Artificial Intelligence Review (2024) 57:145
https://doi.org/10.1007/s10462-024-10757-8

Novel picture fuzzy power partitioned Hamy mean operators with Dempster-Shafer theory and their applications in MCDM

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Accepted: 1 April 2024

Abstract

In some multi-criteria decision-making (MCDM) scenarios, decision makers must address challenges like handling uncertain and incomplete information, managing biases in criteria values, and assessing interrelationships among criteria based on their partitioning as per their characteristics. To tackle these challenges, a picture fuzzy set (PFS) can be utilized to quantify vague information, Hamy mean (HM) can be used to consider criteria interrelationships while the power average (PA) mitigates any kind of biasness. Also, to overcome the limitations of persistence and invariantess in algebraic operations, Dempster-Shafer theory (DST) is employed. By integrating the conventional HM with the traditional PA under partitioning, this paper first introduced the novel power partitioned Hamy mean ($PPtHM^q$) operator. Then, this operator is extended for picture fuzzy numbers (PFNs) with DST and two novel operators are introduced, which are named as picture fuzzy power partitioned Hamy mean ($PFPPtHM^q_{DST}$) and picture fuzzy weighted power partitioned Hamy mean ($PFWPPtHM^q_{DST}$) with some desirable properties. Moreover, based on these operators, a new method for MCDM in the PFS environment has been designed. The paper illustrates their application in selecting the best hotel among four alternatives (B_1, B_2, B_3, B_4) based on five criteria, which are partitioned into two sets. Results indicate that the best and worst alternatives under these operators are hotels B_1 and B_4, respectively. Sensitivity analysis explores the impact of granularity parameter variations, and comparative analysis demonstrates the effectiveness of the presented operators. Overall, the study concludes that these operators offer flexibility, generality, and consistency for analyzing MCDM problems in PFS environments.

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Published online: 16 May 2024

Keywords Picture fuzzy set · Power average · Hamy mean · Partitioned average · Dempster-Shafer theory · Multi criteria decision making

1 Introduction

The broad objective of any multi-criteria decision-making (MCDM) method is to explicitly evaluate multiple criteria and provide the desirable alternative from the available alternatives considering the restrictions. It usually consists of two phases. In the first phase, criteria and alternatives are identified and then a decision matrix with the help of experts having assessment values is prepared. In the second phase, assessment values are used to determine the most desirable alternative, either by classical methods or by aggregation operators.

Thus, in the initial phase of decision-making and data analysis, it is common to encounter assessment values obtained from various sources. These values inherently carry a degree of uncertainty, making precise decision-making challenging. To address this challenge, Zadeh (1965) introduced the concept of the fuzzy set (FS) in 1965. The FS is a valuable mathematical tool that aids in quantifying uncertainty by utilizing the concept of a membership degree. It allows to represent the extent to which an element belongs to a set, providing a flexible means of dealing with uncertainty. However, FS is effective for many applications, but it has limitations. Notably, it does not distinguish between non-membership degree and refusal degree, making it less suitable for scenarios where such distinctions are crucial. Recognizing these limitations, Atanassov (1986) extended the concept of the FS and introduced the intuitionistic fuzzy set (IFS) in 1986. IFS incorporates both membership and non-membership degrees, enabling a more comprehensive representation of uncertainty. To calculate the refusal or indeterminacy degree in an IFS, simply subtracts the sum of the membership and non-membership degrees from one. The utility of IFS extends across a wide range of applications, with researchers employing it to quantify uncertainty in MCDM. It has proven particularly valuable in addressing problems related to operational laws (Wang and Liu 2012), similarity measures (Liu 2017), distance measures (Jin et al. 2009), ranking methods (Zhang et al. 2019), classical MCDM methods (Zhang et al. 2020), aggregation operators (Xu 2008), and many other areas. Researchers have made significant strides in effectively applying IFS across various domains. However, it has become evident that IFS may not be the ideal tool for addressing certain real-life problems that necessitate answers in the format of 'yes', 'no', 'abstain', or 'refusal'. Consider, for instance, the context of voting questions, where all voting outcomes can be categorized into four distinct groups: 'vote for', 'abstain', 'vote against', and 'refuse to vote'. Such scenarios demand a more nuanced and versatile approach. In response to this challenge, the concept of picture fuzzy set (PFS), as introduced by Cuong (2014) in 2014, emerged as a promising alternative. PFS includes three essential degrees: membership degree, non-membership degree, and neutral degree, with the stipulation that the sum of these three degrees equals one or less. This innovative framework not only accommodates the complications of decision-making but also offers a more comprehensive perspective for handling complex problems that across the limitations of traditional IFS. PFS's ability to capture nuances and uncertainties inherent in decision processes has made it a valuable tool in diverse applications, ranging from voting systems to risk assessment and beyond. Similar to IFS, PFS has also gained importance and popularity in the fields of operational laws (Khalil et al. 2019; Seikh and Mandal 2021), similarity measures (Ganie et al. 2020; Ganie and Singh 2021; Wei 2017a), distance measures (Dutta 2018), classical MCDM methods (Biswas and Pamucar 2023; Lin et al. 2020; Singh and

Kumar 2020; Yildirim and Yildirim 2022), and aggregation operators (Akram et al. 2021c; Ahmad et al. 2023; Si et al. 2021; Wang et al. 2017; Yang et al. 2023), etc.

In the second phase, the extracted information is used to evaluate any MCDM problem either by adopting classical methods or aggregation operators based approaches. Generally, classical MCDM methods provide rankings of alternatives after completing a set of predefined steps, but aggregation operators (Ali 2023a, b; Ali and Garg 2023; Ali and Naeem 2023a, b; Ali et al. 2023) based approaches can offer aggregated values for each alternative and then produce rankings of alternatives (Dagistanli 2023; Dincer et al. 2023). So, based on this assertion, the present study considers the aggregation operators based approach to combine the quantitative evaluation of criteria for each alternative and then rank the alternatives in accordance with the aggregated results. In the real-world MCDM problems, the aggregation of values for criteria is a challenging process where the preferences of the decision makers may differ. A good aggregation operator should be all-encompassing, versatile enough to accommodate such variation, consider interrelationships between criteria. Additionally, the preferences of decision makers are a crucial input to MCDM problems because they have some subjectivity. Among different preferences, risk attitudes of decision makers (such as pessimistic, neutral, and optimistic) are a significant type. Such risk attitudes should be able to be captured by an ideal aggregation operator. A literature survey related to aggregation operators and their characteristics for the applicability of PFS in MCDM has been provided in the next section.

1.1 Literature survey

There are typically two types of methods for analyzing MCDM problems. The first type of method employs traditional techniques such as Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS), ELimination Et Choice Translating REality (ELECTRE), VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR), Decision-Making Trial and Evaluation Laboratory (DEMATEL), and so on, whereas the second employs aggregation operators. In general, classical MCDM techniques can only produce rankings of alternatives by following specified steps, whereas aggregation operators based approaches can provide aggregated values for each alternative, and then rankings of alternatives can be done (Sahoo and Goswami 2023). Wei (2017b) presented weighted averaging and weighted geometric operators under the PFS environment. Similarly, Garg (2017) introduced the archimedean weighted averaging and ordered weighted averaging operators for PFS. Khan et al. (2019) introduced Einstein weighted averaging and geometric aggregation operators. Jana et al. (2018) introduced dombi weighted averaging and geometric aggregation operators. Wei (2018) introduced Hamacher weighted averaging and geometric aggregation operators. Apart from these, a number of authors have successfully applied aggregation operators under different fuzzy environments (Akram et al. 2021a, b, 2023a, b, c). A brief summary of some recent research papers based on the characteristics of aggregation operators for PFS are listed in Table 1.

Since the increasing complexity of decision making, we should consider following issues when selecting aggregation operators in PFS environment. The first issue is the operational laws for PFNs. Most of the existing studies mentioned in Table 1 are based on algebraic t-norm and t-conorm operations, which are known to carry certain drawbacks and unfavorable properties. To address these limitations, Dempster (1967); Shafer (1976) introduced the concept of Dempster-Shafer theory (DST). DST serves as a valuable mathematical tool for aggregating information while accounting for possible errors and imprecision. Within the

Table 1 The main characteristics of the existing picture fuzzy aggregation operators

Author(s)	Aggregation operators	t-norms and t-conorms	Whether diminish the negative effect of biased values	Whether consider interrelationship between multiple criteria	Whether consider the partition of the different criteria	Whether quantify the limitations of algebraic operational laws
Wei (2017b)	Picture fuzzy weighted averaging (PFWA)	Algebraic	No	No	No	No
Wei (2017b)	Picture fuzzy weighted geometric (PFWG)	Algebraic	No	No	No	No
Garg (2017)	Picture fuzzy Archimedean weighted averaging (PFAWA)	Archimedean	No	No	No	No
Garg (2017)	Picture fuzzy Archimedean ordered weighted averaging (PFAOWA)	Archimedean	No	No	No	No
Khan et al. (2019)	Picture fuzzy Einstein weighted averaging (PFEWA)	Einstein	No	No	No	No
Khan et al. (2019)	Picture fuzzy Einstein ordered weighted averaging (PFEOWA)	Einstein	No	No	No	No
Jana et al. (2018)	Picture fuzzy Dombi weighted averaging (PFDWA)	Dombi	No	No	No	No
Jana et al. (2018)	Picture fuzzy Dombi weighted geometric (PFDWG)	Dombi	No	No	No	No
Wei (2018)	Picture fuzzy Hamacher weighted averaging (PFHWA)	Hamacher	No	No	No	No
Wei (2018)	Picture fuzzy Hamacher weighted geometric (PFHWG)	Hamacher	No	No	No	No
Wei et al. (2018)	Generalized picture fuzzy Heronian mean (GPFHM)	Algebraic	No	No	No	No
Wei et al. (2018)	Generalized picture fuzzy weighted Heronian mean (GPFWHM)	Algebraic	No	No	No	No

Table 1 continued

Author(s)	Aggregation operators	t-norms and t-conorms	Whether diminish the negative effect of biased values	Whether consider interrelationship between multiple criteria	Whether consider the partition of the different criteria	Whether quantify the limitations of algebraic operational laws
Wang et al. (2018)	Picture fuzzy weighted Muirhead mean (PFWMM)	Algebraic	No	Yes	No	No
Wang et al. (2018)	Picture fuzzy weighted dual Muirhead mean (PFWDMM)	Algebraic	No	Yes	No	No
Ates and Akey (2020)	Picture fuzzy weighted Bonferroni mean (PFWBM)	Algebraic	No	No	No	No
Qin et al. (2021)	Picture fuzzy Archimedean power Maclaurin symmetric mean (PFAPMSM)	Archimedean	Yes	Yes	No	No
Qin et al. (2021)	Picture fuzzy Archimedean power weighted Maclaurin symmetric mean (PFAPWMSM)	Archimedean	Yes	Yes	No	No
Qin et al. (2020)	Picture fuzzy power Muirhead mean ($PFPMM_{DST}$)	DST	Yes	Yes	No	Yes
Qin et al. (2020)	Picture fuzzy weighted power Muirhead mean ($PFWPMM_{DST}$)	DST	Yes	Yes	No	Yes
Luo and Xing (2020)	Picture fuzzy interaction partitioned power Heronian mean (PFIPtHA)	Interaction	No	No	Yes	No
Luo and Xing (2020)	Picture fuzzy weighted interaction partitioned power Heronian mean (PFWIPtHA)	Interaction	No	No	Yes	No
Punetha and Komal (2023)	Picture fuzzy power Muirhead mean (PFPMM)	Algebraic	Yes	Yes	No	No
Punetha and Komal (2023)	Picture fuzzy weighted power Muirhead mean (PFWPMM)	Algebraic	Yes	Yes	No	No

DST framework, basic probability assignment (BPA) quantifies the occurrence probability of criteria in basic events. Furthermore, the belief interval (BI), consisting of the belief function (BF) and the plausibility function (PF), captures the levels of belief and uncertainty associated with a focal element, respectively. Recently, Liu and Gao (2020) developed intuitionstic fuzzy operators under the framework of DST. Then, Qin et al. (2020) extended this work and proposed operators under the framework of DST and applied it to MCDM problem.

The second issue is that in most situations, there are interrelationships among attributes. We should not only consider the attributes themselves but also take into account the interrelationship among attributes. In response to this issue, many aggregation operators under fuzzy information have been proposed by authors (Naseem et al. 2023; Ranjan et al. 2023; Yu 2015). Specifically, in allusion to the interrelationship between or among attributes, some authors developed the Bonferroni mean (BM) (Bonferroni 1950), the Maclaurin Symmetric mean (MSM) (Maclaurin 1729), the Muirhead mean (MM) (Muirhead 1902), and the Hamy mean (HM) (Hara et al. 1998), and applied them to different fuzzy environments. Wei et al. (2018) studied generalized Heronian mean aggregation operators, while Wang et al. (2018) presented Muirhead mean (MM) operators. The BM operators were developed by Ates and Akay (2020). The BM operator only consider the correlation between two attributes. The MM and MSM, are types of aggregation operators that provide the correlation between multiple arguments and the mean value of fused arguments. Hara et al. (1998) established a critical refinement of arithmetic mean (AM) and geometric mean (GM) inequality in 1998, demonstrating that HM and its dual will monotonically lie between AM and GM. The MM and MSM operators do not normalize the correlated values as precisely as HM does, the HM operator presents a more accurate picture of the interrelationship among attributes. It can be found that the HM operator has a greater power and flexibility in addressing information aggregation than the BM operator. By taking advantage of HM, Liu et al. (2019) introduced the HM aggregation operators under interval neutrosophic environment. After that, Liu et al. (2020) introduced partitioned HM under intuitionistic uncertain linguistic environment and applied it to MCDM method for plant location selection.

The third issue is related to weights of criteria. In the fuzzy information aggregation, the computing results are susceptible to extreme values given by biased decision makers and produce low confidence levels for the decision outcomes. To mitigate the negative impact of extreme attribute values, Yager (2001) developed the power average (PA) operator. Theoretical expansions on PA are also quite rich. Xu and Yager (2009) proposed the power geometric (PG) operator, which can analogously model the interactions among aggregated data. The PG operator is considered more suitable for processing the multiplicative preference relationships. Due to this characteristic, the PA operator has been widely utilized in a variety of fields, such as software quality evaluation, multiple attribute (group) decision-making (MADM/MAGDM), and green product development. Using the concept of PA operator, He et al. (2015) proposed the power BM operators under IFS environment. Liu and Liu (2019) integrates PA and MM operators under linguistic q-rung orthopair fuzzy environment. However, Qin et al. (2021) proposed archimedean PMSM operators, while Punetha and Komal (2023) provided PMM operator under PFS environment and applied it to decision making problems.

