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The Krull-Gabriel Dimension of Cycle-Finite Artin Algebras

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Abstract We determine the Krull-Gabriel dimension of the cycle-finite categories of finitely generated modules over artin algebras and derive some consequences.

Keywords Krull-Gabriel dimension · Infinite radical · Auslander-Reiten quiver · Cycle-finite algebra

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1 Introduction and the Main Results

Throughout the paper, by an algebra we mean an artin algebra over a commutative artin ring K. For an algebra A, we denote by $\operatorname{Mod} A$ the category of right A-modules, by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules. The radical rad_A of $\operatorname{mod} A$ is the ideal generated by all nonisomorphisms between modules in $\operatorname{ind} A$. Then the infinite radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of $\operatorname{rad} A$. By a result of Auslander [3], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, there are in $\operatorname{ind} A$ only finite many modules up to isomorphism. On the other hand, if A is of infinite representation type then $(\operatorname{rad}_A^\infty)^2 \neq 0$, by a result proved in [13]. Moreover, we denote by Γ_A the Auslander-Reiten quiver [6] of A and by τ_A the Auslander-Reiten translation $D\operatorname{Tr}$. We do not distinguish between a module X in $\operatorname{ind} A$ and the corresponding vertex $\{X\}$ in Γ_A.

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A prominent role in the representation theory of algebras is played by cycles of indecomposable modules. Recall that a cycle in ind A is a sequence

$$M_0 \xrightarrow{f_1} M_1 \to \cdots \to M_{r-1} \xrightarrow{f_r} M_r = M_0$$

of nonzero nonisomorphisms in ind A [49], and such a cycle is said to be finite if the homomorphisms $f_1, \dots, f_r$ do not belong to $\operatorname{rad}_A^{\infty}$ (see [1, 2]). Following Ringel [49], a module M in ind A which does not lie on a cycle in ind A is said to be directing. It has been proved independently in [39] and [56] that the Auslander-Reiten quiver Γ_A of an algebra A admits at most finitely many τ_A-orbits containing directing modules. Moreover, if all modules in ind A are directing then A is of finite representation type [49]. Following [1, 2], by a cycle-finite algebra we mean an algebra A for which all cycles in ind A are finite. The class of cycle-finite algebras is wide and contains the following distinguished classes of algebras: the algebras of finite representation type, the tame tilted algebras [24, 49], the tame double tilted algebras [46], the tame generalized double tilted algebras [47], the tubular algebras [49, 50], the tame quasi-tilted algebras [63], the tame generalized multicoil algebras [36], and the strongly simply connected algebras of polynomial growths [61]. Moreover, frequently interesting algebras admit Galois coverings by cycle-finite locally bounded categories, and applying covering techniques we may reduce their representation theory to that for the corresponding cycle-finite algebras (see [1, 16, 38, 54, 62, 64, 66]). The study of cycle-finite algebras has attracted much attention. We refer to [9, 10, 32–34, 37, 58–60, 67, 68] for some general results on cycle-finite algebras and their module categories.

Let A be an algebra. We denote by $\mathcal{F}(A)$ the category of all finitely presented contravariant functors from mod A to the category $\mathcal{A}b$ of abelian groups. The category $\mathcal{F}(A)$ was intensively studied over the last 40 years, and is considered to be one of the important topics of the modern representation theory of algebras. It is a hard problem to describe the category $\mathcal{F}(A)$ even if the category mod A is well understood. A natural approach to study the structure of $\mathcal{F}(A)$ is via the associated Krull-Gabriel filtration

$$0 = \mathcal{F}(A)_{-1} \subseteq \mathcal{F}(A)_0 \subseteq \mathcal{F}(A)_1 \subseteq \cdots \subseteq \mathcal{F}(A)_{n-1} \subseteq \mathcal{F}(A)_n \subseteq \cdots$$

of $\mathcal{F}(A)$ by Serre subcategories, where, for each $n \in \mathbb{N}$, $\mathcal{F}(A)_n$ is the subcategory of all functors F in $\mathcal{F}(A)$ which become of finite length in the quotient category $\mathcal{F}(A)/\mathcal{F}(A)_{n-1}$ [18, 40]. Following Geigle [22], we define $KG(A) = \min\{n \in \mathbb{N} \mid \mathcal{F}(A)_n = \mathcal{F}(A)\}$ if such a minimum exists, and set $KG(A) = \infty$ if it is not the case. Then $KG(A)$ is called the Krull-Gabriel dimension of A. We note that it is a finitely presented version of a definition due to Gabriel [21, IV.1]. The interest in the Krull-Gabriel dimension $KG(A)$ is motivated by the fact that the above filtration of $\mathcal{F}(A)$ leads to a hierarchy of exact sequences in mod A, where the almost split sequences form the lowest level (see [22]). Auslander proved that $KG(A) = 0$ if and only if A is of finite representation type (see [4, Corollary 3.14]). It was shown in [26, 11.4] that there exists no algebra A with $KG(A) = 1$. Further, $KG(A) = \infty$ for every wild hereditary algebra [7] and $KG(A) = 2$ for any tame hereditary algebra [22, 4.3]. There exist also tame algebras with infinite Krull-Gabriel dimension [23, 4.1]. Finally, it was shown independently in [12] and [52] that for any natural number $n \geq 2$ there exists a special biserial algebra (over an algebraically closed field K) with Krull-Gabriel dimension n. There is a strong relation between the Krull-Gabriel dimension $KG(A)$ of an algebra A and the transfinite powers $\operatorname{rad}_A^{\alpha}$ of the radical rad_A of mod A (see [27, 51–53]). In particular, Krause proved in [27, Corollary 8.14] that, if an algebra A has finite Krull-Gabriel dimension $KG(A) = n$, then $\operatorname{rad}_A^{\omega(n+1)} = 0$, where ω is the first limit number.

The following general conjecture was posed in [53].

Conjecture 1.1 (Schröer) *Let* $n \geq 2$ *be a natural number and* A *an algebra. Then* $KG(A) = n$ *if and only if* $\mathrm{rad}_A^{\omega(n-1)} \neq 0$ *and* $\mathrm{rad}_A^{\omega n} = 0$.

In particular, for $n = 2$, we have the following special case of Schröer's conjecture: for an algebra A of infinite representation,

$KG(A) = 2$ if and only if $\bigcap_{m\geq 1}(\mathrm{rad}_A^{\infty})^m = 0$.

It has been confirmed for the following classes of algebras: the tilted algebras of Euclidean type [22, 23], the algebras stably equivalent to tame hereditary algebras [23], the algebras with directing indecomposable projective modules [65], the enveloping algebras of restricted Lie algebras [19] (more generally, the infinitesimal group schemes [20]) in odd characteristic, the strongly simply connected algebras [61, 69], the 1-domestic string algebras [44, 45, 51], and recently the tame generalized multicoil algebras [31].

The following main result of the paper provides a characterization of cycle-finite algebras with finite Krull-Gabriel dimension, and confirms the special case of Schröer's conjecture.

Theorem 1.2 *Let* A *be a cycle-finite algebra of infinite representation type. The following statements are equivalent:*

(i) $KG(A) < \infty$.
(ii) $KG(A) = 2$.
(iii) $\bigcap_{m\geq 1}(\mathrm{rad}_A^{\infty})^m = 0$.
(iv) rad_A^{∞} *is nilpotent.*
(v) *All but finitely many components of* Γ_A *are stable tubes of rank one.*
(vi) A *does not admit a tubular quotient algebra.*

An important tool in study of the Krull-Gabriel dimension of an algebra A is the Ziegler spectrum Zg_A of A, introduced by Ziegler in his paper [71] on the model theory of modules. The Ziegler spectrum Zg_A is a topological space whose points are the isomorphism classes of indecomposable pure-injective modules in $\mathrm{Mod}\, A$. Moreover, one associates to A the Cantor-Bendixson rank $\mathrm{CB}(A)$ which measures the degree of isolation of points in the Ziegler spectrum Zg_A. As was noted by Prest [43], for an artin algebra A, the isolated points of Zg_A are exactly the modules from $\mathrm{ind}\, A$. In particular, one obtains that $\mathrm{CB}(A) = 0$ if and only if A is of finite representation type. Further, it has been proved by Ziegler in [71] that if $KG(A) = n < \infty$ then $\mathrm{CB}(A) = n$. Moreover, if A is countable and $\mathrm{CB}(A) = n < \infty$, then $KG(A) = n$. We refer to [41, 42] for model theoretic, topological and geometric aspects of the Ziegler spectrum of an algebra.

