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First Space Cohomology of the Orthosymplectic Lie Superalgebra in the Lie Superalgebra of Superpseudodifferential Operators

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Abstract We investigate the first cohomology space associated with the embedding of the Lie Orthosymplectic superalgebra $\mathfrak{osp}(n|2)$ on the (1,n)-dimensional superspace $\mathbb{R}^{1|n}$ in the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}(n)$ of superpseudodifferential operators with smooth coefficients, where $n = 0, 1, 2$. Following Ovsienko and Roger, we give erxplicit expressions of the basis cocycles.

Keywords Cohomology · Orthosymplectic superalgebra · Superpseudodifferential operators · Poisson superalgebra

Mathematics Subject Classification (2010) 53D55

1 Introduction

Let $\mathbb{R}^{1|n}$ be the superspace with coordinates $(x, \theta_1, \dots, \theta_n)$, where x is an even indeterminate and $\theta_1, \dots, \theta_n$ are odd indeterminates: $\theta_i\theta_j = -\theta_j\theta_i$. This superspace is equipped with the standard contact structure given by the distribution $D = \langle\overline{\eta}_1, \dots, \overline{\eta}_n\rangle$ generated by the vector fields $\overline{\eta}_i = \partial_{\theta_i} - \theta_i\partial_x$. That is, the distribution D is the kernel of the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i.$$

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Consider the superspace $C^\infty(\mathbb{R}^{1|n})$ which is the space of functions F of the form:

$$F = \sum_{1\le i_1<\cdots<i_k\le n} f_{i_1,\dots,i_k}(x)\theta_{i_1}\cdots\theta_{i_k} \quad \text{where} \quad f_{i_1,\dots,i_k} \in C^\infty(\mathbb{R}). \tag{1.1}$$

Of course, even (respectively odd) elements in $C^\infty(\mathbb{R}^{1|n})$ are the functions (1.1) for which the summation is only over even (respectively odd) integer k. Note $p(F)$ the parity of a homogeneous function F. On $C^\infty(\mathbb{R}^{1|n})$, we consider the contact bracket

$$\langle F, G\rangle = FG' - F'G - \frac{1}{2}(-1)^{p(F)}\sum_{i=1}^{n}\overline{\eta}_i(F)\overline{\eta}_i(G) \tag{1.2}$$

where the superscript $'$ stands for $\frac{\partial}{\partial x}$. Consider the superspace $\mathcal{K}(n)$ of contact vector fields on $\mathbb{R}^{1|n}$. That is, $\mathcal{K}(n)$ is the superspace of vector fields on $\mathbb{R}^{1|n}$ preserving the distribution $\langle\overline{\eta}_1, \dots, \overline{\eta}_n\rangle$:

$$\mathcal{K}(n) = \left\{X \in \mathrm{Vect}(\mathbb{R}^{1|n}) \mid [X, \overline{\eta}_i] = F_X\overline{\eta}_i \quad \text{for some } F_X \in C^\infty(\mathbb{R}^{1|n})\right\}.$$

The Lie superalgebra $\mathcal{K}(n)$ is spanned by the vector fields of the form:

$$X_F = F\partial_x - \frac{1}{2}(-1)^{p(F)}\sum_{i=1}^{n}\overline{\eta}_i(F)\overline{\eta}_i, \quad \text{where} \quad F \in C^\infty(\mathbb{R}^{1|n}).$$

The vector field X_F has the same parity as F. The bracket in $\mathcal{K}(n)$ can be written as:

$$[X_F,\ X_G] = X_{\langle F,G\rangle}.$$

Now, consider the 1-parameter action of $\mathcal{K}(n)$ on $C^\infty(\mathbb{R}^{1|n})$ given by the rule:

$$\mathfrak{L}^\lambda_{X_F} = X_F + \lambda F'.$$

We denote this $\mathcal{K}(n)$-module by $\mathfrak{F}^n_\lambda$, the space of all weighted densities on $\mathbb{R}^{1|n}$ of weight λ:

$$\mathfrak{F}^n_\lambda = \left\{F\alpha_n^\lambda \mid F \in C^\infty(\mathbb{R}^{1|n})\right\}.$$

The superspace of the supercommutative algebra of superpseudodifferential symbols on $\mathbb{R}^{1|n}$ with its natural multiplication is spanned by the series

$$\mathcal{SP}(n) = \left\{F = \sum_{k\ge -M}\sum_{\epsilon=(\epsilon_1,\dots,\epsilon_n)} a_{k,\epsilon}(x,\theta)\xi^{-k}\bar{\theta}_1^{\epsilon_1}\dots\bar{\theta}_n^{\epsilon_n} : a_{k,\epsilon} \in C^\infty(\mathbb{R}^{1|n}); \epsilon_i = 0, 1;\ M \in \mathbb{N}\right\}, \tag{1.3}$$

where ξ corresponds to ∂_x and $\bar{\theta}_i$ corresponds to ∂_{θ_i} $(p(\bar{\theta}_i) = 1)$.

The space $\mathcal{SP}(n)$ has a structure of the Poisson Lie superalgebra given by the following bracket :

$$\{F,\ G\} = \frac{\partial(F)}{\partial\xi}\frac{\partial(G)}{\partial x} - \frac{\partial(F)}{\partial x}\frac{\partial(G)}{\partial\xi} - (-1)^{p(F)}\sum_{i=1}^{n}\Big(\frac{\partial(F)}{\partial\theta_i}\frac{\partial(G)}{\partial\bar{\theta}_i} + \frac{\partial(F)}{\partial\bar{\theta}_i}\frac{\partial(G)}{\partial\theta_i}\Big). \tag{1.4}$$

The associative superalgebra of superpseudodifferential operators $\mathcal{S}\Psi\mathcal{DO}(n)$ over $\mathbb{R}^{1|n}$ has the same underlying vector space as $\mathcal{SP}(n)$ but the multiplication is now defined by the following rule:

$$F \circ G = \sum_{k\geq 0,\ \nu_i=0,\ 1} \frac{(-1)^{p(F)+1}}{k!}(\partial_\xi^k \partial_{\bar{\theta}_i}^{\nu_i} F)(\partial_x^k \partial_{\theta_i}^{\nu_i} G). \tag{1.5}$$

This composition rule induces the supercommutator defined by:

$$[F,\ G] = F \circ G - (-1)^{p(F)p(G)} G \circ F. \tag{1.6}$$

Ovsienko and Roger [10] calculated the space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{vect}(S^1), \Psi\mathcal{DO})$. They used the results of Fuchs [7] on $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{vect}(S^1), \mathcal{F}_\lambda)$. To compute $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{vect}(S^1), \Psi\mathcal{DO})$ Ovsienko and Roger applied the theory of spectral seqences to a filtered module over a Lie algebra.

In paper [1, 2], using the same methods as in the paper [10] the authors computed $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1), \mathcal{S}\Psi\mathcal{DO}(1))$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(2), \mathcal{S}\Psi\mathcal{DO}(2))$.

