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χ-linguistic sets and its application for the linguistic multi-attribute group decision making

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Abstract

The information in the real world often contains many properties such as fuzziness, randomness, and approximation. Although existing linguistic collections attempt to solve these problems, with the emergence of more and more constraints and challenges, this information cannot fully express the problem, leading to an increasing demand for methods that can contain multiple uncertain information. In this paper, we comprehensively consider the various characteristics of information including membership degree, credibility and approximation based on rough sets, and propose the concept of χ-linguistic sets (χLSs), which depend on original data rather than prior knowledge and effectively solve the problem of incomplete information representation. At the same time, the corresponding theories such as the comparison method and operational rules have also been proposed. Subsequently, we construct a new χ-linguistic VIKOR (χLVIKOR) method for multi-attribute group decision making (MAGDM) problem with χLSs, and apply it to the risk assessment of COVID-19. Through comparative analysis, we discuss the effectiveness and superiority of χLSs.

Keywords χ-Linguistic sets · MAGDM · Operators · χLVIKOR · COVID-19

1 Introduction

Due to the complexity and uncertainty of the environment, the information in the real word such as the COVID-19 cannot be accurately described. The handling of uncertain information is a widely concerned problem in decision making. Language is a way and tool for human communication and expression. Zadeh (1975) proposed the concept of linguistic variables to represent uncertain information. Compared with traditional values, linguistic variables such as "good" and "high" are more in line with decision maker (DM)'s cognition

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and intuition. And linguistic variables can express natural linguistic information as computable mathematical symbols (Herrera et al. 1996; Gou 2017; Zhang et al. 2023; Krishankumaar et al. 2022).

Subsequently, other linguistic representation methods have been continuously proposed and widely used in decision making. Wang and Li (2010) added linguistic evaluation on the basis of intuitionistic fuzzy sets (IFSs) (Atanassov 1986) and proposed intuitionistic linguistic sets, which contains three parts: linguistic evaluation, membership degree, and non-membership degree. Du and Zuo (2011) and Yu et al. (2018) respectively proposed the extended TOPSIS method and TODIM method based on intuitionistic linguistic numbers (ILNs), and applied them to decision making. Some generalized dependent aggregation operators were also proposed by Liu (2013). Considering DMs may hesitate between several linguistic terms, Rodriguez et al. (2012) put forward the hesitant fuzzy linguistic term set (HFLTS). This concept is proposed based on the hesitant fuzzy set (HFS) (Torra 2010) and the linguistic term set (LTS) (Zadeh 1975), and it can better express the preferences of DMs. Beg and Rashid (2013) extended the TOPSIS method to HFLTSs. Wei et al. (2013) proposed some operators and comparison methods based on HFLTSs. Besides, HFLTSs have been applied in practical decision making problems, such as the strategic management of liquor brands (Liao et al. 2020), the selection of eco-friendly cities (Boyaci 2020), and the selection of specific medicines for the COVID-19 (Ren et al. 2020). However, the weights of different linguistic terms in HFLTSs are not all the same. To express preference, Pang et al. (2016) proposed probabilistic linguistic term sets (PLTSs), which adds probabilistic information to the linguistic terms. Subsequently, its operational rules (Gou and Xu 2016; Liao et al. 2019) and comparison methods (Xian et al. 2019; Bai et al. 2017) were further discussed. PLTSs (Pang et al. 2016; Wu et al. 2018; Liu and Li 2019; Luo et al. 2020; Lin et al. 2021; Gou et al. 2021), its extended sets (Chai et al. 2021; Wei et al. 2020; Krishankumar et al. 2020; Zhang et al. 2020; Jin et al. 2019), and probabilistic linguistic preference relations (PLPRs) (Liu et al. 2019; Gao et al. 2019, 2019; Song and Hu 2019; Li et al. 2020; Opricovic 1998) have very good effects in group decision making (GDM). When expressing decision making information, the credibility of the information is often ignored. Therefore, Zadeh (2011) took it into consideration and proposed the Z-number. Wang et al. (2017) and Xian et al. (2019) combined language and Z-number and proposed linguistic Z-numbers and Z-linguistic variables, respectively. Because of its superiority in the completeness of information representation, it is widely used in decision making (Krohling et al. 2019; Peng et al. 2019; Dai et al. 2008; Tao et al. 2020; Xian et al. 2021).

However, the existing linguistic representation methods are simple to the form of linguistic representation. In practical problems, DMs' cognition results of information may have some or all characteristics such as fuzziness, ambiguity, and randomness. And most of the decision making problems we encounter are one-dimensional, but there are also multi-dimensional ones.

Example 1 An expert evaluates the development potential of a project, and he needs to evaluate it from both the domestic market and the foreign market. So, this is a two-dimensional problem. For the evaluation of the domestic market potential, the degree belongs to "very good" is 0.8, and the degree does not belong to "very good" is 0.1. The credibility of the evaluation is "90% possible". As for foreign markets, the degree belongs to "good" is 0.6, and the degree does not belong to "good" is 0.2. The credibility of the evaluation is "70% possible". Then, the above contents can be expressed as

$$<[(very\ good, 0.8, 0.1); 0.9], [(good, 0.6, 0.2); 0.7]> .$$

Therefore, linguistic variable, ambiguity(include membership degree and non-membership degree), and randomness(probability, credibility) appear simultaneously in the evaluation. However, the existing linguistic representation methods cannot completely represent these information. It is very necessary to propose a new linguistic representation method.

In addition, the decision information in the existing methods is DMs' subjective evaluation, which leads to the lack of objectivity and inaccuracy of the decision results. The characterization of the upper and lower approximations in the rough sets (RSs) (Pawlak 1982) relies on the original data and does not rely on prior knowledge. With this idea, these data can be aggregated into objective and consistent group judgments, so as to obtain an evaluation close to the real ranking.

Example 2 Using the information in Example 1, and inspired by the approximations in the rough set, the lower and upper approximations A, B and A', B' can be obtained by calculation. The approximations are used to express the existing information in Example 1 as

$$<\{[(very\ good, 0.8, 0.1); 0.9]; (A, B)\}, \{[(good, 0.6, 0.2); 0.7]; (A', B')\}> .$$

In the MADGM, Saaty (1980) first proposed the famous Analytic Hierarchy Process (AHP), which is based on the principle of hierarchical decision-making problems. Hwang and Yoon (1981) proposed a TOPSIS method based on the ideal point principle. This method first determines an ideal point and selects the closest solution to the ideal point as the optimal solution. Opricovic (1998) proposed the VIKOR decision-making method, which is a compromise ranking method that uses maximizing group utility and minimizing individual regret values to compromise the ranking of finite decision options. Among the above methods, the VIKOR method has an additional decision-making mechanism coefficient compared to the TOPSIS method, which can enable decision-makers to make more aggressive or conservative decisions. The use of the VIKOR model increases the reflection of the importance of the decision-maker's own needs. With the development of fuzzy mathematics, research on uncertain multi-attribute decision-making problems has begun again, and some achievements have been made one after another. In recent years, Khan et al. (2020) proposed a new technique for Pythagorean cubic fuzzy multi-criteria decision-making using the Topsis method, and based on this, proposed an MCDM method that utilizes PCF information.Meniz (2023) introduced fuzzy AHP-VIKOR to calculate "ideal vaccination". Tazzit et al. (2023) proposes an evaluation method based on mixed weight determination and extended VIKOR model.

The COVID-19 is a major public health crisis that poses a great threat to the lives of citizens around the world and has a major impact on the world economy. Since the outbreak of the COVID-19, scholars have conducted extensive and in-depth research on it. From a spatial perspective, Melin et al. (2020a) used self-organizing maps to analyze the global COVID-19 pandemic. A multiple ensemble neural network model with fuzzy response aggregation (Melin et al. 2020b) was proposed to predict the time series of COVID-19 in Mexico. Berekaa (2021) published his insights on the COVID-19 pandemic from the origin, pathogenesis, diagnosis and therapeutic intervention. Mustafa (2021) also conducted research and statistics on COVID-19. Pejic-Bach (2021) pointed out that electronic commerce met the communication needs of individuals and businesses during the epidemic. Because of the risk of inflow of some confirmed cases, the risk of medical resources, etc., the epidemic situation in some countries is repeated. Sporadic outbreaks still occur in better controlled countries. As things stand, the COVID-19 will last for a while yet. Therefore, assessing the risk of the COVID-19 in cities can help local governments to accurately prevent and control the epidemic.

Combined with the above four writing motivations, the contributions of this paper are as follows:

First, taking the fuzziness, randomness and approximation of information into consideration, the concept of χLSs is proposed in this paper. It overcomes the problem of incomplete information representation in previous methods and is more flexible in dimension.

Second, the comparison method, normalization, operational rules and distance measure of χLSs are also further studied.

Finally, we propose a χLVIKOR method based on χLSs for MAGDM problem. The subsequent use of this method to address COVID-19 risk assessment in cities demonstrates the effectiveness of the method.

The rest of the paper is distributed as follows: Sect. 2 is a review of existing linguistic representation methods. In Sect. 3, a concept of χLSs and related basic theories are put forward. A new χLVIKOR method for the MAGDM problem with χLSs is constructed in Sect. 4. In Sect. 5, a numerical case on the risk assessment of COVID-19 in cities illustrates the effectiveness of this method. In Sect. 6, conclude the entire text.

2 Preliminaries

This section mainly introduces several existing language set studies, providing a sufficient theoretical basis for the subsequent research methods of this article.

2.1 Linguistic term sets

For language is more in line with people's cognition, the LTSs have more advantages in expressing uncertain information and have a wide range of applications.

Definition 1 (Herrera etal. 1995) The expression of the additive LTS whose subscripts are all non-negative numbers is as follows:

$$S = \{s_i | i = 0, 1, \ldots, \tau\}, \tag{1}$$

where s_i is a linguistic term, τ is an even number, s_0 and s_τ are the upper and lower limits of the LTS. S satisfies the following conditions:

(1) If $i > j$, then $s_i > s_j$;
(2) There is a negative operator $neg(s_i) = s_j$, that makes $i + j = \tau$.

To facilitate calculation and avoid losing information, Dai et al. (2008) expanded the discrete additive LTS.

Definition 2 (Dai etal. 2008) Let $S = \{s_i | i = 0, 1, \ldots, \tau\}$ be an additive LTS, then an extended additive LTS is expressed as follows:

$$\bar{S} = \{s_i | i \in [0, q], q > \tau\}. \tag{2}$$

2.2 Intuitionistic linguistic sets

In order to reduce the limitations of linguistic or vague linguistic and better characterize non membership and decision-maker hesitation. On the basis of IFSs, Wang and Li (2010) proposed intuitionistic linguistic sets to reduce the limitations of fuzzy linguistic.

Definition 3 (Wang and Li 2010) Let $\bar{S}$ be an extended additive LTS, $s_{\theta(x)} \in \bar{S}$, and X is the given universe of discourse, then the intuitionistic linguistic set (ILS) is expressed as follows:

$$A = \{ < x, [s_{\theta(x)}, \mu(x), v(x)] > | x \in X\}, \tag{3}$$

where $\mu(x)$ and $v(x) : X \to [0, 1]$ denote the degree to which x belongs and does not belong to $s_{\theta(x)}$, respectively. $\mu(x) + v(x) \leq 1$, and when $\mu(x) = 1, v(x) = 0$, the intuitionistic linguistic set degenerates into the LTS.

Definition 4 (Wang and Li 2010) Let $A = \{ < x, [s_{\theta(x)}, \mu(x), v(x)] > | x \in X\}$ be an ILS, then the triplet $< s_{\theta(x)}, \mu(x), v(x) >$ is an intuitionistic linguistic number (ILN).

Definition 5 (Wang and Li 2010) Let $a_1 = < [s_{\theta(a_1)}, \mu(a_1), v(a_1)] >$ and $a_2 = < [s_{\theta(a_2)}, \mu(a_2), v(a_2)] >$ be two ILNs, and $\lambda \geq 0$, then

(1) $a_1 + a_2 = < s_{\theta(a_1)+\theta(a_2)}, \frac{\theta(a_1)\mu(a_1)+\theta(a_2)\mu(a_2)}{\theta(a_1)+\theta(a_2)}, \frac{\theta(a_1)v(a_1)+\theta(a_2)v(a_2)}{\theta(a_1)+\theta(a_2)} >$;

(2) $a_1 \cdot a_2 = < s_{\theta(a_1)\theta(a_2)}, \mu(a_1)\mu(a_2), v(a_1) + v(a_2) >$;

(3) $\lambda a_1 = < s_{\lambda\theta(a_1)}, \mu(a_1), v(a_1) >$;

(4) $a_1{}^{\lambda} = < s_{\theta(a_1)^{\lambda}}, \mu(a_1)^{\lambda}, 1 - (1 - v(a_1))^{\lambda} >$.

Definition 6 (Wang and Li 2010) Let $a = < [s_{\theta(a)}, \mu(a), v(a)] >$ be an ILN, then its compromise expectation is:

$$E(a) = \frac{s_{\theta(x)} \cdot (\mu(a) + 1 - v(a))}{2}. \tag{4}$$

2.3 Hesitant fuzzy linguistic term sets

To address the hesitation of experts in evaluating multiple values such as indicators, choices, and variables, the HFLTS proposed by Rodriguez et al. (2012) contains several possible linguistic term values, which indicate the hesitation of DMs.