These operators are successfully applied in the decision making process with the assumption that each attribute is related to the rest of the attributes. In real-life decision making situation it may not be always happened. For example, consider a mobile phone selection problem (Rawat et al. 2023), where the best alternative among a number of mobile phone options is selected based on five attributes: basic requirements (A_1), physical characteristic (A_2), technical features (A_3), brand choice (A_4) and customer excitement (A_5). The attributes are partitioned in two classes: $P_1 = \{A_1, A_2, A_3\}$ and $P_2 = \{A_4, A_5\}$. It is found that attributes A_1, A_2 and A_3 are interrelated therefore, they belong to the same class P_1, similar interpretation can be provided for the members of P_2. However, there is no relationship among the members of P_1 and P_2. The expressed interrelationship structure among the attributes is intrinsically connected to the corresponding input arguments. To effectively deal with these cases, Dutta and Guha (2015) proposed the partitioned BM (PtBM) operator to deal with this situation in which parts of attributes are interrelated and others do not have any relationship. Using the concept of partitioning, Liu et al. (2020) provided interval-valued intuitionistic uncertain linguistic multi attribute decision-making method for plant location selection. After that, Luo and Xing (2020) extended this work and presented partitioned Heronian mean operators with interactional operational laws and applied in MCDM problem. Then, Qin et al. (2020) proposed power PtBM operators using Archimedean operations under q-rung orthopair FS environment. Recently, Ali (2022) proposed PtMSM operators under hesitant FS environment.

From the above literature review, the following challenges in context of PFS are noticed.

(a) Criteria weights play crucial role in the aggregation process that must be calculated methodically to reduce subjective randomness caused by human intervention (He et al. 2015; Liu and Liu 2019; Punetha and Komal 2023; Qin et al. 2021).
(b) Some more generalized operators are required that could provide the interrelationship between criteria more efficiently (Liu et al. 2019, 2020).
(c) As different criteria have different characteristics, so partitioning of criteria as per their characteristics can be integrated in MCDM process (Ali 2022; Luo and Xing 2020).
(d) The algebraic operational laws such as addition and scalar multiplication are not invariant and persistent i.e. they have few drawbacks and undesirable properties, which can be effectively resolved by integrating the concept of DST in MCDM process (Liu and Gao 2020; Qin et al. 2020).

Based on the noted challenges, the detailed motivation and contribution of the present study is given in the following section.

1.2 Motivation and contribution of the study

Based on the reviewed literature as presented in Table 1, it has been observed that in some real-life decision-making problems, a few complications arise during the optimal alternative evaluation process. Some of the noted complications along with their resolving approaches are listed as follows:

(a) In some instances, decision-makers input criteria values that can be either extremely low or high or might have a negative effect on final ranking results. To address this challenge, Yager (2001) introduced the PA operator. Since its introduction, the PA operator has found widespread application across different fuzzy contexts (Liu and Liu 2019; Punetha and Komal 2023).
(b) Many MCDM problems exhibit intricate interrelationships among their criteria. Consequently, it becomes essential to thoroughly examine and assess these interdependencies among criteria values. In order to effectively tackle this challenge, a range of aggregation operators, including but not limited to BM, MM, HM, and MSM, can be employed. The choice of the most suitable operator among these options depends on the specific requirements and characteristics of the problem (Hara et al. 1998; Liu et al. 2019). These operators offer versatile tools to capture and incorporate the nuanced interrelationships

among criteria values, facilitating a more comprehensive and accurate decision-making process.

(c) In certain real-life MCDM scenarios, criteria often exhibit distinct characteristics, necessitating a method to categorize and handle them accordingly. One effective approach to address this challenge involves the partitioning of criteria based on their unique characteristics. Dutta and Guha (2015) introduced partitioned BM operators, which consider all possible criterion partitions of a designated size. Researchers have also explored the integration of partitioning concepts with various operators in different fuzzy contexts (Ali 2022; Qin et al. 2020).

(d) Table 1 highlights that a significant portion of aggregation operators relies on algebraic t-norm and t-conorm operations, which are known to carry certain drawbacks and unfavorable properties (Liu and Gao 2020). Since the algebraic operational laws are not invariant and persistent for addition and scalar multiplication, respectively. To address these limitations, Dempster and Shafer introduced the DST. In accordance with findings from (Qin et al. 2020), there exists a close relationship between PFS and DST. This insight allows for the direct application of DST aggregation rules to combine criteria represented by PFNs in real decision-making scenarios. Consequently, PFNs can seamlessly transition into BPAs within the DST framework, preserving all pertinent information, while the operational rules of PFNs can be transformed into operational laws governing BIs. These transformations eliminate the shortcomings and constraints associated with the algebraic operational laws of PFNs, leading to more coherent and compelling aggregated results.

Based on the comprehensive literature review meticulously presented in Table 1 and the motivations outlined earlier, it has become evident that, to the best of our knowledge, no prior research has attempted to concurrently address the complex complexities inherent in the scope of MCDM. These complexities involved various aspects, including but not limited to the integration of PA, the consideration of HM, and the partitioning of criteria, all of which are conclusive in real-world decision-making scenarios. By integrating these concepts, a novel aggregation operator named $PPtHM^q$ is introduced. Moreover, the innovation does not stop here. The paper extends the application of this ground breaking aggregation operator into the domain of PFS, introducing a novel operators. The integration with DST is a transformative step forward and stands as the central thrust of our research endeavor. This innovation represents the core contribution of this paper, as it equips decision-makers with a powerful and versatile tool to address the multifaceted challenges that confront real-world MCDM problems. Ultimately, our goal is to provide a comprehensive framework that empowers decision-makers to make more informed, accurate, and robust decisions in complex decision-making contexts. This article will achieve the following objectives:

(a) A novel $PPtHM^q$ operator has been developed by integrating the concepts of PA, HM, and partitioning of criteria.

(b) The $PPtHM^q$ operator is then extended for the PFS environment, and two novel operators named $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ with DST, as well as some desirable properties, are introduced.

(c) A MCDM approach based on the developed $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ operators has been presented.

(d) An example of the best hotel selection illustrates the effectiveness of the proposed approach. To demonstrate the efficiency of the proposed operators, sensitivity and comparative analyzes were carried out.

The remainder of the paper is divided into the following six sections. The basic definitions are provided in Section 2. The $PPtHM^q$, $PFPPtHM^q_{DST}$, and $PFWPPtHM^q_{DST}$ operators are developed in Section 3, and their desirable properties are also discussed. A MCDM method based on proposed operators has been designed in Section 4. In Section 5, a numerical example of the best hotel selection is presented, along with sensitivity and comparative analyzes, to demonstrate the efficiency and stability of the proposed operators. Finally, the paper ends with some concluding remarks and provides limitations and some possible future directions. The meanings of the symbols used in this article are summarized in Table 2 for easy reading.

2 Some preliminary concepts

This section briefly recalls some of the basic concepts such as definition of PFS, operational laws, score and accuracy functions, distance and support functions for PFNs and DST framework along with PA, HM, PHM and PtHM operators.

2.1 Picture fuzzy set

Definition 1 Cuong (2014) Let X be a universal set, then PFS on X is as follows:

$$K = \{(x, \mu_K(x), \eta_K(x), \nu_K(x)) : x \in X\} \tag{1}$$

where, $\mu_K(x), \eta_K(x), \nu_K(x) \in [0, 1]$ are called as the degrees of positive, neutral and negative membership of x in K respectively with condition $0 \leq \mu_K(x)+\eta_K(x)+\nu_K(x) \leq 1$, $\forall x \in X$. Then for $x \in X, \pi_K(x) = 1-(\mu_K(x)+\eta_K(x)+\nu_K(x))$ could be called the degree of refusal membership of x in K. For sake of convenience, we can call $k = (\mu_k, \eta_k, \nu_k)$ as a picture fuzzy number (PFN).

Table 2 Mathematical symbols with description

Symbols	Description	Symbols	Description
K	Picture fuzzy set	$d_{DST}(x, y)$	Distance function for DST
$\tilde{K}$	Picture fuzzy set with DST	Sup	Support function
k	Picture fuzzy number	Sup_{DST}	Support function for DST
$\tilde{k}$	Picture fuzzy number with DST	T	Sum of support functions
μ	Membership degree	q	Granularity parameter vector
η	Neutral membership degree	P_t	t^{th} partition of criteria set
ν	Non-membership degree	d	Number of partition
π	Refusal membership degree	$\tau_i, \bar{\tau}_i$	Power weights
S	Score function	B	Set of alternative
S_{DST}	Score function for DST	G	Set of criteria
H	Accuracy function	D	Picture fuzzy decision matrix
H_{DST}	Accuracy function for DST	Q	Normalized picture fuzzy decision matrix
$d(x, y)$	Distance function	Q_{DST}	Normalized picture fuzzy decision matrix for DST

Definition 2 Wei (2017b) Let $k = (\mu, \eta, \nu)$, $k_1 = (\mu_1, \eta_1, \nu_1)$ and $k_2 = (\mu_2, \eta_2, \nu_2)$ be three PFNs, and let λ be a positive real number. Then algebraic operational laws for PFNs are as follows:

(i) $k_1 \oplus k_2 = (\mu_1 + \mu_2 - \mu_1\mu_2, \eta_1\eta_2, \nu_1\nu_2)$;
(ii) $k_1 \otimes k_2 = (\mu_1\mu_2, \eta_1 + \eta_2 - \eta_1\eta_2, \nu_1 + \nu_2 - \nu_1\nu_2)$;
(iii) λk=$(1 - (1 - \mu)^\lambda, \eta^\lambda, \nu^\lambda)$;
(iv) k^λ=$(\mu^\lambda, 1 - (1 - \eta)^\lambda, 1 - (1 - \nu)^\lambda)$.

Definition 3 Wei (2017b) Let $k = (\mu, \eta, \nu)$ be a PFN then the score $(S(k))$ and accuracy $(H(k))$ functions of k are defined as $S(k) = \mu - \nu$ and $H(k) = \mu + \eta + \nu$ respectively. Let k_1 and k_2 be two PFNs, then using the definitions of their score and accuracy functions, the ranking of these PFNs can be done as followings.

(a) if $S(k_1) > S(k_2)$, then $k_1 \succ k_2$
(b) if $S(k_1) = S(k_2)$, then
   (i) if $H(k_1) > H(k_2)$, then $k_1 \succ k_2$,
   (ii) if $H(k_1) = H(k_2)$, then $k_1 = k_2$.

Definition 4 Dutta (2018) If $k_i = (\mu_i, \eta_i, \nu_i)$ and $k_j = (\mu_j, \eta_j, \nu_j)$ are two PFNs, the normalized Hamming distance between them is calculated as follows:

$$d(k_i, k_j) = \frac{1}{2}(|\mu_i - \mu_j| + |\eta_i - \eta_j| + |\nu_i - \nu_j| + |\pi_i - \pi_j|) \tag{2}$$

where, π_i=$(1 - \mu_i - \eta_i - \nu_i)$ and π_j=$(1 - \mu_j - \eta_j - \nu_j)$.

Using Hamming distance, the support between k_i and k_j is defined as follows:

$$\mathrm{Sup}(k_i, k_j) = 1 - d(k_i, k_j) \tag{3}$$

The support function satisfies following properties:

(a) $\mathrm{Sup}(k_i, k_j) \in [0, 1]$,
(b) $\mathrm{Sup}(k_i, k_j) = \mathrm{Sup}(k_j, k_i)$,
(c) If $d(k_i, k_j) < d(k_l, k_h)$, then $\mathrm{Sup}(k_i, k_j) > \mathrm{Sup}(k_l, k_h)$.

2.2 Dempster-Shafer theory

The DST proposed by Dempster (1967); Shafer (1976). It is a successful and well-liked tool for combining ambiguous or imprecise information. In the DST framework, BPA describes the occurrence rate of criteria in basic events, and the BI, which is made up of the BF and the PF, reflects the belief and uncertainty of the focus element, respectively. This section goes over fundamental ideas including BPA, BF, PF, and BI for DST. Also, this section extended the concepts of PFS, operational laws of PFNs, score and accuracy functions for PFNs, distance and support functions for PFNs in the framework of DST.

Definition 5 Let X denotes a set of n mutually exclusive and exhaustive objects, then BPA on X denotes a mapping $E : P(X) \to [0, 1]$ that satisfies the following conditions:

$$\sum_{Y \subseteq P(X)} E(Y) = 1 \text{ and } E(\phi) = 0. \tag{4}$$

Definition 6 Let a BPA E on X, then the BF is as follows:

$$Bel(X) = \sum_{Y \subseteq X} E(Y) \tag{5}$$

where, $E(X) > 0$.

Definition 7 Let a BPA E on X, then the PF is as follows:

$$Pl(X) = \sum_{Y \cap X \neq \phi} E(Y) = 1 - Bel(\bar{Y}) \tag{6}$$

The complementary set of X is denoted by $\bar{Y}$.

Definition 8 Let a BPA E on X, then the BI is defined as follows:

$$BI(X) = [Bel(X), Pl(X)] \tag{7}$$

Definition 9 Qin et al. (2020) Let X be a universal set then PFS with DST on X is defined as:

$$\tilde{K} = \{(x, BI_{\tilde{K}}(x)) : x \in X\}, \tag{8}$$

where $BI_{\tilde{K}}(x)$=$[Bel_{\tilde{K}}(x), Pl_{\tilde{K}}(x)]$=$[\mu_{\tilde{K}}(x), 1-\eta_{\tilde{K}}(x)-\nu_{\tilde{K}}(x)]$ and $\mu_{\tilde{K}}(x), \eta_{\tilde{K}}(x), \nu_{\tilde{K}}(x) \in [0, 1]$ are called as the degrees of positive, neutral and negative membership of x in $\tilde{K}$ respectively with condition $0 \le \mu_{\tilde{K}}(x) \le 1-\eta_{\tilde{K}}(x)-\nu_{\tilde{K}}(x) \le 1, \forall x \in X$. Then for $x \in X$, $\pi_{\tilde{K}}(x) = 1 - (\mu_{\tilde{K}}(x) + \eta_{\tilde{K}}(x) + \nu_{\tilde{K}}(x))$ could be called the degree of refusal membership of x in $\tilde{K}$. For sake of convenience, we can call $\tilde{k} = BI_{\tilde{k}} = [Bel_{\tilde{k}}, Pl_{\tilde{k}}] = [\mu_{\tilde{k}}, 1 - \eta_{\tilde{k}} - \nu_{\tilde{k}}]$ a PFN with DST.

Definition 10 Qin et al. (2020) Let $\tilde{k} = BI = [Bel, Pl] = [\mu, 1 - \eta - \nu]$ and $\tilde{k}_1 = BI_1 = [Bel_1, Pl_1] = [\mu_1, 1 - \eta_1 - \nu_1]$, $\tilde{k}_2 = BI_2 = [Bel_2, Pl_2] = [\mu_2, 1 - \eta_2 - \nu_2]$,...,$\tilde{k}_n = BI_n = [Bel_n, Pl_n] = [\mu_n, 1 - \eta_n - \nu_n]$ be $n + 1$ PFNs with DST, then we obtain the following expressions:

1. $\tilde{k}_1 \oplus \tilde{k}_2 \oplus ... \oplus \tilde{k}_n = \left[\frac{\sum_{i=1}^n Bel_i}{n}, \frac{\sum_{i=1}^n Pl_i}{n}\right] = \left[\frac{\sum_{i=1}^n \mu_i}{n}, \frac{\sum_{i=1}^n (1-\eta_i-\nu_i)}{n}\right]$;
2. $\tilde{k}_1 \otimes \tilde{k}_2 = [Bel_1 Bel_2, Pl_1 Pl_2] = [\mu_1\mu_2, (1 - \eta_1 - \nu_1)(1 - \eta_1 - \nu_1)]$;
3. $\delta\tilde{k} = [Bel, Pl] = [\delta\mu, \delta(1 - \eta - \nu)], \delta \in [0, 1]$;
4. $\tilde{k}^\lambda = [Bel, Pl] = [\mu^\lambda, (1 - \eta - \nu)^\lambda], \lambda \in [0, \infty)$.

