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# Erratum to: "Integral Theory for Hopf Algebroids" [Algebra Represent. Theory (2005) 8:563–599]

**Gabriella Böhm**

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**Abstract** Because of similar errors in Lemma 2.6 and Theorem 4.2, we need to add stronger assumptions in several statements.

At several points in the paper we repeated essentially the same incorrect step, thanks to which the to-be-dual-bases in the proof of Lemma 2.6 (1.a)⇔(1.b) and (2.a)⇔(2.b), and also the map $[{}^L A \otimes \omega \circ (A_{\mathbf{L}} \otimes \tau)] \circ (\gamma_R \otimes {}_{\mathbf{L}}M) \circ \tau$ in the proof of Theorem 4.2, are ill-defined. As a consequence of these errors, the published proofs of Lemma 2.6, Theorem 4.2 and Proposition 4.4(2) are not correct. Since in this way also some unjustified claims were used to prove them, Corollary 4.5, Proposition 4.6, some considerations on page 587, Theorems 4.7 and 5.2 become valid only by adding some finitely generated projectivity assumptions. Although the obtained results are somewhat weaker than their (eventually incorrect) versions in the original paper, the most interesting cases are still covered.

Below we go through the necessary corrections.

In Lemma 2.6, only the equivalences (1.a)⇔(2.b) and (1.b)⇔(2.a) are justified, for a Hopf algebroid with a bijective antipode.

Theorem 4.2 is replaced by Theorem 1 below. In its formulation Sweedler's index notation $\tau(m) = m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}$ (with implicit summation) is used, for the left coaction $\tau : M \to A_L \otimes {}_L M$ of the constituent left $L$-bialgebroid $\mathcal{A}_L$ in a Hopf algebroid $\mathcal{A}$, on a left $\mathcal{A}_L$-comodule $M$ and $m \in M$.

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Presented by Susan Montgomery.

G. Böhm (✉)
Research Institute for Particle and Nuclear Physics, 1525 Budapest 114, P.O.B. 49, Budapest, Hungary
e-mail: G.Bohm@rmki.kfki.hu

**Theorem 1** *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid and $\mathcal{W}$ be the $A$-coring* (4.1). *Assume that the kernel of the maps*

$$M \to A_L \otimes {}_L M, \qquad m \mapsto \left(m_{\langle -1\rangle} \otimes m_{\langle 0\rangle}\right) - (1_A \otimes m) \tag{1}$$

*is preserved by the functor ${}^L A \otimes - : \mathcal{M}_L \to \mathcal{M}_L$, for any $M \in {}^{\mathcal{W}}\mathcal{M}$ (e.g. ${}^L A$ is a flat module). Then the functors* (4.2) *and* (4.3) *are inverse equivalences.*

*Proof* We only need to check that $\alpha_M^{-1}(m) = {m_{\langle -1\rangle}}^{(1)} \otimes S({m_{\langle -1\rangle}}^{(2)}) \cdot m_{\langle 0\rangle}$ belongs to ${}^L A \otimes \mathrm{Coinv}(M)_L$, the rest of the proof in the paper is valid. By the assumption that the kernel of Eq. 1 is preserved by the functor ${}^L A \otimes - : \mathcal{M}_L \to \mathcal{M}_L$, we need to show only that

$$\begin{aligned} &{m_{\langle -1\rangle}}^{(1)} \otimes \left(S\left({m_{\langle -1\rangle}}^{(2)}\right) \cdot m_{\langle 0\rangle}\right)_{\langle -1\rangle} \otimes \left(S\left({m_{\langle -1\rangle}}^{(2)}\right) \cdot m_{\langle 0\rangle}\right)_{\langle 0\rangle} \\ &\quad = {m_{\langle -1\rangle}}^{(1)} \otimes 1_A \otimes S\left({m_{\langle -1\rangle}}^{(2)}\right) \cdot m_{\langle 0\rangle}, \end{aligned} \tag{2}$$