The following corollary is a direct consequence of Theorem 1.2 and the Ziegler's result.

Corollary 1.3 *Let* A *be a cycle-finite algebra of infinite representation type and without a tubular quotient algebra. Then* $\mathrm{CB}(A) = 2$.

From the remarkable Tame and Wild Theorem of Drozd [17] (see also [14]) the class of finite dimensional algebras over an algebraically closed field K may be divided into two disjoint classes. The first class is formed by the tame algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional algebras over K.

Accordingly, we may realistically hope to classify the indecomposable finite dimensional modules only for the tame algebras. More precisely, a finite dimensional K-algebra A over an algebraically closed field K is called tame if for any dimension d, there exists a finite number of $K[x]$-A-bimodules M_i, $1 \le i \le n_d$, which are free of finite rank as left modules over the polynomial algebra $K[x]$ in one variable and all but finitely many isomorphism classes of modules in ind A of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Moreover, let $\mu_A(d)$ be the least number of $K[x]$-A-bimodules satisfying the above condition for d. Then A is said to be domestic if there exists a positive integer m such that $\mu_A(d) \le m$ for any $d \ge 1$ (see [15], [48], [55]). We also note that A is of finite representation type if and only if A is domestic with $\mu_A(d) = 0$ for any $d \ge 1$ (by the validity of the second Brauer-Thrall Conjecture [8, 11]).

We have the following conjecture (see [53, p.420]):

Conjecture 1.4 (Prest) *A finite dimensional algebra A over an algebraically closed field K is domestic if and only if $KG(A) < \infty$.*

The following direct consequence of Theorem 1.2 and [60, Theorem 5.1] confirms this conjecture for the cycle-finite algebras.

Corollary 1.5 *Let A be a finite dimensional cycle-finite algebra over an algebraically closed field K. Then A is domestic if and only if $KG(A) < \infty$.*

In the course of our proof of Theorem 1.2 we establish also the following result on the structure of module categories of cycle-finite algebras of finite Krull-Gabriel dimension.

Theorem 1.6 *Let A be a cycle-finite algebra of infinite representation type and without a tubular quotient algebra. Then there exist tame concealed quotient algebras $C_1, \dots, C_m$ of A and domestic generalized multicoil enlargements $B_1, \dots, B_m$ of $C_1, \dots, C_m$, respectively, such that the following statements hold:*

(i) *$B_1, \dots, B_m$ are quotient algebras of A.*
(ii) mod $B_1, \dots,$ mod B_m *are functorially finite subcategories of* mod A.
(iii) *All but finitely many modules in* ind A *belong to* $\bigcup_{i=1}^{m}$ ind B_i.

Recall that a full subcategory $\mathcal{X}$ of mod A is called contravariantly finite (respectively, covariantly finite) if, for each module M in mod A, the family of all homomorphisms $f : X \to M$ (respectively, $g : M \to X$) in mod A with X in $\mathcal{X}$ is finitely generated [4, 5]. Then $\mathcal{X}$ is called functorially finite if it is both contravariantly and covariantly finite.

The paper is organized as follows. In Section 2 we present some concepts and results on module categories of algebras, essential for further considerations. Section 3 contains known results on the module categories of cycle-finite algebras, which are applied in the proofs of our main theorems. Sections 4 and 5 are devoted to the proofs of Theorems 1.6 and 1.2, respectively.

2 Preliminaries

We recall some notation, concepts and results on algebras and module categories needed in our further considerations.

Throughout the article we will assume (without loss of generality) that algebras are basic. Recall that an algebra A is basic if $A/\operatorname{rad} A$ is a product of division algebras.

Let A be an algebra and $e_1, \dots, e_n$ be a set of pairwise orthogonal primitive idempotents of A such that $e_1 + \cdots + e_n$ is the identity of A. Then $S_i = e_i A / e_i \operatorname{rad} A$, $i \in \{1, \dots, n\}$, form a complete set of pairwise nonisomorphic simple modules in $\operatorname{mod} A$. For each $i \in \{1, \dots, n\}$, let $F_i = \operatorname{End}_A(S_i)$ be the associated division algebra. The quiver Q_A of A is the valued quiver defined as follows:

- the vertices of Q_A are the indices $1, \dots, n$ of the chosen set $e_1, \dots, e_n$ of primitive idempotents of A;
- for two vertices i and j in Q_A, there is an arrow $i \to j$ from i to j in Q_A if and only if $e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j \neq 0$. Moreover, one associates to an arrow $i \to j$ in Q_A the valuation (d_{ij}, d'_{ij}), so we have in Q_A the valued arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j,$$

where the valuation numbers are $d_{ij} = \dim_{F_j} \big(e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j \big)$ and $d'_{ij} = \dim_{F_i} \big(e_i(\operatorname{rad} A)e_j / e_i(\operatorname{rad} A)^2 e_j \big)$.

We denote by $K_0(A)$ the Grothendieck group of A. Given a module M in $\operatorname{mod} A$, we denote by $[M]$ its image in $K_0(A)$. We note that $[S_1], \dots, [S_n]$ form a $\mathbb{Z}$-basis of $K_0(A)$. Hence, for two modules M and N in $\operatorname{mod} A$, $[M] = [N]$ if and only if M and N have the same composition factors including the multiplicities. Moreover, for a module X in $\operatorname{mod} A$, $|X|$ denotes the length of X over the commutative artin ring K.

The following fact from [59, Proposition 4.1] will be essential for our proof of Theorem 1.2.

Proposition 2.1 *Let A be an algebra, M, N, X be modules in* $\operatorname{ind} A$*, and assume that* $[M] = [N]$. *Then the following equalities hold:*

(i) $|\operatorname{Hom}_A(X, M)| - |\operatorname{Hom}_A(M, \tau_A X)| = |\operatorname{Hom}_A(X, N)| - |\operatorname{Hom}_A(N, \tau_A X)|$.

(ii) $|\operatorname{Hom}_A(M, X)| - |\operatorname{Hom}_A(\tau_A^{-1} X, M)| = |\operatorname{Hom}_A(N, X)| - |\operatorname{Hom}_A(\tau_A^{-1} X, N)|$.

Let A be an algebra. By a component of Γ_A we mean a connected component of the quiver Γ_A. A component $\mathcal{C}$ of Γ_A is called regular if $\mathcal{C}$ contains neither a projective module nor an injective module. It has been proved independently by Liu [29] and Zhang [70] that a regular component $\mathcal{C}$ of Γ_A contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube, that is, a component of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some $r \geq 1$, called the rank of $\mathcal{C}$. The τ_A-orbit of a stable tube $\mathcal{C}$ of Γ_A formed by the modules having exactly one direct predecessor is called the mouth of $\mathcal{C}$. For a module X lying in a stable tube $\mathcal{C}$ of Γ_A, there is exactly one sectional path $X_1 \to X_2 \to \cdots \to X_s = X$ with X_1 lying on the mouth of $\mathcal{C}$, and s is called the quasi-length of X in $\mathcal{C}$ which we shall denote by $\operatorname{ql}(X)$. Moreover, Liu proved in [30] that a component $\mathcal{C}$ without injective modules (respectively, projective modules) contains an oriented cycle if and only if $\mathcal{C}$ is a ray tube (respectively, a coray tube), which is obtained from a stable tube by a finite number (possibly zero) ray insertions (respectively, coray insertions). A component $\mathcal{C}$ of Γ_A is called almost cyclic if all but finitely many modules in $\mathcal{C}$ are cyclic (lie on oriented cycles).

Further, a component $\mathcal{C}$ of Γ_A is called coherent [35] if the following two conditions are satisfied:

(C1) For each projective module P in $\mathcal{C}$ there is an infinite sectional path

$$P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \dots$$

(C2) For each injective module I in $\mathcal{C}$ there is an infinite sectional path

$$\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I.$$

We note that all ray tubes and coray tubes are almost cyclic and coherent. It has been proved in [35] that a component $\mathcal{C}$ of Γ_A is almost cyclic and coherent if and only if $\mathcal{C}$ is a generalized multicoil, which is obtained from a finite family of stable tubes by a finite sequence of admissible operations (ad1)–(ad5) and their duals (ad1*)–(ad5*). Following [57], a subqiver $\mathcal{D}$ of Γ_A is said to be generalized standard if $\operatorname{rad}_A^\infty(X, Y) = 0$ for all indecomposable modules X and Y in $\mathcal{D}$. Finally, an indecomposable module M in Γ_A is said to be left stable (respectively, right stable) if $\tau_A^n M$ is nonzero for any $n > 0$ (respectively, for any $n < 0$).