Definition 11 Qin et al. (2020) Let $\tilde{k} = BI = [\mu, 1 - \eta - \nu]$ be a PFN with DST then the score $(S(\tilde{k}))$ and accuracy $(H(\tilde{k}))$ functions are as follows:

$$S_{DST}(\tilde{k}) = \frac{(Bel + Pl)}{2} = \frac{(1 + \mu - \eta - \nu)}{2}$$
$$H_{DST}(\tilde{k}) = (Pl - Bel) = (1 - \mu - \eta - \nu)$$

Let $\tilde{k}_1 = BI_1 = [\mu_1, 1 - \eta_1 - \nu_1]$ and $\tilde{k}_2 = BI_2 = [\mu_2, 1 - \eta_2 - \nu_2]$ be two PFNs, then using the definitions of their score and accuracy functions, the ranking of these PFNs can be done as followings.

(a) if $S_{DST}(\tilde{k}_1) > S_{DST}(\tilde{k}_2)$ then $\tilde{k}_1 > \tilde{k}_2$;
(b) if $S_{DST}(\tilde{k}_1) = S_{DST}(\tilde{k}_2)$, then

   (i) if $H_{DST}(\tilde{k}_1) > H_{DST}(\tilde{k}_2)$ then $\tilde{k}_1 < \tilde{k}_2$;

(ii) if $H_{DST}(\tilde{k}_1) = H_{DST}(\tilde{k}_2)$ then $\tilde{k}_1 = \tilde{k}_2$.

Definition 12 Qin et al. (2020) If $\tilde{k}_i = BI_i = [\mu_i, 1 - \eta_i - \nu_i]$ and $\tilde{k}_j = BI_j = [\mu_j, 1 - \eta_j - \nu_j]$ are two PFNs, the normalized Hamming distance between them is calculated as follows:

$$d_{DST}(\tilde{k}_i, \tilde{k}_j) = \frac{1}{2}(|\mu_i - \mu_j| + |\eta_i - \eta_j| + |\nu_i - \nu_j|) \tag{9}$$

Using Hamming distance, the support between $\tilde{k}_i$ and $\tilde{k}_j$ is as follows:

$$\text{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j) = 1 - d_{DST}(\tilde{k}_i, \tilde{k}_j) \tag{10}$$

The support function satisfies following properties:

(a) $\text{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j) \in [0, 1]$,
(b) $\text{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j) = \text{Sup}_{DST}(\tilde{k}_j, \tilde{k}_i)$,
(c) If $d_{DST}(\tilde{k}_i, \tilde{k}_j) < d_{DST}(\tilde{k}_l, \tilde{k}_h)$, then $\text{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j) > \text{Sup}_{DST}(\tilde{k}_l, \tilde{k}_h)$.

2.3 Power average, Hamy mean, power Hamy mean, partitioned Hamy mean operators

This section briefly provides the definitions of PA, HM, PHM and PtHM operators.

Definition 13 Yager (2001) Let $a_i,\ i = 1, 2, ..., n$ be a set of n crisp numbers, then the PA operator is defined as:

$$PA(a_1, a_2, ...a_n) = \frac{\sum_{i=1}^{n}(1 + T(a_i))a_i}{\sum_{i=1}^{n}(1 + T(a_i))} \tag{11}$$

where, $T(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \text{Sup}(a_i, a_j)$ and $\text{Sup}(a_i, a_j)$ is the support for a_i and a_j, and also satisfies the properties as mentioned in Definition 4 for crisp numbers. The PA operator is a non-linear weighted average aggregation operator, and the weight $\frac{\sum_{i=1}^{n}(1+T(a_i))a_i}{\sum_{i=1}^{n}(1+T(a_i))}$ of the argument a_i depends on all the input arguments $a_i (i = 1, 2, ..., n)$ and allows the argument values to support each other in the aggregation process. Moreover, the support measure is possibly a similarity index. The closer two values are, the more they support each other. The PA operator satisfies following properties:

1. **Idempotency:** $\text{PA}(a_1, a_2, ...a_n) = a_i$ if $a_i = a$ for all i;
2. **Boundedness:** $\min(a_i) \leq \text{PA}(a_1, a_2, ...a_n) \geq \max(a_i)$ for all i;
3. **Commutativity:** Any permutation of the arguments has the same PA operator;
4. **Non-monotonous:** An increase in one of the arguments can result in a decrease in the PA operator.

Definition 14 Hara et al. (1998) Let $a_i,\ i = 1, 2, ..., n$ be a set of n crisp numbers, then their HM is defined as:

$$\text{HM}^q(a_1, a_2, ..., a_n) = \frac{\sum\limits_{1 \leq i_1 < ... < i_q \leq n} \left(\prod\limits_{j=1}^{q} a_{i_j}\right)^{\frac{1}{q}}}{C_n^q} \tag{12}$$

where, q is a parameter ($q = 1, 2, ..., n$); $i_1, i_2, ..., i_q$ are q integer values taken from the set $\{1, 2, ..., n\}$ and C_n^q is the Binomial coefficient i.e., $C_n^q = \frac{n!}{q!(n-q)!}$.

Definition 15 Liu et al. (2019) Let $a_i,\ i = 1, 2, ..., n$ be a set of n crisp numbers, then the PHM is defined as:

$$\mathrm{PHM}^q(a_1, a_2, ..., a_n) = \frac{\sum_{1\le i_1 < ... < i_q \le n} \prod_{j=1}^{q} \left(\frac{n(1+T(a_i))a_{i_j}}{\sum_{j=1}^{n}(1+T(a_j))} \right)^{\frac{1}{q}}}{C_n^q} \tag{13}$$

where, q is a parameter ($q = 1, 2, ..., n$) and $i_1, i_2, ..., i_q$ are q integer values taken from the set $\{1, 2, ..., n\}$ while $C_n^q = \frac{n!}{q!(n-q)!}$. The $T(a_i) = \sum_{\substack{j=1 \\ j\ne i}}^{n} \mathrm{Sup}(a_i, a_j)$ and $\mathrm{Sup}(a_i, a_j)$ are the supports for a_i and a_j respectively, and also satisfy the properties as mentioned in Definition 4 for crisp numbers.

Definition 16 Liu et al. (2020) Let $A = (a_1, a_2, ..., a_n)$ be a set of n crisp numbers, and the set A is partitioned into d-number of subsets $P_1, P_2, ..., P_d$ of arguments a_i's with two important partitioning conditions $P_i \cap P_j = \phi$ and $\bigcup_{t=1}^{d} P_t = A$. The parameter $q_1, q_2, ..., q_t, ..., q_d$ take values $q_t = 1, 2, ..., |P_t|$, and $|P_t|$ is the cardinality of P_t $(t = 1, 2, ..., d)$. Also $q = (q_1, q_2, ..., q_t, ..., q_d)$ is the granularity parameter vector. Then the $PtHM$ operator is defined as follows:

$$PtHM^q(a_1, a_2, ..., a_n) = \frac{1}{d}\sum_{t=1}^{d} \left(\frac{\sum_{1\le i_1 < ... < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} a_{i_j} \right)^{\frac{1}{q_t}}}{C_{|P_t|}^{q_t}} \right) \tag{14}$$

where $i_1, i_2, ..., i_{q_t}$ traverses all the q_t-tuple combination of $\{1, 2, ..., |P_t|\}$, and $C_{|P_t|}^{q_t} = \frac{|P_t|!}{q_t!(|P_t|-q_t)!}$ is the Binomial coefficient. The process of partitioning is shown in Fig. 1.

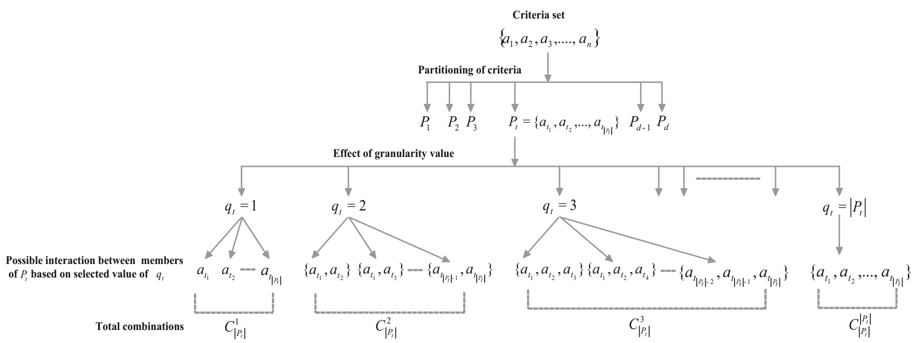


Fig. 1 Process of partitioning for criteria

3 Novel picture fuzzy power partitioned Hamy mean operators with Dempster-Shafer theory

This section, first proposed a novel $PPtHM^q$ operator then this operator is extended for PFS environment. Two novel operators named as $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ are introduced by integrating PFS, PA, HM, and partitioning with DST.

3.1 Power partitioned Hamy mean operator

Definition 17 Let $A = (a_1, a_2, ..., a_n)$ be a collection of n-crisp numbers, and the arguments of a_i $(i = 1, 2, .., n)$ are partitioned into d-number of sets $P_1, P_2, ..., P_d$ with $P_i \cap P_j = \phi$ and $\bigcup_{t=1}^{d} P_t = A$. The parameter $q_1, q_2, ..., q_t, ..., q_d$ take values $q_t = 1, 2, ..., |P_t|$, and $|P_t|$ is the cardinality of P_t $(t = 1, 2, ..., d)$. Also $q = (q_1, q_2, ..., q_t, ..., q_d)$ is the granularity parameter vector. Therefore, the $PPtHM^q$ operator of n-dimension is defined as follows:

$$PPtHM^q(a_1, a_2, ..., a_n) = \frac{1}{d} \bigoplus_{t=1}^{d} \left(\frac{\bigoplus_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\bigotimes_{j=1}^{q_t} \left(\frac{n(1 + T(a_i))a_{i_j}}{\sum_{j=1}^{n}(1 + T(a_j))} \right)^{\frac{1}{q_t}} \right)}{C^{q_t}_{|P_t|}} \right) \tag{15}$$

where $i_1, i_2, ..., i_{q_t}$ traverses all the q_t-tuple combination of $\{1, 2, ..., |P_t|\}$, and $C^{q_t}_{|P_t|} = \frac{|P_t|!}{q_t!(|P_t|-q_t)!}$ is the Binomial coefficient. Meanwhile, $T(a_i) = \sum_{\substack{j=1 \\ j \ne i}}^{n} \text{Sup}(a_i, a_j)$ and $\text{Sup}(a_i, a_j)$ is the support for a_i and a_j, and also satisfy the properties as mentioned in Definition 4 for crisp numbers.

3.2 Picture fuzzy power partitioned Hamy mean operator

Definition 18 Let $\tilde{K} = \{\tilde{k}_i : \tilde{k}_i = [\mu_i, 1 - \eta_i - \nu_i], i = 1, 2, ..., n\}$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$, and the arguments of $\tilde{k}_i$ $(i = 1, 2, .., n)$ are partitioned into d-number of sets $P_1, P_2, ..., P_d$ with $P_i \cap P_j = \phi$ and $\bigcup_{t=1}^{d} P_t = \tilde{K}$. The parameter $q_1, q_2, ..., q_t, ..., q_d$ is with $q_t = 1, 2, ..., |P_t|$, and $|P_t|$ is the cardinality of P_t $(t = 1, 2, ..., d)$. Also $q = (q_1, q_2, ..., q_t, ..., q_d)$ is the granularity parameter vector. Therefore, the $PFPPtHM^q_{DST}$ operator of n-dimension is given as:

$$PFPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \frac{1}{d} \bigoplus_{t=1}^{d} \left(\frac{\bigoplus_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\bigotimes_{j=1}^{q_t} \left(\frac{n(1 + T(\tilde{k}_i))\tilde{k}_{i_j}}{\sum_{j=1}^{n}(1 + T(\tilde{k}_j))} \right)^{\frac{1}{q_t}} \right)}{C^{q_t}_{|P_t|}} \right) \tag{16}$$

where $i_1, i_2, ..., i_{q_t}$ traverses all the q_t-tuple combination of $\{1, 2, ..., |P_t|\}$, and $C_{|P_t|}^{q_t} = \frac{|P_t|!}{q_t!(|P_t|-q_t)!}$ is the Binomial coefficient. Meanwhile, $T(\tilde{k}_i) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j)$ and $\mathrm{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j)$ is the support for $\tilde{k}_i$ and $\tilde{k}_j$, and also satisfy the properties as mentioned in Definition 12.

For convenience, let

$$\tau_i = \frac{(1+T(\tilde{k}_i))}{\sum_{j=1}^{n}(1+T(\tilde{k}_j))} \tag{17}$$

where, $(\tau_1, \tau_2, ..., \tau_n)^T$ is called as power weighting vector (PWV), which satisfy the conditions $\tau_i \in [0, 1]$ and $\sum_{i=1}^{n} \tau_i = 1$.

Then (16) can be written as:

$$PFPPtHM_{DST}^{q}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \frac{1}{d} \bigoplus_{t=1}^{d} \left(\frac{\bigoplus_{1\leq i_1<...<i_{q_t}\leq|P_t|} \left(\bigotimes_{j=1}^{q_t} (n\tau_{i_j}\tilde{k}_{i_j}) \right)^{\frac{1}{q_t}}}{C_{|P_t|}^{q_t}} \right) \tag{18}$$

Theorem 1 *Let* $\tilde{K} = \{\tilde{k}_i : \tilde{k}_i = [\mu_i, 1-\eta_i-\nu_i], i = 1, 2, ..., n\}$ *be the corresponding BIs of PFNs* $K = \{k_i : k_i = (\mu_i, \eta_i, \nu_i), i = 1, 2, ..., n\}$. *The value aggregated by the* $PFPPtHM_{DST}^{q}$ *operator is still a BI and is defined as follows:*

$$PFPPtHM_{DST}^{q}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \left[\frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1\leq i_1<...<i_{q_t}\leq|P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j}\mu_{i_j}) \right)^{\frac{1}{q_t}} \right) \right), \right.$$
$$\left. \frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1\leq i_1<...<i_{q_t}\leq|P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right) \right] \tag{19}$$

Proof On the basis of operational laws for PFNs with DST we have, $n\tau_{i_j}\tilde{k}_{i_j} = [n\tau_{i_j}\mu_{i_j}, n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j})]$.

