

# Remarks on the McKay Conjecture

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**Abstract** The McKay Conjecture (MC) asserts the existence of a bijection between the (inequivalent) complex irreducible representations of degree coprime to  $p$  ( $p$  a prime) of a finite group  $G$  and those of the subgroup  $N$ , the normalizer of Sylow  $p$ -subgroup. In this paper we observe that MC implies the existence of analogous bijections involving various pairs of algebras, including certain crossed products, and that MC is *equivalent* to the analogous statement for (twisted) quantum doubles. Using standard conjectures in orbifold conformal field theory, MC is *equivalent* to parallel statements about holomorphic orbifolds  $V^G, V^N$ . There is a uniform formulation of MC covering these different situations which involves quantum dimensions of objects in pairs of ribbon fusion categories.

**Keywords** McKay correspondence · Quantum double

**Mathematics Subject Classification (2000)** 20C05

## 1 Introduction

The following notation will be used throughout the paper:  $G$  is a finite group,  $p$  a prime,  $P$  a Sylow  $p$ -subgroup of  $G$ ,  $N = N(P)$  the *normalizer* of  $P$  in  $G$ ,  $G'$  the *commutator subgroup*,  $X$  a (finite, non-empty, left-)  $G$ -set,  $\phi$  the Euler phi-function.

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All algebras and modules are finite-dimensional and defined over  $\mathbb{C}$ .  $\mathbb{C}[G]$  is the group algebra of  $G$  and  $\mathbb{C}[G]^*$  the dual group algebra.

For an algebra  $A$ , let

$$\mu(A) = \# \text{ inequivalent simple } A\text{-modules of dimension coprime to } p. \quad (1)$$

We say that a pair of algebras  $(A, B)$  is an  $M$ -pair in case  $\mu(A) = \mu(B)$ . The McKay Conjecture (MC) is the assertion that  $(\mathbb{C}[G], \mathbb{C}[N])$  is an  $M$ -pair. The reader may consult the paper [7] of Isaacs, Malle and Navarro for the current status of this conjecture. The idea of the present paper is to *extend* MC beyond its original formulation for groups. First we show how it may be extended to large classes of algebras that are not group algebras. Examples include *crossed product* algebras, where we show that  $(\mathbb{C}[H]^* \#_{\sigma} \mathbb{C}[G], \mathbb{C}[H]^* \#_{\sigma} \mathbb{C}[N])$  is an  $M$ -pair. Here,  $G$  acts on the group  $H$  and  $\sigma$  is a certain 2-cocycle. (See [8, 11] for background.) A particularly interesting case is that of quantum doubles  $D(G)$  (see [4, 9] and below for more details). In this case we establish

$$\text{MC is true if, and only if, } (D(G), D(N)) \text{ is an } M\text{-pair for all } G \text{ and } N. \quad (2)$$

Note that quantum doubles  $D(G)$  are generally not group algebras (unless  $G$  is abelian).

For a multiplicative 3-cocycle  $\omega \in Z^3(G, \mathbb{C}^*)$ , we show that MC implies the same result for *twisted quantum doubles*. That is,  $(D^{\omega}(G), D^{\omega}(N))$  is an  $M$ -pair. Now there is a standard Ansatz in orbifold conformal field theory (CFT) due to Dijkgraaf et al. [6] which, when interpreted appropriately, says that the tensor category  $D^{\omega}(G)\text{-Mod}$  is equivalent to the module category  $V^G\text{-Mod}$  of a so-called holomorphic  $G$ -orbifold for a suitable vertex operator algebra  $V$  admitting  $G$  as automorphisms. Therefore, granted the DPR conjecture, MC is *equivalent* to a CFT-formulation involving a bijection between certain sets of simple modules for  $V^G$  and  $V^N$ . It is not necessary for the reader to be familiar with this language; the point is simply that modules for  $V^G$  are infinite-dimensional and the idea of an  $M$ -pair based on definition (1) makes no sense. In fact, all three types of  $M$ -pairs that we have discussed (i.e. for groups, (quasi-)Hopf algebras and orbifolds) may be uniformly described in the following setting: a pair of ribbon categories admitting a bijection between objects whose quantum dimension is integral and coprime to  $p$ .