In this paper, we restrict ourselves to the cases $n = 0,\ 1,\ 2$ and we restrict the action to the orthosymplectic Lie (super)algebra $\mathfrak{osp}(n|2)$ and we consider the spaces $\mathcal{SP}(n)$ as $\mathfrak{osp}(n|2)$-modules. Of course, the case $n = 0$ corresponds to the classical setting: $\mathcal{K}(0) = \mathfrak{vect}(\mathbb{R})$ and the corresponding orthosymplectic Lie algebra $\mathfrak{osp}(0|2)$ is nothing but the classical Lie algebra $\mathfrak{sl}(2)$ which is isomorphic to the Lie subalgebra of $\mathfrak{vect}(\mathbb{R})$ generated by

$$\mathfrak{sl}(2) = \mathrm{Span}\left(\frac{d}{dx},\ x\frac{d}{dx},\ x^2\frac{d}{dx}\right),$$

$\mathcal{SP}(0)$ is the classical spaces of symbols, usually denoted

$$\mathcal{P} = \left\{F(x,\xi),\ F(x,\xi) = \sum_{-\infty}^{m} f_k(x)\xi^k\right\}. \tag{1.7}$$

and $\mathcal{S}\Psi\mathcal{DO}(0)$ is the classical associative algebra of pseudodifferential operators, usually denoted $\Psi\mathcal{DO}$.

The Lie superalgebra $\mathfrak{osp}(1|2)$ can realized as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2) = \mathrm{Span}(X_1,\ X_x,\ X_{x^2},\ X_{x\theta_1},\ X_{\theta_1}).$$

While, the Lie superalgebra $\mathfrak{osp}(2|2)$ is realized as a subalgebra of $\mathcal{K}(2)$:

$$\mathfrak{osp}(2|2) = \mathrm{Span}(X_1,\ X_x,\ X_{x^2},\ X_{x\theta_1},\ X_{x\theta_2},\ X_{\theta_1},\ X_{\theta_2},\ X_{\theta_1\theta_2}).$$

Obviously, the Lie superalgebra $\mathfrak{osp}(n-1|2)$ can be considered as a subalgebra of $\mathfrak{osp}(n|2)$, therefore, the spaces of symbols $\mathcal{SP}(n)$ are also $\mathfrak{osp}(n-1|2)$-modules.

We compute the cohomology spaces $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}(n))$ and $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{S}\Psi\mathcal{DO}(n))$, where $n = 0,\ 1,\ 2$. We show that non vanishing spaces . To be more precise, let $d_n = \dim(\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{osp}(n|2), \mathcal{SP}(n))) = \dim(\mathrm{H}^1\ (\mathfrak{osp}(n|2), \mathcal{S}\Psi\mathcal{DO}(n))$, we show that

$$d_0 = d_1 = 2$$

$$d_2 = 6.$$

These cohomology spaces are closely related to the deformation theory, see e.g. [4–6, 9, 10]. These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{osp}(n|2)$-modules. We hope to be able to describe in the future all the deformations of these modules $\mathcal{S}\Psi\mathcal{DO}(n)$.

2 $\mathfrak{osp}(n|2)$– Modules

2.1 $\mathfrak{sl}(2)$-module

Consider the space $\mathbb{C}[\xi, \xi^{-1}]]$ of (formal) Laurent series of finite order in ξ:

$$a(\xi) = \sum_{-\infty}^{m} a_k \xi^k.$$

We put

$$\mathcal{P} := \mathcal{C}^\infty(\mathbb{R}) \otimes \mathbb{C}[\xi, \xi^{-1}]],$$

with its natural multiplication and the Poisson bracket defined as follows. Any element of $\mathcal{P}$ can be written in the following form:

$$F(x, \xi) = \sum_{-\infty}^{m} f_k(x)\xi^k,$$

where $f_k(x) \in \mathcal{C}^\infty(\mathbb{R})$, then for $F, G \in \mathcal{P}$,

$$\{F, G\} = \frac{\partial F}{\partial \xi}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial \xi}.$$

The associative algebra of pseudodifferential symbols $\Psi\mathcal{DO}$ has the same underlying vector space, but the multiplication is now defined by the following formula

$$F \circ G = \sum_{k \geq 0} \frac{1}{k!}\frac{\partial^k F}{\partial \xi^k}\frac{\partial^k G}{\partial x^k}. \tag{2.8}$$

As usual, one can consider this associative algebra $\Psi D\mathcal{O}$ as a Lie algebra with the commutator

$$[F, G] = F \circ G - G \circ F.$$

The order of pseudodifferential operators defines a natural filtration on $\Psi D\mathcal{O}$. Recall, that the order is defined by $\mathrm{ord}(F) = \sup\{k \in \mathbb{Z}/f_k(x) \neq 0\}$, for every $F(x, \xi) = \sum_{k\in\mathbb{Z}} f_k(x)\xi^k$. One sets $\mathbf{F}_n = \{F \in \Psi D\mathcal{O}| \quad \mathrm{ord}(F) \leq -n\}$, where $n \in \mathbb{Z}$. Thus, one has a decreasing filtration [10],

$$\cdots \subset \mathbf{F}_{n+1} \subset \mathbf{F}_n \subset \cdots , \tag{2.9}$$

compatible with multiplication, if $F \in \mathbf{F}_n$ and $G \in \mathbf{F}_m$, then $F \circ G \in \mathbf{F}_{n+m}$, and $\{F, G\} \in \mathbf{F}_{n+m-1}$ (check the symbolic terms!). This filtration makes $\Psi D\mathcal{O}$ an associative filtered algebra, one can consider, as usual, the associated graded algebra. Each quotient space $\mathbf{F}_n/\mathbf{F}_{n+1}$ is canonically isomorphic to $\mathcal{C}^\infty(\mathbb{R})$, any function

$f \in \mathcal{C}^\infty(\mathbb{R})$ induces a pseudodifferential symbol $f\xi^{-n}$ which has a well-defined image in $\mathbf{F}_n/\mathbf{F}_{n+1}$. The associated graded algebra is then $\mathrm{Gr}(\Psi DO) = \widetilde{\bigoplus}_{n\in\mathbb{Z}}\mathbf{F}_n/\mathbf{F}_{n+1}$, where, $\widetilde{\bigoplus_{n\in\mathbb{Z}}} = (\bigoplus_{n<0}) \oplus \prod_{n\geq 0}$.