Then,

$$\bigotimes_{j=1}^{q_t} (n\tau_{i_j}\tilde{k}_{i_j}) = \left[\prod_{j=1}^{q_t} (n\tau_{i_j}\mu_{i_j}), \prod_{j=1}^{q_t} (n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j})) \right]$$

$$\left(\bigotimes_{j=1}^{q_t} (n\tau_{i_j}\alpha_{i_j}) \right)^{\frac{1}{q_t}} = \left[\left(\prod_{j=1}^{q_t} (n\tau_{i_j}\mu_{i_j}) \right)^{\frac{1}{q_t}}, \left(\prod_{j=1}^{q_t} (n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j})) \right)^{\frac{1}{q_t}} \right]$$

$$\bigoplus_{1\leq i_1<...<i_{q_t}\leq|P_t|} \left(\bigotimes_{j=1}^{q_t} (n\tau_{i_j}\tilde{k}_{i_j}) \right)^{\frac{1}{q_t}}$$

$$= \left[\frac{1}{C_{|P_t|}^{q_t}} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right), \right.$$

$$\left. \frac{1}{C_{|P_t|}^{q_t}} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} (1 - \eta_{i_j} - \nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right]$$

Now,

$$\frac{1}{C_{|P_t|}^{q_t}} \left(\bigoplus_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\bigotimes_{j=1}^{q_t} (n\tau_{i_j} \tilde{k}_{i_j}) \right)^{\frac{1}{q_t}} \right) = \left[\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right), \right.$$

$$\left. \frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} (1 - \eta_{i_j} - \nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right]$$

Then,

$$\bigoplus_{t=1}^{d} \left(\frac{\bigoplus_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\bigotimes_{j=1}^{q_t} (n\tau_{i_j} \tilde{k}_{i_j}) \right)^{\frac{1}{q_t}}}{C_{|P_t|}^{q_t}} \right) = \left[\frac{1}{d} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right) \right), \right.$$

$$\left. \frac{1}{d} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} (1 - \eta_{i_j} - \nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right) \right]$$

Finally,

$$\frac{1}{d} \bigoplus_{t=1}^{d} \left(\frac{\bigoplus_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\bigotimes_{j=1}^{q_t} (n\tau_{i_j} \tilde{k}_{i_j}) \right)^{\frac{1}{q_t}}}{C_{|P_t|}^{q_t}} \right) = \left[\frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right) \right), \right.$$

$$\left. \frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} (1 - \eta_{i_j} - \nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right) \right]$$

Hence the proof is complete.

The following Theorem states the commutativity of the $PFPPtHM_{DST}^{q}$ operator: □

Property 1 (Commutativity)**:** Let $\tilde{k}_i = [\mu_i, 1 - \eta_i - \nu_i]$ $(i = 1, 2, ..., n)$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$. If $(\tilde{k}'_1, \tilde{k}'_2, ..., \tilde{k}'_n)$ is any permutation of $(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n)$, $\tilde{k}'_i = [\mu'_i, 1 - \eta'_i - \nu'_i]$, then

$$PFPPtHM_{DST}^{q}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = PFPPtHM_{DST}^{q}(\tilde{k}'_1, \tilde{k}'_2, ..., \tilde{k}'_n). \tag{20}$$

Proof Given $(\tilde{k}'_1, \tilde{k}'_2, ..., \tilde{k}'_n)$ is any permutation of $(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n)$, based on Theorem 1, it is not difficult to obtain following expression:

$$PFPPtHM_{DST}^{q}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \left[\frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C_{|P_t|}^{q_t})^2} \left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\tau_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right) \right), \right.$$

$$\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\Bigg]$$

$$=\Bigg[\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu'_{i_j})\right)^{\frac{1}{q_t}}\right)\right),$$

$$\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta'_{i_j}-\nu'_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\Bigg]$$

$$= PFPPtHM_{DST}^{q}(\tilde{k}'_1,\tilde{k}'_2,...,\tilde{k}'_n)$$

Thus we can obtain

$$PFPPtHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n) = PFPPtBM_{DST}^{q}(\tilde{k}'_1,\tilde{k}'_2,...,\tilde{k}'_n)$$

It is also noted that $PFPPtHM_{DST}^{q}$ operator does not satisfied idempotency and boundedness (Liu and Gao 2020; Qin et al. 2020). However, if a small modification is done by taking multiplication of the aggregated value with $dC_{|P_t|}^{q_t}$ then the modified $PFPPtHM_{DST}^{q}$ operator is denoted by $MPFPPtHM_{DST}^{q}$ is defined below which have idempotency and boundedness.

$$MPFPtPHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n) = \Bigg[\frac{1}{d}\sum_{t=1}^{d}\left(\frac{1}{C_{|P_t|}^{q_t}}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right),$$

$$\frac{1}{d}\sum_{t=1}^{d}\left(\frac{1}{C_{|P_t|}^{q_t}}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\Bigg]$$

□

Property 2 (Idempotency)**:** Let $\tilde{k}_i = [\mu_i, 1-\eta_i-\nu_i]$ $(i = 1, 2, ..., n)$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$. Also let $\tilde{k}_i = \tilde{k} = [\mu, 1-\eta-\nu]$, then

$$MPFPtPHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n) = \tilde{k}.$$

Proof Given that $\tilde{k}_i = \tilde{k} = [\mu, 1-\eta-\nu]$ for all i , then

$$MPFPPtHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n) = \Bigg[\frac{1}{d}\sum_{t=1}^{d}\left(\frac{1}{C_{|P_t|}^{q_t}}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right),$$

$$\frac{1}{d}\sum_{t=1}^{d}\left(\frac{1}{C_{|P_t|}^{q_t}}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\Bigg]$$

Because,

$$\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j})\right) = 1$$

We can obtain, $MPFPPtHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)$

$$=\Bigg[\frac{1}{d}\sum_{t=1}^{d}\left(\frac{1}{C_{|P_t|}^{q_t}}\left(\sum_{1\le i_1<\ldots<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right),$$

$$\frac{1}{d}\sum_{t=1}^{d}\left(\frac{1}{C_{|P_t|}^{q_t}}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(1-\eta_{i_j}-\nu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\Bigg]$$
$$=[\mu,1-\eta-\nu]=\tilde{k}$$

□

Property 3 (Boundedness)**:** Let $\tilde{k}_i=[\mu_i,1-\eta_i-\nu_i]$ $(i=1,2,...,n)$ be the corresponding BIs of PFNs $k_i=(\mu_i,\eta_i,\nu_i)$ $(i=1,2,...,n)$. If $\tilde{k}^-=[\min(\mu_i),\min(1-\eta_i-\nu_i)]$ and $\tilde{k}^+=[\max(\mu_i),\max(1-\eta_i-\nu_i)]$ for $(i=1,2,...,n)$. Then

$$\tilde{k}^-\leq MPFPPtHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)\leq\tilde{k}^+ \tag{21}$$

Proof Since $\min(\mu_i)\leq(\mu_i)\leq\max(\mu_i)$, it is easy to obtain

$$d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(\min\mu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\right)$$
$$\leq d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)$$
$$\leq d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(\max\mu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\right)$$

Because,

$$\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(m\tau_{i_j})\right)=1$$

We can obtain,

$$d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(\min\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)$$
$$\leq d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)$$
$$\leq d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(\max\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)$$

and

$$\min(\mu_i)\leq d(C_{|P_t|}^{q_t})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C_{|P_t|}^{q_t})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)\leq\max(\mu_i)$$

Similarly, we can obtain

$$\min(1-\eta_i-\nu_i) \leq d(C^{q_t}_{|P_t|})\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\right)$$
$$\leq \max(1-\eta_i-\nu_i)$$

Thus we can obtain,

$$\frac{\min(\mu_i)+\min(1-\eta_i-\nu_i)}{2}$$
$$\leq \frac{d(C^{q_t}_{|P_t|})}{2}\left(\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)\right.$$
$$\left.+\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\right)\right)$$
$$\leq \frac{\max(\mu_i)+\max(1-\eta_i-\nu_i)}{2}$$

According to Definition 11, we have

$$S_{DST}(\tilde{k}^-) = \frac{\min(\mu_i)+\min(1-\eta_i-\nu_i)}{2}$$
$$S_{DST}(\tilde{k}^+) = \frac{\max(\mu_i)+\max(1-\eta_i-\nu_i)}{2}$$
$$S_{DST}(MPFPtPHM^q_{DST}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n))$$
$$= \frac{d(C^{q_t}_{|P_t|})}{2}\left(\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right)\right)\right)$$
$$+\frac{d(C^{q_t}_{|P_t|})}{2}\left(\left(\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\leq i_1<\ldots<i_{q_t}\leq|P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\right)\right)$$

Thus, we have

$$S_{DST}(\tilde{k}^-) \leq MPFPPtHM^q_{DST}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n) \leq S_{DST}(\tilde{k}^+).$$

According to comparison rules, we have

$$\tilde{k}^- \leq MPFPPtHM^q_{DST}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n) \leq \tilde{k}^+.$$

In the following, an example is provided to illustrate the calculation process. □

Example 1 Let $\tilde{k}_1 = (0.53, 0.33, 0.09)$, $\tilde{k}_2 = (0.89, 0.08, 0.03)$, $\tilde{k}_3 = (0.42, 0.35, 0.18)$, $\tilde{k}_4 = (0.08, 0.89, 0.02)$ and $\tilde{k}_5 = (0.73, 0.12, 0.08)$ be five PFNs corresponding to the criteria G_1, G_2, G_3, G_4, G_5 respectively, which is partitioned into 2-groups $P_1 = \{G_1, G_3\}$ and $P_2 = \{G_2, G_4, G_5\}$ i.e., the cardinality of these partitions are $|P_1| = 2$ and $|P_2| = 3$. Then, utilizing $PFPPtHM^q_{DST}$ operator, the aggregated value is obtained by using the following steps.

Step 1. Convert these PFNs into DST form of PFNs using Definition 9. Then, these numbers are converted into $\tilde{k}_1 = [0.53, 0.58]$, $\tilde{k}_2 = [0.89, 0.89]$, $\tilde{k}_3 = [0.42, 0.47]$, $\tilde{k}_4 = [0.08, 0.09]$ and $\tilde{k}_5 = [0.73, 0.80]$.

Step 2. Calculate $\text{Sup}(\tilde{k}_i, \tilde{k}_j)$, where $i, j = 1, 2, 3, 4, 5$, using (10). Thus, we have

$$\begin{aligned}
&\text{Sup}(\tilde{k}_1, \tilde{k}_2) = 0.665, \text{Sup}(\tilde{k}_1, \tilde{k}_3) = 0.890, \text{Sup}(\tilde{k}_1, \tilde{k}_4) = 0.460, \text{Sup}(\tilde{k}_1, \tilde{k}_5) = 0.790\\
&\text{Sup}(\tilde{k}_2, \tilde{k}_1) = 0.665, \text{Sup}(\tilde{k}_2, \tilde{k}_3) = 0.555, \text{Sup}(\tilde{k}_2, \tilde{k}_4) = 0.185, \text{Sup}(\tilde{k}_2, \tilde{k}_5) = 0.875\\
&\text{Sup}(\tilde{k}_3, \tilde{k}_1) = 0.890, \text{Sup}(\tilde{k}_3, \tilde{k}_2) = 0.555, \text{Sup}(\tilde{k}_3, \tilde{k}_4) = 0.480, \text{Sup}(\tilde{k}_3, \tilde{k}_5) = 0.680\\
&\text{Sup}(\tilde{k}_4, \tilde{k}_1) = 0.460, \text{Sup}(\tilde{k}_4, \tilde{k}_2) = 0.185, \text{Sup}(\tilde{k}_4, \tilde{k}_3) = 0.480, \text{Sup}(\tilde{k}_4, \tilde{k}_5) = 0.260\\
&\text{Sup}(\tilde{k}_5, \tilde{k}_1) = 0.790, \text{Sup}(\tilde{k}_5, \tilde{k}_2) = 0.875, \text{Sup}(\tilde{k}_5, \tilde{k}_3) = 0.680, \text{Sup}(\tilde{k}_5, \tilde{k}_4) = 0.260
\end{aligned}$$

Step 3. Calculate $T(\tilde{k}_i)$, where $i = 1, 2, 3, 4, 5$. Thus, we have

$$\begin{aligned}
&T(\tilde{k}_1) = Sup(\tilde{k}_1, \tilde{k}_2) + Sup(\tilde{k}_1, \tilde{k}_3) + Sup(\tilde{k}_1, \tilde{k}_4) + Sup(\tilde{k}_1, \tilde{k}_5) = 0.665 + 0.890 + 0.460 + 0.790 = 2.805,\\
&T(\tilde{k}_2) = Sup(\tilde{k}_2, \tilde{k}_1) + Sup(\tilde{k}_2, \tilde{k}_3) + Sup(\tilde{k}_2, \tilde{k}_4) + Sup(\tilde{k}_2, \tilde{k}_5) = 0..665 + 0.555 + 0.185 + 0.875 = 2.280,\\
&T(\tilde{k}_3) = Sup(\tilde{k}_3, \tilde{k}_1) + Sup(\tilde{k}_3, \tilde{k}_2) + Sup(\tilde{k}_3, \tilde{k}_4) + Sup(\tilde{k}_3, \tilde{k}_5) = 0.890 + 0.555 + 0.480 + 0.680 = 2.605,\\
&T(\tilde{k}_4) = Sup(\tilde{k}_4, \tilde{k}_1) + Sup(\tilde{k}_4, \tilde{k}_2) + Sup(\tilde{k}_4, \tilde{k}_3) + Sup(\tilde{k}_4, \tilde{k}_5) = 0.460 + 0.185 + 0.480 + 0.260 = 1.385,\\
&T(\tilde{k}_5) = Sup(\tilde{k}_5, \tilde{k}_1) + Sup(\tilde{k}_5, \tilde{k}_2) + Sup(\tilde{k}_5, \tilde{k}_3) + Sup(\tilde{k}_5, \tilde{k}_4) = 0.790 + 0.875 + 0.680 + 0.260 = 2.605.
\end{aligned}$$

Step 4. Calculate the τ_i, where $i = 1, 2, 3, 4, 5$, using (17). Thus, we have

$$\begin{aligned}
\tau_1 &= \frac{1 + T(\tilde{k}_1)}{(1 + T(\tilde{k}_1)) + (1 + T(\tilde{k}_2)) + (1 + T(\tilde{k}_3)) + (1 + T(\tilde{k}_4)) + (1 + T(\tilde{k}_5))}\\
&= \frac{1 + 2.805}{(1 + 2.805) + (1 + 2.280) + (1 + 2.605) + (1 + 1.385) + (1 + 2.605)}\\
&= 0.2281
\end{aligned}$$

Similarly, we obtain $\tau_2 = 0.1966$, $\tau_3 = 0.2161$, $\tau_4 = 0.1430$ and $\tau_5 = 0.2161$.