All of the proofs in this paper are elementary and involve nothing beyond a few facts about finite groups, their representations, and their cohomology. In Section 2 we discuss some algebras  $D_X(G)$  constructed from  $G$  and a  $G$ -set  $X$  and show that MC implies that  $(D_X(G), D_X(N))$  is an  $M$ -pair. We also establish Eq. 2. In Section 3 we carry out the twisted analog of this construction. Together, these results cover several of the connections with crossed products and twisted quantum doubles mentioned above. In Section 4 we discuss the connections with CFT and ribbon categories. We assure the reader that no knowledge of CFT is required to understand the contents of this paper.

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## 2 The Algebras $D_X(G)$

We use the following additional notation: for  $H \leq G$ ,  $g \in G$ ,  $H^g = \{g^{-1}hg \mid h \in H\}$ . For  $x \in X$ ,  $\text{Stab}_G(x) = \{g \in G \mid g.x = x\}$ .

We now introduce the algebras  $D_X(G)$ , which were mentioned briefly in [9]. Let  $\mathbb{C}[X]^*$  be the space of complex-valued functions on  $X$ . One sees that it is a  $G$ -module algebra, as follows. The algebra structure is pointwise multiplication, with basis the Dirac delta functions

$$e(x) : y \mapsto \delta_{x,y} \quad x, y \in X.$$

Thus

$$e(x)e(y) = \delta_{x,y}e(x).$$

$G$  acts on the left of  $\mathbb{C}[X]^*$  as algebra automorphisms via

$$g : e(x) \mapsto e(g.x).$$

Consider the linear space

$$D_X(G) = \mathbb{C}[X]^* \otimes_{\mathbb{C}} \mathbb{C}[G]. \tag{3}$$

It becomes an algebra via the product

$$\begin{aligned} (e(x) \otimes g)(e(y) \otimes h) &= e(x)e(g.y) \otimes gh \\ &= \delta_{x,g.y}e(x) \otimes gh \end{aligned} \tag{4}$$

for  $x, y \in X$  and  $g, h \in G$ . One readily checks that this is *associative*. There is a decomposition into 2-sided ideals

$$D_X(G) = \bigoplus_Y D_Y(G) \tag{5}$$

where  $Y$  ranges over the (transitive)  $G$ -orbits of  $X$ . For the most part, this reduces questions about  $D_X(G)$  for general  $X$  to the transitive case.

A special example of this construction is the quantum double of  $G$ . Here, we take  $X = G_{\text{conj}}$ , i.e.  $X = G$  and the left action of  $G$  is left conjugation  $g : x \mapsto gxg^{-1}$ . In this case we write  $D(G)$  in place of  $D_{G_{\text{conj}}}(G)$ .  $D(G)$  is in fact a Hopf algebra, but at the moment we only require the algebra structure.

Next we describe the category of (left-)  $D_X(G)$ -modules (cf. [6, 8, 9]). For  $x \in X$  and a left  $\text{Stab}_G(x)$ -module  $V$ , set  $V_x = e(x) \otimes V$ . This is a left  $\mathbb{C}[X]^* \otimes \text{Stab}_G(x)$ -module via

$$(e(y) \otimes g).(e(x) \otimes v) = e(y)e(x) \otimes g.v = \delta_{x,y}e(x) \otimes g.v, \quad g \in \text{Stab}_G(x). \tag{6}$$

From Eq. 4 it follows that  $\mathbb{C}[X]^* \otimes \text{Stab}_G(x)$  is a subalgebra of  $D_X(G)$ , so that  $D_X(G) \otimes_{\mathbb{C}[X]^* \otimes \text{Stab}_G(x)} V_x$  is a left  $D_X(G)$ -module.