Definition 7 (Rodriguez etal. 2012) Let $S=\{s_i|i=0,1,\ldots,\tau\}$ be a LTS, then the HFLTS b_S is a continuous ordered finite subset of S.

Definition 8 (Zhu and Xu 2013) Let $b_1=\{{b_1}^k|k=1,2,\ldots,\#b_1\}$ and $b_2=\{{b_2}^k|k=1,2,\ldots,\#b_2\}$ be two HFLTSs, $\#b_1$ and $\#b_2$ are the number of linguistic terms in b_1 and b_2, $\#b_1=\#b_2$, and $\lambda\geq 0$, then the the operational rules are as follows:

(1) $b_1\oplus b_2=\bigcup_{{b_1}^{\delta(k)}\in b_1,{b_2}^{\delta(k)}\in b_2}\{{b_1}^{\delta(k)}\oplus{b_2}^{\delta(k)}\}$;

(2) $\lambda b_1=\bigcup_{{b_1}^{\delta(k)}\in b_1}\{\lambda{b_1}^{\delta(k)}\}$,

where ${b_1}^{\delta(k)}$ and ${b_2}^{\delta(k)}$ are the kth linguistic terms in b_1 and b_2, respectively.

2.4 Z-linguistic sets

Considering the lack of comprehensive consideration for fuzziness, hesitation, and randomness in the aforementioned linguistic, Xian et al. (2019) proposed Z-linguistic sets on the basis of Z-numbers, which can simultaneously represent linguistic information and its credibility.

Definition 9 (Xian etal. 2019) Let X be a non-empty set, then a Z-linguistic set is defined as follows:

$$L(z(x))=\{x,(<s_{\theta(x)},f_{\sigma(x)}>,r_{\rho(x)})|x\in X\},\tag{5}$$

where $s_\theta:X\to S,x\mapsto s_{\theta(x)}\in S$, $f_\sigma:X\to F,x\mapsto f_{\sigma(x)}\in F$, and $r_\rho:X\to R,x\mapsto r_{\rho(x)}\in R$. S, F, and R are three linguistic scales. $f_{\sigma(x)}$ is the membership of $s_{\theta(x)}$, and $r_{\rho(x)}$ is the credibilty of $<s_{\theta(x)},f_{\sigma(x)}>$.

2.5 Fuzzy rough sets

In order to solve the problem of incomplete and uncertain information, Dubois and Prade (1990) proposed the concept of fuzzy rough sets to solve this problem.

Definition 10 (Dubois and Prade 1990) Let (X, R) be an Pawlak approximate space, and A is a fuzzy set on X, then the lower approximation $\underline{A}$ and the upper approximation $\bar{A}$ of A on (X, R) are fuzzy sets. Their membership functions are:

$$\underline{A}_R=\inf\{A(y)|y\in[x]_R\},x\in X,\tag{6}$$

$$\bar{A}_R=\sup\{A(y)|y\in[x]_R\},x\in X,\tag{7}$$

where $[x]_R$ is the equivalent class of x under R.

3 χ-Linguistic sets

In this section, we comprehensively consider the various characteristics of information including membership degree, credibility and approximation based on rough sets, and propose the concept of χ-linguistic sets (χLSs) and its related theories.

3.1 The concept of χLSs

Definition 11 Let $\bar{S}$ be an extended additive LTS, and X be a non-empty set, then an χ-linguistic set (χLS) is defined as follows:

$$L\chi(x) = \{ <x, \{[\widetilde{s}_x, \widetilde{f}_x; \widetilde{p}_x]; \widetilde{g}_x\} > | x \in X\}, \tag{8}$$

where $\widetilde{s}=s^n_\theta : X \to \bar{S}^n, x \mapsto s^n_{\theta(x)} \in \bar{S}^n$, $\widetilde{f}=f^n_\sigma : X \to F^n, x \mapsto f^n_{\sigma(x)} \in F^n$, $\widetilde{p}=p^n_\rho : X \to P^n$, $x \mapsto p^n_{\rho(x)} \in P^n$, and $\widetilde{g}=g^n_\varsigma : X \to G^n, x \mapsto g^n_{\varsigma(x)} \in G^n$. $\bar{S}^n = \bar{S} \times \bar{S} \times \cdots \times \bar{S}$ is a n-dimensional extended additive LTS, $F^n = F \times F \times \cdots \times F$ is a n-dimensional linguistic ambiguity vector set, $P^n = P \times P \times \cdots \times P$ is a possibility(credibility) vector set of n-dimensional ambiguity, and $G^n = G \times G \times \cdots \times G$ is an approximation vector set of n-dimensional linguistic ambiguity and its possibility. $f^n_{\sigma(x)}$ is the linguistic ambiguity (include the membership degree $\mu^n_{\sigma(x)}$ and the non-membership degree $v^n_{\sigma(x)}$) vector of $s^n_{\theta(x)}$, $p^n_{\rho(x)}$ is the probability (reliability) vector of the fuzzy linguistic information $(s^n_{\theta(x)}, f^n_{\sigma(x)})$, and $g^n_{\varsigma(x)}$ is the approximation (include the lower approximation and the upper approximation, which are introduced in detail in Definitions 13 and 14) vector of the fuzzy linguistic information $[(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)}]$. $\mu^n_{\sigma(x)}, v^n_{\sigma(x)}, p^n_{\rho(x)} \in [0,1]$ and $\mu^n_{\sigma(x)} + v^n_{\sigma(x)} \leq 1$.

Definition 12 Let $L\chi(x) = \{ <x, \{[(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)}]; g^n_{\varsigma(x)}\} > | x \in X\}$ be an χLS, then the quaternion $<[(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)}]; g^n_{\varsigma(x)}>$ is an χ-linguistic variable (χLV), and it is denoted as $l\chi(x)$.

In this way, the χLS can also be represented as $L\chi(x) = \{ <x, l\chi(x) > | x \in X\}$.

The solution to the approximation $g^n_{\varsigma(x)}$ (include the lower approximation and the upper approximation) is introduced below.

Definition 13 Let $l'\chi(x) = <(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)}>$ be an IPLV, and $L'\chi(x) = \{ <x, l'\chi(x) > | x \in X\}$ be an IPLS, then the lower approximation of the χLV $l\chi(x)$ is defined as:

$$\underline{apr}(s^n_{\theta(x)}) = \cup\{\dot{s}^n \in \bar{S}^n | L'\chi(x)(\dot{s}^n) \leq s^n_{\theta(x)}, x \in X\}, \tag{9}$$

$$\begin{aligned} \underline{apr}(f^n_{\sigma(x)}) &= \cup\{\dot{f}^n \in F^n | L'(\chi(x)(\dot{f}^n) \leq f^n_{\sigma(x)}, x \in X\} \\ &= \cup\{\dot{\mu}^n, \dot{v}^n \in F^n | L'\chi(x)(\dot{\mu}^n) - L'\chi(x)(\dot{v}^n) \leq \mu^n_{\sigma(x)} - v^n_{\sigma(x)}, x \in X\}, \end{aligned} \tag{10}$$

$$\underline{apr}(p^n_{\rho(x)}) = \cup\{\dot{p}^n \in P^n | L'\chi(x)(\dot{p}^n) \le p^n_{\rho(x)}, x \in X\}, \tag{11}$$

where $L'\chi(x)(\dot{s}^n)$, $L'\chi(x)(\dot{f}^n)$, and $L'\chi(x)(\dot{p}^n)$ are $\dot{s}^n$, $\dot{f}^n$, and $\dot{p}^n$ in $L'\chi(x)$, respectively.

Definition 14 Let $l'\chi(x) = <(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)}>$ be an IPLV, and $L'\chi(x) = \{<x, l'\chi(x)> | x \in X\}$ be an IPLS, then the upper approximation of the ILV $l\chi(x)$ is defined as:

$$\overline{apr}(s^n_{\theta(x)}) = \cup\{\dot{s}^n \in \bar{S}^n | L'\chi(x)(\dot{s}^n) \ge s^n_{\theta(x)}, x \in X\}, \tag{12}$$

$$\begin{aligned} \overline{apr}(f^n_{\sigma(x)}) &= \cup\{\dot{f}^n \in F^n | L'\chi(x)(\dot{f}^n) \ge f^n_{\sigma(x)}, x \in X\} \\ &= \cup\{\dot{\mu}^n, \dot{v}^n \in F^n | L'\chi(x)(\dot{\mu}^n) - L'\chi(x)(\dot{v}^n) \ge \mu^n_{\sigma(x)} - v^n_{\sigma(x)}, x \in X\}, \end{aligned} \tag{13}$$

$$\overline{apr}(p^n_{\rho(x)}) = \cup\{\dot{p}^n \in P^n | L'\chi(x)(\dot{p}^n) \ge p^n_{\rho(x)}, x \in X\}, \tag{14}$$

where $L'\chi(x)(\dot{s}^n)$, $L'\chi(x)(\dot{f}^n)$, and $L'\chi(x)(\dot{p}^n)$ are $\dot{s}^n$, $\dot{f}^n$, and $\dot{p}^n$ in $L'\chi(x)$, respectively.

In order to facilitate calculation, we use the lower and the upper approximation to find the lower and upper limits of $l\chi(x)$.

Definition 15 Let $<\left(\underline{apr}(s^n_{\theta(x)}), \underline{apr}(f^n_{\sigma(x)})\right); \underline{apr}(p^n_{\rho(x)})>$ be the lower approximation of $l\chi(x)$, then the lower limit of $l\chi(x)$ is defined as:

$$\underline{\lim}(s^n_{\theta(x)}) = \frac{1}{N_{ls}} \sum_{j=1}^{N_{ls}} \dot{s}^n \in \underline{apr}(s^n_{\theta(x)}), \tag{15}$$

$$\underline{\lim}(f^n_{\sigma(x)}) = \frac{1}{N_{lf}} \sum_{j=1}^{N_{lf}} \dot{f}^n \in \underline{apr}(f^n_{\sigma(x)}) = \left(\underline{\lim}(\mu^n_{\sigma(x)}), \underline{\lim}(v^n_{\sigma(x)})\right), \tag{16}$$

$$\underline{\lim}(p^n_{\rho(x)}) = \frac{1}{N_{lp}} \sum_{j=1}^{N_{lp}} \dot{p}^n \in \underline{apr}(p^n_{\rho(x)}), \tag{17}$$

where N_{ls}, N_{lf}, and N_{lp} are the number of elements in $\underline{apr}(s^n_{\theta(x)})$, $\underline{apr}(f^n_{\sigma(x)})$, and $\underline{apr}(p^n_{\rho(x)})$.

Definition 16 Let $<\left(\overline{apr}(s^n_{\theta(x)}), \overline{apr}(f^n_{\sigma(x)})\right); \overline{apr}(p^n_{\rho(x)})>$ be the upper approximation of $l\chi(x)$, then the upper limit of $l\chi(x)$ is defined as:

$$\overline{\lim}(s^n_{\theta(x)}) = \frac{1}{N_{us}}\sum_{j=1}^{N_{us}} \dot{s}^n \in \overline{apr}(s^n_{\theta(x)}), \tag{18}$$

$$\overline{\lim}(f^n_{\sigma(x)}) = \frac{1}{N_{uf}}\sum_{j=1}^{N_{uf}} \dot{f}^n \in \overline{apr}(f^n_{\sigma(x)}) = \left(\overline{\lim}(\mu^n_{\sigma(x)}), \overline{\lim}(v^n_{\sigma(x)})\right), \tag{19}$$

$$\overline{\lim}(p^n_{\rho(x)}) = \frac{1}{N_{up}}\sum_{j=1}^{N_{up}} \dot{p}^n \in \overline{apr}(p^n_{\rho(x)}), \tag{20}$$

where N_{us}, N_{uf}, and N_{up} are the number of elements in $\overline{apr}(s^n_{\theta(x)})$, $\overline{apr}(f^n_{\sigma(x)})$, and $\overline{apr}(p^n_{\rho(x)})$.

In this way, $g^n_{\varsigma(x)}$ can also be expressed in the form of limits:

$$\begin{aligned} g^n_{\varsigma(x)} = (&< \left(\underline{\lim}(s^n_{\theta(x)}), \underline{\lim}(f^n_{\sigma(x)})\right); \underline{\lim}(p^n_{\rho(x)}) > \ , \\ &< \left(\overline{\lim}(s^n_{\theta(x)}), \overline{\lim}(f^n_{\sigma(x)})\right); \overline{\lim}(p^n_{\rho(x)}) >). \end{aligned} \tag{21}$$

The boundary of $l\chi(x)$ is:

$$\begin{aligned} &bnd(l\chi(x)) \\ &= < \begin{pmatrix} \underline{\lim}(s^n_{\theta(x)}) - \overline{\lim}(s^n_{\theta(x)}), \\ \underline{\lim}(f^n_{\sigma(x)}) - \overline{\lim}(f^n_{\sigma(x)}) \end{pmatrix}; \underline{\lim}(p^n_{\rho(x)}) - \overline{\lim}(p^n_{\rho(x)}) > . \end{aligned} \tag{22}$$

Example 3 Use the linguistic terms in Example 2 to express the linguistic information in Example 1, then it is expressed as an IPLV: $l_1{}'\chi(x) = <[(s_6, 0.8, 0.1); 0.9], [(s_5, 0.6, 0.2); 0.7] >$.