Step 5. Suppose $q_1, q_2 = 2$ are the granularity values which are used in the criteria weights evaluation process during proposed operator implementation. Then using above computed power weights τ_i and the definition of the proposed $PFPPtHM^{q}_{DST}$ operator (18), then the aggregated value is computed as follows:

$$PFPPtHM^{q_1,q_2}_{DST}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5) = \frac{1}{d}\bigoplus_{t=1}^{d}\left(\frac{\bigoplus_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|}\left(\bigotimes_{j=1}^{q_t}(n\tau_{i_j}\tilde{k}_{i_j})\right)^{\frac{1}{q_t}}}{C^{q_t}_{|P_t|}}\right)$$

$$\begin{aligned}
&= \Bigg[\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right),\\
&\quad \frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1 \le i_1 < \ldots < i_{q_t} \le |P_t|}\left(\prod_{j=1}^{q_t}(n\tau_{i_j}(1 - \eta_{i_j} - \nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\Bigg]\\
&= \Bigg[\frac{1}{2^2}\Bigg\{\frac{1}{(C^2_2)^2}\Big(((5 \times 0.2281 \times 0.53) \times (5 \times 0.2161 \times 0.42))\Big)^{\frac{1}{2}}
\end{aligned}$$

$$+\frac{1}{(C_3^2)^2}\Big(((5\times 0.1966\times 0.89)\times(5\times 0.1430\times 0.08))$$

$$((5\times0.1966\times0.89)\times(5\times0.2161\times0.73))((5\times0.1430\times0.08)\times(5\times0.2161\times0.73))\Big)^{\frac{1}{2}}\Big\},$$

$$\frac{1}{2^2}\Big\{\frac{1}{(C_2^2)^2}\Big(\Big\{(5\times 0.2281\times(1-0.33-0.09))\times(5\times 0.2161(1-0.35-0.18))\Big\}\Big)^{\frac{1}{2}}$$

$$+\frac{1}{(C_3^2)^2}\Big(\Big\{(5\times 0.1966\times(1-0.08-0.03))\times(5\times 0.1430\times(1-0.89-0.02))\Big\}$$

$$\Big\{(5\times 0.1966\times(1-0.08-0.03))\times(5\times 0.2161\times(1-0.12-0.08))\Big\}$$

$$\Big\{(5\times 0.1430\times(1-0.89-0.02))\times(5\times 0.2161\times(1-0.12-0.08))\Big\}\Big)^{\frac{1}{2}}\Big\}\Big]$$

$$=[0.1661, 0.1822].$$

Special cases of $PFPPtHM_{DST}^{q}$ operator: In addition, the special cases of $PFPPtHM_{DST}^{q}$ operator are provided as follows:

Case 1: For $d=1$, i.e., $q_t=q$ and $|P_t|=n$, the $PFPPtHM_{DST}^{q}$ operator converts to the picture fuzzy power Hmay mean (PFPHM) operator.

$$PFPPtHM_{DST}^{q}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)=\frac{1}{1}\bigoplus_{t=1}^{1}\left(\frac{\bigoplus\limits_{1\le i_1<...<i_q\le n}\left(\bigotimes\limits_{j=1}^{q}(n\tau_{i_j}\tilde{k}_{i_j})\right)^{\frac{1}{q}}}{C_n^q}\right)$$

$$=\frac{1}{C_n^q}\left(\bigoplus_{1\le i_1<...<i_q\le n}\left(\bigotimes_{j=1}^{q}(n\tau_{i_j}\tilde{k}_{i_j})\right)^{\frac{1}{q}}\right)$$

$$=PFPHM(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)$$

Case 2: For $d=1$ and $q_t=q=1$, the $PFPPtHM_{DST}^{q}$ operator converts to the picture fuzzy power averaging (PFPA) operator (Yager 2001).

$$PFPPtHM_{DST}^{q=1}(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)=\frac{1}{1}\bigoplus_{t=1}^{1}\left(\frac{\bigoplus\limits_{1\le i_1<...<i_1\le n}\left(\bigotimes\limits_{j=1}^{1}(n\tau_{i_j}\tilde{k}_{i_j})\right)^{\frac{1}{1}}}{C_n^1}\right)$$

$$=\bigoplus_{i=1}^{n}\tau_i\tilde{k}_i=PFPA(\tilde{k}_1,\tilde{k}_2,...,\tilde{k}_n)$$

Case 3: For $d = 1$ and $q_t = q = n$, the $PFPPtHM^q_{DST}$ operator converts into (Yang and Garg 2021):

$$PFPPtHM^{q=n}_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \frac{1}{1}\bigoplus_{t=1}^{n}\left(\frac{\bigoplus_{1\leq i_1<...<i_n\leq n}\left(\bigotimes_{j=1}^{n}(n\tau_{i_j}\tilde{k}_{i_j})\right)^{\frac{1}{n}}}{C_n^n}\right)$$

$$= \bigotimes_{i=1}^{n}(n\tau_i\tilde{k}_i)^{\frac{1}{n}}$$

The discussed $PFPPtHM^q_{DST}$ operator accounts interaction between arguments, partitioning of criteria, reduction of biased values of criteria and quantify the limitations of algebraic operational laws. However, still $PFPPtHM^q_{DST}$ operator does not count the importance of criteria in aggregation of information, therefore next subsection introduces the $PFWPPtHM^q_{DST}$ operator which considers the corresponding importance of criteria in terms of weights to aggregate the PFNs.

3.3 Picture fuzzy weighted power partitioned Hamy mean operator

Definition 19 Let $\tilde{K} = \{\tilde{k}_i : \tilde{k}_i = [\mu_i, 1-\eta_i-\nu_i], i = 1, 2, ..., n\}$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$. The arguments $\tilde{k}_i$ $(i = 1, 2, .., n)$ are partitioned into d-number of sets $P_1, P_2, ..., P_d$ with conditions $P_i \cap P_j = \phi$ and $\bigcup_{t=1}^{d} P_t = \tilde{K}$. The parameter $q_1, q_2, ..., q_t, ..., q_d$ is with $q_t = 1, 2, ..., |P_t|$, and $|P_t|$ is the cardinality of P_t $(t = 1, 2, ..., d)$. Also $q = (q_1, q_2, ..., q_t, ..., q_d)$ is the granularity parameter vector. The $\omega_i (i = 1, 2, .., n)$ is the weight of the corresponding input arguments with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n}\omega_i = 1$. Therefore, the $PFWPPtHM^q_{DST}$ operator of n-dimension is defined as follows:

$$PFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \frac{1}{d}\bigoplus_{t=1}^{d}\left(\frac{\bigoplus_{1\leq i_1<...<i_{q_t}\leq |P_t|}\left(\bigotimes_{j=1}^{q_t}\left(\frac{n\omega_{i_j}(1+T(\tilde{k}_{i_j}))\tilde{k}_{i_j}}{\sum_{j=1}^{n}(\omega_j(1+T(\tilde{k}_j)))}\right)^{\frac{1}{q_t}}\right)}{C^{q_t}_{|P_t|}}\right) \tag{22}$$

where $i_1, i_2, ..., i_{q_t}$ traverses all the q_t-tuple combination of $\{1, 2, ..., |P_t|\}$, and $C^{q_t}_{|P_t|} = \frac{|P_t|!}{q_t!(|P_t|-q_t)!}$ is the Binomial coefficient. Meanwhile, $T(\tilde{k}_i) = \sum_{\substack{j=1\\ j\neq i}}^{n} \mathrm{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j)$ and $\mathrm{Sup}_{DST}(\tilde{k}_i, \tilde{k}_j)$ is the support for $\tilde{k}_i$ and $\tilde{k}_j$, and also satisfy the properties as mentioned in Definition 12.

Let

$$\bar{\tau}_i = \frac{\omega_i(1+T(\tilde{k}_i))}{\sum_{j=1}^{n}\omega_j(1+T(\tilde{k}_j))} \tag{23}$$

where, $(\bar{\tau}_1, \bar{\tau}_2, ..., \bar{\tau}_n)^T$ is called as power weighting vector (PWV), which satisfy the conditions $\bar{\tau}_i \in [0, 1]$ and $\sum_{i=1}^{n} \bar{\tau}_i = 1$.

Then (22) converted into,

$$PFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \frac{1}{d} \bigoplus_{t=1}^{d} \left(\frac{\bigoplus_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\bigotimes_{j=1}^{q_t} (n\bar{\tau}_{i_j} \tilde{k}_{i_j}) \right)^{\frac{1}{q_t}}}{C^{q_t}_{|P_t|}} \right) \tag{24}$$

Theorem 2 *Let $\tilde{K} = \{\tilde{k}_i : \tilde{k}_i = [\mu_i, 1 - \eta_i - \nu_i], i = 1, 2, ..., n\}$ be the corresponding BIs of PFNs $K = \{k_i : k_i = (\mu_i, \eta_i, \nu_i), i = 1, 2, ..., n\}$. The value aggregated by the $PFWPPtHM^q_{DST}$ operator is still a BI and is given as follows:*

$$PFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \left[\frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C^{q_t}_{|P_t|})^2} \left(\sum_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\bar{\tau}_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right) \right), \right.$$
$$\left. \frac{1}{d^2} \sum_{t=1}^{d} \left(\frac{1}{(C^{q_t}_{|P_t|})^2} \left(\sum_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\bar{\tau}_{i_j} (1 - \eta_{i_j} - \nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right) \right] \tag{25}$$

Proof Similar as Theorem 1.

The commutativity of the $PFWPPtHM^q_{DST}$ operator is as follows: □

Property 4 (Commutativity) Let $\tilde{k}_i = [\mu_i, 1 - \eta_i - \nu_i]$ $(i = 1, 2, ..., n)$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$. If $(\tilde{k}'_1, \tilde{k}'_2, ..., \tilde{k}'_n)$ is any permutation of $(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n)$, $\tilde{k}'_i = [\mu'_i, 1 - \eta'_i - \nu'_i]$, then

$$PFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = PFWPPtHM^q_{DST}(\tilde{k}'_1, \tilde{k}'_2, ..., \tilde{k}'_n). \tag{26}$$

Proof Proof is similar to Property 1.

It must be noted that $PFWPPtHM^q_{DST}$ does not satisfied idempotency and boundedness (Liu and Gao 2020; Qin et al. 2020). However, if a small modification (multiply by $dC^{q_t}_{|P_t|}$) is considered, the modified $PFWPPtHM^q_{DST}$ ($MPFWPPtHM^q_{DST}$) corresponds to an idempotency and boundedness operator as follows:

$$MPFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \left[\frac{1}{d} \sum_{t=1}^{d} \left(\frac{1}{C^{q_t}_{|P_t|}} \left(\sum_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\bar{\tau}_{i_j} \mu_{i_j}) \right)^{\frac{1}{q_t}} \right) \right), \right.$$
$$\left. \frac{1}{d} \sum_{t=1}^{d} \left(\frac{1}{C^{q_t}_{|P_t|}} \left(\sum_{1 \le i_1 < ... < i_{q_t} \le |P_t|} \left(\prod_{j=1}^{q_t} (n\bar{\tau}_{i_j} (1 - \eta_{i_j} - \nu_{i_j})) \right)^{\frac{1}{q_t}} \right) \right) \right]$$

□

Property 5 (Idempotency). Let $\tilde{k}_i = [\mu_i, 1 - \eta_i - \nu_i]$ $(i = 1, 2, ..., n)$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$. Also let $\tilde{k}_i = \tilde{k} = [\mu, 1 - \eta - \nu]$, then

$$MPFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \tilde{k} \tag{27}$$

Proof Proof is similar to Property 2. □

Property 6 (Boundedness). Let $\tilde{k}_i = [\mu_i, 1-\eta_i-\nu_i]$ $(i = 1, 2, ..., n)$ be the corresponding BIs of PFNs $k_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, 2, ..., n)$. If $\tilde{k}^- = [\min(\mu_i), \min(1-\eta_i-\nu_i)]$ and $\tilde{k}^+ = [\max(\mu_i), \max(1-\eta_i-\nu_i)]$ for $(i = 1, 2, ..., n)$. Then

$$\tilde{k}^- = MPFWPPtHM^q_{DST}(\tilde{k}_1, \tilde{k}_2, ..., \tilde{k}_n) = \tilde{k}^+ \tag{28}$$

Proof Proof is similar to Property 3. □

Example 2 Let $\tilde{k}_1 = (0.53, 0.33, 0.09)$, $\tilde{k}_2 = (0.89, 0.08, 0.03)$, $\tilde{k}_3 = (0.42, 0.35, 0.18)$, $\tilde{k}_4 = (0.08, 0.89, 0.02)$ and $\tilde{k}_5 = (0.73, 0.12, 0.08)$ be five PFNs corresponding to the criteria G_1, G_2, G_3, G_4, G_5 respectively, which is partitioned into 2-groups $P_1 = \{G_1, G_3\}$ and $P_2 = \{G_2, G_4, G_5\}$ i.e., the cardinality of these partitions are $|P_1| = 2$ and $|P_2| = 3$, and the weighting vector is ω=(0.1, 0.2, 0.3, 0.25, 0.15). Then, utilizing $PFWPPtHM^q_{DST}$ operator, the aggregated value is obtained by using the following steps.

In this example the steps 1, 2, and 3 are same as Example 1. From Step 3, we obtain $T(\tilde{k}_1)$=2.805, $T(\tilde{k}_2) = 2.280$, $T(\tilde{k}_3) = 2.605$, $T(\tilde{k}_4) = 1.385$ and $T(\tilde{k}_3) = 2.605$.

Step 4. Calculate the $\bar{\tau}_i$, where $i = 1, 2, 3, 4, 5$, using (23). Thus, we have $\bar{\tau}_1$

$$= \frac{\omega_1(1+T(\tilde{k}_1))}{\omega_1(1+T(\tilde{k}_1)) + \omega_2(1+T(\tilde{k}_2)) + \omega_3(1+T(\tilde{k}_3)) + \omega_4(1+T(\tilde{k}_4)) + \omega_5(1+T(\tilde{k}_5))}$$
$$= \frac{0.1(1+2.805)}{0.1(1+2.805) + 0.2(1+2.280) + 0.2(1+2.605) + 0.25(1+1.385) + 0.15(1+2.605)}$$
$$= 0.1169$$

Similarly, we obtain $\bar{\tau}_2 = 0.2015$, $\bar{\tau}_3 = 0.3323$, $\bar{\tau}_4 = 0.1832$ and $\bar{\tau}_5 = 0.1661$.

Step 5. Suppose $q_1, q_2 = 2$ are the granularity values which are used in the criteria weights evaluation process during proposed operator implementation. Then using above computed power weights $\bar{\tau}_i$ and the definition of the proposed $PFWPPtHM^q_{DST}$ operator (24), then the aggregated value is computed as follows:

$$PFWPPtHM^{q_1,q_2}_{DST}(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4, \tilde{k}_5) = \frac{1}{d}\bigoplus_{t=1}^{d}\left(\frac{\bigoplus\limits_{1\le i_1<...<i_{q_t}\le|P_t|}\left(\bigotimes\limits_{j=1}^{q_t}(n\bar{\tau}_{i_j}\tilde{k}_{i_j})\right)^{\frac{1}{q_t}}}{C^{q_t}_{|P_t|}}\right)$$

$$= \left[\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\le i_1<...<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\bar{\tau}_{i_j}\mu_{i_j})\right)^{\frac{1}{q_t}}\right)\right),\right.$$
$$\left.\frac{1}{d^2}\sum_{t=1}^{d}\left(\frac{1}{(C^{q_t}_{|P_t|})^2}\left(\sum_{1\le i_1<...<i_{q_t}\le|P_t|}\left(\prod_{j=1}^{q_t}(n\bar{\tau}_{i_j}(1-\eta_{i_j}-\nu_{i_j}))\right)^{\frac{1}{q_t}}\right)\right)\right]$$
$$= \left[\frac{1}{2^2}\left\{\frac{1}{(C^2_2)^2}\Big(\big((5\times0.1169\times0.53)\times(5\times0.3323\times0.42)\big)\Big)^{\frac{1}{2}}\right.\right.$$
$$+\frac{1}{(C^2_3)^2}\Big(\big((5\times0.2015\times0.89)\times(5\times0.1832\times0.08)\big)$$

$$\left((5\times0.2015\times0.89)\times(5\times0.1661\times0.73)\right)\left((5\times0.1832\times0.08)\times(5\times0.1661\times0.73)\right)\Big)^{\frac{1}{2}}\Big\},$$

$$\frac{1}{2^2}\left\{\frac{1}{(C_2^2)^2}\left(\left\{\left(5\times0.1169\times(1-0.33-0.09)\right)\times\left(5\times0.1661(1-0.35-0.18)\right)\right\}\right)^{\frac{1}{2}}\right.$$

$$+\frac{1}{(C_3^2)^2}\left(\left\{\left(5\times0.2015\times(1-0.08-0.03)\right)\times\left(5\times0.1832\times(1-0.89-0.02)\right)\right\}\right.$$

$$\left\{\left(5\times0.2015\times(1-0.08-0.03)\right)\times\left(5\times0.1661\times(1-0.12-0.08)\right)\right\}$$

$$\left.\left.\left\{\left(5\times0.1832\times(1-0.89-0.02)\right)\times\left(5\times0.1661\times(1-0.12-0.08)\right)\right\}\right)^{\frac{1}{2}}\right\}\Bigg]$$

$$=[0.1495, 0.1641].$$

4 A new multi criteria decision making method based on the proposed operators

A MCDM method depend on the developed $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ operators for PFNs is presented in this section.