**Proposition 2.1** *Suppose that  $X$  is a transitive  $G$ -set and  $x \in X$ . Then the map*

$$\begin{aligned} \mathbb{C}[\text{Stab}_G(x)]\text{-Mod} &\rightarrow D_X(G)\text{-Mod}, \\ V &\mapsto D_X(G) \otimes_{\mathbb{C}[X]^* \otimes \text{Stab}_G(x)} V_x, \end{aligned} \tag{7}$$

*is a Morita equivalence.*

*Proof* For this and more, see [9] and Section 3 of [8]. In these references  $X$  is a group, but this is not necessary and the proofs go through without change. Via the natural identification  $\mathbb{C}[\text{Stab}_G(x)] \xrightarrow{\cong} e(x) \otimes \mathbb{C}[\text{Stab}_G(x)]$ ,  $g \mapsto e(x) \otimes g$ , the object map inverse to (7) is  $W \mapsto (e(x) \otimes 1)W$ .  $\square$

**Proposition 2.2** *Suppose that  $X$  is a transitive  $G$ -set and  $x \in X$ . The following hold:*

- (a) *If  $p$  does not divide  $|X|$  then  $(D_X(G), \text{Stab}_G(x))$  is an  $M$ -pair.*
- (b) *If  $p$  divides  $|X|$  then  $\mu(D_X(G)) = 0$ .*

*Proof* Let  $T$  be a set of right coset representatives in  $G$  for  $\text{Stab}_G(x)$ , so that there is a disjoint union  $G = \cup_{t \in T} t\text{Stab}_G(x)$ . Because  $X$  is transitive then  $X = \{t.x \mid t \in T\}$ . From Eq. 4, observe that  $D_X(G)$  is a free right  $\mathbb{C}[X]^* \otimes \text{Stab}_G(x)$ -module with free basis  $\{e(t.x) \otimes t \mid t \in T\}$ . Using this observation, it follows that for a left  $\text{Stab}_G(x)$ -module  $V$ ,

$$\begin{aligned} \dim(D_X(G) \otimes_{\mathbb{C}[X]^* \otimes \text{Stab}_G(x)} V_x) &= |T| \dim V \\ &= |X| \dim V. \end{aligned}$$

Therefore, in the Morita equivalence (7) modules of dimension  $d$  are mapped to modules of dimension  $d|X|$ . Parts (a) and (b) both follow immediately from this.  $\square$

**Lemma 2.3**  $\mu(D_X(G)) = \sum_Y \mu(D_Y(G))$  where  $Y$  ranges over the  $G$ -orbits of  $X$ .

*Proof* This follows from the decomposition (5) into 2-sided ideals.  $\square$

We will also need the following standard result.

**Lemma 2.4** *The number of  $G$ -orbits of  $X$  of cardinality coprime to  $p$  is equal to the number of  $N$ -orbits of  $X$  of cardinality coprime to  $p$ .*

*Proof* By considering the decomposition of  $X$  into  $G$ -orbits, we see that we must prove the following assertion:

If  $X$  is a transitive  $G$ -set, then either (a)  $p$  divides  $|X|$  and  $N$  has no orbits of cardinality coprime to  $p$ , or (b)  $p$  does not divide  $|X|$  and there is a unique  $N$ -orbit of cardinality coprime to  $p$ . (8)

Assume, then, that  $X$  is a transitive  $G$ -set, and that  $x, y \in X$  lie in  $N$ -orbits of cardinality coprime to  $p$ . In such an  $N$ -orbit,  $P$  must fix at least one, and therefore all, elements in the  $N$ -orbit. In particular,  $P$  lies in the stabilizers of both  $x$  and  $y$ . By transitivity there is  $g \in G$  with  $g.x = y$ . Then  $P$  and  $P^g$  are both Sylow  $p$ -subgroups of  $\text{Stab}_G(x)$  and by Sylow's theorem there is  $t \in \text{Stab}_G(x)$  such that  $P^{gt} = P$ . Then  $gt \in N$  and  $(gt).x = y$ . This shows that  $x, y$  lie in the same  $N$ -orbit, so that there is at most one  $N$ -orbit of cardinality coprime to  $p$ . Equation (8) is easily deduced from this, and the lemma is proved.  $\square$

Consider the following statements:

- MC1 :  $(\mathbb{C}[G], \mathbb{C}[N])$  is an M-pair for all  $G$ ,
- MCD :  $(D(G), D(N))$  is an M-pair for all  $G$ ,
- MCX :  $(D_X(G), D_X(N))$  is an M-pair for all  $G$  and all  $X$ ,
- MCT :  $(D_X(G), D_X(N))$  is an M-pair for all  $G$  and all *transitive*  $X$ .