The defined filtration is also a filtration of $\Psi\mathcal{DO}$ as a $\mathfrak{vect}(1)$-module (i.e. compatible with the natural action of $\mathfrak{vect}(1)$ on $\Psi\mathcal{DO}$). Indeed, if $X \in \mathfrak{vect}(1)$ and $F \in \mathbf{F}_n(\Psi DO)$, then $X.F = [X, F] \in \mathbf{F}_n$. One induces an action of $\mathfrak{vect}(1)$ on the associative algebra $Gr(\Psi DO) = \mathcal{P}$ and a simple computation shows that we recover the canonical action of $\mathfrak{vect}(1)$ on the Poisson algebra $\mathcal{P}$ [10]. More explicitly, as a $\mathfrak{vect}(1)$-module, $\mathbf{F}_n/\mathbf{F}_{n+1} = \mathcal{F}_n$, where $\mathcal{F}_n$ is the space of tensor densities of degree n on $\mathbb{R}$:

$$\mathcal{F}_n = \{f(x)dx^n | f \in \mathcal{C}^\infty(\mathbb{R})\}$$

and the action of $\mathfrak{vect}(1)$ reads

$$L^n_{X_g}(fdx^n) = (gf' + ng'f)dx^n.$$

Note, that the expression in the right hand side is just the standard Lie derivative of a tensor density along a vector field. Here, we restrict ourselves to the subalgebra $\mathfrak{sl}(2)$, thus we obtain a $\mathfrak{sl}(2)$-modules still denoted by $\mathcal{P}$.

2.2 $\mathfrak{osp}(1|2)$-module

The superspace of the supercommutative algebra of superpseudodifferential symbols on $\mathbb{R}^{1|1}$ with its natural multiplication is spanned by the series(see [1])

$$\mathcal{SP}(1) = \left\{F = \sum_{k=-M}^{\infty} a_{k,\epsilon}(x,\theta_1)\xi^{-k}\bar{\theta}_1^{\,\epsilon} : \; a_{k,\epsilon} \in C^\infty(\mathbb{R}^{1|1});\; \epsilon = 0,\, 1;\; M \in \mathbb{N}\right\}, \quad (2.10)$$

where ξ corresponds to ∂_x and $\bar{\theta}_1$ corresponds to ∂_{θ_1} $(p(\bar{\theta}_1) = 1)$.

The space $\mathcal{SP}(1)$ has a structure of the Poisson Lie superalgebra given by the following bracket (cf, [8]):

$$\{F,\; G\} = \frac{\partial(F)}{\partial\xi}\frac{\partial(G)}{\partial x} - \frac{\partial(F)}{\partial x}\frac{\partial(G)}{\partial\xi} - (-1)^{p(F)}\Big(\frac{\partial(F)}{\partial\theta_1}\frac{\partial(G)}{\partial\bar{\theta}_1} + \frac{\partial(F)}{\partial\bar{\theta}_1}\frac{\partial(G)}{\partial\theta_1}\Big). \quad (2.11)$$

The space $\mathcal{SP}(1)$ has a structure of associative superalgebra defined by the following rule:

$$F \circ G = \sum_{k\geq 0,\, \nu=0,\,1} \frac{(-1)^{p(F)+1}}{k!}(\partial^k_\xi \partial^\nu_{\bar{\theta}_1} F)(\partial^k_x \partial^\nu_{\theta_1} G). \quad (2.12)$$

This composition rule induces the supercommutator defined by:

$$[F,\; G] = F \circ G - (-1)^{p(F)p(G)} G \circ F. \quad (2.13)$$

The natural embedding of $\mathcal{K}(1)$ into $\mathcal{SP}(1)$ given by $\mathcal{K}(1)$:

$$\pi : X_F \longmapsto F\xi - \frac{(-1)^{p(F)}}{2}\bar{\eta}_1(F)\bar{\zeta}_1. \quad (2.14)$$

where $\bar{\zeta}_1 = \bar{\theta}_1 - \theta_1\xi$, induces a $\mathcal{K}(1)$-module structure on it.

Setting the degree of x, θ_1 be zero and the degree of ξ, ζ_1 be 1 we introduce a $\mathbb{Z}$-grading in the Poisson superalgebra $\mathcal{SP}(1)$ which will be simply denoted $\mathcal{SP}(1)$. Then we have

$$\mathcal{SP}(1) = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathcal{SP}_n(1) := \left(\bigoplus_{n<0} \mathcal{SP}_n(1)\right) \oplus \left(\prod_{n\geq 0} \mathcal{SP}_n(1)\right), \tag{2.15}$$

where $\mathcal{SP}_n(1) = \{F\xi^{-n} + G\xi^{-n-1}\zeta_1 \ : \ F, G \in C^\infty(\mathbb{R}^{1|1})\}$ is the homogeneous subspace of degree $-n$.

Each element of $\mathcal{S}\Psi\mathcal{DO}(1)$ can be expressed as

$$F = \sum_{k\in\mathbb{Z}} (F_k + G_k \eta_1^{-1})\eta_1^{2k}, \ \text{ where } \ F_k, \ G_k \in C^\infty(\mathbb{R}^{1|1}). \tag{2.16}$$

We define the order of F to be

$$\mathrm{ord}(F) = \sup\{k \ : \ F_k(x,\theta) \neq 0 \text{ or } G_k(x,\theta) \neq 0\}. \tag{2.17}$$

This definition of order equips $\mathcal{S}\Psi\mathcal{DO}(1)$ with a decreasing filtration as follows: set

$$\mathbf{F}_n = \{F \in \mathcal{SP}(1) \ : \ \mathrm{ord}(F) \leq -n\}, \ \text{ where } n \in \mathbb{Z}. \tag{2.18}$$

So one has (see [1])

$$\ldots \subset \mathbf{F}_{n+1} \subset \mathbf{F}_n \subset \ldots. \tag{2.19}$$

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for $F \in \mathbf{F}_n$ and $G \in \mathbf{F}_m$, one has $F \circ G \in \mathbf{F}_{n+m}$ and $\{F, G\} \in \mathbf{F}_{n+m-1}$, after as a vector spaces we identify $\mathcal{SP}(1)$ with $\mathcal{S}\Psi\mathcal{DO}(1)$. This filtration makes $\mathcal{SP}(1)$ an associative filtered superalgebra. Moreover, this filtration is compatible with the natural action of $\mathcal{K}(1)$ on $\mathcal{SP}(1)$. Indeed, if $X_F \in \mathcal{K}(1)$ and $G \in \mathbf{F}_n$, then

$$X_F \cdot G = [X_F, G] \in \mathbf{F}_n. \tag{2.20}$$

The induced $\mathcal{K}(1)$-action on the quotient $\mathbf{F}_n/\mathbf{F}_{n+1}$ is isomorphic to the $\mathcal{K}(1)$-action on $\mathcal{SP}_n$. Therefore, the $\mathcal{K}(1)$-action on the associated graded space of the filtration (2.24), is isomorphic to the graded $\mathcal{K}(1)$-module $\mathcal{SP}(1)$, that is (see [1])

$$\mathcal{SP}(1) \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathbf{F}_n/\mathbf{F}_{n+1}. \tag{2.21}$$

If we restrict ourselves to the Lie subalgebra $\mathfrak{osp}(1|2)$ of $\mathcal{K}(1)$, we get a family of infinite dimensional $\mathfrak{osp}(1|2)$ modules, still denoted $\mathfrak{F}^1_\lambda$ and $\mathcal{SP}(1)$.