4.1 Problem description

Consider a MCDM problem with m number of alternatives $B = \{B_1, B_2, ..., B_m\}$ and n number of criteria $G = \{G_1, G_2, ..., G_n\}$. On the basis of collected information, a picture fuzzy decision matrix $D = [k_{ij}]_{m\times n}$ $(i = 1, 2, ..., m;\ j = 1, 2, ..., n)$ is constructed, where $k_{ij} = (\mu_{ij}, \eta_{ij}, \nu_{ij})$ is a PFN for the alternative B_i with respect to criteria G_j. The problem now is to find the best alternative from the set of alternatives B based on criteria G and decision matrix D. The method is explained in the next section.

4.2 Decision making procedure

In this section, we provide a detailed description of the procedural steps for the MCDM method, which is based on the utilization of the proposed operators, namely, $PFPPtHM^q DST$ and $PFWPPtHM^q DST$. These operators have been specifically designed to enhance the decision-making process by incorporating the concept of DST. The steps outlined below demonstrate how these operators are employed to guide the decision-making process effectively.

Step 1. In the first step, normalization of picture fuzzy decision matrix D is done with the help of normalization process. The normalized decision matrix is denoted by $Q = [k_{ij}]_{m\times n}$ and its entries are given as follows:

$$k_{ij} = \begin{cases} (\mu_{ij}, \eta_{ij}, \nu_{ij}), & \text{if } G_j \text{ is the benifit criterion} \\ (\nu_{ij}, \eta_{ij}, \mu_{ij}), & \text{if } G_j \text{ is the cost criterion} \end{cases} \tag{29}$$

Step 2. In this step, conversion of the PFNs in the matrix Q to the PFNs with DST is done based on the definition 9. The resulting converted matrix Q_{DST} is as follows:

$$Q_{DST} = [\tilde{k}_{ij}]_{m\times n} = \left[[\mu_{ij}, 1-\eta_{ij}-\nu_{ij}]\right]_{m\times n} \tag{30}$$

Step 3. Calculate the support values as follows:

$$\mathrm{Sup}_{DST}(\tilde{k}_{ij}, \tilde{k}_{it}) = 1 - d_{DST}(\tilde{k}_{ij}, \tilde{k}_{it}); \quad i = 1, 2, ..., m; \quad j, t = 1, 2, ..., n; \quad j \neq t, \tag{31}$$

where, $d_{DST}(\tilde{k}_{ij}, \tilde{k}_{it})$ is the distance between two PFNs $\tilde{k}_{ij}$ and $\tilde{k}_{it}$ with DST.

Step 4. Calculate sum of support values $T(\tilde{k}_{ij})$ as follows:

$$T(\tilde{k}_{ij}) = \sum_{\substack{t=1\\ j\neq t}}^{n} \mathrm{Sup}_{DST}(\tilde{k}_{ij}, \tilde{k}_{it}); \quad i = 1, 2, ..., m; \quad j, t = 1, 2, ..., n; \quad j \neq t. \tag{32}$$

Step 5. Determine power weights τi and $\bar{\tau} i$ for the operators $PFPPtHM^{q}_{DST}$ and $PFWPPtHM^{q}_{DST}$, respectively, using the following equations:

$$\tau_i = \frac{(1 + T(\tilde{k}_i))}{\sum_{j=1}^{n}(1 + T(\tilde{k}_j))}; \quad i = 1, 2, ..., m; \quad j = 1, 2, ..., n. \tag{33}$$

and

$$\bar{\tau}_i = \frac{\omega_i(1 + T(\tilde{k}_i))}{\sum_{j=1}^{n}\omega_j(1 + T(\tilde{k}_j))}; \quad i = 1, 2, ..., m; \quad j = 1, 2, ..., n. \tag{34}$$

Step 6. Utilize the operators $PFPPtHM^{q}_{DST}$ and $PFWPPtHM^{q}_{DST}$ to assess the overall preference value for each alternative based on the decision matrix $Q_{DST} = [\tilde{k}_{ij}]_{m\times n}$ as described below:

$$\tilde{k}_i = PFPPtHM^{q}_{DST}(\tilde{k}_{i_1}, \tilde{k}_{i_2}, ..., \tilde{k}_{i_n}) \tag{35}$$

$$\tilde{k}_i = PFWPPtHM^{q}_{DST}(\tilde{k}_{i_1}, \tilde{k}_{i_2}, ..., \tilde{k}_{i_n}) \tag{36}$$

Step 7. Calculate the score $S_{DST}(\tilde{k}_i)$ and accuracy $H_{DST}(\tilde{k}_i)$ values for each alternative $B_i(i = 1, 2, ..., m)$ by using following equations:

$$S_{DST}(\tilde{k}) = \frac{(Bel + Pl)}{2} = \frac{(1 + \mu - \eta - \nu)}{2} \tag{37}$$

$$H_{DST}(\tilde{k}_i) = (Pl - Bel) = (1 - \mu_i - \eta_i - \nu_i) \tag{38}$$

Step 8. Rank all the alternatives B_i (where $i = 1, 2, ..., m$) using the criteria provided by $S_{DST}(\tilde{k}i)$ and $HDST(\tilde{k}_i)$, employing the ranking method detailed in Definition 11. Based on the outcomes of this ranking process, identify and choose the best alternative.

The steps of proposed MCDM problem is depicted in Fig. 2.

5 Numerical example

To illustrate the proposed MCDM method, an example adapted from Luo and Xing (2020) is discussed in this section. The example deals with problem of the best hotel selection among the available four hotels B_1, B_2, B_3 and B_4. The selection procedure is based on five criterion includes price(G_1),comfortability (G_2), service (G_3), location (G_4) and convenience (G_5).

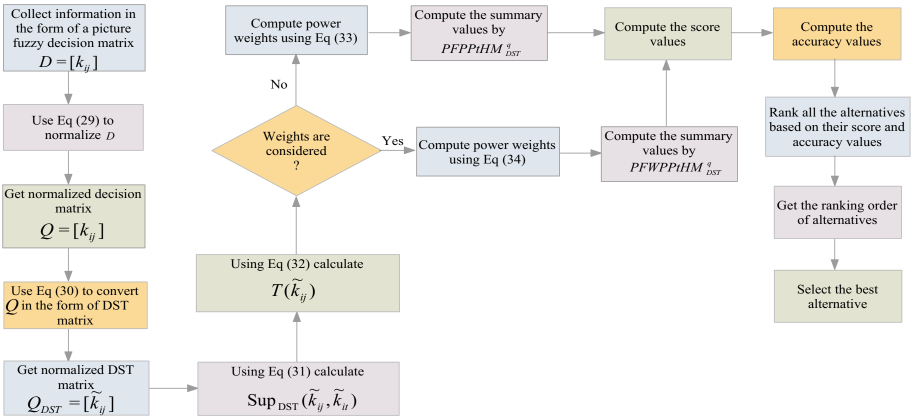


Fig. 2 Flow chart for proposed MCDM method

The weight vector associated with criteria is $\omega = (0.10, 0.20, 0.30, 0.25, 0.15)^T$ and the collected information in the form of a PFN matrix $D = [k_{ij}]_{4\times5}$ is presented in Table 3.

5.1 Implementation of the MCDM method's procedural steps

Step 1. Normalization of the decision matrix D.

Since G_2, G_3, G_4, and G_5 are benefit criteria, and G_1 is a cost criterion, they must be normalized in accordance with (29). Then, the normalized picture fuzzy decision matrix $Q = [k_{ij}]_{m\times n}$ is given in Table 4.

Step 2. Convert the normalized picture fuzzy decision matrix to picture fuzzy decision making with DST by using (30). The computed values are shown in Table 5.

Step 3. Calculate the support values $\mathrm{Sup}_{DST}(\tilde{k}_{ij}, \tilde{k}_{it})$ $(i = 1, 2, 3, 4;\ j, t = 1, 2, 3, 4, 5;\ j \neq t)$ by using (31) and computation is given below.

$$\mathrm{Sup}_{DST}(\tilde{k}_{1j}, \tilde{k}_{1t}) = \mathrm{Sup}_{DST}(\tilde{k}_{1t}, \tilde{k}_{1j}) = \begin{bmatrix} 1.0, & 0.8, & 0.9, & 0.7, & 0.8 \\ 0.8, & 1.0, & 0.9, & 0.9, & 0.8 \\ 0.9, & 0.9, & 1.0, & 0.8, & 0.7 \\ 0.7, & 0.9, & 0.8, & 1.0, & 0.9 \\ 0.8, & 0.8, & 0.7, & 0.9, & 1.0 \end{bmatrix}$$

$$\mathrm{Sup}_{DST}(\tilde{k}_{2j}, \tilde{k}_{2t}) = \mathrm{Sup}_{DST}(\tilde{k}_{2t}, \tilde{k}_{2j})$$

Table 3 Decision matrix D

Alternative	G_1	G_2	G_3	G_4	G_5
B_1	(0.4, 0.2, 0.4)	(0.6, 0.2, 0.2)	(0.4, 0.2, 0.2)	(0.6, 0.0, 0.2)	(0.6, 0.0, 0.4)
B_2	(0.2, 0.4, 0.4)	(0.8, 0.2, 0.0)	(0.4, 0.4, 0.2)	(0.6, 0.4, 0.0)	(0.4, 0.2, 0.2)
B_3	(0.4, 0.4, 0.2)	(0.6, 0.0, 0.4)	(0.8, 0.0, 0.2)	(0.6, 0.2, 0.2)	(0.4, 0.2, 0.2)
B_4	(0.2, 0.4, 0.2)	(0.4, 0.4, 0.2)	(0.6, 0.4, 0.0)	(0.4, 0.2, 0.4)	(0.8, 0.2, 0.0)

Table 4 The normalized decision matrix Q

Alternative	G_1	G_2	G_3	G_4	G_5
B_1	(0.4, 0.2, 0.4)	(0.6, 0.2, 0.2)	(0.4, 0.2, 0.2)	(0.6, 0.0, 0.2)	(0.6, 0.0, 0.4)
B_2	(0.4, 0.4, 0.2)	(0.8, 0.2, 0.0)	(0.4, 0.4, 0.2)	(0.6, 0.4, 0.0)	(0.4, 0.2, 0.2)
B_3	(0.2, 0.4, 0.4)	(0.6, 0.0, 0.4)	(0.8, 0.0, 0.2)	(0.6, 0.2, 0.2)	(0.4, 0.2, 0.2)
B_4	(0.2, 0.4, 0.2)	(0.4, 0.4, 0.2)	(0.6, 0.4, 0.0)	(0.4, 0.2, 0.4)	(0.8, 0.2, 0.0)

$$= \begin{bmatrix} 1.0, & 0.6, & 1.0, & 0.8, & 0.9 \\ 0.6, & 1.0, & 0.6, & 0.8, & 0.7 \\ 1.0, & 0.6, & 1.0, & 0.8, & 0.9 \\ 0.8, & 0.8, & 0.8, & 1.0, & 0.7 \\ 0.9, & 0.7, & 0.8, & 0.7, & 1.0 \end{bmatrix}$$

$$\mathrm{Sup}_{DST}(\tilde{k}_{3j}, \tilde{k}_{3t}) = \mathrm{Sup}_{DST}(\tilde{k}_{3t}, \tilde{k}_{3j}) = \begin{bmatrix} 1.0, & 0.6, & 0.4, & 0.6, & 0.7 \\ 0.6, & 1.0, & 0.8, & 0.8, & 0.7 \\ 0.4, & 0.8, & 1.0, & 0.8, & 0.7 \\ 0.6, & 0.8, & 0.8, & 1.0, & 0.9 \\ 0.7, & 0.7, & 0.7, & 0.9, & 1.0 \end{bmatrix}$$

$$\mathrm{Sup}_{DST}(\tilde{k}_{4j}, \tilde{k}_{4t}) = \mathrm{Sup}_{DST}(\tilde{k}_{4t}, \tilde{k}_{4j}) = \begin{bmatrix} 1.0, & 0.9, & 0.7, & 0.7, & 0.5 \\ 0.9, & 1.0, & 0.8, & 0.8, & 0.6 \\ 0.7, & 0.8, & 1.0, & 0.6, & 0.8 \\ 0.7, & 0.8, & 0.6, & 1.0, & 0.6 \\ 0.5, & 0.6, & 0.8, & 0.6, & 1.0 \end{bmatrix}$$

Step 4. Compute the sum of support values $T(\tilde{k}_{ij})(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4, 5)$ by using (32) and computation is given below.

$$T(\tilde{k}_{ij}) = \begin{bmatrix} 3.2, & 3.4, & 3.3, & 3.3, & 3.2 \\ 3.3, & 2.7, & 3.3, & 3.1, & 3.2 \\ 2.3, & 2.9, & 2.7, & 3.1, & 3.0 \\ 2.8, & 3.1, & 2.9, & 2.7, & 2.5 \end{bmatrix}$$

Table 5 Decision matrix Q with DST

Alternative	G_1	G_2	G_3	G_4	G_5
B_1	[0.4, 0.4]	[0.6, 0.6]	[0.4, 0.6]	[0.6, 0.8]	[0.6, 0.6]
B_2	[0.4, 0.4]	[0.8, 0.8]	[0.4, 0.4]	[0.6, 0.6]	[0.4, 0.6]
B_3	[0.2, 0.2]	[0.6, 0.6]	[0.8, 0.8]	[0.6, 0.6]	[0.4, 0.6]
B_4	[0.2, 0.4]	[0.4, 0.4]	[0.6, 0.6]	[0.4, 0.4]	[0.8, 0.8]

Step 5. Compute the power weights τ_{ij} and $\bar{\tau}_{ij}$, $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4, 5)$ by using (33) and (34), respectively, and computation is given below.

$$\tau_{ij} = \begin{bmatrix} 0.1963, & 0.2056, & 0.2009, & 0.2009, & 0.1963 \\ 0.2087, & 0.1796, & 0.2087, & 0.1990, & 0.2039 \\ 0.1737, & 0.2053, & 0.1947, & 0.2158, & 0.2105 \\ 0.2000, & 0.2158, & 0.2053, & 0.1947, & 0.1842 \end{bmatrix}$$

and

$$\bar{\tau}_{ij} = \begin{bmatrix} 0.0978, & 0.2049, & 0.3003, & 0.2503, & 0.1467 \\ 0.1045, & 0.1798, & 0.3135, & 0.2491, & 0.1531 \\ 0.0858, & 0.2029, & 0.2887, & 0.2666, & 0.1560 \\ 0.0995, & 0.2147, & 0.3063, & 0.2421, & 0.1374 \end{bmatrix}$$

Step 6. Utilizing the proposed operators, calculate the overall collective value of each alternative using (35) and (36). Here, we have divided the criteria set into 2 partitions i.e., $d = 2$. The first partition (P_1) considers three criteria G_1, G_2 and G_3, while the second partition (P_2) contains two criteria G_4 and G_5. For computation, the granularity parameter vector is selected as $q = (q_1, q_2) = (1, 1)$. The computed values are shown in Table 6.