The McKay Conjecture is of course the assertion that MC1 is true.

**Theorem 2.5** *MC1, MCD, MCX and MCT are equivalent statements.*

*Proof*

- MCX  $\Leftrightarrow$  MCT This follows from Lemma 2.4.
- MCT  $\Rightarrow$  MC1 This holds because if  $X = \mathbf{1}$  is the one-element set, then  $D_X(G) = \mathbb{C}[G]$ .
- MC1  $\Rightarrow$  MCT Let  $X$  be a transitive  $G$ -set. If  $p$  divides  $|X|$  then  $\mu(D_X(G)) = 0$  by Proposition 2.2(b). Similarly,  $\mu(D_X(N)) = 0$  by Lemma 2.3, (8)(a), and Proposition 2.2(b) (applied to  $N$ ). Now suppose that  $p$  does *not* divide  $|X|$ . Then  $\mu(D_X(G)) = \mu(\mathbb{C}[\text{Stab}_G(x)])$  for any  $x \in X$  by Proposition 2.2(a). Moreover by Lemma 2.4 there is a *unique*  $N$ -orbit of cardinality coprime to  $p$ , call it  $Y \subseteq X$ . By Lemmas 2.3, (8) and Proposition 2.2 once more we find that  $\mu(D_X(N)) = \mu(D_Y(N)) = \mu(\mathbb{C}[\text{Stab}_N(y)])$  for  $y \in Y$ . Note that because  $|Y|$  is coprime to  $p$  then  $P \leq \text{Stab}_G(y)$  and  $N_{\text{Stab}_G(y)}(P) = \text{Stab}_N(y)$ . The assumption that MC1 holds (applied to  $\text{Stab}_G(y)$ ) tells us that  $\mu(\mathbb{C}[\text{Stab}_G(y)]) = \mu(\mathbb{C}[\text{Stab}_N(y)])$ , whence  $\mu(D_X(G)) = \mu(D_X(N))$ .
- MCX  $\Rightarrow$  MCD Let  $Y_1, \dots, Y_h$  be the  $N$ -orbits of  $G_{\text{conj}}$  of cardinality coprime to  $p$ . We have  $Y_i \subseteq N$  for each index  $i$ , so that they are also the  $N$ -orbits of  $N_{\text{conj}}$  of cardinality coprime to  $p$ . Taking  $X = G_{\text{conj}}$ , MCX together with Lemma 2.3 and Proposition 2.2, we conclude that  $\mu(D(G)) = \mu(D_{G_{\text{conj}}}(N)) = \sum_{i=1}^h \mu(D_{Y_i}(N)) = \mu(D_{N_{\text{conj}}}(N)) = \mu(D(N))$ , as required.
- MCD  $\Rightarrow$  MC1 We prove this using induction on  $|G|$ . Retain the notation of the last paragraph, and choose  $y_i \in Y_i$ . By Lemma 2.4,  $y_1, \dots, y_h$  are representatives for the  $G$ -orbits of  $G_{\text{conj}}$  (i.e., conjugacy classes of  $G$ ) of cardinality coprime to  $p$ . By Proposition 2.2,  $\mu(D(G)) = \sum_{i=1}^h \mu(\mathbb{C}[C_G(y_i)])$ . Since each  $y_i \in N$ , we similarly have  $\mu(D(N)) = \sum_{i=1}^h \mu(\mathbb{C}[C_N(y_i)])$ . If  $C_G(y_i)$  is a *proper* subgroup of  $G$  then by induction  $\mu(\mathbb{C}[C_G(y_i)]) = \mu(\mathbb{C}[C_N(y_i)])$ . Then the assumption MCD tells us that  $\sum_{i'} \mu(\mathbb{C}[G]) = \sum_{i'} \mu(\mathbb{C}[N])$  where  $i'$  ranges over those indices for which  $y_{i'}$  lies in the *center*  $Z(G)$  of  $G$ . We conclude that  $|Z(G)|\mu(\mathbb{C}[G]) = |Z(G)|\mu(\mathbb{C}[N])$ , whence  $\mu(\mathbb{C}[G]) = \mu(\mathbb{C}[N])$ . This completes the proof of the theorem. □