A total of three DMs participated in the evaluation. The evaluations of the remaining two DMs are $l_2{}'\chi(x) = <[(s_4, 0.7, 0.1); 0.6], [(s_6, 0.5, 0.3); 0.8] >$ and $l_3{}'\chi(x) = <[(s_2, 0.7, 0.2); 0.6], [(s_5, 0.9, 0.1); 0.9] >$, then the three evaluations aggregate an IPLS, denoted as

$$\begin{aligned} L'\chi(x) = \{ &<[(s_6, 0.8, 0.1); 0.9], [(s_5, 0.6, 0.2); 0.7] > , \\ &<[(s_4, 0.7, 0.1); 0.6], [(s_6, 0.5, 0.3); 0.8] > , \\ &<[(s_2, 0.7, 0.2); 0.6], [(s_5, 0.9, 0.1); 0.9] > \}. \end{aligned}$$

(1) From Definitions 13 and 15, we use the lower approximation method to find the lower limit:

From Eq. (15), we can obtain the lower approximation of linguistic variables:
$\underline{\lim}(s^1_{\theta(x_1)}) = \frac{1}{3} \times (s_6 \oplus s_4 \oplus s_2) = s_4$

From Eq. (16), we can obtain the lower approximation of linguistic membership degree:
$\underline{\lim}(\mu^1_{\sigma(x_1)}) = \frac{1}{3} \times (0.8 + 0.7 + 0.7) \approx 0.73$
$\underline{\lim}(v^1_{\sigma(x_1)}) = \frac{1}{3} \times (0.1 + 0.1 + 0.2) \approx 0.13$

From Eq. (17), we can obtain a lower approximation of the credibility of uncertain linguistic information:

$\underline{\lim}(p^1_{\rho(x_1)}) = \frac{1}{3} \times (0.9 + 0.6 + 0.6) = 0.7$

(2) From Definitions 14 and 16, we use the upper approximation method to find the upper limit:

From Eq. (18), we can obtain the upper approximation of linguistic variables:

$\overline{\lim}(s^1_{\theta(x_1)}) = \frac{1}{1} \times s_6 = s_6$

From Eq. (19), we can obtain the upper approximation of linguistic membership:

$\overline{\lim}(\mu^1_{\sigma(x_1)}) = \frac{1}{1} \times 0.8 = 0.8$

$\overline{\lim}(v^1_{\sigma(x_1)}) = \frac{1}{1} \times 0.1 = 0.1$

From Eq. (20), we can obtain a upper approximation of the credibility of uncertain linguistic information:

$\overline{\lim}(p^1_{\rho(x_1)}) = \frac{1}{1} \times 0.9 = 0.9$

(3) From Eq. (21), Based on the above calculation results, we can obtain one-dimensional approximations:

$g^1_{\varsigma(x_1)} = (<(s_4, 0.73, 0.13); 0.7> , <(s_6, 0.8, 0.1); 0.9>).$

Similarly, we can obtain a two-dimensional approximation:

$g^2_{\varsigma(x_1)} = (<(s_5, 0.55, 0.25); 0.7> , <(s_{5.3}, 0.75, 0.15); 0.8>).$

(4) Add $g^1_{\varsigma(x_1)}$ and $g^2_{\varsigma(x_1)}$ to the back of the corresponding dimension in $l_1{}'\chi(x)$ and obtain an χLV:

$$l_1\chi(x) = \left\langle \begin{array}{l} \{[(s_6, 0.8, 0.1); 0.9]; \begin{pmatrix} <(s_4, 0.73, 0.13); 0.7> , \\ <(s_6, 0.8, 0.1); 0.9> \end{pmatrix}\}, \\ \{[(s_5, 0.6, 0.2); 0.7]; \begin{pmatrix} <(s_5, 0.55, 0.25); 0.7> , \\ <(s_{5.3}, 0.75, 0.15); 0.8> \end{pmatrix}\} \end{array} \right\rangle.$$

Similarly, we obtain $l_2\chi(x)$ and $l_3\chi(x)$.

Then, an χLS is aggregate as follows:

$$L\chi(x) = \left\{ \begin{array}{l} \left\langle \begin{array}{l} \{[(s_6, 0.8, 0.1); 0.9]; \begin{pmatrix} <(s_4, 0.73, 0.13); 0.7> , \\ <(s_6, 0.8, 0.1); 0.9> \end{pmatrix}\}, \\ \{[(s_5, 0.6, 0.2); 0.7]; \begin{pmatrix} <(s_5, 0.55, 0.25); 0.7> , \\ <(s_{5.3}, 0.75, 0.15); 0.8> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \{[(s_4, 0.7, 0.1); 0.6]; \begin{pmatrix} <(s_3, 0.7, 0.15); 0.6> , \\ <(s_5, 0.75, 0.1); 0.7> \end{pmatrix}\}, \\ \{[(s_6, 0.5, 0.3); 0.8]; \begin{pmatrix} <(s_{5.3}, 0.5, 0.3); 0.75> , \\ <(s_6, 0.67, 0.2); 0.85> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \{[(s_2, 0.7, 0.2); 0.6]; \begin{pmatrix} <(s_2, 0.7, 0.2); 0.6> , \\ <(s_3, 0.73, 0.2); 0.7> \end{pmatrix}\}, \\ \{[(s_5, 0.9, 0.1); 0.9]; \begin{pmatrix} <(s_5, 0.67, 0.2); 0.8> , \\ <(s_{5.3}, 0.9, 0.1); 0.9> \end{pmatrix}\} \end{array} \right\rangle \end{array} \right\}.$$

Next, the operational rules of $g^n_{\varsigma(x)}$ in χLVs are introduced.

Definition 17 Let $g^{n_1}_{\varsigma(x_1)} = (<(\underline{\lim}(s^{n_1}_{\theta(x_1)}), \underline{\lim}(f^{n_1}_{\sigma(x_1)})); \underline{\lim}(p^{n_1}_{\rho(x_1)})> , <(\overline{\lim}(s^{n_1}_{\theta(x_1)}), \overline{\lim}(f^{n_1}_{\sigma(x_1)})); \overline{\lim}(p^{n_1}_{\rho(x_1)})>)$ and $g^{n_2}_{\varsigma(x_2)} = (<\left(\underline{\lim}(s^{n_2}_{\theta(x_2)}), \underline{\lim}(f^{n_2}_{\sigma(x_2)})\right); \underline{\lim}(p^{n_2}_{\rho(x_2)})> , <\left(\overline{\lim}(s^{n_2}_{\theta(x_2)}), \overline{\lim}(f^{n_2}_{\sigma(x_2)})\right); \overline{\lim}(p^{n_2}_{\rho(x_2)})>)$ be two approximations in $l_1\chi(x)$ and $l_2\chi(x)$, respectively. The subscript of $s^{n}_{\theta(x)}$ is represented as $\theta^{n}(x)$, $n_1 = n_2 = n$, and $\lambda \geq 0$. Then, the operational rules are as follows:

(1) $$g^{n}_{\varsigma(x_1)} \oplus g^{n}_{\varsigma(x_2)} = \left(< \left(s_{\underline{\lim}(\theta^{n}(x_1)) + \underline{\lim}(\theta^{n}(x_2))}, \underline{\lim}(f^{n_1}_{\sigma(x_1)}) \oplus \underline{\lim}(f^{n_2}_{\sigma(x_2)}) \right); \frac{\underline{\lim}(p^{n}_{\rho(x_1)}) + \underline{\lim}(p^{n}_{\rho(x_2)})}{2} > , < \left(s_{\overline{\lim}(\theta^{n}(x_1)) + \overline{\lim}(\theta^{n}(x_2))}, \overline{\lim}(f^{n}_{\sigma(x_1)}) \oplus \overline{\lim}(f^{n}_{\sigma(x_2)}) \right); \frac{\overline{\lim}(p^{n}_{\rho(x_1)}) + \overline{\lim}(p^{n}_{\rho(x_2)})}{2} > \right),$$

where
$$\underline{\lim}(f^{n}_{\sigma(x_1)}) \oplus \underline{\lim}(f^{n}_{\sigma(x_2)}) = \left(\begin{array}{c} \dfrac{\underline{\lim}(\theta^{n}(x_1))\underline{\lim}(\mu^{n}_{\sigma(x_1)}) + \underline{\lim}(\theta^{n}(x_2))\underline{\lim}(\mu^{n}_{\sigma(x_2)})}{\underline{\lim}(\theta^{n}(x_1)) + \underline{\lim}(\theta^{n}(x_2))}, \\ \dfrac{\underline{\lim}(\theta^{n}(x_1))\underline{\lim}(v^{n}_{\sigma(x_1)}) + \underline{\lim}(\theta^{n}(x_2))\underline{\lim}(v^{n}_{\sigma(x_2)})}{\underline{\lim}(\theta^{n}(x_1)) + \underline{\lim}(\theta^{n}(x_2))} \end{array} \right),$$

$$\overline{\lim}(f^{n}_{\sigma(x_1)}) \oplus \overline{\lim}(f^{n}_{\sigma(x_2)}) = \left(\begin{array}{c} \dfrac{\overline{\lim}(\theta^{n}(x_1))\overline{\lim}(\mu^{n}_{\sigma(x_1)}) + \overline{\lim}(\theta^{n}(x_2))\overline{\lim}(\mu^{n}_{\sigma(x_2)})}{\overline{\lim}(\theta^{n}(x_1)) + \overline{\lim}(\theta^{n}(x_2))}, \\ \dfrac{\overline{\lim}(\theta^{n}(x_1))\overline{\lim}(v^{n}_{\sigma(x_1)}) + \overline{\lim}(\theta^{n}(x_2))\overline{\lim}(v^{n}_{\sigma(x_2)})}{\overline{\lim}(\theta^{n}(x_1)) + \overline{\lim}(\theta^{n}(x_2))} \end{array} \right),$$

$\underline{\lim}(\theta^{n}(x_1))$, $\underline{\lim}(\theta^{n}(x_2))$, $\overline{\lim}(\theta^{n}(x_1))$, and $\overline{\lim}(\theta^{n}(x_2))$ are the subscripts of $\underline{\lim}(s^{n}_{\theta(x_1)})$, $\underline{\lim}(s^{n}_{\theta(x_2)})$, $\overline{\lim}(s^{n}_{\theta(x_1)})$, and $\overline{\lim}(s^{n}_{\theta(x_2)})$, respectively.

(2) $$g^{n}_{\varsigma(x_1)} \otimes g^{n}_{\varsigma(x_2)} = \begin{array}{l} \left(< \left(\begin{array}{c} s_{\underline{\lim}(\theta^{n}(x_1)) \times \underline{\lim}(\theta^{n}(x_2))}, \\ \underline{\lim}(f^{n}_{\sigma(x_1)}) \otimes \underline{\lim}(f^{n}_{\sigma(x_2)}) \end{array} \right); \underline{\lim}(p^{n}_{\rho(x_1)}) \times \underline{\lim}(p^{n}_{\rho(x_2)}) > , \right. \\ \left. < \left(\begin{array}{c} s_{\overline{\lim}(\theta^{n}(x_1)) \times \overline{\lim}(\theta^{n}(x_2))}, \\ \overline{\lim}(f^{n}_{\sigma(x_1)}) \otimes \overline{\lim}(f^{n}_{\sigma(x_2)}) \end{array} \right); \overline{\lim}(p^{n}_{\rho(x_1)}) \times \overline{\lim}(p^{n}_{\rho(x_2)}) > \right), \end{array}$$

where $\underline{\lim}(f^{n}_{\sigma(x_1)}) \otimes \underline{\lim}(f^{n}_{\sigma(x_2)}) = \left(\underline{\lim}(\mu^{n}_{\sigma(x_1)}) \times \underline{\lim}(\mu^{n}_{\sigma(x_2)}), \underline{\lim}(v^{n}_{\sigma(x_1)}) + \underline{\lim}(v^{n}_{\sigma(x_2)}) \right)$, and $\overline{\lim}(f^{n}_{\sigma(x_1)}) \otimes \overline{\lim}(f^{n}_{\sigma(x_2)}) = \left(\overline{\lim}(\mu^{n}_{\sigma(x_1)}) \times \overline{\lim}(\mu^{n}_{\sigma(x_2)}), \overline{\lim}(v^{n}_{\sigma(x_1)}) + \overline{\lim}(v^{n}_{\sigma(x_2)}) \right)$.