as elements of ${}^L A \otimes A_{\mathbf{L}} \otimes {}_{\mathbf{L}} M_L$, for all $m \in M$. Compose the well defined map

$$A^R \otimes {}^R A_L \otimes {}_L A \to A^R \otimes {}_R A, \qquad a \otimes b \otimes c \mapsto a \otimes S(b)c$$

with the equal maps $(\gamma_R \otimes {}_L A) \circ \gamma_L = (A^R \otimes \gamma_L) \circ \gamma_R : A \to A^R \otimes {}^R A_L \otimes {}_L A$ (cf. (2.18)) in order to conclude that, for any $a \in A$,

$${a_{(1)}}^{(1)} \otimes S\left({a_{(1)}}^{(2)}\right) a_{(2)} = a^{(1)} \otimes S\left({a^{(2)}}_{(1)}\right) {a^{(2)}}_{(2)} = a^{(1)} \otimes s_R \circ \pi_R\left(a^{(2)}\right) = a \otimes 1_A. \tag{3}$$

In Eq. 3, in the second equality (2.20) was used and the last equality follows by the counitality of $\gamma_R$. Using the left $A$-linearity of the coaction $\tau : M \to \mathcal{W} \otimes_A M \cong A_L \otimes {}_L M$, anti-comultiplicativity of the antipode (cf. Proposition 2.3(2)), coassociativity of $\tau$ and $\gamma_R$ and finally Eq. 3, the left hand side of Eq. 2 is computed to be equal to

$$\begin{aligned} &{m_{\langle -2\rangle}}^{(1)} \otimes S\left({m_{\langle -2\rangle}}^{(2)}\right)_{(1)} m_{\langle -1\rangle} \otimes S\left({m_{\langle -2\rangle}}^{(2)}\right)_{(2)} \cdot m_{\langle 0\rangle} \\ &\quad = {m_{\langle -2\rangle}}^{(1)} \otimes S\left({{m_{\langle -2\rangle}}^{(2)}}^{(2)}\right) m_{\langle -1\rangle} \otimes S\left({{m_{\langle -2\rangle}}^{(2)}}^{(1)}\right) \cdot m_{\langle 0\rangle} \\ &\quad = {{m_{\langle -1\rangle (1)}}^{(1)}}^{(1)} \otimes S\left({m_{\langle -1\rangle (1)}}^{(2)}\right) m_{\langle -1\rangle (2)} \otimes S\left({{m_{\langle -1\rangle (1)}}^{(1)}}^{(2)}\right) \cdot m_{\langle 0\rangle} \\ &\quad = {m_{\langle -1\rangle}}^{(1)} \otimes 1_A \otimes S\left({m_{\langle -1\rangle}}^{(2)}\right) \cdot m_{\langle 0\rangle}. \end{aligned}$$

Thus it follows that $\alpha_M^{-1}(m)$ belongs to ${}^L A \otimes \mathrm{Coinv}(M)_L$ for all $m \in M$, as stated. □

In the arXiv version of [1] a more restrictive notion of a comodule of a Hopf algebroid is studied, cf. [1, arXiv version, Definition 2.20]. The total algebra $A$ of any Hopf algebroid $\mathcal{A}$ can be regarded as a monoid in the monoidal category of $\mathcal{A}$-comodules in this more restrictive sense. In this setting, the category of $A$-modules in the category of $\mathcal{A}$-comodules, and the category of modules for the base algebra $L$ of $\mathcal{A}$, were proven to be equivalent without any further (equalizer preserving) assumption, see [1, Theorem 3.27 and Remark 3.28]. That is, in the arXiv version of [1] a weaker statement is proven under weaker assumptions, compared to Theorem 1 above.