Let A be an algebra and $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ a family of components of Γ_A. Then $\mathcal{C}$ is called sincere if every simple module in $\operatorname{mod} A$ is a composition factor of a module in $\mathcal{C}$. Then $\mathcal{C}$ is said to be a separating family in $\operatorname{mod} A$ if the components in Γ_A split into three disjoint families, $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$, such that the following conditions are satisfied:

(S1) $\mathcal{C}^A$ is a sincere family of pairwise orthogonal generalized standard components;

(S2) $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\operatorname{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;

(S3) every homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through the additive category $\operatorname{add}(\mathcal{C}^A)$ of $\mathcal{C}^A$.

Moreover, if $(S1)$, $(S2)$ and the condition

(S3*) every homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through $\operatorname{add}(\mathcal{C}_i)$ for any $i \in I$

are satisfied, then $\mathcal{C}$ is said to be a strongly separating family in $\operatorname{mod} A$ (see [35, 49]). We then say that $\mathcal{C}^A$ separates (respectively, strongly separates) $\mathcal{P}^A$ from $\mathcal{Q}^A$.

Let A be an algebra. Following [35], we denote by ${}_c\Gamma_A$ the cyclic quiver of A obtained from Γ_A by removing all acyclic vertices (vertices not lying on oriented cycles in Γ_A) and the arrows attached to them. Then the connected components of the translation quiver ${}_c\Gamma_A$ are said to be cyclic components of Γ_A. We have the following fact proved in [35, Proposition 5.1].

Proposition 2.2 *Let A be an algebra and M, N be modules in* $\operatorname{ind} A$. *Then M and N belong to a cyclic component Γ of Γ_A if and only if there is an oriented cycle in Γ_A containing M and N.*

Let A be an algebra and Γ be a subquiver of Γ_A. We denote by $\operatorname{ann}_A(\Gamma)$ the intersection of the annihilators $\operatorname{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all indecomposable modules X in Γ. Then $\operatorname{ann}_A(\Gamma)$ is a two-sided ideal of A, and the quotient algebra $B(\Gamma) = A/\operatorname{ann}_A(\Gamma)$ is said to be the faithful algebra of Γ. Observe that Γ is a subquiver of $\Gamma_{B(\Gamma)}$.

3 Cycle-Finite Algebras

In this section we recall known results on the structure of module categories of cycle-finite algebras, playing a prominent role in the proofs of Theorems 1.2 and 1.6.

By a tame concealed algebra we mean a tilted algebra $C = \operatorname{End}_H(T)$, where H is a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_{11}$, $\widetilde{\mathbb{A}}_{12}$, $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{B}}_n$, $\widetilde{\mathbb{C}}_n$, $\widetilde{\mathbb{BC}}_n$, $\widetilde{\mathbb{BD}}_n$, $\widetilde{\mathbb{CD}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, $\widetilde{\mathbb{F}}_{41}$, $\widetilde{\mathbb{F}}_{42}$, $\widetilde{\mathbb{G}}_{21}$, or $\widetilde{\mathbb{G}}_{22}$, and T is a (multiplicity-free) tilting H-module from the additive category of the postprojective component of Γ_H. The Auslander-Reiten quiver Γ_C of a tame concealed algebra C is of the form

$$\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C,$$

where

- $\mathcal{P}^C$ is a postprojective component of Euclidean type containing all indecomposable projective C-modules;
- $\mathcal{Q}^C$ is a preinjective component of Euclidean type containing all indecomposable injective C-modules;
- $\mathcal{T}^C$ is an infinite family of stable tubes, and all but finitely many tubes in $\mathcal{T}^C$ have rank one;
- $\mathcal{T}^C$ strongly separates $\mathcal{P}^C$ from $\mathcal{Q}^C$.

More generally, by a tilted algebra of Euclidean type we mean a tilted algebra $B = \operatorname{End}_H(T)$, where H is a hereditary algebra of Euclidean type and T is a (multiplicity-free) tilting module in $\operatorname{mod} H$. Assume that B is a representation-infinite tilted algebra of Euclidean type. Then one of the following holds:

(1) B is a domestic tubular extension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where

- $\mathcal{P}^B = \mathcal{P}^C$ is the postprojective component of Γ_C;
- $\mathcal{Q}^B$ is a preinjective component of Euclidean type containing all indecomposable injective B-modules and all indecomposable modules from the preinjective component $\mathcal{Q}^C$ of Γ_C;
- $\mathcal{T}^B$ is an infinite family of ray tubes, obtained from the family $\mathcal{T}^C$ of stable tubes of Γ_C by ray insertions;
- $\mathcal{T}^B$ strongly separates $\mathcal{P}^B$ from $\mathcal{Q}^B$;

(2) B is a domestic tubular coextension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where

- $\mathcal{P}^B$ is a postprojective component of Euclidean type containing all indecomposable projective B-modules and all indecomposable modules from the postprojective component $\mathcal{P}^C$ of Γ_C;
- $\mathcal{Q}^B = \mathcal{Q}^C$ is the preinjective component of Γ_C;
- $\mathcal{T}^B$ is an infinite family of coray tubes, obtained from the family $\mathcal{T}^C$ of stable tubes of Γ_C by coray insertions;
- $\mathcal{T}^B$ strongly separates $\mathcal{P}^B$ from $\mathcal{Q}^B$.

By a tubular algebra we mean a tubular extension (equivalently, tubular coextension) of a tame concealed algebra, with the Euler quadratic form positive semidefinite of corank 2 (see [28], [49], [50]). By general theory, a tubular algebra B admits two different tame concealed quotient algebras C_0 and C_∞ such that B is a tubular extension of C_0 and a tubular coextension of C_∞, and the Auslander-Reiten quiver Γ_B has a disjoint union decomposition

$$\Gamma_B = \mathcal{P}_0^B \cup \mathcal{T}_0^B \cup \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \right) \cup \mathcal{T}_\infty^B \cup \mathcal{Q}_\infty^B,$$

where $\mathbb{Q}^+$ is the set of positive rational numbers, and

- $\mathcal{P}_0^B = \mathcal{P}^{C_0}$ is the postprojective component of Γ_{C_0};
- $\mathcal{T}_0^B$ is an infinite family of ray tubes with at least one projective B-module, obtained from the family $\mathcal{T}^{C_0}$ of stable tubes of Γ_{C_0} by ray insertions;
- $\mathcal{Q}_\infty^B = \mathcal{Q}^{C_\infty}$ is the preinjective component of Γ_{C_∞};
- $\mathcal{T}_\infty^B$ is an infinite family of coray tubes with at least one injective B-module, obtained from the family $\mathcal{T}^{C_\infty}$ of stable tubes of Γ_{C_∞} by coray insertions;
- for each $q \in \mathbb{Q}^+$, $\mathcal{T}_q^B$ is an infinite family of stable tubes, containing at least one stable tube of rank bigger than one;
- for each $q \in \mathbb{Q}^+ \cup \{0, \infty\}$, the family $\mathcal{T}_q^B$ strongly separates $\mathcal{P}_0^B \cup (\bigcup_{p<q} \mathcal{T}_p^B)$ from $(\bigcup_{p>q} \mathcal{T}_p^B) \cup \mathcal{Q}_\infty^B$;
- $\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^B$ is the family of all sincere stable tubes of Γ_B and contains all sincere indecomposable C_0-modules from the preinjective component $\mathcal{Q}^{C_0}$ of Γ_{C_0} and all sincere indecomposable C_∞-modules from the postprojective component $\mathcal{P}^{C_\infty}$ of Γ_{C_∞}.

The following characterization of tame concealed algebras has been established in [60, Theorem 4.1].

Theorem 3.1 *Let A be an algebra. The following statements are equivalent:*

(i) *A is cycle-finite and Γ_A admits a sincere stable tube;*
(ii) *A is a tame concealed algebra or a tubular algebra.*

We have also the following consequence of [58, Theorem 4.1].

Theorem 3.2 *Let A be a cycle-finite algebra. The following statements are equivalent:*

(i) *A is a tame concealed algebra;*
(ii) *for every nonzero idempotent e of A, the algebra A/AeA is of finite representation type.*

In particular, we obtain the following useful fact (see [58, Corolary 4.3]).

Corollary 3.3 *Let A be a cycle-finite algebra. The following statements are equivalent:*

(i) *A is of infinite representation type;*
(ii) *there is an idempotent e of A such that A/AeA is a tame concealed algebra.*

We also mention that Theorem 3.1 describes the structure of all regular components of the Auslander-Reiten quivers of cycle-finite algebras (see [9, Proposition 2.4] or [59, Proposition 3.3] for a proof).

