambda + k) B^k \cdot \frac{\eta}{N}. \quad (5.7)$$

On the other hand, we compute $(\partial/\partial t) B^k$ and $(\partial^2/\partial t^2) B^{k+1}$:

$$\begin{aligned}\frac{\partial}{\partial t} B^k &= k B^{k-1} \cdot \frac{-2\eta t + \xi \eta + 1}{N} \\ &= -k B^{k-1} + 2k B^{k-1} \cdot \frac{1 - \eta t}{N},\end{aligned} \quad (5.8)$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2} B^{k+1} &= (k + 1) \left\{ k B^{k-1} \cdot \frac{(-2\eta t + \xi \eta + 1)^2}{N^2} - B^k \cdot \frac{2\eta}{N} \right\} \\ &= (k + 1) \left\{ k B^{k-1} - 2(2k + 1) B^k \cdot \frac{\eta}{N} \right\}.\end{aligned} \quad (5.9)$$

Expressing from (5.8) and (5.9) the second summands in right hand sides, substituting in (5.7) and remembering (5.3), we obtain (5.6) for $X = L_-$. Now for $X = L_1$ and $X = L_+$, we use equality (5.6) with $X = L_-$ already proved and commutation relations – successively the first and the second ones in (5.2), and corresponding relations for operators $S_k(X)$ and $E(X)$ in (5.43) and (5.44). $\square$

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