2.3 $\mathfrak{osp}(2|2)$-module

Let $\mathcal{K}(2)$ be the Lie superagebra of vector fields on $\mathbb{R}^{1|2}$. The natural embedding of $\mathcal{K}(2)$ into $\mathcal{SP}(2)$ defined by

$$\pi(X_F) = F\xi + \frac{(-1)^{p(F)+1}}{2} \sum_{i=1}^{2} \bar{\eta}_i(F)\bar{\zeta}_i, \ \text{ where } \bar{\zeta}_i = \bar{\theta}_i - \theta_i\xi, \tag{2.22}$$

induces a $\mathcal{K}(2)$-module structure on $\mathcal{SP}(2)$.

Setting $\deg x = \deg \theta_i = 0$, $\deg \xi = \deg \bar{\theta}_i = 1$ for all i, we endow the Poisson superalgebra $\mathcal{SP}(2)$ with a $\mathbb{Z}$-grading:

$$\mathcal{SP}(2) = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathcal{SP}_n(2), \tag{2.23}$$

where $\widetilde{\bigoplus}_{n\in\mathbb{Z}} = (\bigoplus_{n<0}) \oplus \prod_{n\geq 0}$ and

$$\mathcal{SP}_n(2) = \left\{ F\xi^{-n} + G\xi^{-n-1}\bar{\theta}_1 + H\xi^{-n-1}\bar{\theta}_2 + T\xi^{-n-2}\bar{\theta}_1\bar{\theta}_2 \mid F,\ G,\ H,\ T \in C^\infty(\mathbb{R}^{1|2}) \right\}$$

is the homogeneous subspace of degree $-n$. Each element of $\mathcal{S}\Psi\mathcal{DO}(2)$ can be expressed as

$$A = \sum_{k\in\mathbb{Z}} (F_k + G_k\xi^{-1}\bar{\theta}_1 + H_k\xi^{-1}\bar{\theta}_2 + T_k\xi^{-2}\bar{\theta}_1\bar{\theta}_2)\xi^{-n},$$

where $F_k,\ G_k,\ H_k,\ T_k \in C^\infty(\mathbb{R}^{1|2})$. We define the *order* of A to be

$$\begin{aligned} \operatorname{ord}(A) = \sup\{k \mid\ & F_k(x,\theta_1,\theta_2) \neq 0 \text{ or } G_k(x,\theta_1,\theta_2) \neq 0 \text{ or } H_k(x,\theta_1,\theta_2) \\ & \neq 0 \text{ or } T_k(x,\theta_1,\theta_2) \neq 0\}. \end{aligned}$$

This definition of order equips $\mathcal{S}\Psi\mathcal{DO}(2)$ with a decreasing filtration as follows: set

$$\mathbf{F}_n = \{A \in \mathcal{S}\Psi\mathcal{DO}(2),\ \operatorname{ord}(A) \leq -n\},$$

where $n \in \mathbb{Z}$. So one has

$$\ldots \subset \mathbf{F}_{n+1} \subset \mathbf{F}_n \subset \ldots \tag{2.24}$$

This filtration is compatible with the multiplication and the Poisson bracket, that is, for $A \in \mathbf{F}_n$ and $B \in \mathbf{F}_m$, one has $A \circ B \in \mathbf{F}_{n+m}$ and $\{A, B\} \in \mathbf{F}_{n+m-1}$. This filtration makes $\mathcal{S}\Psi\mathcal{DO}(2)$ an associative filtered superalgebra. Consider the associated graded space

$$\operatorname{Gr}(\mathcal{S}\Psi\mathcal{DO}(2)) = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathbf{F}_n/\mathbf{F}_{n+1}.$$

The filtration (2.24) is also compatible with the natural action of $\mathcal{K}(2)$ on $\mathcal{S}\Psi\mathcal{DO}(2)$. Indeed, if $X_F \in \mathcal{K}(2)$ and $A \in \mathbf{F}_n$, then

$$X_F.A = [X_F, A] \in \mathbf{F}_n.$$

The induced $\mathcal{K}(2)$-module on the quotient $\mathbf{F}_n/\mathbf{F}_{n+1}$ is isomorphic to the $\mathcal{K}(2)$-module $\mathcal{SP}_n(2)$. Therefore, the $\mathcal{K}(2)$-module $Gr(\mathcal{S}\Psi\mathcal{DO}(2))$, is isomorphic to the graded $\mathcal{K}(2)$-module $\mathcal{SP}(2)$, that is

$$\mathcal{SP}(2) \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathbf{F}_n/\mathbf{F}_{n+1}.$$

If we restrict ourselves to the Lie subalgebra $\mathfrak{osp}(2|2)$ of $\mathcal{K}(2)$, we get a family of infinite dimensional $\mathfrak{osp}(2|2)$ modules, still denoted $\mathfrak{F}_\lambda^2$ and $\mathcal{SP}(2)$.

Recall that a C^∞ function on $\mathbb{R}^{1|2}$ has the form $F = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2$ with $f_0, f_1, f_2, f_{12} \in C^\infty(\mathbb{R})$ and a C^∞ function on $\mathbb{R}_i^{1|1} (i = 1, 2)$, where $\mathbb{R}_i^{1|1}$ is the superligne with local coordinates (x, θ_i), has the form $F = f_0 + f_i\theta_i$ ($f_{12} = f_{3-i} = 0$) with $f_0, f_i \in C^\infty(\mathbb{R})$. Then the Lie superalgebra $\mathfrak{osp}(2|2)$ has two subsuperalgebras $\mathfrak{osp}(1|2)_i$ for $i = 1, 2$ isomorphic to $\mathfrak{osp}(1|2)$ defined by

$$\mathfrak{osp}(1|2)_i = \mathrm{Span}(X_1,\ X_x,\ X_{x^2},\ X_{x\theta_i},\ X_{\theta_i}).$$

Therefore, $\mathcal{SP}(2)$ and $\mathfrak{F}_\lambda^i$ are $\mathfrak{osp}(1|2)_i$-modules.

For $i = 1, 2$, let $\mathfrak{F}_\lambda^1$ be the $\mathfrak{osp}(1|2)_i$-module of weighted densities of weight λ on $\mathbb{R}_i^{1|1}$.

3 Cohomology

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra acting on a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The space of k-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^k(\mathfrak{g}, V) := \mathrm{Hom}(\Lambda^k\mathfrak{g}; V).$$

The *coboundary operator* $\delta_k : C^k(\mathfrak{g}, V) \longrightarrow C^{k+1}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{k-1} = 0$. The kernel of δ_k, denoted $Z^k(\mathfrak{g}, V)$, is the space of *k-cocycles*, among them, the elements in the range of δ_{k-1} are called *k-coboundaries*. We denote $B^k(\mathfrak{g}, V)$ the space of k-coboundaries.