Proposition 3.4 *Let A be a cycle-finite algebra and $\mathcal{C}$ be a regular component of Γ_A. Then the following statements hold:*

(i) *$\mathcal{C}$ is a stable tube;*

(ii) *$B(\mathcal{C})$ is a tame concealed algebra or a tubular algebra.*

Let $C_1, \dots, C_m$ be a family of tame concealed algebras. Following [36], by a generalized multicoil enlargement of $C_1, \dots, C_m$ we mean an algebra A obtained by a sequence of algebra admissible operations of types (ad1)–(ad5) and their duals (ad1*)–(ad5*) using modules from the separating families $\mathcal{T}^{C_1}, \dots, \mathcal{T}^{C_m}$ of stable tubes $\Gamma_{C_1}, \dots, \Gamma_{C_m}$. Then there are a unique quotient algebra $A^{(l)}$ of A being a product of algebras having separating families of coray tubes (the left quasitilted algebra of A) and a unique quotient algebra $A^{(r)}$ of A being a product of algebras having separating families of ray tubes (the right quasitilted algebra of A) such that Γ_A has a disjoint union decomposition (see [36, Theorems C and E])

$$\Gamma_A = \mathcal{P}^A \cup \mathcal{C}^A \cup \mathcal{Q}^A,$$

where

- $\mathcal{P}^A$ is the left part $\mathcal{P}^{A^{(l)}}$ in a decomposition $\Gamma_{A^{(l)}} = \mathcal{P}^{A^{(l)}} \cup \mathcal{T}^{A^{(l)}} \cup \mathcal{Q}^{A^{(l)}}$ of $\Gamma_{A^{(l)}}$, with $\mathcal{T}^{A^{(l)}}$ a family of coray tubes strongly separating $\mathcal{P}^{A^{(l)}}$ from $\mathcal{Q}^{A^{(l)}}$;
- $\mathcal{Q}^A$ is the right part $\mathcal{Q}^{A^{(r)}}$ in a decomposition $\Gamma_{A^{(r)}} = \mathcal{P}^{A^{(r)}} \cup \mathcal{T}^{A^{(r)}} \cup \mathcal{Q}^{A^{(r)}}$ of $\Gamma_{A^{(r)}}$, with $\mathcal{T}^{A^{(r)}}$ a family of ray tubes strongly separating $\mathcal{P}^{A^{(r)}}$ from $\mathcal{Q}^{A^{(r)}}$;
- $\mathcal{C}^A$ is a family of generalized multicoils separating $\mathcal{P}^A$ from $\mathcal{Q}^A$, obtained from the families $\mathcal{T}^{C_1}, \dots, \mathcal{T}^{C_m}$ of stable tubes of $\Gamma_{C_1}, \dots, \Gamma_{C_m}$ by a sequence of translation quiver admissible operations of types (ad1)–(ad5) and their duals (ad1*)–(ad5*), corresponding to the algebra admissible operations of types (ad1)–(ad5) and their duals (ad1*)–(ad5*) leading from $C_1, \dots, C_m$ to A;
- $\mathcal{C}^A$ contains all indecomposable modules of the cyclic parts of $\mathcal{T}^{A^{(l)}}$ and $\mathcal{T}^{A^{(r)}}$;
- $\mathcal{P}^A$ contains all indecomposable modules of $\mathcal{P}^{A^{(r)}}$;
- $\mathcal{Q}^A$ contains all indecomposable modules of $\mathcal{Q}^{A^{(l)}}$.

We also mention that $\operatorname{gl.dim} A \le 3$ and every module X in $\operatorname{ind} A$ satisfies $\operatorname{pd}_A X \le 2$ or $\operatorname{id}_A X \le 2$ (see [36, Theorem E]).

By a tame generalized multicoil algebra we mean a generalized multicoil enlargement A of a finite family of tame concealed algebras such that $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean type or tubular algebras. Moreover, a tame generalized multicoil algebra is said to be domestic if $A^{(l)}$ and $A^{(r)}$ are products of tilted algebras of Euclidean type.

The following theorem provides a characterization of tame generalized multicoil algebras (see [36, Theorems A and F]).

Theorem 3.5 *Let A be an algebra. The following statements are equivalent:*

(i) *A is a tame generalized multicoil algebra;*

(ii) *A is cycle-finite and Γ_A admits a separating family of almost cyclic coherent components.*

The following special case of [34, Theorem 1.2] will be crucial for our investigations.

Theorem 3.6 *Let A be a cycle-finite algebra and Γ be an infinite cyclic component of Γ_A. Then there exist infinite full translation subquivers $\Gamma_1, \ldots, \Gamma_r$ of Γ such that the following statements hold.*

(i) *For each $i \in \{1, \ldots, r\}$, Γ_i is a cyclic coherent subquiver of Γ_A.*
(ii) *For each $i \in \{1, \ldots, r\}$, $B(\Gamma_i)$ is a tame generalized multicoil algebra.*
(iii) *$\Gamma_1, \ldots, \Gamma_r$ are pairwise disjoint full translation subquivers of Γ and $\Gamma^{cc} = \Gamma_1 \cup \cdots \cup \Gamma_r$ is a maximal cyclic coherent and cofinite full translation subquiver of Γ.*
(iv) *$B(\Gamma \setminus \Gamma^{cc})$ is of finite representation type.*

We recall also the following known facts, and their simple proofs.

Lemma 3.7 *Let A be an algebra, B a quotient algebra, $\mathcal{T}$ a stable tube of Γ_B, and $\mathcal{C}$ a stable tube of Γ_A containing all indecomposable modules of $\mathcal{T}$. Then $\mathcal{C} = \mathcal{T}$.*

Proof Observe that there is a coray in $\mathcal{C}$ containing infinitely many modules from $\mathcal{T}$. Then for any indecomposable module X in $\mathcal{C}$ there are a sequence of irreducible monomorphisms

$$X = X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = Y$$

and a sequence of irreducible epimorphisms

$$Z = Y_s \xrightarrow{g_s} Y_{s-1} \to \cdots \to Y_1 \xrightarrow{g_1} Y_0 = Y$$

in mod A between indecomposable modules in $\mathcal{C}$ and with Z from $\mathcal{T}$. Then X is isomorphic to a submodule of Y and Y is isomorphic to a quotient module of Z. Therefore, X is a B-module, because Z is a B-module. □

Lemma 3.8 *Let A be a cycle-finite algebra, B a quotient algebra of A, and $\mathcal{T}$ a stable tube of Γ_B. Then there exists a cyclic component Γ of Γ_A containing all indecomposable modules of $\mathcal{T}$.*

Proof Since $\mathcal{T}$ is a stable tube, for any modules X and Y in $\mathcal{T}$ there exists a cycle of irreducible homomorphisms

$$X \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} Y \xrightarrow{g_s} Y_{s-1} \to \cdots \to Y_1 \xrightarrow{g_1} X$$

in mod B between indecomposable modules in $\mathcal{T}$. Since A is cycle-finite, the homomorphisms $f_1, \ldots, f_r, g_1, \ldots, g_s$ do not belong to rad_A^{∞}, and consequently there exists an oriented cycle of irreducible homomorphisms between modules in ind A containing X and Y. Hence there is an oriented cycle in Γ_A containing X and Y. This shows that all indecomposable modules of the stable tube $\mathcal{T}$ of Γ_B belong to a common cyclic component Γ of Γ_A, by Proposition 2.2. □

The following propositions are consequences of Theorems 3.1, 3.6 , Proposition 3.4, and Lemmas 3.7, 3.8.

Proposition 3.9 *Let A be a cycle-finite algebra, C a tame concealed quotient algebra of A, and $\mathcal{T}^C$ the family of all stable tubes of Γ_C. Then all but finitely many tubes of $\mathcal{T}^C$ are stable tubes of Γ_A.*

Proposition 3.10 *Let A be a cycle-finite algebra, B a tubular quotient algebra of A, and*

$$\Gamma_B = \mathcal{P}_0^B \cup \mathcal{T}_0^B \cup \left(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^B \right) \cup \mathcal{T}_\infty^B \cup \mathcal{Q}_\infty^B$$

the canonical decomposition of Γ_B. Then all tubes in $\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^B$ are stable tubes of Γ_A. In particular, Γ_A contains infinitely many stable tubes of rank bigger than one.

We end this section with some results towards the proof of Theorem 1.2.
The following fact has been proved in [25, 1.5] (see also [61, Proposition 3.5]).