### 3 Twisted Algebras

In this section we explain how to extend the results of the previous section to the *twisted case*, i.e. the incorporation of a cocycle. Let  $\theta \in Z^2(G, \mathbb{C}^*)$  be a (normalized) multiplicative 2-cocycle. Thus  $\theta : G^2 \rightarrow \mathbb{C}^*$  satisfies the identities

$$\begin{aligned} \theta(h, k)\theta(g, hk) &= \theta(gh, k)\theta(g, h), \quad g, h, k \in G, \\ \theta(1, g) &= \theta(g, 1) = 1. \end{aligned}$$

The corresponding twisted group algebra is  $\mathbb{C}^\theta[G]$ . It has the same underlying linear space as  $\mathbb{C}[G]$  with multiplication  $g \circ h = \theta(g, h)gh$  for  $g, h \in G$ . The cocycle identities ensure that this is an associative algebra with identity element 1. For a subgroup  $H \leq G$  we identify  $\theta$  with its *restriction*  $\text{Res}_H^G \theta$  to  $H$ . Then  $\mathbb{C}^\theta[H]$  is a subalgebra of  $\mathbb{C}^\theta[G]$ . For more information on this subject, including results that we use below, see for example [3].

The cohomological analog of Proposition 2.2(b) is the following

**Lemma 3.1** *Suppose that the cohomology class  $[\theta] \in H^2(G, \mathbb{C}^*)$  determined by  $\theta$  has order  $k$ . If  $k$  is divisible by  $p$  then  $\mu(\mathbb{C}^\theta[G]) = \mu(\mathbb{C}^\theta[N]) = 0$ .*

*Proof* One knows (loc.cit.) that there is a central extension

$$1 \rightarrow Z \rightarrow L \xrightarrow{\pi} G \rightarrow 1$$

such that  $\mathbb{Z}_k \cong Z \leq L'$ , and  $\mathbb{C}^\theta[G]$  is the algebra summand of  $\mathbb{C}[L]$  corresponding to the irreducible representations of  $L$  in which a generator  $z$  of  $Z$  acts as multiplication by a prescribed primitive  $k$ th root of unity, say  $\lambda$ . If  $V$  is a simple  $\mathbb{C}^\theta[G]$ -module of dimension  $d$  then the determinant of  $z$  considered as operator on  $V$  is clearly  $\lambda^d$ . On the other hand  $z \in L'$ , so that this determinant is necessarily 1. So  $\lambda^d = 1$ , whence  $k|d$ . In particular, if  $p|k$  then  $\mu(\mathbb{C}^\theta[G]) = 0$ .

Now it is well-known that the restriction map  $\text{Res}_N^G : H^2(G, \mathbb{C}^*) \rightarrow H^2(N, \mathbb{C}^*)$  is an *injection* on the  $p$ -part of  $H^2(G, \mathbb{C}^*)$ . In particular, if  $p|k$  then  $\text{Res}_N^G[\theta]$  is divisible by  $p$ . Then the result of the last paragraph also applies to  $\mathbb{C}^\theta[N]$ , and we obtain  $\mu(\mathbb{C}^\theta[N]) = 0$ . This completes the proof of the lemma. □

The McKay Conjecture implies that the twisted analog is also true. This is the content of