(3) $$\lambda g^{n}_{\varsigma(x_1)} \begin{array}{l} = \left(< \left(\lambda\underline{\lim}(s^{n}_{\theta(x_1)}), \lambda\underline{\lim}(f^{n}_{\sigma(x_1)}) \right); \underline{\lim}(p^{n}_{\rho(x_1)}) > \; < \left(\lambda\overline{\lim}(s^{n}_{\theta(x_1)}), \lambda\overline{\lim}(f^{n}_{\sigma(x_1)}) \right); \overline{\lim}(p^{n}_{\rho(x_1)}) > \right) \\ = \left(< \left(s_{\lambda\underline{\lim}(\theta^{n}(x_1))}, \underline{\lim}(\mu^{n}_{\sigma(x_1)}), \underline{\lim}(v^{n}_{\sigma(x_1)}) \right); \underline{\lim}(p^{n}_{\rho(x_1)}) > \; < \left(s_{\lambda\overline{\lim}(\theta^{n}(x_1))}, \overline{\lim}(\mu^{n}_{\sigma(x_1)}), \overline{\lim}(v^{n}_{\sigma(x_1)}) \right); \overline{\lim}(p^{n}_{\rho(x_1)}) > \right). \end{array}$$

(4) $$\left(g^{n}_{\varsigma(x_1)}\right)^{\lambda} = \left(< \left(s_{\underline{\lim}(\theta^{n}(x_1))^{\lambda}}, \left(\underline{\lim}(f^{n}_{\sigma(x_1)})\right)^{\lambda} \right); \left(\underline{\lim}(p^{n}_{\rho(x_1)})\right)^{\lambda} > \; < \left(s_{\overline{\lim}(\theta^{n}(x_1))^{\lambda}}, \left(\overline{\lim}(f^{n}_{\sigma(x_1)})\right)^{\lambda} \right); \left(\overline{\lim}(p^{n}_{\rho(x_1)})\right)^{\lambda} > \right),$$

where $\left(\underline{\lim}(f^{n}_{\sigma(x_1)})\right)^{\lambda} = \left(\left(\underline{\lim}(\mu^{n}_{\sigma(x_1)})\right)^{\lambda}, 1 - \left(1 - \underline{\lim}(v^{n}_{\sigma(x_1)})\right)^{\lambda} \right)$,

and $\left(\overline{\lim}(f^{n}_{\sigma(x_1)})\right)^{\lambda} = \left(\left(\overline{\lim}(\mu^{n}_{\sigma(x_1)})\right)^{\lambda}, 1 - \left(1 - \overline{\lim}(v^{n}_{\sigma(x_1)})\right)^{\lambda} \right)$.

Example 4 Assume $g_{\varsigma(x_1)} = (<(s_3, 0.7, 0.2); 0.6>, <(s_4, 0.8, 0.2); 0.65>)$ and $g_{\varsigma(x_2)} = (<(s_4, 0.65, 0.1); 0.5>, <(s_5, 0.8, 0.1); 0.55>)$, then

(1) $g_{\varsigma(x_1)} \oplus g_{\varsigma(x_2)} = (<(s_7, 0.67, 0.14); 0.35>, <(s_9, 0.44, 0.14); 0.6>)$;
(2) $g_{\varsigma(x_1)} \otimes g_{\varsigma(x_2)} = (<(s_{12}, 0.46, 0.3); 0.3>, <(s_{20}, 0.64, 0.3); 0.36>)$;
(3) $2g_{\varsigma(x_1)} = (<(s_6, 0.7, 0.2); 0.6>, <(s_8, 0.8, 0.2); 0.65>)$;
(4) $(g_{\varsigma(x_1)})^2 = (<(s_9, 0.49, 0.36); 0.36>, <(s_{16}, 0.64, 0.36); 0.42>)$.

Remark 1 (1) When the upper approximation is equal to the lower approximation, the χ LS degenerates into the intuitionistic probabilistic linguistic set (IPLS), and the χLV degenerates into the intuitionistic probabilistic linguistic variable (IPLV), which is

$$l\chi(x) = <[(s^1, \mu^1, v^1); p^1], [(s^2, \mu^2, v^2); p^2], \ldots, [(s^n, \mu^n, v^n); p^n] > . \tag{23}$$

(2) When the upper approximation is equal to the lower approximation, and $p^1_{\rho(x)} = p^2_{\rho(x)} = \cdots = p^n_{\rho(x)}$, the χLS degenerates into the intuitionistic hesitant linguistic set (IHLS), and the χLV degenerates into the intuitionistic hesitant linguistic variable (IHLV), which is

$$l\chi(x) = <(s^1, \mu^1, v^1), (s^2, \mu^2, v^2), \ldots, (s^n, \mu^n, v^n) > . \tag{24}$$

(3) When the upper approximation is equal to the lower approximation and $n = 1$, the χLS degenerates into the intuitionistic Z-linguistic set (IZLS) (Xian et al. 2019, 2022), and the χLV degenerates into the intuitionistic Z-linguistic variable (IZLV), which is

$$l\chi(x) = [(s, \mu, v); p]. \tag{25}$$

(4) When the upper approximation is equal to the lower approximation, $n = 1$, and $\mu + v = 1$, the χLS degenerates into the Z linguistic set (ZLS) (Xian et al. 2019), and the χLV degenerates into the Z linguistic variable(ZLV), which is

$$l\chi(x) = [(s, \mu); p]. \tag{26}$$

(5) When the upper approximation is equal to the lower approximation, $n = 1, \mu = 1, v = 0$, and $p = 1$, the χLS degenerates into the LTS (Zadeh 1975) and the χLV degenerates into the linguistic term, which is

$$l\chi(x) = s. \tag{27}$$

The approximation of the χLS is similar to the RS, which includes the lower approximation, the upper approximation, and the boundary. It needs to be solved with the known fuzzy information in the χLS. The evaluation of a DM is expressed by the IPLV $l'\chi(x) = <(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)} >$. The evaluations of multiple DMs are aggregated to obtain an IPLS $L\chi(x) = \{<x, l'\chi(x) > |x \in X\}$. After solving the approximation $g^n_{\varsigma(x)}$ and adding it to $l\chi(x)$, the IPLV becomes the χLV $l\chi(x) = <[(s^n_{\theta(x)}, f^n_{\sigma(x)}); p^n_{\rho(x)}]; g^n_{\varsigma(x)} >$, and the IPLS becomes the χLS $L\chi(x) = \{<x, l\chi(x) > |x \in X\}$.

3.2 Comparison method of χLSs

In this section, we propose the comparison methods of χLVs and χLSs, respectively. First, the concepts of the ambiguity expectation, the possibility expectation, and the approximation expectation of the χLV are introduced.

Definition 18 Let $l_k\chi(x) = <[(s^{n_k}_{\theta(x_k)}, \mu^{n_k}_{\sigma(x_k)}, v^{n_k}_{\sigma(x_k)}); p^{n_k}_{\rho(x_k)}]; g^{n_k}_{\varsigma(x_k)} >$ be a χLV, then the ambiguity expectation of $l_k\chi(x)$ is

$$E(f^k) = \frac{\sum_{n=1}^{n_k} \Delta\left(\frac{s^n_{\theta(x_k)}(\mu^n_{\sigma(x_k)} + 1 - v^n_{\sigma(x_k)})}{2}\right)}{n_k}, \tag{28}$$

where Δ is an equivalent transformation function of linguistic terms, $\Delta : [0, \tau] \rightarrow [0, 1]$, and $\Delta(s_{\theta(x)}) = \frac{\theta(x)}{\tau}$.

Definition 19 Let $l_k\chi(x) = <[(s^{n_k}_{\theta(x_k)}, \mu^{n_k}_{\sigma(x_k)}, v^{n_k}_{\sigma(x_k)});\ p^{n_k}_{\rho(x_k)}];\ g^{n_k}_{\varsigma(x_k)} >$ be a χLV, then the possibility expectation of $l_k\chi(x)$ is:

$$E(p^k) = \frac{\sum_{n=1}^{n_k} p^{n_k}_{\rho(x_k)}}{n_k}. \tag{29}$$

Definition 20 Let $l_k\chi(x) = <[(s^{n_k}_{\theta(x_k)}, \mu^{n_k}_{\sigma(x_k)}, v^{n_k}_{\sigma(x_k)});\ p^{n_k}_{\rho(x_k)}];\ g^{n_k}_{\varsigma(x_k)} >$ be a χLV, then the approximation expectation of $l_k\chi(x)$ is:

$$E(g^k) = \frac{\sqrt{A(E(\underline{\lim}f^k))^2 + B(E(\underline{\lim}p^k))^2}}{\sqrt{A\left(E(\overline{\lim}f^k)\right)^2 + B\left(E(\overline{\lim}p^k)\right)^2}}, \tag{30}$$

where $E(\underline{\lim}f^k) = \frac{\sum_{n=1}^{n_k} \Delta\left(\frac{\underline{\lim}(s^{n_k}_{\theta(x)})\left(\underline{\lim}(\mu^{n_k}_{\sigma(x)})+1-\underline{\lim}(v^{n_k}_{\sigma(x)})\right)}{2}\right)}{n_k}$, $E(\overline{\lim}f^k) = \frac{\sum_{n=1}^{n_k} \Delta\left(\frac{\overline{\lim}(s^{n_k}_{\theta(x)})\left(\overline{\lim}(\mu^{n_k}_{\sigma(x)})+1-\overline{\lim}(v^{n_k}_{\sigma(x)})\right)}{2}\right)}{n_k}$, $E(\underline{\lim}p^k) = \frac{\sum_{n=1}^{n_k} \underline{\lim}(p^{n_k}_{\rho(x_k)})}{n_k}$, $E(\overline{\lim}p^k) = \frac{\sum_{n=1}^{n_k} \overline{\lim}(p^{n_k}_{\rho(x_k)})}{n_k}$, and $0 \le A, B \le 1$. Δ is an equivalent transformation function of linguistic terms, $\Delta : [0, \tau] \rightarrow [0, 1]$, and $\Delta(s_{\theta(x)}) = \frac{\theta(x)}{\tau}$.

$E(f^k) \rightarrow [0, 1]$, $E(p^k) \rightarrow [0, 1]$, $E(g^k) \rightarrow [0, 1]$. In this way, we map these three variables of $l_k\chi(x)$ to a three-dimensional coordinate system, and the value range they form is a cube with a side length of 1 in the first quadrant, as shown in Fig. 1. A certain point in this coordinate is denoted as $l_k\chi(x)$. Inspired by the spherical coordinate system, we can find the distance between this point and the origin O and two angles, respectively.

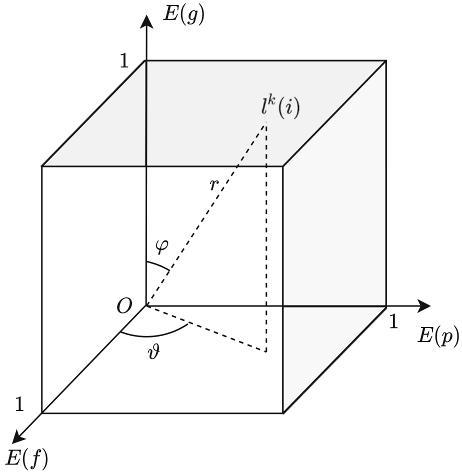

Fig. 1 The χLV $l_k\chi(x)$ in the three-dimensional coordinate system

Definition 21 Let $E(f^k)$, $E(p^k)$, and $E(g^k)$ be the ambiguity expectation, the possibility expectation, and the approximation expectation of the χLV $l_k\chi(x)$, then the distance between $l_k\chi(x)$ and the origin O is

$$r(l_k\chi(x)) = \sqrt{A(E(f^k))^2 + B(E(p^k))^2 + C(E(g^k))^2}, \tag{31}$$

where $0 \leq A, B, C \leq 1$ and $A + B + C = 1$.

Definition 22 Let $E(f^k)$ and $E(p^k)$ be the ambiguity expectation and the possibility expectation of the χLV $l_k\chi(x)$, then the angle between the projection line of the line from the origin to the point $l_k\chi(x)$ on the $E(f^k)E(p^k)$-plane and the positive $E(f^k)$-axis is

$$\vartheta(l_k\chi(x)) = \arctan\frac{\sqrt{B}E(p^k)}{\sqrt{A}E(f^k)}. \tag{32}$$

Definition 23 Let $E(g^k)$ be the approximation expectation of the χLV $l_k\chi(x)$, $r(l_k\chi(x))$ be the distance between $l_k\chi(x)$ and the origin O, then the angle between the line from the origin O to the point $l_k\chi(x)$ and the positive $E(g^k)$-axis is

$$\varphi(l_k\chi(x)) = \arccos\frac{\sqrt{C}E(g^k)}{r(l^k\chi(x))}. \tag{33}$$

In the information evaluation, the importance of the ambiguity, possibility, and approximation decreases in order (Fig. 2). Therefore, the comparison method of χLVs is as follows:

Definition 24 Let $l_1\chi(x)$ and $l_2\chi(x)$ be two ILVs, then:

(I) If $r(l_1\chi(x)) > r(l_2\chi(x))$, then $l_1\chi(x) \succ l_2\chi(x)$;
(II) If $r(l_1\chi(x)) < r(l_2\chi(x))$, then $l_1\chi(x) \prec l_2\chi(x)$;
(III) If $r(l_1\chi(x)) = r(l_2\chi(x))$, then

- (i) If $\vartheta(l_1\chi(x)) < \vartheta(l_2\chi(x))$, then $l_1\chi(x) \succ l_2\chi(x)$;
- (ii) If $\vartheta(l_1\chi(x)) > \vartheta(l_2\chi(x))$, then $l_1\chi(x) \prec l_2\chi(x)$;
- (iii) If $\vartheta(l_1\chi(x)) = \vartheta(l_2\chi(x))$, then

(1) If $\varphi(l_1\chi(x)) < \varphi(l_2\chi(x))$, then $l_1\chi(x) \succ l_2\chi(x)$;
(2) If $\varphi(l_1\chi(x)) > \varphi(l_2\chi(x))$, then $l_1\chi(x) \prec l_2\chi(x)$;
(3) If $\varphi(l_1\chi(x)) = \varphi(l_2\chi(x))$, then $l_1\chi(x) \sim l_2\chi(x)$.