Proposition 3.11 *Let B a tubular algebra. Then $\bigcap_{m \geq 1} (\mathrm{rad}_B^\infty)^m \neq 0$. In particular, rad_B^∞ is not nilpotent.*

We have also the following consequence of [36, Theorems A and C].

Proposition 3.12 *Let B a domestic generalized multicoil algebra. Then $(\mathrm{rad}_B^\infty)^3 = 0$.*

The next proposition is due to Geigle [23, Proposition 4.1].

Proposition 3.13 *Let B a tubular algebra. Then $KG(B) = \infty$.*

We will need also the following result by Malicki [31, Theorem 1.1], extending a result proved in [69, Proposition 2.2].

Proposition 3.14 *Let B a domestic generalized multicoil algebra. Then $KG(B) = 2$.*

Finally, we present the following consequence of recent results proved in [9, Propositions 2.10 and 2.11].

Theorem 3.15 *Let A be a cycle-finite algebra of infinite representation type and without a tubular quotient algebra. Moreover, let C be a tame concealed quotient algebra of A. Then the following statements hold.*

(i) *All but finitely many indecomposable modules from the postprojective component $\mathcal{P}^C$ of Γ_C are directing modules.*
(ii) *All but finitely many indecomposable modules from the preinjective component $\mathcal{Q}^C$ of Γ_C are directing modules in* $\mathrm{ind}\, A$.

4 Proof of Theorem 1.6

Throughout this section we assume that A is a cycle-finite algebra of infinite representation type and without a tubular quotient algebra. Moreover, let $e_1, \ldots, e_n$ be a fixed set of pairwise orthogonal primitive idempotents in A such that $e_1 + \cdots + e_n$ is the identity 1_A of A. It follows from Corollary 3.3 that there is an idempotent f_C in A such that $C = A/Af_CA$ is a tame concealed algebra. Further, it follows from Proposition 3.9 that there is a stable tube $\mathcal{T}$ in Γ_C which remains a stable tube in Γ_A. Moreover, C is the faithful algebra $B(\mathcal{T})$ of $\mathcal{T}$, considered as a component of Γ_A. We note that $\mathcal{T}$ is a cyclic component of Γ_A. Then, applying [34, Theorem 1.2 and Propositions 2.3, 2.4], we conclude that there is a convex full subquiver Δ_C of the quiver Q_A of A such that $f_C = 1_A - e_C$ with $e_C = e_{i_1} + \cdots + e_{i_r}$, for $i_1, \dots, i_r$ being the vertices of Δ_C. Moreover, C is isomorphic to the algebra $e_C A e_C$ (see [34, Corollary 1.3]). Since the quiver Q_A of A has only finitely many convex subquivers, we conclude that A admits only a finite number of tame concealed quotient algebras.

Let $C_1, \dots, C_m$ be the family of all tame concealed quotient algebras of A. Fix $i \in \{1, \dots, m\}$. Let $e^{(i)} = e_{C_i}$ and $f^{(i)} = f_{C_i}$ be idempotents of A such that $1_A = e^{(i)} + f^{(i)}$ and $C_i = A/Af^{(i)}A$. We identify $\operatorname{mod} C_i$ with the full subcategory of $\operatorname{mod} A$ formed by all modules M in $\operatorname{mod} A$ such that $Mf^{(i)} = 0$, or equivalently, $M = Me^{(i)}$. Moreover, let $\operatorname{res}_i : \operatorname{mod} A \to \operatorname{mod} C_i$ be the restriction functor which assigns to a module X in $\operatorname{mod} A$ the module $\operatorname{res}_i(X) = Xe^{(i)}$ in $\operatorname{mod} C_i$ and to a homomorphism $f : X \to Y$ in $\operatorname{mod} A$ the restriction $\operatorname{res}_i(f) : \operatorname{res}_i(X) \to \operatorname{res}_i(Y)$ of f to $Xe^{(i)}$. Consider the decomposition

$$\Gamma_{C_i} = \mathcal{P}^{C_i} \cup \mathcal{T}^{C_i} \cup \mathcal{Q}^{C_i}$$

of the Auslander-Reiten quiver Γ_{C_i} of C_i, where $\mathcal{P}^{C_i}$ is the unique postprojective component of Euclidean type, $\mathcal{Q}^{C_i}$ is the unique preinjective component of Euclidean type, and $\mathcal{T}^{C_i} = (\mathcal{T}_\lambda^{C_i})_{\lambda \in \Lambda_i}$ is an infinite family of stable tubes, strongly separating $\mathcal{P}^{C_i}$ from $\mathcal{Q}^{C_i}$. We denote by $r_\lambda^{(i)}$ the rank of the stable tube $\mathcal{T}_\lambda^{C_i}$, for any $\lambda \in \Lambda_i$. By general theory, we have $r_\lambda^{(i)} = 1$ for all but finitely many $\lambda \in \Lambda_i$. Moreover, by Proposition 3.9, all but finitely many stable tubes in $\mathcal{T}^{C_i}$ are components of Γ_A. Assume now that $\mathcal{T}_\lambda^{C_i}$, for some $\lambda \in \Lambda_i$, is not a component of Γ_A. Then, applying Lemma 3.8, we conclude that there exists a cyclic component $\Gamma_\lambda^{(i)}$ of Γ_A containing all indecomposable modules of $\mathcal{T}_\lambda^{C_i}$, and $\Gamma_\lambda^{(i)}$ is a full translation subquiver of a nonregular component $\mathcal{C}_\lambda^{(i)}$ of Γ_A, by Proposition 3.4 and Lemma 3.7. Hence, there is a cofinite subset Λ_i' of Λ_i such that the stable tubes $\mathcal{T}_\lambda^{C_i}$, $\lambda \in \Lambda_i'$, are all regular components of Γ_A containing an indecomposable module from $\mathcal{T}^{C_i}$. We set $\Lambda_i'' = \Lambda_i \setminus \Lambda_i'$. We note that we may have $\Lambda_i = \Lambda_i'$, and then Λ_i'' is empty. Assume Λ_i'' is nonempty, and take $\lambda \in \Lambda_i''$. Then, according to Theorem 3.6, there is a unique maximal cyclic coherent translation subquiver $\Omega_\lambda^{(i)}$ of the cyclic component $\Gamma_\lambda^{(i)}$ containing all indecomposable modules of the stable tube $\mathcal{T}_\lambda^{C_i}$ of Γ_{C_i}. Further, by general theory (see [35, Section 2] and [34, Proposition 3.4]), the translation quiver $\Omega_\lambda^{(i)}$ admits a left border $\Delta_\lambda^{(i)}$ and a right border $\Sigma_\lambda^{(i)}$ having the following properties:

- $\Delta_\lambda^{(i)}$ and $\Sigma_\lambda^{(i)}$ are finite, disjoint, and unions of sectional paths of Γ_A;
- $\Omega_\lambda^{(i)}$ is a maximal coherent cyclic subquiver of Γ_A consisting of modules which are both successors of modules lying on $\Delta_\lambda^{(i)}$ and predecessors of modules lying on $\Sigma_\lambda^{(i)}$;
- every path in Γ_A from a module in $\mathcal{C}_\lambda^{(i)} \setminus \Omega_\lambda^{(i)}$ to a module in $\Omega_\lambda^{(i)}$ contains a module from $\Delta_\lambda^{(i)}$;
- every path in Γ_A from a module in $\Omega_\lambda^{(i)}$ to a module in $\mathcal{C}_\lambda^{(i)} \setminus \Omega_\lambda^{(i)}$ contains a module from $\Sigma_\lambda^{(i)}$.

We also mention that $\Omega_\lambda^{(i)}$ consists of all indecomposable modules Z in $\Gamma_\lambda^{(i)}$ such that $\operatorname{res}_i(Z)$ contains an indecomposable direct summand from $\mathcal{T}_\lambda^{C_i}$, and equivalently, with $\operatorname{res}_i(Z)$ being an indecomposable module in $\mathcal{T}_\lambda^{C_i}$. We define

$$\Delta^{(i)} = \bigcup_{\lambda \in \Lambda_i''} \Delta_\lambda^{(i)} \qquad \text{and} \qquad \Sigma^{(i)} = \bigcup_{\lambda \in \Lambda_i''} \Sigma_\lambda^{(i)}.$$

In particular, we obtain that, if λ and μ are different elements in Λ_i'', then the quivers $\Omega_\lambda^{(i)}$ and $\Omega_\mu^{(i)}$ are disjoint. We would like to stress that we may have $\Gamma_\lambda^{(i)} = \Gamma_\mu^{(i)}$ for $\lambda \neq \mu$ in Λ_i''. We set also $\Omega_\lambda^{(i)} = \mathcal{T}_\lambda^{C_i}$ for any $\lambda \in \Lambda_i'$, and define

$$\Omega^{(i)} = \bigcup_{\lambda \in \Lambda_i} \Omega_\lambda^{(i)}.$$

We observe that, for any different elements i and j in $\{1, \ldots, m\}$, the translation quivers $\Omega^{(i)}$ and $\Omega^{(j)}$ are disjoint. Moreover, it follows from Theorem 3.6 that all but finitely many indecomposable modules of the cyclic quiver ${}_c\Gamma_A$ belong to the translation quiver

$$\Omega = \bigcup_{i=1}^{m} \Omega^{(i)}.$$

In particular, all finite cyclic components of Γ_A are contained in ${}_c\Gamma_A \setminus \Omega$ (see [34, Theorem 1.2] for the structure of such components).