Step 7. Calculate the score values $S_{DST}(\tilde{k}_i)(i = 1, 2, 3, 4)$ for the alternatives in accordance with the overall preference values $\tilde{k}_i(i = 1, 2, 3, 4)$ using (37). The computed values are shown in Table 7.

Step 8. Rank all the alternatives $B_i,\ i = 1, 2, 3, 4$ on the basis of their score values $S_{DST}(\tilde{k}_i)$ by using the methodology discussed in Definition 11. The ranking orders of alternatives based on the score values and the best alternative for both the proposed operators are shown in Table 8. Table shows that the best alternative for both the proposed operators is B_1.

The above analysis considers the value $q = (q_1, q_2) = (1, 1)$ for the granularity parameter vector. From above analysis, we have observed that the results are stable and consistent for both the operators. However, there are some complexities are observed during the implementation of the proposed approach which includes the complexities due to the evaluation of the power weights, partitioning of criteria, conversion of data into interval during DST implementation, and interdependency between criteria due to HM.

To analyse the effect of variation in $q = (q_1, q_2)$ on final decision making, a sensitivity analysis has been conducted by taking different combinations of q_1 and q_2, and a detailed procedure is provided in the next section.

5.2 Sensitivity analysis

In this section, a sensitivity analysis is provided to examine the stability and effect of variation in values of parameters q_t by using proposed $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ operators. The value of parameter q_t signify all the possible granularity(interrelationship)

Table 6 The overall preference value computed from both the operators

Alternative	$PFPPtHM^q_{DST}$	$PFWPPtHM^q_{DST}$
B_1	[0.2639, 0.3036]	[0.2478, 0.2851]
B_2	[0.2484, 0.2762]	[0.2334, 0.2602]
B_3	[0.2397, 0.2690]	[0.2229, 0.2510]
B_4	[0.2279, 0.2524]	[0.2139, 0.2360]

Table 7 Score values of each alternative

Alternative	$PFPPtHM^q_{DST}$	$PFWPPtHM^q_{DST}$
B_1	0.1227	0.1257
B_2	0.1123	0.1138
B_3	0.1169	0.1256
B_4	0.1078	0.1062

between the q_t number of criteria grouped in the t^{th} partition. In this example, the values of q_1 and q_2 depend on the cardinality of partitions P_1 and P_2, respectively. The possible values of parameters are $q_1 = 1, 2, 3$ and $q_2 = 1, 2$. The evaluated score values and associated ranking order of alternatives computed by $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ operators are shown in Tables 9 and 10 respectively. From Tables 9 and 10, it can be noticed that when $q_1 = 1$ and $q_2 = 1$, then the ranking order of alternatives is $B_1 > B_3 > B_2 > B_4$ for both the operators. Thus, for this combination of q_1 and q_2, the best and worst alternatives are B_1 and B_4, respectively. Similar analysis can be conducted for the rest of the combinations of (q_1, q_2). Based on ranking results, it is concluded that for all the combinations of (q_1, q_2), the best and worst alternatives are always B_1 and B_4, respectively, from both the operators. The computed results are also plotted in Fig. 3(a) (for $PFPPtHM^q_{DST}$) and Fig. 3(b) (for $PFWPPtHM^q_{DST}$). The ranking order of the alternatives is significantly affected by changing the combination of (q_1, q_2), as shown by Tables 9, 10 and Fig. 3(a), (b), and the results are stable and consistent. As a result, in the example under consideration, by adjusting the various combinations of (q_1, q_2) based on actual requirement, we can obtain rational decision results.

5.3 Comparative analysis

To show the efficiency and consistency of the developed operators, a qualitative and quantitative comparison analyses with 23 existing aggregation operators such as PFWA, PFWG, PFAWA, PFAOWA, PFEWA, PFEOWA, PFDWA, PFDWG, PFHWA, PFHWG, GPFHM, GPHWHM, PFWMM, PFWDMM, PFWBM, PFAPMSM, PFAPWMSM, $PFPMM_{DST}$, $PFWPMM_{DST}$, PFIPtHA, PFWIPtHA, PFPMM and PFWPMM have been done. The full detail of these existing aggregation operators are provided in Table 1. The discussion is presented hereafter.

5.3.1 Qualitative comparison

This section provides a qualitative comparative of proposed and considered existing operators based on their characteristics, such as: (1) the power to diminish the negative effect of biased values in aggregation results; (2) the ability to provide interrelationships among multiple

Table 8 Ranking of all the alternatives

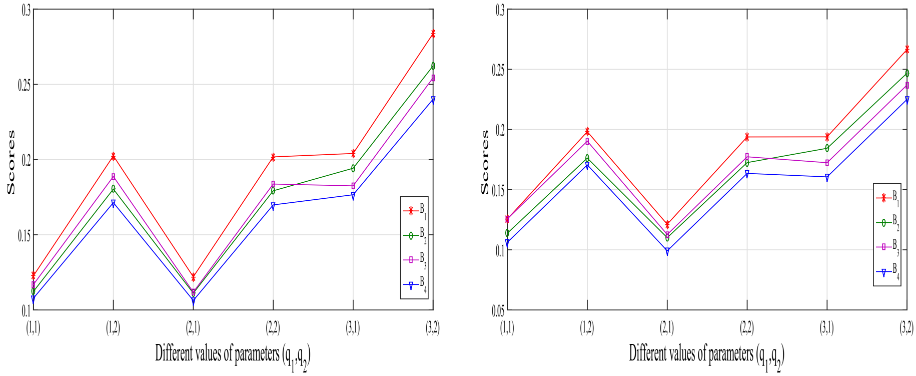
Operator	Ordering	Best alternative
$PFPPtHM^q_{DST}$	$B_1 > B_3 > B_2 > B_4$	B_1
$PFWPPtHM^q_{DST}$	$B_1 > B_3 > B_2 > B_4$	B_1

Table 9 Results obtained by using different combinations of parameters $q = (q_1, q_2)$ in $PFPPtHM_{DST}^{q}$ operator

$q = (q_1, q_2)$	Score value				Ranking results	Best alternative
(1,1)	$S(B_1)$=0.1227,	$S(B_2)$=0.1123,	$S(B_3)$=0.1169,	$S(B_4)$=0.1078	$B_1 > B_3 > B_2 > B_4$	B_1
(1,2)	$S(B_1)$=0.2024,	$S(B_2)$=0.1804,	$S(B_3)$=0.1888,	$S(B_4)$=0.1714	$B_1 > B_3 > B_2 > B_4$	B_1
(2,1)	$S(B_1)$=0.1221,	$S(B_2)$=0.1112,	$S(B_3)$=0.1119,	$S(B_4)$=0.1063	$B_1 > B_3 > B_2 > B_4$	B_1
(2,2)	$S(B_1)$=0.2018,	$S(B_2)$=0.1792,	$S(B_3)$=0.1837,	$S(B_4)$=0.1699	$B_1 > B_3 > B_2 > B_4$	B_1
(3,1)	$S(B_1)$=0.2041,	$S(B_2)$=0.1942,	$S(B_3)$=0.1825,	$S(B_4)$=0.1766	$B_1 > B_2 > B_3 > B_4$	B_1
(3,2)	$S(B_1)$=0.2838,	$S(B_2)$=0.2623,	$S(B_3)$=0.2544,	$S(B_4)$=0.2401	$B_1 > B_2 > B_3 > B_4$	B_1

Table 10 Results obtained by utilizing different combinations of parameters $q = (q_1, q_2)$ in $PFWPPtHM^{q}_{DST}$ operator

$q = (q_1, q_2)$	Score value				Ranking results	Best alternative
(1,1)	$S(B_1)$=0.1256,	$S(B_2)$=0.1138,	$S(B_3)$=0.1257,	$S(B_4)$=0.1062	$B_1 > B_3 > B_2 > B_4$	B_1
(1,2)	$S(B_1)$=0.1982,	$S(B_2)$=0.1762,	$S(B_3)$=0.1903,	$S(B_4)$=0.1706	$B_1 > B_3 > B_2 > B_4$	B_1
(2,1)	$S(B_1)$=0.1213,	$S(B_2)$=0.1101,	$S(B_3)$=0.1127,	$S(B_4)$=0.0991	$B_1 > B_3 > B_2 > B_4$	B_1
(2,2)	$S(B_1)$=0.1938,	$S(B_2)$=0.1725,	$S(B_3)$=0.1773,	$S(B_4)$=0.1635	$B_1 > B_3 > B_2 > B_4$	B_1
(3,1)	$S(B_1)$=0.1939,	$S(B_2)$=0.1844,	$S(B_3)$=0.1724,	$S(B_4)$=0.1606	$B_1 > B_2 > B_3 > B_4$	B_1
(3,2)	$S(B_1)$=0.2665,	$S(B_2)$=0.2468,	$S(B_3)$=0.2370,	$S(B_4)$=0.2250	$B_1 > B_2 > B_3 > B_4$	B_1



(a) $PFPPtHM^q_{DST}$ operator (b) $PFWPPtHM^q_{DST}$ operator

Fig. 3 Plot for score Vs. granularity parameter vector q

criteria; (3) the ability to partition multiple criteria based on their characteristics; and (4) the power to quantify the limitations of algebraic operational laws. The findings are given in Table 11 and are described in the following aspects:

(a) **The power to diminish the negative effect of biased values.** The PA operator is helpful in diminishing the negative effect of biased values of criteria by assigning power weights to them using support degree (Yager 2001). The existing operators PFAPMSM, PFAPWMSM, $PFPMM_{DST}$, $PFWPMM_{DST}$, PFPMM and PFWPMM, as well as the proposed operators, used PA to diminish the negative effect on criteria values, whereas the remaining operators under consideration did not.

(b) **The ability to provide interrelationships among multiple criteria.** In decision making, sometimes multiple criteria are interrelated and their interrelationship is essentially required to be incorporated into the decision making process to get feasible and realistic results (Hara et al. 1998). The existing operators PFWMM, PFDWMM, PFAPMSM, PFAPWMSM, $PFPMM_{DST}$, $PFWPMM_{DST}$, PFPMM and PFWPMM, and proposed operators incorporated the interrelationship among multiple criteria very effectively, while the operators GPFHM, GPFWHM, PFWBM, PFIPtHA, and PFWIPtHA incorporated only the interrelationship between any two criteria. The remaining operators under consideration, such as PFWA, PFWG, PFAWA, PFAOWA, PFEWA, PFEOWA, PFDWA, PFDWG, PFHWA, and PFHWG do not even consider interrelationships between any number of criteria in the aggregation process.

(c) **The ability to partition multiple criteria based on their characteristics.** Since different criteria have different characteristics, it is feasible to partition them based on their characteristics before applying the aggregation process (Dutta and Guha 2015). The existing PFIPtHA and PFWIPtHA, and the proposed operators, used the concept of partitioning of criteria as per their characteristics for getting feasible and realistic results. The remaining operators under consideration do not consider the concept of partitioning of criteria.

(d) **The power to quantify the limitations of algebraic operations.** Since the algebraic operations are not invariant and persistent for addition and scalar multiplication, respectively, the DST, which overcomes these limitations, can be used. Some other advantages of DST include the use of incomplete data in analysis, the inclusion of measures of

Table 11 The results by different operators of the qualitative comparison

Author(s)	Aggregation operators		Whether diminish the negative effect of biased values	Whether consider interrelationship between multiple criteria	Whether considers the partition of the different criteria	Whether quantify the limitations of algebraic operational laws
Wei (2017b)	PFWA,	PFWG	No	No	No	No
Garg (2017)	PFAWA,	PFAOWA	No	No	No	No
Khan et al. (2019)	PFEWA,	PFEOWA	No	No	No	No
Jana et al. (2018)	PFDWA,	PFDWG	No	No	No	No
Wei (2018)	PFHWA,	PFHWG	No	No	No	No
Wei et al. (2018)	GPFHM,	GPHWHM	No	No	No	No
Wang et al. (2018)	PFWMM,	PFWDMM	No	Yes	No	No
Ates and Akey (2020)	PFWBM		No	No	No	No
Qin et al. (2021)	PFAPMSM,	PFAPWMSM	Yes	Yes	No	No
Qin et al. (2020)	$PFPMM_{DST}$,	$PFWPMM_{DST}$	Yes	Yes	No	Yes
Luo and Xing (2020)	PFIPtHA,	PFWIPtHA	No	No	Yes	No
Punetha and Komal (2023)	PFPMM,	PFWPMM	Yes	Yes	No	No
Proposed operators	$PFPPtMM^{q}_{DST}$,	$PFWPPtMM^{q}_{DST}$	Yes	Yes	Yes	Yes

probability into the argument values, and allowing integration of probability theory in finding the values associated with alternatives (Dempster 1967; Qin et al. 2020; Shafer 1976). Based on these features, the framework of DST is used in the existing $PFPMM_{DST}$ and $PFWPMM_{DST}$, and proposed operators. The remaining operators under consideration do not incorporate DST into their analysis.

Thus, it has been observed that the proposed operators have all the four characteristics as mentioned above. In contrast to the listed existing operators under the PFS environment, the proposed operators are therefore more effective, general, consistent, stable, and flexible to analyse any real-world MCDM problem more logically.

5.3.2 Quantitative comparison

Based on the ranking results, this section compares the proposed to 23 other existing aggregation operators quantitatively. The calculated results are shown in Table 12 and plotted in Fig. 4 as a radar graph.