Since the original version of Lemma 2.6 turned out to be incorrect, so is the proof of Proposition 4.4(2) built on it. In order to replace Proposition 4.4(2) by a justified claim, take a Hopf algebroid $\mathcal{A}$, such that the module ${}^R A$ is finitely generated and projective, and let $\{k_j\} \subset A$ and $\{{}^*\kappa^j\} \subset {}^*\!\mathcal{A}$ be dual bases for it. Alternatively to (4.6), a right $\mathcal{A}_R$-comodule structure on $\mathcal{A}^*$ can be introduced by the right $R$-action

$$\mathcal{A}^*{}_R: \quad \phi^* \cdot r := \phi^* \leftharpoondown s_R(r) \qquad \text{for } r \in R,\ \phi^* \in \mathcal{A}^*$$

and the right coaction

$$\tau_R : \mathcal{A}^* \to \mathcal{A}^*{}_R \otimes {}^R A \qquad \phi^* \mapsto \sum_i \chi^{-1}(\pi_L \circ t_R \circ {}^*\kappa^j \circ S)\phi^* \otimes k_j, \tag{4}$$

where $\chi : \mathcal{A}^* \to \mathcal{A}_*$ is the algebra anti-isomorphism (2.22). Note that if both modules $A_L$ and ${}^R A$ are finitely generated and projective, then the dual bases $\{b_i\} \subset A$, $\{\beta_*^i\} \subset \mathcal{A}_*$ for $A_L$, and $\{k_j\} \subset A$, $\{{}^*\kappa^j\} \subset {}^*\!\mathcal{A}$ for ${}^R A$, are related via the identity

$$\sum_i \beta_*^i \otimes S(b_i) = \sum_j \pi_L \circ t_R \circ {}^*\kappa_j \circ S \otimes k_j$$

in $\mathcal{A}_*{}^R \otimes {}^R A$. Hence in this case the coaction (4) is equal to (4.6).

Proposition 4.4 is replaced by the following.

**Proposition 2** *Let $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ be a Hopf algebroid.*

(1) *Introduce the left $A$-module*

$$_A\mathcal{A}^*: \quad a \cdot \phi^* := \phi^* \leftharpoondown S(a) \qquad \textit{for } a \in A,\ \phi^* \in \mathcal{A}^*.$$

*If the module $A_L$ is finitely generated and projective, then $({}_A\mathcal{A}^*, \tau_L)$—where $\tau_L$ is the map* (4.5)*—is a left-left Hopf module over $\mathcal{A}_L$.*

(2) *Introduce the right $A$-module*

$$\mathcal{A}^*{}_A: \quad \phi^* \cdot a := \phi^* \leftharpoondown a \qquad \textit{for } a \in A,\ \phi^* \in \mathcal{A}^*.$$

*If the module ${}^R A$ is finitely generated and projective, then $(\mathcal{A}^*{}_A, \tau_R)$—where $\tau_R$ is the map* (4)*—is a right-right Hopf module over $\mathcal{A}_R$.*

*The coinvariants of both Hopf modules $({}_A\mathcal{A}^*, \tau_L)$ and $(\mathcal{A}^*{}_A, \tau_R)$ are the elements of $\mathcal{L}(\mathcal{A}^*)$.*

*Proof* Part (1) of Proposition 2 is proven in the paper, and part (2) follows by similar steps. □

Since Theorem 1 and Proposition 2 contain stronger assumptions than their original counterparts, we need stronger assumptions than those in the paper to conclude that the maps (4.14) and (4.15) are isomorphisms. Applying Theorem 1 to the Hopf modules in Proposition 2, the following are obtained. The map (4.14) is an isomorphism of left-left Hopf modules over $\mathcal{A}_L$ provided that the module $A_L$ is finitely generated and projective, and the kernel of the map

$${}^L A \otimes \mathcal{A}^{*L} \to {}^{\mathbf{L}}A \otimes A_L \otimes {}_L\mathcal{A}^{*\mathbf{L}}, \qquad a \otimes \phi^* \mapsto a \otimes \tau_L(\phi^*) - a \otimes 1_A \otimes \phi^*$$

is equal to ${}^L A \otimes \mathcal{L}(\mathcal{A}^*)^L$. The map (4.14) is an isomorphism, in particular, if both modules $A_L$ and ${}^L A$ are finitely generated and projective. The map (4.15) is an isomorphism of right-right Hopf modules over $\mathcal{A}_R$ provided that the module ${}^R A$ is finitely generated and projective, and the kernel of

