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The Nodal Cubic is a Quantum Homogeneous Space

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Abstract The cusp was recently shown to admit the structure of a quantum homogeneous space, that is, its coordinate ring B can be embedded as a right coideal subalgebra into a Hopf algebra A such that A is faithfully flat as a B-module. In the present article such a Hopf algebra A is constructed for the coordinate ring B of the nodal cubic, thus further motivating the question which affine varieties are quantum homogeneous spaces.

Keywords Hopf algebra · Quantum homogeneous space · Singular curve · Noncommutative Galois extension

Mathematics Subject Classification (2010) 16T05 · 16T20 · 14H50

Just as quantum groups (Hopf algebras) generalise affine algebraic groups, quantum homogeneous spaces as studied e.g. in [2, 7, 11, 14–19, 21] generalise affine varieties with a transitive action of an algebraic group:

Definition A *quantum homogeneous space* is a right coideal subalgebra B of a Hopf algebra A which is faithfully flat as a left B-module.

There is also an analytic theory of transitive or more generally ergodic actions of compact or locally compact quantum groups, see e.g. [8, 13] and the references therein.

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The best studied examples are deformation quantisations of affine homogeneous spaces, and in particular Podleś' quantum 2-spheres [20]. However, even if A is noncommutative, B can be a commutative algebra, so a natural question to ask is:

Question *Which affine varieties are quantum homogeneous spaces?*

In the operator algebraic setting, the analogue of this question has been raised and studied for example in [12]. Here we consider the purely algebraic setting, working over an algebraically closed field k of characteristic 0.

Unlike a homogeneous space, an affine variety which is a quantum homogeneous space can be singular as the example of the cusp shows (see [15, Section 2.11] and [10, Construction 1.2]). Note that by the results from [9], noncommutative Hopf algebras coacting on commutative algebras are quite restricted. Still, we conjecture that every plane curve can be given the structure of a quantum homogeneous space, and our aim here is to point this out for the nodal cubic given by the equation $y^2 = x^2 + x^3$:

Theorem *Fix $(q, p) \in \kappa^2$ satisfying $p^2 = q^2 + q^3$. Then the unital associative κ-algebra A with generators x, y, a, a^{-1}, b satisfying the relations*

$$aa^{-1} = a^{-1}a = 1, \qquad y^2 = x^2 + x^3, \qquad b^2 = a^3,$$
$$ba = ab, \qquad ya = ay, \qquad bx = xb, \qquad yx = xy, \qquad by = -yb + 2pb^2,$$
$$a^2x = -xa^2 - axa - a^2 + (1 + 3q)\, a^3,$$
$$ax^2 = -ax - xa - x^2a - xax + (2 + 3q)qa^3.$$

admits a Hopf algebra structure whose coproduct Δ, counit ε and antipode S satisfy

$$\Delta(x) = 1 \otimes (x - qa) + x \otimes a, \qquad \Delta(y) = 1 \otimes (y - pb) + y \otimes b,$$
$$\Delta(a) = a \otimes a, \quad \Delta(b) = b \otimes b, \quad \varepsilon(x) = q, \quad \varepsilon(y) = p, \quad \varepsilon(a) = \varepsilon(b) = 1,$$
$$S(x) = q - (x - q)\, a^{-1}, \quad S(y) = p - (y - p)\, b^{-1}, \quad S(a) = a^{-1}, \quad S(b) = b^{-1}.$$

Furthermore, the right coideal subalgebra $B \subset A$ generated by x, y is the coordinate ring of the nodal cubic, and A is free and in particular faithfully flat as a B-module.

Observe that the commutation relations in A are chosen in such a way that

$$(y - pb)^2 = y^2 - p^2b^2, \qquad (x - qa)^2 + (x - qa)^3 = x^2 + x^3 - (q^2 + q^3)a^3$$

so that

$$(y - pb)^2 = (x - qa)^2 + (x - qa)^3.$$

Thus informally speaking each coordinate on the curve becomes perturbed by some additional group-like "quantum coordinate" and the perturbed coordinates still satisfy the defining relation of the curve. As $x - qa$ and $y - pb$ are twisted primitive, the Hopf algebra A is generated by group-likes and twisted primitives. This implies:

Proposition *The Hopf algebra A is pointed.*

The point (q, p) on the curve is the point the quantum orbit of which it is presented as. To use these observations as starting point of a general study of quantum homogeneous space structures on affine varieties seems a promising future research direction.

The proof of the theorem consists of a straightforward (albeit tedious) verification that the formulas for the coproduct, counit and antipode are compatible with the defining relations of A, followed by a similarly straigthforward application of Bergman's diamond lemma [1] yielding a vector space basis of A that implies the freeness over B:

Proposition *The set*

$$\{x^i y^j (ax)^l a^m b^n \mid i, l \in \mathbb{N}, j \in \{0, 1\}, m \in \mathbb{Z}, n \in \{0, 1\}\}$$

is a vector space basis of A, and the GK-dimension of A equals 3.

Using this basis, one also easily observes that like the nonstandard Podleś spheres, the algebra extension $B \subset A$ is an example of a coalgebra Galois extension [3–6] rather than of a Hopf-Galois extension: the coalgebra is $C := A/B^+A$, where $B^+ := B \cap \ker \varepsilon$. The canonical projection $\pi : A \to C$ defines a left C-coaction

$$\lambda\colon A \to C \otimes A, \quad f \mapsto f_{(-1)} \otimes f_{(0)} := \pi(f_{(1)}) \otimes f_{(2)}$$

and we have (as a consequence of the faithful flatness of A over B)

$$B = \{f \in A \mid f_{(-1)} \otimes f_{(0)} = \pi(1) \otimes f\}.$$

That C is not a Hopf algebra quotient of A follows from $B^+A \neq AB^+$ (cf. [18, Lemma 1.4]); for example, we have

$$AB^+ \ni a^2(x - q) = -xa^2 - axa - (1 + q)a^2 + (1 + 3q)a^3 \notin B^+A.$$

We finally remark that some properties of the algebra A are better understood when using a slightly different set of generators: if we abbreviate

$$c := 3x - (1 + 3q)a + 1, \quad d := 3y - 6pb, \quad e := ac + rca$$

where r is a primitive 6th root of 1 (so that $r + r^{-1} = 1$), then the defining relations of A in terms of the generators $a^{\pm 1}, b, c, d, e$ read

$$aa^{-1} = a^{-1}a = 1, \quad ab = ba, \quad ac + rca = e, \quad ad = da, \quad ae + r^{-1}ea = 0,$$
$$bc = cb, \quad bd = -db, \quad be = eb, \quad b^2 = a^3, \quad cd = dc, \quad r^{-1}ce + ec = 3(a - a^3),$$
$$de = ed, \quad 3d^2 = c^3 - 3c + 2 + (1 + 3q)(-2 + 6q + 9q^2)a^3.$$

Using these generators, one easily verifies for example:

Proposition *The units in A are of the form $\alpha a^i b^j$, $\alpha \in k$, $i \in \mathbb{Z}$, $j \in \{0, 1\}$.*

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