Based on the tabulated and plotted results, the noted findings are as follows:

(a) The ranking order obtained from different operators depends upon the type of the aggregation operator and the algebraic operations applied. The ranking order is also influenced by the inclusion of partitioning of criteria, granularity power of taking interrelationships between multiple criterion, and the power to diminish the negative effect of biased values of criteria. The existing PFWA, PFWG, PFAWA, PFAOWA, PFEWA, PFEOWA, PFDWA, PFHWA, and PFHWG (Garg 2017; Jana et al. 2018; Khan et al. 2019; Wei 2017b, 2018) operators provide the same ranking order as $B_2 > B_4 > B_3 > B_1$, which differs from the proposed method because these existing operators do not take into

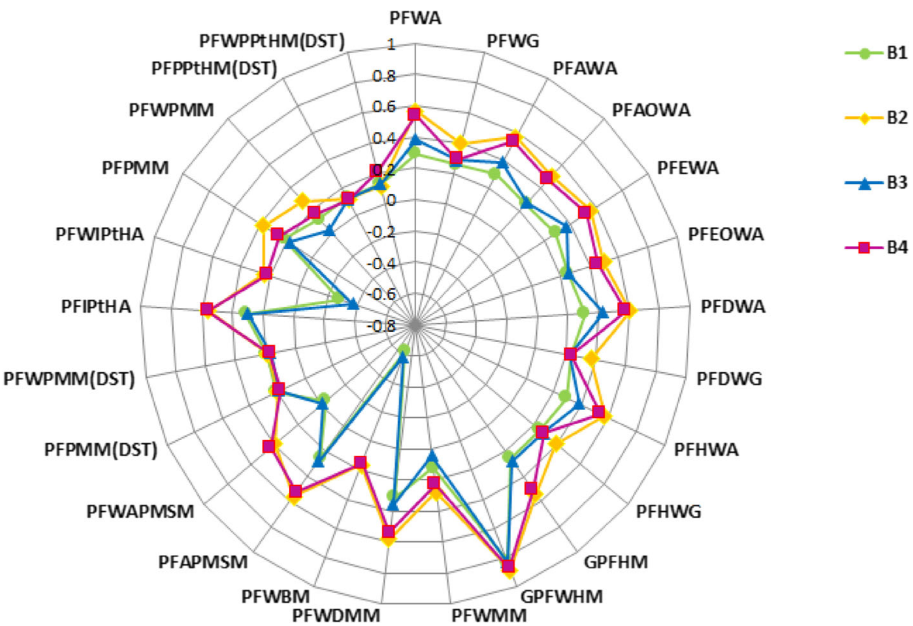


Fig. 4 Radar graph for quantitative comparison in which scale of grid is -1 to 1 representing score values

Table 12 The aggregating results by different operators of the quantitative comparison

	Operator	Parameter vector	Parameter value	Score value	Ranking results	Best alternative
Existing operators which do not consider parameter	PFWA (Wei 2017b)	–	–	$S(B_1)$=0.2917, $S(B_2)$=0.5648, $S(B_3)$=0.3837, $S(B_4)$=0.5363	$B_2 > B_4 > B_3 > B_1$	B_2
	PFWG (Wei 2017b)	–	–	$S(B_1)$=0.2547, $S(B_2)$=0.3930, $S(B_3)$=0.2853, $S(B_4)$=0.2908	$B_2 > B_4 > B_3 > B_1$	B_2
	PFAWA (Garg 2017)	–	–	$S(B_1)$=0.2917, $S(B_2)$=0.5648, $S(B_3)$=0.3837, $S(B_4)$=0.5363	$B_2 > B_4 > B_3 > B_1$	B_2
	PFAOWA (Garg 2017)	–	–	$S(B_1)$=0.2560, $S(B_2)$=0.5043, $S(B_3)$=0.2684, $S(B_4)$=0.4673	$B_2 > B_4 > B_3 > B_1$	B_2
	PFEWA (Khan et al. 2019)	–	–	$S(B_1)$=0.2870, $S(B_2)$=0.5553, $S(B_3)$=0.3719, $S(B_4)$=0.5253	$B_2 > B_4 > B_3 > B_1$	B_2
	PFEOWA (Khan et al. 2019)	–	–	$S(B_1)$=0.2503, $S(B_2)$=0.4968, $S(B_3)$=0.2557, $S(B_4)$=0.4541	$B_2 > B_4 > B_3 > B_1$	B_2
Existing operators which consider parameter	PFDWA (Jana et al. 2018)	R	1	$S(B_1)$=0.3099, $S(B_2)$=0.6066, $S(B_3)$=0.4314, $S(B_4)$=0.5789	$B_2 > B_4 > B_3 > B_1$	B_2
	PFDWG (Jana et al. 2018)	R	1	$S(B_1)$=0.2385, $S(B_2)$=0.3689, $S(B_3)$=0.2273, $S(B_4)$=0.2378	$B_2 > B_4 > B_1 > B_3$	B_2
	PFHWA (Wei 2018)	R	1	$S(B_1)$=0.2917, $S(B_2)$=0.5648, $S(B_3)$=0.3837, $S(B_4)$=0.5363	$B_2 > B_4 > B_3 > B_1$	B_2
	PFHWG (Wei 2018)	R	1	$S(B_1)$=0.2547, $S(B_2)$=0.3930, $S(B_3)$=0.2853, $S(B_4)$=0.2908	$B_2 > B_4 > B_3 > B_1$	B_2
	GPFHM (Wei et al. 2018)	(p, q)	(1,1)	$S(B_1)$=0.2549, $S(B_2)$=0.5390, $S(B_3)$=0.2764, $S(B_4)$=0.5079	$B_2 > B_4 > B_3 > B_1$	B_2
	GPFWHM (Wei et al. 2018)	(p, q)	(1,1)	$S(B_1)$=0.8268, $S(B_2)$=0.8860, $S(B_3)$=0.8329, $S(B_4)$=0.8708	$B_2 > B_4 > B_3 > B_1$	B_2
	PFWMM (Wang et al. 2018)	R	(1,1,1,1,1)	$S(B_1)$=0.1207, $S(B_2)$=0.2787, $S(B_3)$=0.0404, $S(B_4)$=0.2298	$B_2 > B_4 > B_1 > B_3$	B_2
	PFWDMM (Wang et al. 2018)	R	(1,1,1,1,1)	$S(B_1)$=0.3113, $S(B_2)$=0.5843, $S(B_3)$=0.3580, $S(B_4)$=0.5416	$B_2 > B_4 > B_3 > B_1$	B_2

Table 12 continued

	Operator	Parameter vector	Parameter value	Score value	Ranking results	Best alternative
	PFWBM (Ates and Akay 2020)	(p, q)	(1,1)	$S(B_1)$=-0.6160, $S(B_2)$=0.1633, $S(B_3)$=-0.5796, $S(B_4)$=0.1544	$B_2 > B_4 > B_3 > B_1$	B_2
	PFAPMSM (Qin et al. 2021)	R	1	$S(B_1)$=0.2575, $S(B_2)$=0.5566, $S(B_3)$=0.2804, $S(B_4)$=0.5241	$B_2 > B_4 > B_3 > B_1$	B_2
	PFWAPMSM (Qin et al. 2021)	R	1	$S(B_1)$=-0.0366, $S(B_2)$=0.3935, $S(B_3)$=-0.0114, $S(B_4)$=0.4311	$B_4 > B_2 > B_3 > B_1$	B_4
	$PFPMM_{DST}$ (Qin et al. 2020)	R	(1,1,1,1,1)	$S(B_1)$=0.2104, $S(B_2)$=0.1990, $S(B_3)$=0.1877, $S(B_4)$=0.1786	$B_1 > B_2 > B_3 > B_4$	B_1
	$PFWPMM_{DST}$ (Qin et al. 2020)	R	(1,1,1,1,1)	$S(B_1)$=0.1954, $S(B_2)$=0.1852, $S(B_3)$=0.1686, $S(B_4)$=0.1708	$B_1 > B_2 > B_4 > B_3$	B_1
	PFIPtHA (Luo and Xing 2020)	(q_1, q_2)	(1,1)	$S(B_1)$=0.3144, $S(B_2)$=0.5597, $S(B_3)$=0.2978, $S(B_4)$=0.5607	$B_4 > B_2 > B_1 > B_3$	B_4
	PFWIPtHA (Luo and Xing 2020)	(q_1, q_2)	(1,1)	$S(B_1)$=-0.2787, $S(B_2)$=0.2436, $S(B_3)$=-0.3691, $S(B_4)$=0.2220	$B_2 > B_4 > B_1 > B_3$	B_2
	PFPMM (Punetha and Komal 2023)	R	(1,1,1,1,1)	$S(B_1)$=0.2221, $S(B_2)$=0.3804, $S(B_3)$=0.1751, $S(B_4)$=0.2597	$B_2 > B_4 > B_1 > B_3$	B_2
	PFWPMM (Punetha and Komal 2023)	R	(1,1,1,1,1)	$S(B_1)$=0.1230, $S(B_2)$=0.2816, $S(B_3)$=0.0262, $S(B_4)$=0.1640	$B_2 > B_4 > B_1 > B_3$	B_2
Proposed operators	$PFPPtHM^q_{DST}$	(q_1, q_2)	(1,1)	$S(B_1)$=0.1227, $S(B_2)$=0.1123, $S(B_3)$=0.1169, $S(B_4)$=0.1078	$B_1 > B_3 > B_2 > B_4$	B_1
	$PFWPPtHM^q_{DST}$	(q_1, q_2)	(1,1)	$S(B_1)$=0.1257, $S(B_2)$=0.1138, $S(B_3)$=0.1256, $S(B_4)$=0.1062	$B_1 > B_3 > B_2 > B_4$	B_1

account interrelationships between multiple criteria, partitioning of different criteria, and even do not remove the negative effect of biased values of criteria in the aggregation process.

(b) The ranking orders obtained from GPFHM, GPFWHM and PFWBM (Ates and Akay 2020; Wei et al. 2018) operators is same as $B_2 > B_4 > B_3 > B_1$ which is different from the proposed operators because they considered only interrelationship between any two criteria without taking partitioning of different criteria and removing the negative effect of biased values of criteria in the aggregation process.

(c) The ranking orders obtained from the existing operators PFIPtHA and PFWIPtHA (Luo and Xing 2020) are different from the proposed methods because they only consider interrelationship between any two criteria and taking partitioning of different criteria as per the characteristics, however they do not still remove the negative effect of biased values of criteria in the aggregation process.

(d) Furthermore, the suggested technique using $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ operators is compared to the Qin et al. (2021) method using parameter value $R = 1$. The ranking order obtained by PFAPMSM and PFWAPMSM operators are $B_2 > B_4 > B_3 > B_1$ and $B_4 > B_2 > B_3 > B_1$, respectively. The results indicate that the B_2 and B_4 are best alternatives for PFAPMSM and PFWAPMSM operators, respectively, whereas B_1 is the worst alternative for both the operators. We observed that, the ranking results obtained by the proposed operators are totally different from the existing PFAPMSM and PFWAPMSM operators because the proposed operators involve the concepts of partitioning and DST.

(e) The ranking orders obtained from the existing PFPMM and PFWPMM operators (Punetha and Komal 2023) is $B_2 > B_4 > B_1 > B_3$, which is differ from the proposed operators because they consider interrelationship between multiple criteria and mitigate the negative effect of biased values, however they still do not deal with the limitation of algebraic operations in the aggregation process.

(f) The ranking orders obtained from the proposed $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$ operators is same as $B_1 > B_3 > B_2 > B_4$ which is different from most of the existing operators under consideration because these proposed operators overcome the shortcoming in the existing methods and include all the four characteristics as discussed in qualitative analysis in the aggregation process. The best and the worst alternatives obtained from the proposed operators are B_1 and B_4 respectively. Thus, proposed operators are more efficient, general, consistent, stable and flexible to solve any real life MCDM problem more accurately in comparison to some exiting operators under PFS environment.

6 Conclusion

This paper introduces a novel MCDM method based on the proposed operators for PFNs. It extends the PtHM operator to create $PPtHM^q$ by integrating it with the PA operator to mitigate bias. This operator is then adapted for PFNs with DST, resulting in $PFPPtHM^q_{DST}$ and $PFWPPtHM^q_{DST}$, which exhibit desirable properties. The use of the framework of DST in PFNs quantifies the incomplete data in analysis and the limitations of algebraic operational laws by including measures of probability in the argument values and integrating probability theory in finding the values associated with alternatives. Some advantages of the proposed operators are:

- They mitigate negative effect of biased values in criteria by applying power weights through the PA operator.
- They consider the interrelationship between multiple criteria with the help of HM.
- They consider partitioning of the different criteria based on their characteristic.
- They quantify the limitations of incomplete information and algebraic t-norm and t-conorm based operational laws through DST.

To identify the feasibility of the proposed operators, a numerical example is presented to select the best hotel among the available options. It has been concluded that the best and worst alternatives for proposed operators are the hotels B_1 and B_4, respectively. A sensitivity analysis has been discussed to examine the effect of variation in the granularity parameter vector q on the aggregated values. The sensitivity results concluded that, for each choice of granularity parameter vector $q = (q_1, q_2)$ values, the best and worst alternatives are always B_1 and B_4, which shows the stability and consistency of the proposed operators. Qualitative and quantitative comparative analyses demonstrated that the developed operators outperformed the existing ones in the PFS environment. The results indicate that the developed operators, as compared to the examined existing aggregation operators under the PFS environment, are more adaptable, general, consistent, and stable to solve any real-life MCDM problem more effectively.

6.1 Limitations

Some of the noted limitations of this study are as follows:

- The proposed decision-making approach has limitations related to the decision space because the proposed MCDM approach is based on PFNs and is unable to apply to problems where the decision space needs some extended space, such as spherical fuzzy sets, T-spherical fuzzy sets, complex T-spherical fuzzy sets, etc.
- The proposed approach uses DST operations. Due to the involvement of DST operations, our proposed operators do not satisfy the property of idempotency. However, in the literature, there are Archimedean-type generalized operations available that may be used to get more flexible results.
- The proposed approach is also unable to address the issue of group decision-making. Additionally, if the group of decision makers is large, the level of agreement among them would influence the final decision, and as a result, the concept of consensus should be integrated with the proposed approach.
- Another drawback is that in order to identify the best hotel, this study relies on secondary data (Luo and Xing 2020). The approach can be developed further and then used to evaluate the best hotel with real-time data, with the goal of trying to suggest some preventative measures to help decision-makers make sensible and workable choices.

6.2 Future scope

In future research, the proposed MCDM approach can be extended further in several directions, such as:

- The method can be integrated with other MCDM techniques like TOPSIS, AHP, WASPAS, MABAC, MEREC, VIKOR, EDAS, DEMATEL, etc.

- The strategy can be applied to other types of fuzzy environments, such as complex picture fuzzy sets, complex pythagorean fuzzy sets, complex q-rung fuzzy sets, complex t-spherical fuzzy sets, etc.
- This method can be extended by using other types of methods, such as distance measures, entropy measures, similarity measures, the maximizing deviation method, the best-worst method, gray relation analysis, etc., to calculate the weights of both the DMs and criteria.
- The method can be used to solve some other types of MCDM issues, such as supply chain management, municipal solid waste management, sewage treatment, the evaluation of renewable energy sources, human resource management, and the evaluation of air quality monitoring systems, etc.

Acknowledgements The authors are thankful to the Editor, Associate Editor and anonymous referees for providing their valuable comments and suggestions to improve the quality of the paper.

Author Contributions Conceptualization, methodology and software Tanuja Punetha, Komal and Dragan Pamucar; validation, Tanuja Punetha and Komal; formal analysis, Tanuja Punetha; investigation, Dragan Pamucar; resources, Tanuja Punetha; data curation, Komal; writing-original draft preparation, writing-review and editing, Tanuja Punetha, Komal and Dragan Pamucar; visualization, Tanuja Punetha and Komal; supervision, Komal and Dragan Pamucar; All authors have read and agreed to the published version of the manuscript.

Funding During this research no funding was received.

Declarations

Ethical Approval This article does not contain any studies with human participants or animals preformed by any of the authors.

Conflict of Interest The authors declare that they have no conflict of interest.

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