**Proposition 3.2** *Suppose that MC1 holds. Then  $(\mathbb{C}^\theta[G]), \mathbb{C}^\theta[N]$  is an  $M$ -pair for all  $G$  and all  $\theta$ .*

*Proof* Let the notation be as in Lemma 3.1. Although it is not really necessary to do so, because of Lemma 3.1 we may, and shall, assume that  $k$  is not divisible by  $p$ . Let  $P_1$  be a Sylow  $p$ -subgroup of  $L$  with  $\pi : P_1 \xrightarrow{\cong} P$ . Applying MC1 to pairs  $(L/Z_0, N_L(P_1)/Z_0)$  with  $Z_0 \leq Z$ , we see that the number  $l$  of irreducible representations of both  $\mathbb{C}[L]$  and  $\mathbb{C}[N_L(P_1)]$  which have degree coprime to  $p$  and in which  $z$  acts as *some* primitive  $k$ th root of unity are equal. Since  $\mathbb{C}^\theta[G]$  is the algebra

summand of  $\mathbb{C}[L]$  corresponding to  $\lambda$ , then  $\mu(\mathbb{C}^\theta[G]) = l/\phi(k)$ . On the other hand,  $N_L(P_1) = ZK$  where  $K \leq L$  is such that  $\mathbb{C}^\theta[N]$  is the algebra summand of  $\mathbb{C}[K]$  corresponding to  $\lambda^t$ , where  $t|k$  and  $k/t$  is the order of  $\text{Res}_N^G[\theta]$ . By slightly modifying the previous argument, we also find that  $\mu(\mathbb{C}^\theta[N]) = l/\phi(k)$ , and the Proposition is proved.  $\square$

We can now treat the twisted version of  $D_X(G)$ . Let  $U = U(\mathbb{C}[X]^*)$  be the group of units in  $\mathbb{C}[X]^*$ . Then

$$U = \left\{ \sum \lambda_x e(x) \mid \lambda_x \neq 0 \right\}$$

is a *multiplicative* left  $G$ -module. Let  $\alpha \in Z^2(G, U)$  be a normalized 2-cocycle with coefficients in  $U$ , and set  $\alpha(g, h) = \sum_{x \in X} \alpha_x(g, h)e(x)$ . Here, the cocycle property amounts to the identity

$$\alpha_x(g, h)\alpha_x(gh, k) = \alpha_x(g, hk)\alpha_{g^{-1}x}(h, k). \tag{9}$$

Define  $D_X^\alpha(G)$  to be the linear space  $D_X(G)$  with multiplication being the twisted version of Eq. 4. That is,

$$(e(x) \otimes g)(e(y) \otimes h) = \alpha_x(g, h)\delta_{x, g.y}e(x) \otimes gh. \tag{10}$$

Equation 9 is exactly what is needed to show that Eq. 10 is *associative*. Note also from Eq. 9 that for fixed  $x \in X$ ,  $\alpha_x$  defines an element in  $Z^2(\text{Stab}_G(x), \mathbb{C}^*)$  and that as a subalgebra of  $D_X^\alpha(G)$ ,  $e(x) \otimes \mathbb{C}[\text{Stab}_G(x)] \cong \mathbb{C}^{\alpha_x}[\text{Stab}_G(x)]$ . The proof of Proposition 2.1 still applies in this situation (cf. [8]). It provides a Morita equivalence of categories

$$\mathbb{C}^{\alpha_x}[\text{Stab}_G(x)]\text{-Mod} \xrightarrow{\sim} D_X^\alpha(G)\text{-Mod}.$$

The proof of the twisted version of Theorem 2.5 then goes through too. We just state a part of this as

**Theorem 3.3** *Suppose that MC1 holds. Then  $(D_X^\alpha(G), (D_X^\alpha(N))$  is an  $M$ -pair for all  $G, X$  and  $\alpha$ .*

Special cases of  $D_X^\alpha(G)$  include certain kinds of crossed products and abelian extensions of Hopf algebras. See, for example, [8] for further details.