Example 5 Let $S = \{s_i | i = 0, 1, \ldots, 6\}$ be an additive χTS, $l_1\chi(x) = \{[(s_3, 0.8, 0.2);\ 0.6];\ \begin{pmatrix} <(s_3, 0.7, 0.2);\ 0.6>, \\ <(s_4, 0.8, 0.2);\ 0.65> \end{pmatrix}\}$ and $l_2\chi(x) = \{[(s_5, 0.8, 0.1);\ 0.5];\ \begin{pmatrix} <(s_4, 0.65, 0.1);\ 0.5>, \\ <(s_5, 0.8, 0.1);\ 0.55> \end{pmatrix}\}$ be two χLVs, and $A = 0.5, B = 0.3, C = 0.2$.

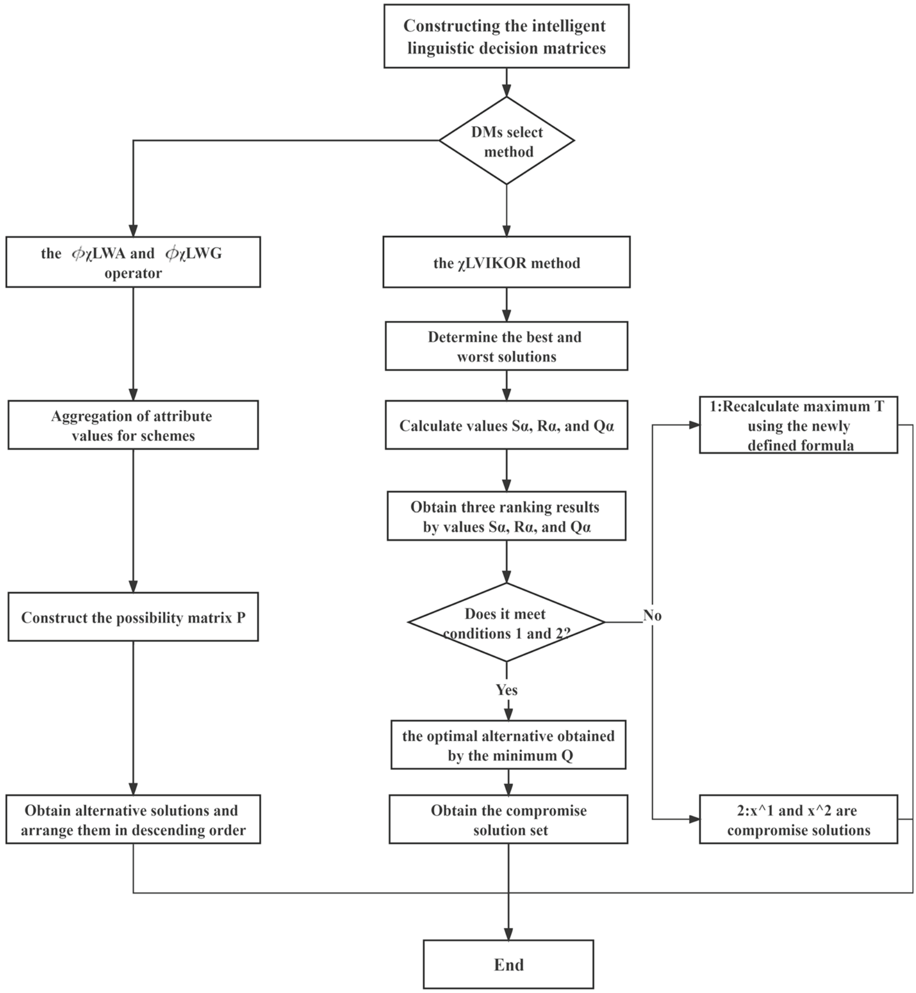

Fig. 2 The $\phi_{\chi LWA}$, $\phi_{\chi LWG}$ operator and the χLVIKOR method decision-making process

(1) From Definitions 18-20, we obtain:

$$E(f^1) = 0.4, \;\; E(p^1) = 0.6, \;\; E(g^1) \approx 0.82.$$
$$E(f^2) = 0.71, \;\; E(p^2) = 0.5, \;\; E(g^2) \approx 0.78.$$

(2) From Definitions 21 and 24, based on the above calculation results, we obtain:

$$r(l_1\chi(x)) \approx 0.57, \;\; r(l_2\chi(x)) \approx 0.67.$$

(3) From

$$r(l_1\chi(x)) < r(l_2\chi(x)),$$

it can be concluded that

$$l_1\chi(x) \prec l_2\chi(x).$$

The concept of the probability degree is introduced before comparing χLSs.

Definition 25 Let $L_1\chi(x) = \{<x, l_1^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_1\chi(x)\}$ and $L_2\chi(x) = \{<x, l_2^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_2\chi(x)\}$ be two χLSs, $l_1^k\chi(x)$ and $l_2^k\chi(x)$ are the χLVs in $L_1\chi(x)$ and $L_2\chi(x)$, $\#L_1\chi(x)$ and $\#L_2\chi(x)$ are the numbers of χLVs in $L_1\chi(x)$ and $L_2\chi(x)$, then the possibility degree of $L_1\chi(x) \geq L_2\chi(x)$ is

$$\begin{aligned} &p(L_1\chi(x) \geq L_2\chi(x)) \\ &= \frac{\max\left(0, E(L_1\chi(x)^+) - E(L_2\chi(x)^-)\right) - \max\left(0, E(L_1\chi(x)^-) - E(L_2\chi(x)^+)\right)}{\left(E(L_1\chi(x)^+) - E(L_1\chi(x)^-)\right) + \left(E(L_2\chi(x)^+) - E(L_2\chi(x)^-)\right)}, \end{aligned} \tag{34}$$

where $L_t\chi(x)^+ = \max_k\{l_t^k\chi(x)\}$, $L_t\chi(x)^- = \min_k\{l_t^k\chi(x)\}(t = 1, 2)$. $r(l_t^k\chi(x))$, $\vartheta(l_t^k\chi(x))$, and $\varphi(l_t^k\chi(x))$ are the distance between $l_t^k\chi(x)(t = 1, 2)$ and the origin O, and two angles of $l_t^k\chi(x)(t = 1, 2)$, respectively. Then, the value $E(l_t^k\chi(x))$ is divided into three cases:

(1) If all χLVs in $L_1\chi(x)$ and $L_2\chi(x)$ have the same $r(l_t^k\chi(x))$, then $E(l_t^k\chi(x)) = \vartheta(l_t^k\chi(x))$;
(2) If all χLVs in $L_1\chi(x)$ and $L_2\chi(x)$ have the same $r(l_t^k\chi(x))$ and $\vartheta(l_t^k\chi(x))$, then $E(l_t^k\chi(x)) = \varphi(l_t^k\chi(x))$;
(3) If neither of the above two cases is satisfied, then $E(l_t^k\chi(x)) = r(l_t^k\chi(x))$.

Therefore, the comparison method of χLSs is as follows:

Definition 26 Let $L_1\chi(x)$ and $L_2\chi(x)$ be two χLSs, then:

(I) If $p(L_1\chi(x) \geq L_2\chi(x)) > p(L_2\chi(x) \geq L_1\chi(x))$, then $L_1\chi(x) \succ L_2\chi(x)$;
(II) If $p(L_1\chi(x) \geq L_2\chi(x)) = 1$, then $L_1\chi(x) > L_2\chi(x)$;
(III) If $p(L_1\chi(x) \geq L_2\chi(x)) = 0.5$, then $L_1\chi(x) \sim L_2\chi(x)$.

Theorem 1 (*Boundness*) $0 < p(L_1\chi(x) \geq L_2\chi(x)) < 1$.

Proof According to Eq. (34), it is obvious.

Theorem 2 (*Complementarity*) $p(L_1\chi(x) \geq L_2\chi(x)) + p(L_2\chi(x) \geq L_1\chi(x)) = 1$; *especially, if* $L_1\chi(x) = L_2\chi(x)$, *then* $p(L_1\chi(x) \geq L_2\chi(x)) = p(L_2\chi(x) \geq L_1\chi(x)) = 0.5$.

The proof of Theorem 2 *is shown in Appendix* 1.

Example 6 Let $S = \{s_i | i = 0, 1, \ldots, 6\}$ be an additive LTS,

$$L_1\chi(x) = \left\{ \begin{array}{l} \{[(s_3, 0.8, 0.2);\ 0.6];\ \begin{pmatrix} <(s_3, 0.7, 0.2);\ 0.6>, \\ <(s_4, 0.8, 0.2);\ 0.65> \end{pmatrix}\}, \\ \{[(s_5, 0.6, 0.2);\ 0.7];\ \begin{pmatrix} <(s_4, 0.6, 0.2);\ 0.65>, \\ <(s_5, 0.7, 0.2);\ 0.7> \end{pmatrix}\} \end{array} \right\},$$

$$L_2\chi(x) = \left\{ \begin{array}{l} \{[(s_5, 0.8, 0.1);\ 0.5];\ \begin{pmatrix} <(s_4, 0.65, 0.1);\ 0.5>, \\ <(s_5, 0.8, 0.1);\ 0.55> \end{pmatrix}\}, \\ \{[(s_3, 0.5, 0.1);\ 0.6];\ \begin{pmatrix} <(s_3, 0.5, 0.1);\ 0.55>, \\ <(s_4, 0.65, 0.1);\ 0.6> \end{pmatrix}\} \end{array} \right\}$$

be two χLSs, and $A = 0.5, B = 0.3, C = 0.2$. From Definition 25, we obtain:

$$p(L_1\chi(x) \geq L_2\chi(x)) \approx 0.57 > 0.43 \approx p(L_2\chi(x) \geq L_1\chi(x))$$
$$\Rightarrow L_1\chi(x) \succ L_2\chi(x).$$

3.3 The normalization of χLSs

In decision making, evaluations may be different dimensions and numbers of χLVs. To facilitate the operations of χLVs and χLSs, we need to normalize them. For χLVs, the normalization is to unify the dimensions of each χLV.

Definition 27 Let $l_1\chi(x) = <[(s^{n_1}_{\theta(x_1)}, \mu^{n_1}_{\sigma(x_1)}, v^{n_1}_{\sigma(x_1)});\ p^{n_1}_{\rho(x_1)}];\ g^{n_1}_{\varsigma(x_1)}>$ and $l_2\chi(x) = <[(s^{n_2}_{\theta(x_2)}, \mu^{n_2}_{\sigma(x_2)}, v^{n_2}_{\sigma(x_2)});\ p^{n_2}_{\rho(x_2)}];\ g^{n_2}_{\varsigma(x_2)}>$ be two χLVs, n_1 and n_2 are the dimensions of $l_1\chi(x)$ and $l_2\chi(x)$, respectively. If $n_1 > n_2$, then add $n_1 - n_2$-dimensional χLV to $l_2\chi(x)$ so that the dimensions of $l_1\chi(x)$ and $l_2\chi(x)$ are unified. The added χLV is $n_1 - n_2$-dimensional $\{[(s_0, 0, 0);\ 0];\ \begin{pmatrix} <(s_0, 0, 0);\ 0>, \\ <(s_0, 0, 0);\ 0> \end{pmatrix}\}$.

For χLSs, the normalization refers to the double unity of the dimensions and the number of χLVs it contains.

Definition 28 Let $L_1\chi(x) = \{<x, l_1^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_1\chi(x)\}$ and $L_2\chi(x) = \{<x, l_2^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_2\chi(x)\}$ be two χLSs, $l_1^k\chi(x)$ and $l_2^k\chi(x)$ are the normalized χLVs, $\#L_1\chi(x)$ and $\#L_2\chi(x)$ are the numbers of χLVs in $L_1\chi(x)$ and $L_2\chi(x)$, respectively. If $\#L_1\chi(x) > \#L_2\chi(x)$, then add $\#L_1\chi(x) - \#L_2\chi(x)$ χLVs to $\#L_2\chi(x)$ so that the numbers of χLVs of $L_1\chi(x)$ and $L_2\chi(x)$ are unified. The added χLVs depend on the attitudes of DMs. There are two cases:

(1) If DMs take a positive attitude, then the added χLVs are $L_2\chi(x)^+ = \max_k\{l_2^k\chi(x)\}$;

(2) If DMs take a negative attitude, then the added χLVs are $L_2\chi(x)^- = \min_k\{l_2^k\chi(x)\}$.