Since A does not admit a tubular quotient algebra, applying Proposition 3.4, we conclude that the stable tubes $\Omega_\lambda^{(i)} = \mathcal{T}_\lambda^{C_i}$, for $i \in \{1, \ldots, m\}$ and $\lambda \in \Lambda_i'$, form the family of all regular components of Γ_A.

We shall discuss now the structure of left stable acyclic subquivers and right stable acyclic subquivers of Γ_A. Fix $i \in \{1, \ldots, m\}$. We know from Theorem 3.15 that all but finitely many indecomposable modules from the postprojective component $\mathcal{P}^{C_i}$ and the preinjective component $\mathcal{Q}^{C_i}$ of Γ_{C_i} are directing modules in ind A. Then it follows from the proofs of [9, Propositions 3.2 and 3.4] that there exist indecomposable modules M_i in $\mathcal{P}^{C_i}$ and N_i in $\mathcal{Q}^{C_i}$ such that the following statements hold:

- the full translation subquiver $\mathcal{D}_i = (M_i \rightarrow)$ of Γ_A formed by all successors of M_i in Γ_A is right stable, acyclic, and contains all but finitely many indecomposable modules of $\mathcal{P}^{C_i}$;
- the full translation subquiver $\mathcal{E}_i = (\rightarrow N_i)$ of Γ_A formed by all predecessors of N_i in Γ_A is left stable, acyclic, and contains all but finitely many indecomposable modules of $\mathcal{Q}^{C_i}$.

We note that $\mathcal{D}_i$ and $\mathcal{E}_i$ consist entirely of directing modules, because every cycle in ind A is finite. In particular, $\mathcal{D}_i$ and $\mathcal{E}_i$ have only finitely many τ_A-orbits. In fact, there exists a hereditary algebra H_i of Euclidean type and a tilting module T_i in mod H_i without indecomposable direct summands from the postprojective component $\mathcal{P}^{H_i}$ of Γ_{H_i} such that the tilted algebra $B_i = \mathrm{End}_{H_i}(T_i)$ is a quotient algebra of A and $\mathcal{D}_i$ is the image of $\mathcal{P}^{H_i}$ via the functor $F_i = \mathrm{Ext}^1_{H_i}(T_i, -) : \mathrm{mod}\, H_i \rightarrow \mathrm{mod}\, B_i$ (see the dual of [33, Theorem 2.2]). In particular, the images of the indecomposable projective modules in $\mathcal{P}^{H_i}$ via F_i form a Euclidean section Φ_i in $\mathcal{D}_i$ having the module M_i as a unique source. Similarly, there exists a hereditary algebra H_i^* of Euclidean type and a tilting module T_i^* in mod H_i^* without indecomposable direct summands from the preinjective component $\mathcal{Q}^{H_i^*}$ of $\Gamma_{H_i^*}$ such that the tilted algebra $B_i^* = \mathrm{End}_{H_i^*}(T_i^*)$ is a quotient algebra of A and $\mathcal{E}_i$ is the image of $\mathcal{Q}^{H_i^*}$ via the functor $F_i^* = \mathrm{Hom}_A(T_i^*, -) : \mathrm{mod}\, H_i^* \rightarrow \mathrm{mod}\, B_i^*$ (see [33, Theorem 2.2]). Moreover, the images of the indecomposable injective modules in $\mathcal{Q}^{H_i^*}$ via F_i^* form a Euclidean section Ψ_i in $\mathcal{E}_i$ having the module N_i as a unique sink. By general theory, B_i is a tubular coextension of C_i and B_i^* is a tubular extension of C_i, and hence there are at most finitely many isomorphism classes of modules Z in ind B_i (respectively, in ind B_i^*) with $\mathrm{res}_i(Z) = 0$. Therefore, we may assume that $\mathrm{res}_i(Z) \neq 0$ for all modules X in $\mathcal{D}_i$ and $\mathrm{res}_i(Y) \neq 0$ for all modules Y in

$\mathcal{E}_i$. It follows from [33, Theorem 2] and its dual that all but finitely many acyclic modules of Γ_A (equivalently, directing modules in ind A) belong to the union of translation quivers

$$\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_m \cup \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_m.$$

Fix $i \in \{1, \ldots, m\}$. Let X be a module in ind A which does not belong to $\Omega^{(i)}$. We claim that $\mathrm{res}_i(X)$ belongs to the additive category $\mathrm{add}(\mathcal{P}^{C_i} \cup \mathcal{Q}^{C_i})$. Assume to the contrary that $\mathrm{res}_i(X)$ is nonzero and admits an indecomposable direct summand U from $\mathcal{T}_\lambda^{C_i}$, for some $\lambda \in \Lambda_i$. Observe that then $\mathrm{Hom}_A(U, X) = \mathrm{Hom}_{C_i}(U, \mathrm{res}_i(X)) \neq 0$. Since C_i is a quotient algebra of A, we may consider the largest right C_i-submodule Y of X. Clearly, then $\mathrm{Hom}_A(U, X) = \mathrm{Hom}_{C_i}(U, Y)$, and hence Y is nonzero. Let $Y = V \oplus W$ be a decomposition of Y in $\mathrm{mod}\, C_i$ such that W is a maximal direct summand of Y with all indecomposable direct summands from the family $\mathcal{T}^{C_i}$. Recall that $\Gamma_{C_i} = \mathcal{P}^{C_i} \cup \mathcal{T}^{C_i} \cup \mathcal{Q}^{C_i}$, where $\mathcal{T}^{C_i} = (\mathcal{T}_\varrho^{C_i})_{\varrho \in \Lambda_i}$ strongly separates $\mathcal{P}^{C_i}$ from $\mathcal{Q}^{C_i}$. Since Λ_i' is infinite, we may choose an element $\mu \in \Lambda_i' \setminus \{\lambda\}$ such that $\mathrm{Hom}_{C_i}(R, W) = 0$ for any indecomposable module R in $\mathcal{T}_\mu^{C_i}$. By general theory, the stable tubes $\mathcal{T}_\lambda^{C_i}$ and $\mathcal{T}_\mu^{C_i}$ have common composition factors, that is, there exist indecomposable modules M in $\mathcal{T}_\lambda^{C_i}$ and N in $\mathcal{T}_\mu^{C_i}$ with $[M] = [N]$. We note that then $r_\lambda^{(i)}$ divides $\mathrm{ql}(M)$ and $r_\mu^{(i)}$ divides $\mathrm{ql}(N)$ (see [59, Corollary 4.6]). Moreover, we may choose M and N with $[M] = [N]$ such that $\mathrm{Hom}_{C_i}(M, U) \neq 0$, and consequently $\mathrm{Hom}_A(M, X) \neq 0$ (see [59, Theorem 4.3 and Corollary 4.6]). Let L be an indecomposable module in $\mathcal{Q}^{C_i}$. Then $\tau_{C_i}^{-1} L$ is either 0 or an indecomposable module in $\mathcal{Q}^{C_i}$, and hence $\mathrm{Hom}_{C_i}(\tau_{C_i}^{-1} L, M) = 0$ and $\mathrm{Hom}_{C_i}(\tau_{C_i}^{-1} L, N) = 0$. Then it follows from Proposition 2.1 that $|\mathrm{Hom}_{C_i}(M, L)| = |\mathrm{Hom}_{C_i}(N, L)|$. Clearly, we have $\mathrm{Hom}_{C_i}(M, T) = 0$ and $\mathrm{Hom}_{C_i}(N, T) = 0$ for any indecomposable module T in $\mathcal{P}^{C_i}$. Therefore, we obtain that $|\mathrm{Hom}_{C_i}(M, V)| = |\mathrm{Hom}_{C_i}(N, V)|$. Observe that then $\mathrm{Hom}_A(\tau_A^{-1} X, M) = 0$. Indeed, if $\mathrm{Hom}_A(\tau_A^{-1} X, M) \neq 0$, then we have in ind A a cycle of the form