By definition, the k^{th} cohomology space is the quotient space

$$\mathrm{H}^k(\mathfrak{g}, V) = Z^k(\mathfrak{g}, V)/B^k(\mathfrak{g}, V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0, 1 and 2: for $v \in C^0(\mathfrak{g}, V) = V$, $\delta v(x) := (-1)^{p(x)p(v)}x \cdot v$, for $\Upsilon \in C^1(\mathfrak{g}, V)$,

$$\delta(\Upsilon)(x,\ y) := (-1)^{p(x)p(\Upsilon)}x \cdot \Upsilon(y) - (-1)^{p(y)(p(x)+p(\Upsilon))}y \cdot \Upsilon(x) - \Upsilon([x,\ y]) \quad (3.25)$$

In this paper, we study the differential cohomology spaces $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{SP}(n))$ and $\mathrm{H}^1(\mathfrak{osp}(n|2), \mathcal{S}\Psi\mathcal{DO}(n))$, where $n = 0,\ 1,\ 2$.

3.1 The Spectral Sequence for a Filtered Module Over a Lie (super)algebra

Let $\mathfrak{g}$ be a Lie (super)algebra and M a filltered module with decreasing filtration $\{M_n\}_{n\in\mathbb{Z}}$ so that $M_{n+1} \subset M_n$, $\cup_{n\in\mathbb{Z}} M_n = M$ and $\mathfrak{g}.M_n \subset M_n$. Let

$$Gr(M) = \widetilde{\bigoplus}_{n\in\mathbb{Z}} M_n/M_{n+1}$$

be the associated graded $\mathfrak{g}$-module and $Gr^n(M) = M_n/M_{n+1}$. One can then naturally construct a filtration on the space of cochains by setting $F_n(C^*(\mathfrak{g}; M)) = C^*(\mathfrak{g}; M_n)$, this filtration being obviously compatible with the Chevalley-Eilenberg differential. Then we have: $dF_n(C^*(\mathfrak{g}, M)) \subset F_n(C^*(\mathfrak{g}, M))$ (i.e., the filtration is preserved by d); $F_{n+1}(C^*(\mathfrak{g}, M)) \subset F_n(C^*(\mathfrak{g}, M))$ (i.e. the filtration is decreasing).

Hence there is a spectral sequence $(E_k^{*,*}, d_k)$ for $k \in \mathbb{N}$ with dr of degree $(k, 1-k)$ and

$$E_0^{p,q} = F_p(C^{p+q}(\mathfrak{g}; M))/F_{p-1}(C^{p+q}(\mathfrak{g}; M))$$

and

$$E_1^{p,q} = H^{p+q}(\mathfrak{g}; Gr^p(M)),$$

where $Gr^p(M)$

We define

$$Z_k^{p,q} = F_pC^{p+q} \bigcap d^{-1}(F_{p+k}C^{p+q+1}),$$

$$B_k^{p,q} = F_pC^{p+q} \bigcap d(F_{p-k}C^{p+q-1}),$$

$$E_k^{p,q} = Z_k^{p,q}/(Z_{k-1}^{p+1,q-1} + B_{k-1}^{p,q}).$$

The differential d maps $Z_k^{p,q}$ into $Z_k^{p+k,q-k+1}$, and hence includes a homomorphism

$$d_k : E_k^{p,q} \longrightarrow E_k^{p+k,q-k+1}$$

The spectral sequence converges to $H^*(C, d)$, that is

$$E_\infty^{p,q} \simeq F_pH^{p+q}(C, d)/F_{p+1}H^{p+q}(C, d),$$

where $F_pH^*(C, d)$ is the image of the map $H^*(F_pC, d) \to H^*(C, d)$ induced by the inclusion $F_pC \to C$ (see [2, 10]).

3.2 Cohomology of $\mathfrak{sl}(2)$

We compute the space $\mathrm{H}^1(\mathfrak{sl}(2), \mathcal{P})$. The result is the following:

Theorem 3.1

$$\mathrm{H}^1(\mathfrak{sl}(2), \mathcal{P}) \simeq \mathbb{R}^2. \tag{3.26}$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{sl}(2), \mathcal{P}))$ *are spanned by the cohomology classes of the* 1-*cocycles* χ_n *defined by:*

$$\chi_0(X_f) = f' \quad and \quad \chi_1(X_f) = f''\xi^{-1}.$$

In our case we can use the results in [3]. One has:

$$\mathrm{H}^1(\mathfrak{sl}(2), \mathcal{F}_\lambda) \simeq \begin{cases} \mathbb{R} \text{ if } & \lambda = 0, 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.27}$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{sl}(2), \mathcal{F}_\lambda))$ are spanned by the cohomology classes of the 1-cocycles β_λ defined by:

$$\beta_0(X_f) = f' \quad \text{and} \quad \beta_1(X_f) = f''dx.$$

Proposition 3.2 [10] *As a* $\mathfrak{sl}(2)$*-module we have*

$$\mathcal{P} \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathcal{F}_n.$$

To prove Proposition 3.2, we need the following result (see [10]).

Proof According to Proposition 3.2 and using the cohomology space (3.27), we obtain that the space of $\mathfrak{sl}(2)$ with coefficients in the space of symbols $\mathcal{P}$ has the following structure

$$\mathrm{H}^1(\mathfrak{sl}(2),\ \mathcal{P}) = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathrm{H}^1(\mathfrak{sl}(2),\ \mathcal{F}_n) = \mathbb{R}^2. \tag{3.28}$$

It is generated by the non-trivial cohomology classes of $\chi_0(X_f) = f'$ and $\chi_1(X_f) = f''\xi^{-1}$ corresponding β_0 and β_1 respectively. □

Corollary 3.1

$$\mathrm{H}^1(\mathfrak{sl}(2), \Psi\mathcal{DO}) \simeq \mathbb{R}^2. \tag{3.29}$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{sl}(2), \Psi\mathcal{DO})$ *are spanned by the cohomology classes of the* 1*-cocycles* Ξ_n *defined by:*

$$\Xi_0(X_f) = f' \quad \textit{and} \quad \Xi_1(X_f) = f''\xi^{-1}.$$

Proof The filtration (2.24) is compatible with the multiplication and the Poisson bracket, that is, for $F \in \mathbf{F}_n$ and $G \in \mathbf{F}_m$, one has $F \circ G \in \mathbf{F}_{n+m}$ and $\{F, G\} \in \mathbf{F}_{n+m-1}$, after we identify $\mathcal{P}$ with $\Psi\mathcal{DO}$. Then $\mathrm{H}^1(\mathfrak{sl}(2)\,, \Psi\mathcal{DO}) = \mathrm{H}^1(\mathfrak{sl}(2), \mathcal{P})$. Now, as Ovsienko and Roger, we will use the constructed spectral sequence to compute the explicit expressions of the basis cocycles.

Indeed, the differential d_1 is

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}.$$

One can check it in the following way: consider a cocycle with values in $\mathcal{P}$, but compute its boundary as if it was with values in $\Psi\mathcal{DO}$ and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image by d_1. The higher order differentials

$$d_k : E_k^{p,q} \to E_k^{p+k,q-k+1}$$

can be constructed by iteration of this procedure, the space $E_k^{p+k,q-k+1}$ contains the subspace coming from $H^{p+q+1}(\mathfrak{sl}(2); Gr^{p+1}(\Psi\mathcal{DO}))$, where $Gr^p(\Psi\mathcal{DO})$ is isomorphic, as $\mathfrak{sl}(2)$-module to the space of weighted densities $\mathcal{F}_p$.