In this way, we summarize the normalization process of χLSs. Let $L_1\chi(x) = \{<x, l_1^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_1\chi(x)\}$ and $L_2\chi(x) = \{<x, l_2^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_2\chi(x)\}$ be two χLSs, $n_{1k} (k = 1, 2, \ldots, \#L_1\chi(x))$ and $n_{2k} (k = 1, 2, \ldots, \#L_2\chi(x))$ are the dimensions of $L_1\chi(x)$ and $L_2\chi(x)$, then:

Step 1. If $n_{1k} \neq n_{2k}$, then according to Definition 27, we add χLVs of some dimensions to the χLVs with less dimension.

Step 2. If $\#L_1\chi(x) \neq \#L_2\chi(x)$, then according to Definition 28, we add some χLVs to the χLS with the lower number of χLVs.

The normalized χLVs and χLSs are still denoted as $l_1^k\chi(x)$, $l_2^k\chi(x)$, $L_1\chi(x)$, and $L_2\chi(x)$.

3.4 Some operational rules of χLSs

In this section, we introduce some basic operational rules and related theorems of χLVs and χLSs. We assume all χLVs and χLSs have been normalized.

Definition 29 Let $l_1\chi(x) = <[(s^{n_1}_{\theta(x_1)}, \mu^{n_1}_{\sigma(x_1)}, v^{n_1}_{\sigma(x_1)});\ p^{n_1}_{\rho(x_1)}];\ g^{n_1}_{\varsigma(x_1)}>$ and $l_2\chi(x) = <[(s^{n_2}_{\theta(x_2)}, \mu^{n_2}_{\sigma(x_2)}, v^{n_2}_{\sigma(x_2)});\ p^{n_2}_{\rho(x_2)}];\ g^{n_2}_{\varsigma(x_2)}>$ be two χLVs, $n_1 = n_2$, the subscript of $s^n_{\theta(x)}$ is represented as $\theta^n(x)$, and $\lambda \geq 0$, then the operational rules of χLVs are as follows:

(1) $l_1\chi(x) \oplus l_2\chi(x) = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\theta^n(x_1)+\theta^n(x_2)}, \frac{\theta^n(x_1)\mu^n_{\sigma(x_1)}+\theta^n(x_2)\mu^n_{\sigma(x_2)}}{\theta^n(x_1)+}\ \theta^n(x_2),\ \frac{\theta^n(x_1)v^n_{\sigma(x_1)}+\theta^n(x_2)v^n_{\sigma(x_2)}}{\theta^n(x_1)+\theta^n(x_2)});$ $\frac{p^n_{\rho(x_1)}+p^n_{\rho(x_2)}}{2}];\ g^n_{\varsigma(x_1)} \oplus g^n_{\varsigma(x_2)}\} > .$

(2) $l_1\chi(x) \otimes l_2\chi(x) = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\theta^n(x_1)\times\theta^n(x_2)}, \mu^n_{\sigma(x_1)}\mu^n_{\sigma(x_2)}, v^n_{\sigma(x_1)} + v^n_{\sigma(x_2)});\ p^n_{\rho(x_1)} \times p^n_{\rho(x_2)}];\ g^n_{\varsigma(x_1)} \otimes$ $g^n_{\varsigma(x_2)}\} > .$

(3) $\lambda l_1\chi(x) = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\lambda\theta^n(x_1)}, \mu^n_{\sigma(x_1)}, v^n_{\sigma(x_1)});\ p^n_{\rho(x_1)}];\ \lambda g^n_{\varsigma(x_1)}\} > .$

(4) $(l_1\chi(x))^\lambda = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{(\theta^n(x_1))^\lambda}, \left(\mu^n_{\sigma(x_1)}\right)^\lambda, 1 - \left(1 - v^n_{\sigma(x_1)}\right)^\lambda;$ $\left(p^n_{\rho(x_1)}\right)^\lambda];\ \left(g^n_{\varsigma(x_1)}\right)^\lambda\} > .$

Theorem 3 *Let* $l_1\chi(x) = <[(s^{n_1}_{\theta(x_1)}, \mu^{n_1}_{\sigma(x_1)}, v^{n_1}_{\sigma(x_1)});\ p^{n_1}_{\rho(x_1)}];\ g^{n_1}_{\varsigma(x_1)}>$ *and* $l_2\chi(x) = <[(s^{n_2}_{\theta(x_2)}, \mu^{n_2}_{\sigma(x_2)}, v^{n_2}_{\sigma(x_2)});\ p^{n_2}_{\rho(x_2)}];\ g^{n_2}_{\varsigma(x_2)}>$ *be two* χ*LVs, and* $\lambda, \lambda_1, \lambda_2 \geq 0$, *then*

(1) $l_1\chi(x) \oplus l_2\chi(x) = l_2\chi(x) \oplus l_1\chi(x);$
(2) $l_1\chi(x) \otimes l_2\chi(x) = l_2\chi(x) \otimes l_1\chi(x);$
(3) $\lambda_1 l_1\chi(x) \oplus \lambda_2 l_1\chi(x) = (\lambda_1 + \lambda_2) l_1\chi(x).$

The proof of Theorem 3 is shown in Appendix 2.

Example 7 Use the χLVs $l_1\chi(x)$ and $l_2\chi(x)$ in Example 4, then

(1) $l_1\chi(x) \oplus l_2\chi(x) = \{[(s_8, 0.8, 0.14);\ 0.55];\ \begin{pmatrix} <(s_7, 0.67, 0.14);\ 0.35>, \\ <(s_9, 0.44, 0.14);\ 0.6> \end{pmatrix}\}.$

(2) $l_1\chi(x) \otimes l_2\chi(x) = \{[(s_{15}, 0.64, 0.3);\ 0.3];\ \begin{pmatrix} <(s_{12}, 0.46, 0.3);\ 0.3>, \\ <(s_{20}, 0.64, 0.3);\ 0.36> \end{pmatrix}\}.$

(3) $2l_1\chi(x) = \{[(s_6, 0.8, 0.2);\ 0.6];\ \begin{pmatrix} <(s_6, 0.7, 0.2);\ 0.6>, \\ <(s_8, 0.8, 0.2);\ 0.65> \end{pmatrix}\}.$

(4) $(l_1\chi(x))^2 = \{[(s_9, 0.64, 0.36);\ 0.36];\ \begin{pmatrix} <(s_9, 0.49, 0.36);\ 0.36>, \\ <(s_{16}, 0.64, 0.36);\ 0.42> \end{pmatrix}\}.$

Next, some basic operational rules of χLSs are introduced.

Definition 30 Let $L_1\chi(x) = \{<x, l^k_1\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_1\chi(x)\}$ and $L_2\chi(x) = \{<x, l^k_2\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_2\chi(x)\}$ be two χLVs, $\#L_1\chi(x) = \#L_2\chi(x)$, and $\lambda, \lambda_1, \lambda_2 \geq 0$, then the operational rules of ILSs are as follows:

(1) $L_1\chi(x) \oplus L_2\chi(x) = \bigcup_{l_1^{\delta(k)}\chi(x) \in L_1\chi(x),\ l_2^{\delta(k)}\chi(x) \in L_2\chi(x)} \{l_1^{\delta(k)}\chi(x) \oplus l_2^{\delta(k)}\chi(x)\};$

(2) $L_1\chi(x) \otimes L_2\chi(x) = \bigcup_{l_1^{\delta(k)}\chi(x)\in L_1\chi(x),\ l_2^{\delta(k)}\chi(x)\in L_2\chi(x)} \{l_1^{\delta(k)}\chi(x) \otimes l_2^{\delta(k)}\chi(x)\}$;

(3) $\lambda L_1\chi(x) = \bigcup_{l_1^{\delta(k)}\chi(x)\in L_1\chi(x)} \{\lambda l_1^{\delta(k)}\chi(x)\}$;

(4) $(L_1\chi(x))^\lambda = \bigcup_{l_1^{\delta(k)}\chi(x)\in L_1\chi(x)} \{\left(l_1^{\delta(k)}\chi(x)\right)^\lambda\}$, where $l_1^{\delta(k)}\chi(x)$ and $l_2^{\delta(k)}\chi(x)$ are the kth largest χLVs in $l_1^k\chi(x)$ and $l_2^k\chi(x)$.

Theorem 4 *Let* $L_1\chi(x) = \{<x, l_1^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_1\chi(x)\}$, $L_2\chi(x) = \{<x, l_2^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_2\chi(x)\}$, *and* $L_3\chi(x) = \{<x, l_3^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_3\chi(x)\}$ *be three* χ*LVs*, $\#L_1\chi(x) = \#L_2\chi(x) = \#L_3\chi(x)$, *and* $\lambda, \lambda_1, \lambda_2 \geq 0$, *then*

(1) $L_1\chi(x) \oplus L_2\chi(x) = L_2\chi(x) \oplus L_1\chi(x)$;
(2) $(L_1\chi(x) \oplus L_2\chi(x)) \oplus L_3\chi(x) = L_1\chi(x) \oplus (L_2\chi(x) \oplus L_3\chi(x))$;
(3) $L_1\chi(x) \otimes L_2\chi(x) = L_2\chi(x) \otimes L_1\chi(x)$;

Proof The above theorem is easy to prove, so we omit the proof here.

3.5 The distance measure of χLSs

In this section, the distance between χLVs and the distance between χLSs are introduced. We assume all χLVs and ILSs have been normalized.

Definition 31 Let $l_1\chi(x)$ and $l_2\chi(x)$ be two χLVs. The ambiguity expectation, the possibility expectation, and the approximation expectation of $l_1\chi(x)$ are $E(f^1)$, $E(p^1)$, and $E(g^1)$, respectively. The ambiguity expectation, the possibility expectation, and the approximation expectation of $l_2\chi(x)$ are $E(f^2)$, $E(p^2)$, and $E(g^2)$, respectively. Then, the generalized distance between $l_1\chi(x)$ and $l_2\chi(x)$ is:

$$\begin{aligned} &d(l_1\chi(x), l_2\chi(x)) \\ &= \left(A\left|E(f^1) - E(f^2)\right|^\lambda + B\left|E(p^1) - E(p^2)\right|^\lambda + C\left|E(g^1) - E(g^2)\right|^\lambda\right)^{1/\lambda}, \end{aligned} \tag{35}$$

where $0 \leq A, B, C \leq 1$, $A + B + C = 1$, and $\lambda > 0$.

Remark 2

(1) If $\lambda = 1$, then the generalized distance degenerates into the Hamming distance:

$$\begin{aligned} &d(l_1\chi(x), l_2\chi(x)) \\ &= A\left|E(f^1) - E(f^2)\right| + B\left|E(p^1) - E(p^2)\right| + C\left|E(g^1) - E(g^2)\right|. \end{aligned} \tag{36}$$

(2) If $\lambda = 2$, then the generalized distance degenerates into the Euclidean distance:

$$\begin{aligned} &d(l_1\chi(x), l_2\chi(x)) \\ &= \left(A\left|E(f^1) - E(f^2)\right|^2 + B\left|E(p^1) - E(p^2)\right|^2 + C\left|E(g^1) - E(g^2)\right|^2\right)^{1/2}. \end{aligned} \tag{37}$$

(3) If $E(f^2) = E(p^2) = E(g^2) = 0$, and $\lambda = 2$, then the generalized distance is equivalent to the distance between $l_1\chi(x)$ and the origin O proposed in Definition 21:

$$d(l_1\chi(x), l_2\chi(x)) = \left(A\left|E(f^1)\right|^2 + B\left|E(p^1)\right|^2 + C\left|E(g^1)\right|^2\right)^{1/2} = r(l_1\chi(x)). \tag{38}$$

Theorem 5 *Let $l_1\chi(x)$ and $l_2\chi(x)$ be two χLVs, then*:

(1) $0 \le d(l_1\chi(x), l_2\chi(x)) \le 1$.
(2) $d(l_1\chi(x), l_2\chi(x)) = 0 \Rightarrow l_1\chi(x) \sim l_2\chi(x)$.
(3) $d(l_1\chi(x), l_2\chi(x)) = d(l_2\chi(x), l_1\chi(x))$.

The proof of Theorem 5 is shown in Appendix 3.

Next, we introduce the distance between χLSs.

Definition 32 Let $L_1\chi(x) = \{<x, l_1^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_1\chi(x)\}$ and $L_2\chi(x) = \{<x, l_2^k\chi(x)> | x \in X, k = 1, 2, \ldots, \#L_2\chi(x)\}$ be two χLSs, and $\#L_1\chi(x) = \#L_2\chi(x)$, then the distance between $L_1\chi(x)$ and $L_2\chi(x)$ is:

$$d(L_1\chi(x), L_2\chi(x)) = \frac{1}{\#L_1\chi(x)} \sum_{k=1}^{\#L_1\chi(x)} d(l_1^k\chi(x), l_2^k\chi(x)), \tag{39}$$

where $d(l_1^k\chi(x), l_2^k\chi(x))$ is the distance between $l_1^k\chi(x)$ and $l_2^k\chi(x)$.