$$M \to X \to E \to \tau_A^{-1} X \to M,$$

and hence a contradiction, because M belongs to $\Omega_\lambda^{(i)}$, X does not belong to $\Omega_\lambda^{(i)}$, and A is cycle-finite. Applying Proposition 2.1 again, we obtain the equality

$$|\mathrm{Hom}_A(M, X)| = |\mathrm{Hom}_A(N, X)| - |\mathrm{Hom}_A(\tau_A^{-1} X, N)|,$$

and hence $\mathrm{Hom}_A(N, X) \neq 0$. We note that X does not belong to $\mathcal{T}_\mu^{C_i} = \Omega_\mu^{(i)}$, because $\mathrm{Hom}_A(M, X) \neq 0$ and the stable tubes $\mathcal{T}_\lambda^{C_i}$ and $\mathcal{T}_\mu^{C_i}$ are orthogonal. Then we infer as above that $\mathrm{Hom}_A(\tau_A^{-1} X, N) = 0$. Therefore, we have $|\mathrm{Hom}_A(M, X)| = |\mathrm{Hom}_A(N, X)|$. We have also the equalities

$$\begin{aligned} |\mathrm{Hom}_A(M, X)| &= |\mathrm{Hom}_{C_i}(M, Y)| = |\mathrm{Hom}_{C_i}(M, V)| + |\mathrm{Hom}_{C_i}(M, W)|, \\ |\mathrm{Hom}_A(N, X)| &= |\mathrm{Hom}_{C_i}(N, Y)| = |\mathrm{Hom}_{C_i}(N, V)| + |\mathrm{Hom}_{C_i}(N, W)|, \end{aligned}$$

and hence $|\mathrm{Hom}_{C_i}(M, W)| = |\mathrm{Hom}_{C_i}(N, W)| = 0$, by the choice of μ. This is a contradiction because U is a direct summand of W and $\mathrm{Hom}_A(M, U) \neq 0$. Therefore, indeed $\mathrm{res}_i(X)$ is a module in $\mathrm{add}(\mathcal{P}^{C_i} \cup \mathcal{Q}^{C_i})$.

Consider now the faithful algebra $B(\Omega)$ of the cyclic quiver Ω. Then it follows from Theorem 3.6, the theory of generalized multicoil algebras [36], and the assumption imposed on A, that $B(\Omega)$ is a tame generalized multicoil enlargement (not necessarily indecomposable) of the family $C_1, \ldots, C_m$ of all tame concealed quotient algebras of A such that the following statements hold:

- the left quasitilted algebra $B(\Omega)^{(l)}$ of $B(\Omega)$ is a product $B_1^{(l)} \times \cdots \times B_m^{(l)}$ of tilted algebras $B_1^{(l)}, \ldots, B_m^{(l)}$ of Euclidean type such that, for any $i \in \{1, \ldots, m\}$, the translation quiver $\mathcal{D}_i = F_i(\mathcal{P}^{H_i})$ is a cofinite full translation subquiver of the postprojective component $\mathcal{P}^{B_i^{(l)}}$ of $\Gamma_{B_i^{(l)}}$, which is closed under successors in Γ_A;
- the right quasitilted algebra $B(\Omega)^{(r)}$ of $B(\Omega)$ is a product $B_1^{(r)} \times \cdots \times B_m^{(r)}$ of tilted algebras $B_1^{(r)}, \ldots, B_m^{(r)}$ of Euclidean type such that, for any $i \in \{1, \ldots, m\}$, the translation quiver $\mathcal{E}_i = F_i^*(\mathcal{Q}^{H_i^*})$ is a cofinite full translation subquiver of the preinjective component $\mathcal{Q}^{B_i^{(r)}}$ of $\Gamma_{B_i^{(r)}}$, which is closed under predecessors in Γ_A;
- Ω is the cyclic part of the family $\mathcal{C}^{B(\Omega)}$ of generalized multicoils of $\Gamma_{B(\Omega)}$, separating $\mathcal{P}^{B(\Omega)} = \mathcal{P}^{B(\Omega)^{(l)}} = \mathcal{P}^{B_1^{(l)}} \cup \cdots \cup \mathcal{P}^{B_m^{(l)}}$ from $\mathcal{Q}^{B(\Omega)} = \mathcal{Q}^{B(\Omega)^{(r)}} = \mathcal{Q}^{B_1^{(r)}} \cup \cdots \cup \mathcal{Q}^{B_m^{(r)}}$.

Moreover, it follows from the structure of generalized multicoils [35, 36] that, for any $i \in \{1, \ldots, m\}$, there is a quotient algebra B_i of $B(\Omega)$ such that the following statements hold:

- B_i is a generalized multicoil enlargement of the tame concealed algebra C_i;
- $B_i^{(l)}$ is the left quasitilted algebra of B_i;
- $B_i^{(r)}$ is the right quasitilted algebra of B_i;
- $\Omega^{(i)} = \bigcup_{\lambda \in \Lambda_i} \Omega_\lambda^{(i)}$ is the family of all maximal cyclic coherent translation subquivers of the Auslander-Reiten quiver Γ_{B_i} of B_i.

In particular, we obtain that the translation quiver

$$\mathcal{B}_i = \mathcal{D}_i \cup \Omega^{(i)} \cup \mathcal{E}_i$$

is a cofinite full translation subquiver of Γ_{B_i}.

For each $i \in \{1, \ldots, m\}$, we denote by $\mathcal{X}_i$ the full additive subcategory of mod A generated by all indecomposable modules in $\mathcal{B}_i$. Moreover, denote by $\mathcal{X}_0$ the full additive subcategory of mod A generated by the family $\mathcal{B}_0$ of all indecomposable modules in mod A which are not in $\mathcal{B}_1, \ldots, \mathcal{B}_m$. Observe that $\mathcal{B}_0$ contains only finitely many indecomposable modules. In particular, we obtain that $\mathcal{X}_0$ is a functorially finite subcategory of mod A. We will prove now that $\mathcal{X}_1, \ldots, \mathcal{X}_m$ are functorially finite subcategories of mod A.

Fix $i \in \{1, \ldots, m\}$. We denote by R_i the direct sum of all indecomposable modules in $\Omega^{(i)}$ lying on the union $\Sigma^{(i)}$ of the right borders $\Sigma_\lambda^{(i)}$ of the maximal cyclic coherent translation quivers $\Omega_\lambda^{(i)}$, $\lambda \in \Lambda_i''$, if Λ_i'' is not empty, and $R_i = 0$ otherwise. Moreover, we denote by V_i the direct sum of all indecomposable modules in $\mathcal{E}_i$ lying on the Euclidean subquiver Ψ_i. Dually, we denote by L_i the direct sum of all indecomposable modules in $\Omega^{(i)}$ lying on the union $\Delta^{(i)}$ of the left borders $\Delta_\lambda^{(i)}$ of the maximal cyclic coherent translation quivers $\Omega_\lambda^{(i)}$, $\lambda \in \Lambda_i''$, if Λ_i'' is not empty, and $L_i = 0$ otherwise. Finally, we denote by U_i the direct sum of all indecomposable modules in $\mathcal{D}_i$ lying on the Euclidean subquiver Φ_i.