It is easy to see that the cocycles χ_0 and χ_1 will survive in the same form, we will denote them Ξ_0 and Ξ_1 when seen as cocycles with values in $\Psi\mathcal{DO}$. □

3.3 Cohomology of $\mathfrak{osp}(1|2)$

The main result in this subsection is the following:

Theorem 3.3

$$\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{SP}(1)) \simeq \mathbb{R}^2.$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{SP}(1))$ *are spanned by the following* 1*-cocycles:*

$$\varepsilon_0(X_F) = F' \quad \textit{and} \quad \varepsilon_1(X_F) = \eta_1^3(F)\xi^{-1}\zeta_1 - 2\theta_1\eta_1^3(F).$$

Proposition 3.4 [3]

$$\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{F}^1_\lambda) \simeq \begin{cases} \mathbb{R} & \textit{if} \quad \lambda = 0, \frac{1}{2} \\ 0 & \textit{otherwise.} \end{cases} \tag{3.30}$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathfrak{F}^1_\lambda)$ *are spanned by the following* 1*-cocycles:*

$$\epsilon_0(X_F) = F' \quad \textit{and} \quad \epsilon_{\frac{1}{2}}(X_F) = \eta_1^3(F)\alpha^{\frac{1}{2}}.$$

To prove Proposition 3.4, we need the following result (see [3]).

Proposition 3.5 [1] *As a* $\mathfrak{osp}(1|2)$*-module we have*

$$\mathcal{SP}(1) \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} (\mathfrak{F}^1{}_n \oplus \mathfrak{F}^1{}_{n+\frac{1}{2}}).$$

To prove Proposition 3.5, we need the following result (see [1]).

Proof According to Proposition 3.4 and Eq. 2.23, we obtain that the space of $\mathfrak{osp}(1|2)$ with coefficients in the space of symbols $\mathcal{SP}_n$ has the following structure

$$\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{SP}_n(1)) \simeq \begin{cases} \mathbb{R}^2 & \text{if} \quad n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using to Proposition 3.5, we obtain that the space of $\mathfrak{osp}(1|2)$ with coefficients in the space of symbols $\mathcal{SP}(1)$ has the following structure

$$\mathrm{H}^1(\mathfrak{osp}(1|2)\,, \mathcal{SP}(1)) = \widetilde{\bigoplus_{n\in\mathbb{Z}}} \mathrm{H}^1(\mathfrak{osp}(1|2)\,, \mathcal{SP}_n(1)) = \mathbb{R}^2.$$

It is generated by the non-trivial cohomology classes of $\bar{\varepsilon}_0 = \varepsilon_0 + \varepsilon_1$ and ε_1 corresponding ϵ_0 and $\epsilon_{\frac{1}{2}}$ respectively. □

Corollary 3.2

$$\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{S}\Psi\mathcal{DO}(1)) \simeq \mathbb{R}^2. \tag{3.31}$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{S}\Psi\mathcal{DO}(1))$ *are spanned by the cohomology classes of the* 1*-cocycles* Λ_n *defined by:*

$$\Lambda_0(X_F) = F' \quad \textit{and} \quad \Lambda_1(X_F) = \eta_1^3(F)\bar{\eta}_1^{-3} + \bar{\eta}^4(F)\bar{\eta}_1^{-2}.$$

Proof The cohomology space $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{S}\Psi\mathcal{DO}(1))$ is obviously upper-bounded by $\mathrm{H}^1(\mathfrak{osp}(1|2), \mathcal{SP}(1))$, we have to find explicit expressions for the non trivial cocycles generating the former cohomology. To constrict these cocycles, we follow Ovsienko and Roger in [10] based on the computations of successive differentials of the spectral sequences corresponding to the complex $C^*(\mathfrak{osp}(1|2), \mathcal{SP}(1))$ to compute the explicit expressions of the basis cocycles: consider a cocycle with values in $\mathcal{SP}$, but compute its boundary as if it was with values in $\mathcal{S}\Psi\mathcal{DO}(1)$ and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image by d_1. It is generated by the non-trivial cohomology classes of $\Lambda_0(X_F)=F'$ and $\Lambda_1(X_F)=\eta_1^3(F)\bar{\eta}_1^{-3}+\bar{\eta}^4(F)\bar{\eta}_1^{-2}$ corresponding ε_0 and ε_1 respectively. □

3.4 Cohomology of $\mathfrak{osp}(2|2)$

As $\mathfrak{osp}(1|2)_i$-isomorphism:

$$\mathfrak{osp}(2|2)\simeq\mathfrak{osp}(1|2)_i\oplus\Pi(\mathcal{H}_i),$$

where $\mathcal{H}_i$ is the subspace of $\mathfrak{F}^1_{-\frac{1}{2}}$ spanned by $\{\theta_i\alpha_1^{-\frac{1}{2}},\ x\alpha_1^{-\frac{1}{2}},\ \alpha_1^{-\frac{1}{2}}\}$ where $i=1,2$. To be more precise, any element X_F is decomposed into $X_F=X_{F_i}+X_{F_{3-i}\theta_{3-i}}$ where $\partial_{3-i}F_i=\partial_{3-i}F_{3-i}=0$, and then $X_{F_i}\in\mathfrak{osp}(1|2)_i$ and $X_{F_{3-i}\theta_{3-i}}$ can be identified to $\Pi(F_{3-i}\alpha_1^{-\frac{1}{2}})\in\Pi(\mathcal{H}_i)$. Moreover, we can see easily that

$$[\mathfrak{osp}(1|2)_i,\Pi(\mathcal{H}_i)]\subset\Pi(\mathcal{H}_i)\quad\text{and}\quad[\Pi(\mathcal{H}_i),\Pi(\mathcal{H}_i)]\subset\mathfrak{osp}(1|2)_i. \tag{3.32}$$

The first cohomology space $\mathrm{H}^1(\mathfrak{osp}(1|2)_i,\mathfrak{F}^2_\lambda)$ was computed in [3]. The result is the following:

$$\mathrm{H}^1(\mathfrak{osp}(1|2)_i,\mathfrak{F}^2_\lambda)\simeq\begin{cases}\mathbb{R}^2 & \text{if }\ \lambda=0,\\ \mathbb{R} & \text{if }\ \lambda=-\frac{1}{2},\frac{1}{2}\\ 0 & \text{otherwise .}\end{cases}$$

The respective nontrivial 1-cocycles are

$$\begin{aligned}&C_1(X_{F_i})=F_i',\ \ C_2(X_F)=\eta_i(F_i')\theta_{3-i} && \text{if }\lambda=0,\\ &C_3(X_{F_i})=F_i'\theta_{3-i}\alpha_2^{-\frac{1}{2}} && \text{if }\lambda=-\frac{1}{2},\\ &C_4(X_{F_i})=\eta_i^3(F_i)\alpha_2^{\frac{1}{2}} && \text{if }\lambda=\frac{1}{2},\end{aligned}\tag{3.33}$$

where $X_{F_i}\in\mathfrak{osp}(1|2)_i$.