$$ {}^R\mathcal{A}^* \otimes A_R \to {}^{\mathbf{R}}\mathcal{A}^*{}_R \otimes {}^R A \otimes A_{\mathbf{R}}, \qquad \phi^* \otimes a \mapsto \tau_R(\phi^*) \otimes a - \phi^* \otimes 1_A \otimes a $$

is equal to ${}^R\mathcal{L}(\mathcal{A}^*) \otimes A_R$. The map (4.15) is an isomorphism, in particular, if both modules ${}^R A$ and $A_R$ are finitely generated and projective.

The proofs of Corollary 4.5, Proposition 4.6, considerations about depth 2 properties on page 587, Theorems 4.7 and 5.2 all resort to the bijectivity of (4.14) and/or (4.15). Hence, as it is explained above, they are valid only under more restrictive assumptions. In each case, it has to be added to the hypothes is that

$\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$ *is a Hopf algebroid such that all of the modules* $A^R$, ${}^R A$, ${}_L A$ *and* $A_L$ *are finitely generated and projective.*

Accordingly, all claims about finitely generated projectivity of these modules need to be removed in the statements (together with their verifications in the proofs).

Based on the corrected form of Theorem 4.7, we obtain the following generalization of Theorem 4.1.

**Corollary 3** *For any Hopf algebroid* $\mathcal{A} = (\mathcal{A}_L, \mathcal{A}_R, S)$, *the following assertions are equivalent.*

(1.a) *Both maps* $s_R : R \to A$ *and* $t_R : R^{op} \to A$ *are Frobenius extensions of k-algebras.*
(1.b) *Both maps* $s_L : L \to A$ *and* $t_L : L^{op} \to A$ *are Frobenius extensions of k-algebras.*
(2.a) *The module* $A^R$ *is finitely generated and projective and there exists an element* $\lambda^* \in \mathcal{L}(\mathcal{A}^*)$ *such that the map* $\mathcal{F} : A \to \mathcal{A}^*, \quad a \mapsto \lambda^* \leftharpoonup a$ *is bijective.*
(2.b) *S is bijective, the module* ${}^R A$ *is finitely generated and projective and there exists an element* ${}^*\lambda \in \mathcal{L}({}^*\!\mathcal{A})$ *such that the map* $A \to {}^*\!\mathcal{A}, \quad a \mapsto {}^*\lambda \leftharpoondown a$ *is bijective.*
(2.c) *The module* ${}_L A$ *is finitely generated and projective and there exists an element* ${}_*\rho \in \mathcal{R}({}_*\mathcal{A})$ *such that the map* $A \to {}_*\mathcal{A}, \quad a \mapsto a \rightharpoondown {}_*\rho$ *is bijective.*
(2.d) *S is bijective, the module* $A_L$ *is finitely generated and projective and there exists an element* $\rho_* \in \mathcal{R}(\mathcal{A}_*)$ *such that the map* $A \to \mathcal{A}_*, \quad a \mapsto a \rightharpoonup \rho_*$ *is bijective.*
(3.a) *There exists a* non-degenerate *left integral, that is an element* $\ell \in \mathcal{L}(\mathcal{A})$ *such that both maps* $\mathcal{F}^* : \mathcal{A}^* \to A, \quad \phi^* \mapsto \phi^* \rightharpoonup \ell$ *and* ${}^*\mathcal{F} : {}^*\!\mathcal{A} \to A, \quad {}^*\phi \mapsto {}^*\phi \rightharpoondown \ell$ *are bijective.*
(3.b) *There exists a* non-degenerate right integral, *that is, an element* $\wp \in \mathcal{R}(\mathcal{A})$ *such that both maps* ${}_*\mathcal{A} \to A, \quad {}_*\phi \mapsto \wp \leftharpoondown {}_*\phi$ *and* $\mathcal{A}_* \to A, \quad \phi_* \mapsto \wp \leftharpoonup \phi_*$ *are bijective.*

*Proof* $(1.a) \Leftrightarrow (1.b)$: This follows by the same reasoning used to prove (1.a)⇔(1.d) and (1.b)⇔(1.c) in Theorem 4.7.