Once again the case of the quantum double, when  $X = G_{\text{conj}}$ , is of special interest (cf. [2, 5, 6, 9] for more details and further background.) Here, one twists  $D(G)$  by a normalized *three cocycle*  $\omega \in Z^3(G, \mathbb{C}^*)$ . The resulting object is denoted by  $D^\omega(G)$ . It is a quasi-Hopf algebra, but not a Hopf algebra in general. To connect with previous paragraphs, we observe that there is a map [6]

$$Z^3(G, \mathbb{C}^*) \rightarrow Z^2(G, G_{\text{conj}})$$

for which

$$\alpha_x(g, h) = \frac{\omega(x, g, h)\omega(g, h, (gh)^{-1}x(gh))}{\omega(g, g^{-1}xg, h)}. \tag{11}$$

There is a natural interpretation of this map in terms of the loop space  $LBG$ , but we will not need it. The twisted product in  $D^\omega(G)$  is as in Eq. 10 using Eq. 11. This gives the algebra structure, and as before leads to

**Theorem 3.4** *Suppose that MC1 holds. Then  $(D^\omega(G), D^\omega(N))$  is an M-pair for all  $G$  and  $\omega$ .*

The statement and proof of Theorem 3.4 only require the algebra structure of  $D^\omega(G)$ . However, we will make use of other structural features of  $D^\omega(G)$  in the next section.

#### 4 Orbifolds and Ribbon Categories

We refer the interested reader to [5] for background concerning vertex operator algebras. Let  $V$  be a holomorphic vertex operator algebra admitting  $G$  as a group of automorphisms, with  $V^G$  the subalgebra of  $G$ -invariants. One expects that the module category  $V^G\text{-Mod}$  is a (braided, ribbon) tensor category and that it is equivalent to the tensor category  $D^\omega(G)\text{-Mod}$  for a 3-cocycle  $\omega$  which describes the associativity constraint in  $V^G\text{-Mod}$ . If this is so, we deduce from Theorem 3.4 that there are bijections between the simple objects of  $V^G\text{-Mod}$  and  $V^N\text{-Mod}$  which themselves correspond to the simple modules of  $D^\omega(G)\text{-Mod}$  and  $D^\omega(N)\text{-Mod}$  respectively which have dimension coprime to  $p$ .

We seek a direct definition of an M-pair for modules over orbifolds such as  $V^G$  and  $V^N$ . We cannot use Eq. 1 as it stands because it makes no sense for infinite-dimensional spaces such as a module over a vertex operator algebra. Instead, we can make use of the expected structure of  $V^G\text{-Mod}$  as a ribbon tensor category, whereby the objects have a *quantum dimension*. Indeed,  $D^\omega(G)\text{-Mod}$  has a *canonical* ribbon structure (cf. [1, 10]), and the quantum dimensions of simple objects are the usual dimensions. Granted the equivalence of  $V^G\text{-Mod}$  and  $D^\omega(G)\text{-Mod}$ , it follows that the quantum dimension of simple objects in  $V^G\text{-Mod}$  are also integers. Then the definition of an M-pair makes sense if we use quantum dimension in place of dimension.

Thus we arrive at the following situation: a pair of ribbon fusion categories  $\mathcal{G}, \mathcal{N}$  whose simple objects have quantum dimensions that are rational integers. We say that  $(\mathcal{G}, \mathcal{N})$  is an M-pair if  $\mu(\mathcal{G}) = \mu(\mathcal{N})$ , where we use (1) with quantum dimension in place of dimension in order to define  $\mu$ . As we have seen, taking  $\mathcal{G}$  to be  $\mathbb{C}[G]\text{-Mod}$ ,  $D^\omega(G)\text{-Mod}$  or  $V^G\text{-Mod}$  and  $\mathcal{N}$  to be  $\mathbb{C}[N]\text{-Mod}$ ,  $D^\omega(N)\text{-Mod}$  or  $V^N\text{-Mod}$  respectively results (conjecturally) in an M-pair. Furthermore, the three versions of MC for groups, quantum doubles of groups, and holomorphic orbifolds, are *equivalent*.

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