Proposition 3.6 [2]

1) *As a $\mathfrak{osp}(1|2)_i$-module, $i=1,\ 2$, we have*

$$\mathcal{SP}_n(2)\simeq\mathfrak{F}^2_n\oplus\Pi(\mathfrak{F}^2_{n+\frac{1}{2}}\oplus\mathfrak{F}^2_{n+\frac{1}{2}})\oplus\mathfrak{F}^2_{n+1}\ \textit{for}\ \ n=0,-1.$$

2) *For $n\neq 0,-1$:*

a) *The following subspace of $\mathcal{SP}_n(2)$:*

$$\mathcal{SP}_{n,\ i}(2)=\left\{\begin{array}{l}B_F^{(n,i)}=F\theta_{3-i}\bar{\theta}_{3-i}\xi^{-n-1}+\theta_{3-i}(\bar{\eta}_{3-i}-\frac{1}{2}\bar{\eta}_i)(F)\bar{\zeta}_i\bar{\zeta}_{3-i}\xi^{-n-2}\mid\\ \qquad F\in C^\infty(\mathbb{R}^{1|2})\end{array}\right\}\tag{3.34}$$

is a $\mathfrak{osp}(1|2)_i$- module, $i=1,2$, isomorphic to $\mathfrak{F}^2_{n+1}$.

b) *As a $\mathfrak{osp}(1|2)_i$-module we have*

$$\mathcal{SP}_n(2)/\mathcal{SP}_{n,\ i}(2)\simeq\mathfrak{F}^2_n\oplus\Pi\left(\mathfrak{F}^2_{n+\frac{1}{2}}\oplus\mathfrak{F}^2_{n+\frac{1}{2}}\right),\ \ i=1,2.$$

To prove Proposition 3.6, we need the following result (see [2]).

The space $\mathrm{H}^1(\mathfrak{osp}(2|2),\ \mathcal{SP}(2))$ inherits the grading (2.23) of $\mathcal{SP}(2)$, so it suffices to compute it in each degree. The main result in this paper is the following:

Theorem 3.7

$$\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{SP}(2)) \simeq \mathbb{R}^6.$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{SP}(2))$ *are spanned by the following* 1*-cocycles:*

$\Upsilon_1(X_F) = (-1)^{p(F)}\eta_1\eta_2(F)\xi^{-1}\bar{\zeta}_1\bar{\zeta}_2$
$\Upsilon_2(X_F) = F'\xi^{-1}\bar{\zeta}_1\bar{\zeta}_2$
$\Upsilon_3(X_F) = F'$
$\Upsilon_4(X_F) = (-1)^{p(F)}\eta_1\eta_2(F)$
$\Upsilon_5(X_F) = (-1)^{p(F)}\Big(\bar{\eta}_1^3(F)\bar{\zeta}_1 + \bar{\eta}_2^3(F)\bar{\zeta}_2\Big)\xi^{-1}$
$\Upsilon_6(X_F) = F''\xi^{-2}\bar{\zeta}_1\bar{\zeta}_2 + (-1)^{p(F)}\Big(\bar{\eta}_1^3(F)\bar{\zeta}_2 - \bar{\eta}_2^3(F)\bar{\zeta}_1\Big)\xi^{-1}$

We know that any element $\Upsilon \in Z^1(\mathfrak{osp}(2|2), \mathcal{SP}_n(2))$ is decomposed into $\Upsilon = \Upsilon' + \Upsilon''$ where $\Upsilon' \in Z^1(\mathfrak{osp}(1|2)_i, \mathcal{SP}_n(2))$ and $\Upsilon'' \in \mathrm{Hom}(\Pi(\mathcal{H}_i), \mathcal{SP}_n(2))$.

Proof To prove Theorem 3.7, we need first to prove the following lemma:

Lemma 3.8 *The* 1*-cocycle* $\Upsilon \in \mathrm{Z}^1(\mathfrak{osp}(2|2), \mathcal{SP}_n(2)),\ n \in \mathbb{Z}$ *is a coboundary if and only if* Υ' *is a coboundary.*

Proof It is easy to see that if Υ is a coboundary for $\mathfrak{osp}(2|2)$ then Υ' is a coboundary over $\mathfrak{osp}(1|2)_i$. Now assume that Υ' is a coboundary over $\mathfrak{osp}(1|2)_i$, that is, there exists $A \in \mathcal{SP}_n(2)$ such that for all X_{F_i}

$$\Upsilon(X_{F_i}) = \{X_{F_i}\ ,\ A\}.$$

Using the condition of a 1-cocycle, we prove that

$$\Upsilon(X_{\theta_1\theta_2}) = \{X_{\theta_1\theta_2}\ ,\ A\}.$$

We deduce that $\Upsilon(X_F) = \{X_F\ ,\ A\}$, for any $X_F \in \mathfrak{osp}(2|2)$, and therefore Υ is a coboundary of $\mathfrak{osp}(2|2)$. □

Proof of Theorem 3.7 First, according to Lemma 3.8, the restriction of any nontrivial 1-cocycle of $\mathfrak{osp}(2|2)$ with coefficients in $\mathcal{SP}_n(2)$ to $\mathfrak{osp}(1|2)_1$ or to $\mathfrak{osp}(1|2)_2$ is a nontrivial 1-cocycle.

As in [2],using Propositions 3.6 and 3.4, we obtain:

$$\mathrm{H}^1(\mathfrak{osp}(1|2)_i, \mathcal{SP}_n(2)) \simeq \begin{cases} \mathbb{R}^4 & \text{if } n = -1, \\ \mathbb{R}^4 & \text{if } n = 0, \\ 0 & \text{otherwise}. \end{cases}$$

So, we see that if $n \neq 0, -1$, then by Lemma 3.8, the corresponding cohomology $\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{SP}_n(2))$ vanishes.