$(1.a) \Rightarrow (2.a)$: Since $s_R : R \to A$ is a Frobenius extension by assumption, the modules $A^R$ and ${}_R A$ (hence also $A_L$) are finitely generated and projective by definition. Similarly, since $t_R : R^{op} \to A$ is a Frobenius extension, the modules ${}^R A$ and ${}_L A$ are finitely generated and projective. Thus this implication follows by (the corrected form of) Theorem 4.7 (1.a)⇒(3.a).

(2.$a$) $\Rightarrow$ (3.$a$) *and* $S$ *is bijective*: This is proven by repeating the same steps used to prove (3.a)$\Rightarrow$(4.a) and (4.b) in Theorem 4.7.

(3.$a$) $\Rightarrow$ (1.$a$) *and* $S$ *is bijective*: Putting $\lambda^* := \mathcal{F}^{*-1}(1_A)$, the map $a \mapsto (\lambda^* \leftharpoonup a) \rightharpoonup \ell$ is checked to be the inverse of $S$.

For any $r \in R$, $\mathcal{F}^*(r\lambda^*(-)) = t_R(r) = \mathcal{F}^*(\lambda^* \leftharpoonup s_R(r))$. So by the bijectivity of $\mathcal{F}^*$, we conclude that $\lambda^*$ is a left $R$-module map ${}_RA \to R$. Therefore the module ${}^RA$ is finitely generated and projective with dual basis $\lambda^*(S(-)\ell^{(1)}) \otimes \ell^{(2)} \in {}^*\!\mathcal{A}_R \otimes {}^RA$. Since the antipode is bijective, the module $A_L$ is finitely generated and projective by Lemma 2.6 (2.b)$\Rightarrow$(1.a). Applying the same reasoning to the Hopf algebroid $\mathcal{A}_{cop}$, we conclude that by the bijectivity of ${}^*\mathcal{F}$ also the modules $A^R$ and ${}_LA$ are finitely generated and projective. Hence the claim follows by (the corrected form of) Theorem 4.7, (4.a)$\Rightarrow$(1.a) and (1.b).

(1.$b$) $\Leftrightarrow$ (2.$c$) $\Leftrightarrow$ (3.$b$): This follows by applying (1.a)$\Leftrightarrow$(2.a)$\Leftrightarrow$(3.a) to the Hopf algebroid $\mathcal{A}_{cop}^{op}$.

(1.$a$) $\Leftrightarrow$ (2.$b$): Since we proved that (1.a) implies the bijectivity of the antipode, we can apply (1.a)$\Leftrightarrow$(2.a) to the Hopf algebroid $\mathcal{A}_{cop}$.

(1.$b$) $\Leftrightarrow$ (2.$d$): This follows by applying (1.a)$\Leftrightarrow$(2.b) to the Hopf algebroid $\mathcal{A}_{cop}^{op}$.

□

Finally, we would like to correct some regrettable typos in the paper.

In both parts of Lemma 5.1, instead of the condition $\sum_{i,k} au_i^k \otimes v_i^k = \sum_{i,k} u_i^k \otimes v_i^k a$, for all $a \in A$, the conditions $\sum_i au_i^k \otimes v_i^k = \sum_i u_i^k \otimes v_i^k a$ need to hold, for all possible values of $k$ and all $a \in A$. (In the published version also the summation symbols are missing on the right hand sides.)

In the computation on page 595, the first and the last expressions are interchanged.

In the penultimate line on page 597, the phrase “and vice versa” has to be erased.

## References

1. Böhm, G.: Galois extensions over commutative and non-commutative base. In: Caenepeel, S., Van Oystaeyen, F. (eds.) New Techniques in Hopf Algebras and Graded Ring Theory. http://arxiv.org/abs/math/0701064v2 (2006)