Remark 3 Let f_1^k, f_2^k, p_1^k, p_2^k, g_1^k and g_2^k be the ambiguity expectations, the possibility expectations, and the approximation expectations of $l_1^k\chi(x)$ and $l_2^k\chi(x)$, respectively. $0 \le A, B, C \le 1$, $A + B + C = 1$, and $\lambda > 0$. Then:

(1) If $d(l_1^k\chi(x), l_2^k\chi(x))$ is the generalized distance, then $d(L_1\chi(x), L_2\chi(x))$ is the generalized distance between $L_1\chi(x)$ and $L_2\chi(x)$:

$$d(L_1\chi(x), L_2\chi(x)) = \frac{1}{\#L_1\chi(x)} \sum_{k=1}^{\#L_1\chi(x)} \left(A\left|E(f_1^k) - E(f_2^k)\right|^\lambda + B\left|E(p_1^k) - E(p_2^k)\right|^\lambda + C\left|E(g_1^k) - E(g_2^k)\right|^\lambda\right)^{1/\lambda}. \tag{40}$$

(2) If $d(l_1^k\chi(x), l_2^k\chi(x))$ is the Hamming distance, then $d(L_1\chi(x), L_2\chi(x))$ is the Hamming distance between $L_1\chi(x)$ and $L_2\chi(x)$:

$$d(L_1\chi(x), L_2\chi(x)) = \frac{1}{\#L_1\chi(x)} \sum_{k=1}^{\#L_1\chi(x)} \left(A\left|E(f_1^k) - E(f_2^k)\right| + B\left|E(p_1^k) - E(p_2^k)\right| + C\left|E(g_1^k) - E(g_2^k)\right|\right). \tag{41}$$

(3) If $d(l_1^k\chi(x), l_2^k\chi(x))$ is the Euclidean distance, then $d(L_1\chi(x), L_2\chi(x))$ is the Euclidean distance between $L_1\chi(x)$ and $L_2\chi(x)$:

$$d(L_1\chi(x), L_2\chi(x))$$
$$= \frac{1}{\#L_1\chi(x)} \sum_{k=1}^{\#L_1\chi(x)} \left(A\left|E(f_1^k) - E(f_2^k)\right|^2 + B\left|E(p_1^k) - E(p_2^k)\right|^2 + C\left|E(g_1^k) - E(g_2^k)\right|^2 \right)^{1/2}. \tag{42}$$

Theorem 6 *Let $L_1\chi(x)$ and $L_2\chi(x)$ be two χLVs, then*:

(1) $0 \le d(L_1\chi(x), L_2\chi(x)) \le 1$;
(2) $d(L_1\chi(x), L_2\chi(x)) = 0 \Rightarrow L_1\chi(x) = L_2\chi(x)$;
(3) $d(L_1\chi(x), L_2\chi(x)) = d(L_2\chi(x), L_1\chi(x))$.

Proof The above theorem is easy to prove, so we omit the proof here.

4 A new χLVIKOR method based on χ-linguistic sets for MAGDM

χ- linguistic information can be applied to MAGDM problems. Assume $X = \{x_1, x_2, \ldots, x_t\}$ is a set of t alternatives, $A = \{a_1, a_2, \ldots, a_q\}$ is a set of q attributes, $\omega = (\omega_1, \omega_2, \ldots, \omega_q)^T$ is the known attribute weight vector, $\omega_\beta \in [0,1] (\beta = 1, 2, \ldots, q)$, and $\sum_{\beta=1}^{q} \omega_\beta = 1$. The evaluation of multiple DMs can be represented by an χLS $L_{\alpha\beta}(I) = \{l_{\alpha\beta}^k\chi(x) |\ k = 1, 2, \ldots, \#L_{\alpha\beta}\chi(x)\}$, which represents the decision information under the attribute $a_\beta(\beta = 1, 2, \ldots, q)$ of the alternative $x_\alpha(\alpha = 1, 2, \ldots, t)$. All the χLSs constitute an χ-linguistic decision matrix:

$$R = [L_{\alpha\beta}\chi(x)]_{t\times q} = \begin{bmatrix} L_{11}\chi(x) & L_{12}\chi(x) & \cdots & L_{1q}\chi(x) \\ L_{21}\chi(x) & L_{22}\chi(x) & \cdots & L_{2q}\chi(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{t1}\chi(x) & L_{t2}\chi(x) & \cdots & L_{tq}\chi(x) \end{bmatrix}, \tag{43}$$

where $L_{\alpha\beta}\chi(x) = \{l_{\alpha\beta}^k\chi(x) | k = 1, 2, \ldots, \#L_{\alpha\beta}\chi(x)\} (\alpha = 1, 2, \ldots, t, \beta = 1, 2, \ldots, q)$, and $l_{\alpha\beta}^k\chi(x)$ is the kth χLV in $L_{\alpha\beta}\chi(x)$. We assume all χLSs and χLVs in this section are normalized.

4.1 Some basic operators for χLSs

To better aggregate the decision information, we propose some basic operators for χLSs.

Definition 33 Let $L_\beta\chi(x) = \{l_\beta^k\chi(x) | k = 1, 2, \ldots, L_\beta\chi(x)\}$ $(\beta = 1, 2, \ldots, q)$ be q χLSs, then the χ- linguistic weighted averaging(χLWA) operator is defined as follows:

$$\begin{aligned}&\phi_{\chi LWA}(L_1\chi(x),L_2\chi(x),\ldots,L_q\chi(x))\\&=\omega_1L_1\chi(x)\oplus\omega_2L_2\chi(x)\oplus\cdots\oplus\omega_qL_q\chi(x)\\&=\bigcup_{l_1^{\delta(k)}\chi(x)\in L_1\chi(x)}\{\omega_1l_1^{\delta(k)}\chi(x)\}\oplus\bigcup_{l_2^{\delta(k)}\chi(x)\in L_2\chi(x)}\{\omega_2l_2^{\delta(k)}\chi(x)\}\oplus\\&\cdots\oplus\bigcup_{l_q^{\delta(k)}\chi(x)\in L_q\chi(x)}\{\omega_ql_q^{\delta(k)}\chi(x)\},\end{aligned}\tag{44}$$

where $l_\beta^{\delta(k)}\chi(x)(\beta=1,2,\ldots,q)$ is the kth largest ILVs in $l_\beta^k\chi(x)$, $\omega=(\omega_1,\omega_2,\ldots,\omega_q)^T$ is the weight vector, $\omega_\beta\in[0,1]$, and $\sum_{\beta=1}^q\omega_\beta=1$.

Remark 4 When $\omega=(\frac{1}{q},\frac{1}{q},\ldots,\frac{1}{q})^T$, the χLWA operator degenerates into the χ-linguistic averaging(χLA) operator:

$$\begin{aligned}&\phi_{\chi LA}(L_1\chi(x),L_2\chi(x),\ldots,L_q\chi(x))\\&=\frac{1}{q}L_1\chi(x)\oplus\frac{1}{q}L_2\chi(x)\oplus\cdots\oplus\frac{1}{q}L_q\chi(x)\\&=\frac{1}{q}(\bigcup_{l_1^{\delta(k)}\chi(x)\in L_1(I),l_2^{\delta(k)}\chi(x)\in L_2\chi(x),\ldots,l_q^{\delta(k)}\chi(x)\in L_q\chi(x)}\\&\{l_1^{\delta(k)}\chi(x)\oplus\ l_2^{\delta(k)}\chi(x)\oplus\cdots\oplus l_q^{\delta(k)}\chi(x)\}),\end{aligned}\tag{45}$$

where $l_\beta^{\delta(k)}\chi(x)(\beta=1,2,\ldots,q)$ is the kth largest χLVs in $l_\beta^k\chi(x)$.

Definition 34 Let $L_\beta\chi(x)=\{l_\beta^k\chi(x)|k=1,2,\ldots,L_\beta\chi(x)\}$ $(\beta=1,2,\ldots,q)$ be q χLSs, then the χ-linguistic weighted geometric(χLWG) operator is defined as follows:

$$\begin{aligned}&\phi_{\chi LWG}(L_1\chi(x),L_2\chi(x),\ldots,L_q\chi(x))\\&=(L_1\chi(x))^{\omega_1}\otimes(L_2\chi(x))^{\omega_2}\otimes\cdots\otimes(L_q\chi(x))^{\omega_q}\\&=\bigcup_{l_1^{\delta(k)}\chi(x)\in L_1\chi(x)}\{(l_1^{\delta(k)}\chi(x))^{\omega_1}\}\otimes\bigcup_{l_2^{\delta(k)}\chi(x)\in L_2\chi(x)}\{(l_2^{\delta(k)}\chi(x))^{\omega_2}\}\otimes\\&\cdots\otimes\bigcup_{l_q^{\delta(k)}\chi(x)\in L_q\chi(x)}\{(l_q^{\delta(k)}\chi(x))^{\omega_q}\},\end{aligned}\tag{46}$$

where $l_\beta^{\delta(k)}\chi(x)(\beta=1,2,\ldots,q)$ is the kth largest χLVs in $l_\beta^k\chi(x)$, $\omega=(\omega_1,\omega_2,\ldots,\omega_q)^T$ is the weight vector, $\omega_\beta\in[0,1]$, and $\sum_{\beta=1}^q\omega_\beta=1$.

Remark 5 When $\omega=(\frac{1}{q},\frac{1}{q},\ldots,\frac{1}{q})^T$, the ILWG operator degenerates into the intelligent linguistic geometric(ILG) operator:

$$\begin{aligned}
&\phi_{\chi LG}(L_1\chi(x), L_2\chi(x), \ldots, L_q\chi(x)) \\
&= (L_1\chi(x))^{\frac{1}{q}} \otimes (L_2\chi(x))^{\frac{1}{q}} \otimes \cdots \otimes (L_q\chi(x))^{\frac{1}{q}} \\
&= \bigcup_{l_1^{\delta(k)}\chi(x)\in L_1\chi(x), l_2^{\delta(k)}\chi(x)\in L_2\chi(x), \ldots, l_q^{\delta(k)}\chi(x)\in L_q\chi(x)} \\
&\{(l_1^{\delta(k)}\chi(x))^{\frac{1}{q}} \otimes (l_2^{\delta(k)}\chi(x))^{\frac{1}{q}} \otimes \cdots \otimes (l_q^{\delta(k)}\chi(x))^{\frac{1}{q}}\},
\end{aligned} \tag{47}$$

where $l_\beta^{\delta(k)}\chi(x)(\beta = 1, 2, \ldots, q)$ is the kth largest χLVs in $l_\beta^k\chi(x)$.

4.2 The χLVIKOR method

The VIKOR method was proposed by Opricovic (1998). This method determines a compromise solution that provide a maximum group utility for most people and minimum of an individual regret for opponents. It also takes into account the relative importance of the distances among the compromise solution, the ideal solution, and the negative ideal solution. It is widely used in MAGDM problems. In this section, we propose the χLVIKOR method of χLSs.

Definition 35 Let $R = [L_{\alpha\beta}\chi(x)]_{t\times q}$ be an intelligent linguistic decision matrix with $L_{\alpha\beta}\chi(x) = \{l_{\alpha\beta}^k\chi(x)|k = 1, 2, \ldots, \#L_{\alpha\beta}\chi(x)\}(\alpha = 1, 2, \ldots, t, \beta = 1, 2, \ldots, q)$. Then, the best one of all $L_\beta\chi(x)$ is denoted as $L_\beta\chi(x)^*$:

$$L_\beta\chi(x)^* = \{l_\beta^k\chi(x)^*|k = 1, 2, \ldots, \#L_{\alpha\beta}\chi(x)\}, \tag{48}$$

where $l_\beta^k\chi(x)^* = <[({s_{\theta(x_\beta)}^{n_k}}^*, {\mu_{\sigma(x_\beta)}^{n_k}}^*, {v_{\sigma(x_\beta)}^{n_k}}^*); {p_{\rho(x_\beta)}^{n_k}}^*]; \ {g_{\varsigma(x_\beta)}^{n_k}}^* >$, ${s_{\theta(x_\beta)}^{n_k}}^* = \max_\alpha\{s_{\theta(x_{\alpha\beta})}^{n_k}\}$, ${\mu_{\sigma(x_\beta)}^{n_k}}^*, {v_{\sigma(x_\beta)}^{n_k}}^* = \{\mu_{\sigma(x_{\alpha\beta})}^{n_k}, v_{\sigma(x_{\alpha\beta})}^{n_k}| \max_\alpha(\mu_{\sigma(x_{\alpha\beta})}^{n_k} - v_{\sigma(x_{\alpha\beta})}^{n_k})\}$, ${p_{\rho(x_\beta)}^{n_k}}^* = \max_\alpha\{p_{\rho(x_{\alpha\beta})}^{n_k}\}$ and ${g_{\varsigma(x_\beta)}^{n_k}}^* = \{g_{\varsigma(x_{\alpha\beta})}^{n_k}| \max_\alpha E(g_{\varsigma(x_{\alpha\beta})}^{n_k})\}$. $E(g_{\varsigma(x_{\alpha\beta})}^{n_k})$ is the approximation expectation of $l_{\alpha\beta}^k\chi(x)$.