Let X be a module in ind A which does not belong to $\mathcal{X}_i$. Moreover, let Y be an indecomposable module in $\mathcal{X}_i$ and $g : Y \to X$ a nonzero homomorphism in mod A. We will prove that g factors through a module of the form $(R_i \oplus V_i)^s$, for some positive integer s. Let Z be the image of g. Consider a decomposition $Z = Z_1 \bigoplus Z_2$, where Z_1 is a maximal direct summand of Z with $\mathrm{res}_i(Z_1) = 0$, and the homomorphisms $g_1 : Y \to Z_1$ and $g_2 : Y \to Z_2$ induced by g. It follows from the previous considerations that $\mathrm{res}_i(X)$ is a module from $\mathrm{add}(\mathcal{P}^{C_i} \cup \mathcal{Q}^{C_i})$, because X does not belong to $\Omega^{(i)}$. Assume first that Y belongs to $\mathcal{D}_i$. Then Z is a right $B_i^{(l)}$-submodule of X. Moreover, there is a positive integer a such that, for any integer $b \geq a$, g factors through a direct sum of modules lying on the Euclidean subquiver $\tau_A^{-b}\Phi_i$ of $\mathcal{D}_i$. In particular, we conclude that $\mathrm{res}_i(Z_2)$ is a direct sum of indecomposable modules from $\mathcal{Q}^{C_i}$. Then we conclude that g_1 factors through a module R_i^k and g_2 factors through a module V_i^t for some positive integers k and t. Assume now that Y belongs to $\Omega^{(i)}$. Then $\mathrm{res}_i(Z_2)$ is a direct sum of modules lying in $\mathcal{Q}^{C_i}$, because $\mathrm{Hom}_{B_i}(\Omega^{(i)}, \mathcal{P}^{B_i^{(l)}}) = 0$, and hence Z_2 is a right $B_i^{(r)}$-module from the additive category of $\mathcal{Q}^{B_i^{(r)}}$. Thus we conclude that g_2 factors through a module V_i^t for some positive integer t. Clearly, we have $\mathrm{res}_i(g_1) = 0$. Assume $g_1 \neq 0$. Then Y belongs to $\Omega_\lambda^{(i)}$ for some $\lambda \in \Lambda_i''$. But then g_1 factors through a module of the form $R_i^k \oplus V_i^t$ for some positive integers k and t. Finally, assume that Y belongs to $\mathcal{E}_i$. Obviously then g factors through a module V_i^t for some positive integer t. Summing up, we conclude that g factors through a module of the form $(R_i \oplus V_i)^s$ for some positive integer s. Moreover, $\mathrm{Hom}_A(R_i \oplus V_i, X)$ is a finitely generated module over the commutative artin ring K. Then we conclude that the restriction $\mathrm{Hom}_A(-, X)|_{\mathcal{X}_i}$ of the contravariant functor $\mathrm{Hom}_A(-, X) : (\mathrm{mod}\, A)^{\mathrm{op}} \to \mathcal{A}b$ to $\mathcal{X}_i$ is a finitely generated functor, being an epimorphic image of a contravariant functor $\mathrm{Hom}_A\big(-, (R_i \oplus V_i)^p\big) : \mathcal{X}_i^{\mathrm{op}} \to \mathcal{A}b$, for some positive integer p. Therefore, $\mathcal{X}_i$ is a contravariantly finite subcategory of $\mathrm{mod}\, A$. Dually, we conclude that every nonzero homomorphism $f : X \to Y$ in $\mathrm{mod}\, A$ with Y an indecomposable module in $\mathcal{X}_i$ factors through a module of the form $(L_i \oplus U_i)^t$, for some positive integer t. Moreover, $\mathrm{Hom}_A(X, L_i \oplus U_i)$ is a finitely generated module over the commutative artin ring K. Then we conclude that the restriction $\mathrm{Hom}_A(X, -)|_{\mathcal{X}_i}$ of the covariant functor $\mathrm{Hom}_A(X, -) : \mathrm{mod}\, A \to \mathcal{A}b$ to $\mathcal{X}_i$ is a finitely generated functor, being an epimorphic image of a covariant functor $\mathrm{Hom}_A\big((L_i \oplus U_i)^q, -\big) : \mathcal{X}_i \to \mathcal{A}b$, for some positive integer q. Hence, $\mathcal{X}_i$ is a covariantly finite subcategory of $\mathrm{mod}\, A$. Therefore, we proved that $\mathcal{X}_i$ is a functorially finite subcategory of $\mathrm{mod}\, A$. Finally, since $\mathcal{X}_i$ is a full subcategory of $\mathrm{mod}\, B_i$ containing all but finitely many modules of $\mathrm{ind}\, B_i$, we obtain that $\mathrm{mod}\, B_i$ is also a functorially finite subcategory of $\mathrm{mod}\, A$.

Summing up, we conclude that $\mathrm{mod}\, B_1, \ldots, \mathrm{mod}\, B_n$ are functorially finite subcategories of $\mathrm{mod}\, A$ and all but finitely many modules in $\mathrm{ind}\, A$ belong to $\bigcup_{i=1}^m \mathrm{ind}\, B_i$. This finishes the proof of Theorem 1.6.

5 Proof of Theorem 1.2

The implications (v) ⇒ (vi), (iii) ⇒ (vi), (i) ⇒ (vi) follow from Propositions 3.10, 3.11, 3.13, respectively. The implications (ii) ⇒ (i) and (iv) ⇒ (iii) are obvious. Hence, it remains to prove that (vi) implies (ii), (iv), (v).

Let A be a cycle-finite algebra of infinite representation type and without a tubular quotient algebra. Let $C_1, \ldots, C_m$ be the family of all tame concealed quotient algebras of A. We keep the notation introduced in Section 4.

It follows from Proposition 3.4, and the assumption imposed on A, that the stable tubes $\Omega_\lambda^{(i)} = \mathcal{T}_\lambda^{C_i}$, for $i \in \{1, \ldots, m\}$ and $\lambda \in \Lambda_i'$, form the family of all regular components of Γ_A. Moreover, all but finitely many stable tubes in this family have rank one. Since the number of nonregular components of Γ_A is finite, we conclude that all but finitely many components of Γ_A are stable tubes of rank one. Hence, (vi) implies (v).

We know from Section 4 that there exist domestic generalized multicoil enlargements $B_1, \ldots, B_m$ of $C_1, \ldots, C_m$, respectively, satisfying the the following properties:

- for each $i \in \{1, \ldots, m\}$, the translation quiver $\mathcal{B}_i = \mathcal{D}_i \cup \Omega^{(i)} \cup \mathcal{E}_i$ is a cofinite full translation subquiver of Γ_{B_i};
- the family $\mathcal{B}_0$ of all modules in ind A which do not belong to $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$ is finite;
- for each $i \in \{0, 1, \ldots, m\}$, the additive category $\mathcal{X}_i$ of mod A generated by the indecomposable modules in $\mathcal{B}_i$ is contravariantly finite.

We will show now that $KG(A) = 2$, and hence (vi) implies (ii). Observe first that the projective cover of any functor in $\mathcal{F}(A)$ is a functor $\mathrm{Hom}_A(-, M)$ for a module M in mod A. Hence it is enough to consider contravariant Hom-functors. It follows from Proposition 3.14 that $KG(B_i) = 2$, and hence $KG(\mathcal{X}_i) = 2$ for any $i \in \{1, \ldots, m\}$. Clearly, we have $KG(\mathcal{X}_0) = 0$. Take now an indecomposable module X in ind A and the associated contravariant functor $\mathrm{Hom}_A(-, X)$ from mod A to $\mathcal{A}b$. Since $\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_m$ are contravariantly finite subcategories of mod A, generating the whole category mod A, we conclude that F becomes of finite length in the quotient category $\mathcal{F}(A)/\mathcal{F}(A)_1$. This shows that $KG(A) \leq 2$. On the other hand, we have $2 = KG(\mathcal{B}_i) \leq KG(A)$, for any $i \in \{1, \ldots, m\}$. Therefore, indeed $KG(A) = 2$.

Finally, we prove that rad_A^{∞} is nilpotent, and hence (vi) implies (iv). Let W_0 be the direct sum of all indecomposable modules in $\mathcal{X}_0$. Then $\mathrm{End}_A(W_0)$ is an artin algebra, and hence the radical of $\mathrm{End}_A(W_0)$ is nilpotent, say $(\mathrm{rad}\,\mathrm{End}_A(W_0))^l = 0$ for some positive integer l. Let n be a natural number such that $n \geq l + 4m$. We claim that $(\mathrm{rad}_A^{\infty})^n = 0$. Suppose it is not the case. Then there exists a sequence

$$Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \to \cdots \to Z_{n-1} \xrightarrow{f_n} Z_n$$

of homomorphisms in rad_A^{∞} between modules in ind A such that $f_n \ldots f_2 f_1 \neq 0$. It follows from the choice of n that there are $p < q < r < s$ in $\{0, 1, \ldots, n\}$ such that Z_p, Z_q, Z_r, Z_s belong to $\mathcal{X}_i$, for some $i \in \{1, \ldots, m\}$. Let $u = f_q \ldots f_{p+1} : Z_p \to Z_q$, $v = f_r \ldots f_{q+1} : Z_q \to Z_r$, $w = f_s \ldots f_{r+1} : Z_r \to Z_s$. Since $\mathcal{X}_i$ is a subcategory of mod B_i, the homomorphisms u, v, w belong to $\mathrm{rad}_{B_i}^{\infty}$, and hence wvu is a nonzero homomorphism in $(\mathrm{rad}_{B_i}^{\infty})^3$. This contradicts Proposition 3.12. Therefore, rad_A^{∞} is nilpotent.

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