Case 1 $n=-1$, the space $\mathrm{H}^1(\mathfrak{osp}(1|2)_i, \mathcal{SP}_{-1}(2))$ is spanned by the following 1-cocyles:

$$\Phi_1^i(X_F) = F_i'\theta_{3-i}\bar{\theta}_{3-i} + \theta_{3-i}\left(\bar{\eta}_{3-i} - \frac{1}{2}\bar{\eta}_i\right)(F_i')\bar{\zeta}_i\bar{\zeta}_{3-i}\xi^{-1},$$

$$\Phi_2^i(X_F) = \theta_{3-i}\left(\bar{\eta}_{3-i} - \frac{1}{2}\bar{\eta}_i\right)(\eta_i(F_i')\theta_{3-i})\bar{\zeta}_i\bar{\zeta}_{3-i}\xi^{-1},$$

$$\Phi_3^i(X_F) = F_i'\bar{\zeta}_i - (\theta_{3-i}\bar{\eta}_i + \theta_i\partial_{\theta_{3-i}})(F_i')\bar{\theta}_{3-i} - (-1)^{p(F_i)}\partial_{\theta_{3-i}}(F_i')\bar{\theta}_i\bar{\theta}_{3-i}\xi^{-1},$$

$$\widetilde{\Phi}_3^i(v_F) = F_i'\bar{\zeta}_i + (1-\theta_{3-i}\bar{\eta}_i)(F_i')\bar{\theta}_{3-i}.$$

Case 2 $n=0$, the space $\mathrm{H}^1(\mathfrak{osp}(1|2)_i, \mathcal{SP}_0(2))$ is spanned by the following 1-cocyle:

$$\Phi_4^i(X_F) = F_i',$$

$$\Phi_5^i(X_F) = \eta_i(F')\theta_{3-i},$$

$$\begin{aligned}\Phi_6^i(X_F) &= \theta_i\Pi(C_4(X_F)) - (1-2\theta_{3-i}\partial_{\theta_{3-i}})(\Pi(C_4(X_F)))\bar{\theta}_i\xi^{-1}\\ &-\theta_{3-i}\partial_{\theta_i}(\Pi(C_4(X_F)))\bar{\theta}_{3-i}\xi^{-1} + (\Pi(C_4(X_F)))'\theta_{3-i}\bar{\theta}_i\bar{\theta}_{3-i}\xi^{-2},\\ \widetilde{\Phi}_6^i(X_F) &= \theta_i(\partial_{\theta_{3-i}} - 2\partial_{\theta_i} + 2\theta_{3-i}\partial_{\theta_{3-i}}\partial_{\theta_i})(\Pi(C_4(X_F)))\bar{\theta}_{3-i}\xi^{-1}\\ &+\frac{1}{2}(3-(-1)^{p(F)})(\Pi(C_4(X_F)))\bar{\theta}_{3-i}\xi^{-1}\\ &+(-1)^{p(\Pi(C_4(X_F)))}(\partial_{\theta_{3-i}} - \partial_{\theta_i} + \theta_i\partial_x)(\Pi(C_4(X_F)))\bar{\theta}_i\bar{\theta}_{3-i}\xi^{-2}(\Pi(C_4(X_F))),\end{aligned}$$

where the cocycle C_4 is defined by the formulae (3.33).

Now, consider a nontrivial 1-cocycle of $\mathfrak{osp}(2|2)$ with coefficients in $\mathcal{SP}_n(2)$ can be decomposed as $\Upsilon = (\Upsilon', \ \Upsilon'')$ and

$$\begin{cases}\Upsilon' : \mathfrak{osp}(1|2)_i \longrightarrow \mathcal{SP}_n(2),\\ \Upsilon'' : \Pi(\mathcal{H}_i) \longrightarrow \mathcal{SP}_n(2),\end{cases}$$

where $\Upsilon', \ \Upsilon''$ are linear maps.

The space $\mathrm{H}^1(\mathfrak{osp}(1|2)_i, \mathcal{SP}_n(2))$, $i=1, \ 2$, determines the linear maps Υ'. Then $\Upsilon' = \Phi^i$. More precisely, we get:

Case 1 $n=-1$, $\Upsilon' = \nu_1\Phi_1^i + \nu_2\Phi_2^i + \nu_3\Phi_3^i + \nu_4\widetilde{\Phi}_3^i$, where the coefficients ν_k are constants.

Case 2 $n=0$, $\Upsilon' = \nu_5\Phi_4^i + \nu_6\Phi_5^i + \nu_7\Phi_6^i + \nu_8\widetilde{\Phi}_6^i$.

In each case, the 1-cocycle conditions determines Υ''. We obtain for $n=-1$, $\Upsilon = \nu_1\Upsilon_1 + \nu_3\Upsilon_3$ and for $n=0$, $\Upsilon = \nu_5\Upsilon_3 + (\nu_6+\nu_8)\Upsilon_4 + \nu_7\Upsilon_5 + \nu_8\Upsilon_6$.

Thus, the space $\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{SP}_{-1}(2))$ is generated by the nontrivial cocycles Υ_1 and Υ_2 and the space $\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{SP}_0(2))$ is spanned by the nontrivial cocycles Υ_3, Υ_4, Υ_5 and Υ_6.

Theorem 3.7 is proved. □

Corollary 3.3

$$\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{S}\Psi\mathcal{DO}(2)) \simeq \mathbb{R}^6. \tag{3.35}$$

The nontrivial spaces $\mathrm{H}^1(\mathfrak{osp}(2|2), \mathcal{S}\Psi\mathcal{DO}(2))$ *are spanned by the cohomology classes of the* 1*-cocycles* Θ_n *defined by:*

$$\begin{aligned}
\Theta_1(X_F) &= (-1)^{p(F)}\eta_1\eta_2(F)\xi^{-1}\bar{\zeta}_1\bar{\zeta}_2\\
\Theta_2(X_F) &= F'\xi^{-1}\bar{\zeta}_1\bar{\zeta}_2\\
\Theta_3(X_F) &= F'\\
\Theta_4(X_F) &= (-1)^{p(F)}\eta_1\eta_2(F)\\
\Theta_5(X_F) &= \Big((-1)^{p(F)}\Big(\bar{\eta}_1^3(F)\bar{\zeta}_1 + \bar{\eta}_2^3(F)\bar{\zeta}_2\Big) - F''\Big)\xi^{-1}\\
\Theta_6(X_F) &= F''\xi^{-2}\bar{\zeta}_1\bar{\zeta}_2 + (-1)^{p(F)}\Big(\bar{\eta}_1^3(F)\bar{\zeta}_2 - \bar{\eta}_2^3(F)\bar{\zeta}_1\Big)\xi^{-1}
\end{aligned}$$

Proof of Theorem 3.7 Following Ovsienko and Roger in [10], based on the computations of successive differentials of the spectral sequences corresponding to the complex $C^*(\mathfrak{osp}(2|2), \mathcal{SP}(2))$ to compute the explicit expressions of the basis cocycles: consider a cocycle in $\mathcal{SP}(2)$, but compute its differential as if it were with values in $\mathcal{S}\Psi\mathcal{DO}(2)$ and keep the symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one, and its class will represent its image under d_1. The higher order differentials d_k can be calculated by iteration of this procedure, the space $E_k^{p+k,q-k+1}$ contains the subspace coming from $H^{p+q+1}(\mathfrak{osp}(2|2); Gr^{p+1}(\mathcal{S}\Psi\mathcal{DO}(2)))$, where $Gr^p(\mathcal{S}\Psi\mathcal{DO}(2))$ is isomorphic, as $\mathfrak{osp}(2|2)$-module to the space of weighted densities $\mathfrak{F}_p^2$. We give explicit expressions of the basis cocycles. □

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