Definition 36 Let $R = [L_{\alpha\beta}\chi(x)]_{t\times q}$ be an intelligent linguistic decision matrix with $L_{\alpha\beta}\chi(x) = \{l_{\alpha\beta}^k\chi(x)|k = 1, 2, \ldots, \#L_{\alpha\beta}\chi(x)\}(\alpha = 1, 2, \ldots, t, \beta = 1, 2, \ldots, q)$. Then, the worst one of all $L_\beta\chi(x)$ is denoted as $L_\beta\chi(x)^\#$:

$$L_\beta\chi(x)^\# = \{l_\beta^k\chi(x)^\#|k = 1, 2, \ldots, \#L_{\alpha\beta}\chi(x)\}, \tag{49}$$

where $l_\beta^k\chi(x)^\# = <[({s_{\theta(x_\beta)}^{n_k}}^\#, {\mu_{\sigma(x_\beta)}^{n_k}}^\#, {v_{\sigma(x_\beta)}^{n_k}}^\#); {p_{\rho(x_\beta)}^{n_k}}^\#]; {g_{\varsigma(x_\beta)}^{n_k}}^\# >$, ${s_{\theta(x_\beta)}^{n_k}}^\# = \min_\alpha\{s_{\theta(x_{\alpha\beta})}^{n_k}\}$, $\{{\mu_{\sigma(x_\beta)}^{n_k}}^\#, {v_{\sigma(x_\beta)}^{n_k}}^\#\} = \{\mu_{\sigma(x_{\alpha\beta})}^{n_k}, v_{\sigma(x_{\alpha\beta})}^{n_k}| \min_\alpha(\mu_{\sigma(x_{\alpha\beta})}^{n_k} - v_{\sigma(x_{\alpha\beta})}^{n_k})\}$, ${p_{\rho(x_\beta)}^{n_k}}^\# = \min_\alpha\{p_{\rho(x_{\alpha\beta})}^{n_k}\}$ and ${g_{\varsigma(x_\beta)}^{n_k}}^\# = \{g_{\varsigma(x_{\alpha\beta})}^{n_k}| \min_\alpha E(g_{\varsigma(x_{\alpha\beta})}^{n_k})\}$. $E(g_{\varsigma(x_{\alpha\beta})}^{n_k})$ is the approximation expectation of $l_{\alpha\beta}^k\chi(x)$.

After determining the best solution $L\chi(x)^* = \{L_1\chi(x)^*, L_2\chi(x)^*, \ldots, L_q\chi(x)^*\}$ and the worst solution $L\chi(x)^\# = \{L_1\chi(x)^\#, L_2\chi(x)^\#, \ldots, L_q\chi(x)^\#\}$ with Definitions 35 and 36, we calculate values $\mathcal{S}_\alpha$ and $\mathcal{R}_\alpha(\alpha = 1, 2, \ldots, t)$:

$$\mathcal{S}_\alpha = \sum_{\beta=1}^{q} \omega_\beta d(L_\beta\chi(x)^*, L_{\alpha\beta}\chi(x))/d(L_\beta\chi(x)^\#, L_{\alpha\beta}\chi(x)), \tag{50}$$

$$\mathcal{R}_\alpha = \max_\beta \left\{ \omega_\beta d(L_\beta\chi(x)^*, L_{\alpha\beta}\chi(x))/d(L_\beta\chi(x)^\#, L_{\alpha\beta}\chi(x)) \right\}, \tag{51}$$

where $d(L_\beta\chi(x)^*, L_{\alpha\beta}\chi(x))$ is the distance between $L_\beta\chi(x)^*$ and $L_{\alpha\beta}\chi(x)$, and $d(L_\beta\chi(x)^\#, L_{\alpha\beta}\chi(x))$ is the distance between $L_\beta\chi(x)^\#$ and $L_{\alpha\beta}\chi(x)$.

Then, the value $\mathcal{Q}_\alpha (\alpha = 1, 2, \ldots, t)$ is calculated:

$$\mathcal{Q}_\alpha = v \frac{\mathcal{S}_\alpha - \mathcal{S}^\#}{\mathcal{S}^* - \mathcal{S}^\#} + (1 - v) \frac{\mathcal{R}_\alpha - \mathcal{R}^\#}{\mathcal{R}^* - \mathcal{R}^\#}, \tag{52}$$

where $\mathcal{S}^\# = \min_\alpha\{\mathcal{S}_\alpha\}$, $\mathcal{S}^* = \max_\alpha\{\mathcal{S}_\alpha\}$, $\mathcal{R}^\# = \min_\alpha\{\mathcal{R}_\alpha\}$, and $\mathcal{R}^* = \max_\alpha\{\mathcal{R}_\alpha\}$. v is the weight of the strategy of "the majority of attribute". We assume $v = 0.5$.

Rank the alternatives with the values $\mathcal{S}$, $\mathcal{R}$, and $\mathcal{Q}$. And obtain three ranking results. The smaller the value is, the higher the ranking is.

Next, determine a compromise solution. When the following two conditions are satisfied, the best alternative x^1 obtained by $\mathcal{Q}$ (minimum) is the compromise solution. The conditions are as follows:

Condition 1. Acceptable advantage.

$$\mathcal{Q}(x^2) - \mathcal{Q}(x^1) \geq \frac{1}{t-1}, \tag{53}$$

where x^2 is the second-ranked alternative obtained by $\mathcal{Q}$, and t is the number of alternatives.

Condition 2. Acceptable stability in decision making.

The alternative x^1 must be the best ranked by $\mathcal{S}$ or $\mathcal{R}$.

If one of the above conditions is not satisfied, a compromise solution set is proposed:

(1) If condition 2 is not satisfied, then both x^1 and x^2 are compromise solutions.
(2) If condition 1 is not satisfied, then obtain maximum T with the equation:

$$\mathcal{Q}(x^T) - \mathcal{Q}(x^1) < \frac{1}{t-1}. \tag{54}$$

Alternatives $x^1, x^2, \ldots, x^T$ are close to the compromise solution.

4.3 A new χLVIKOR approach to decision making with χ-linguistic information

In this section, we are going to develop a new χLVIKOR method to MAGDM problem with theχ-linguistic environment, and comparative analysis with the $\phi_{\chi LWA}$ and $\phi_{\chi LWG}$ operator.

Step 1. Analyze the MAGDM problem and determine t alternatives $X = \{x_1, x_2, \ldots, x_t\}$, q attributes $A = \{a_1, a_2, \ldots, a_q\}$, and the attribute weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_q)^T$. For the attribute $\beta(\beta = 1, 2, \ldots, q)$ under alternative $\alpha(\alpha = 1, 2, \ldots, t)$, DMs use the linguistic term $S = \{s_i | i = 0, 1, \ldots, \tau\}$ to represent their evaluation and give the evaluation's membership, non-membership, and credibility with numbers between 0 and 1. Multiple DMs' evaluation information is assembled into an IPLS, and a χLS $L_{\alpha\beta}\chi(x) = \{l^k_{\alpha\beta}\chi(x) | k = 1, 2, \ldots, \# L_{\alpha\beta}\chi(x)\}$ is obtained by adding the approximation from Definitions 13-16. Multiple χLSs are constructed into an intelligent linguistic decision matrix $R = [L_{\alpha\beta}\chi(x)]_{t\times q}$. If DMs choose the $\phi_{\chi LWA}$ and $\phi_{\chi LWG}$ operator, then go to Step 2; if DMs choose the χLVIKOR method, then go to Step 5.

Step 2. Use the $\phi_{\chi LWA}$ operator (Eq. (44)) or the $\phi_{\chi LWG}$ operator (Eq. (46)) to aggregate the attribute values of the alternatives $x_\alpha(\alpha = 1, 2, \ldots, t)$ to obtain $L_\alpha\chi(x)(\alpha = 1, 2, \ldots, t)$.

Step 3. Construct a possibility degree matrix $P = (p_{\alpha k})_{t\times t}$, where $p_{\alpha k} = p(L_\alpha\chi(x) \geq L_k\chi(x))$ is calculated by Eq. (34), $p_{\alpha k} + p_{k\alpha} = 1$, $p_{\alpha\alpha} = 0.5$, and $\alpha, k = 1, 2, \ldots, t$.

Step 4. Rank the alternatives $x_\alpha(\alpha = 1, 2, \ldots, t)$ with the values of $p_\alpha = \sum_{k=1}^{t} p_{\alpha k}, (\alpha = 1, 2, \ldots, t)$ in descending order and select the optimal alternative. Then, go to Step 10.

Step 5. Determine the best solution $L\chi(x)^*$ and the worst solution $L\chi(x)^\#$ with Definitions 35 and 36.

Step 6. Calculate values $\mathcal{S}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{Q}_\alpha(\alpha = 1, 2, \ldots, t)$ with Eqs.(50-52).

Step 7. Obtain three ranking results by values $\mathcal{S}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{Q}_\alpha(\alpha = 1, 2, \ldots, t)$.

Step 8. If both conditions 1 and 2 are satisfied, the optimal alternative obtained by the minimum $\mathcal{Q}$ is the compromise solution. And go to Step 9. If any one of the conditions is not satisfied, then go to Step 8.

Step 9. Obtain the compromise solution set.

Step 10. End.

The $\phi_{\chi LWA}$ and $\phi_{\chi LWG}$ operator are easy to calculate. Compared with it, the χLVIKOR method is more complex but can reduce the loss of information.

5 A case study

5.1 A numerical example

It is now more than one year since COVID-19 first broke out. However, the epidemic situation in cities around the world has not been completely eliminated, and even rebounded. Therefore, it is necessary to make an effective epidemic risk assessment. Three DMs formed an expert group to assess the risk of cities from the second COVID-19 shock. We obtained data on the prevention and control of the COVID-19 in three cities through research. Experts assess the risks of the three cities(x_1, x_2, x_3), rank the cities, select the most risky city, and gave early warning to the cities with high risk, so as to help the local government adjust the prevention and control mechanism and prepare the response plan in advance. The evaluation considers the following three attributes: (1) the risk of inflow of confirmed cases (domestic and overseas); (2) the risk of medical resources; (3) the risk of accumulated experience in the epidemic. The attribute weight vector is $\omega = (0.4, 0.3, 0.3)^T$. We assume $A = 0.33$, $B = 0.33$, $C = 0.33$, and $\lambda = 2$.

Table 1 Symbol description

Symbol	Interpretative statement
$\bar{S}^n$	N-dimensional extended additive LTS
F^n	N-dimensional linguistic ambiguity vector set
P^n	Possibility(Credibility) vector set of n-dimensional ambiguity
G^n	Approximation vector set of n-dimensional linguistic ambiguity and its possibility
$s^n_{\theta(x)}$	Linguistic variables
$f^n_{\sigma(x)}$	Linguistic membership degree of $s^n_{\theta(x)}$
$p^n_{\rho(x)}$	Credibility of uncertain linguistic information $(s^n_{\theta(x)}; f^n_{\sigma(x)})$
$g^n_{\varsigma(x)}$	Approximation of the fuzzy linguistic information

Table 2 The intuitionistic probabilistic linguistic decision matrix of DMs

DM	x	a_1	a_2	a_3
DM1	x_1	$<[(s_5, 0.6, 0.3); 0.9], [(s_4, 0.5, 0.4); 0.7]>$	$[(s_3, 0.8, 0.2); 0.6]$	$[(s_5, 0.8, 0.1); 0.5]$
	x_2	$<[(s_3, 0.7, 0.2); 0.7], [(s_5, 0.4, 0.3); 0.8]>$	$[(s_2, 0.6, 0.2); 0.7]$	$[(s_6, 0.7, 0.1); 0.4]$
	x_3	$<[(s_2, 0.7, 0.1); 0.9], [(s_4, 0.7, 0.3); 0.7]>$	$[(s_4, 0.5, 0.1); 0.8]$	$[(s_4, 0.8, 0.2); 0.7]$
DM2	x_1	$<[(s_3, 0.7, 0.3); 0.4], [(s_6, 0.4, 0.2); 0.6]>$	$[(s_5, 0.6, 0.2); 0.7]$	$[(s_3, 0.5, 0.1); 0.6]$
	x_2	$<[(s_5, 0.8, 0.1); 0.8], [(s_6, 0.4, 0.1); 0.7]>$	$[(s_5, 0.5, 0.1); 0.8]$	$[(s_2, 0.8, 0.1); 0.9]$
	x_3	$<[(s_4, 0.7, 0.1); 0.7], [(s_5, 0.8, 0.1); 0.5]>$	$[(s_6, 0.5, 0.5); 0.7]$	$[(s_1, 0.6, 0.2); 0.5]$
DM3	x_1	$<[(s_2, 0.6, 0.3); 0.6], [(s_4, 0.6, 0.2); 0.8]>$	$[(s_3, 0.4, 0.2); 0.5]$	$[(s_4, 0.7, 0.2); 0.5]$
	x_2	$<[(s_6, 0.5, 0.1); 0.4], [(s_4, 0.6, 0.4); 0.8]>$	$[(s_4, 0.6, 0.3); 0.4]$	$[(s_4, 0.8, 0.2); 0.8]$
	x_3	$<[(s_3, 0.6, 0.3); 0.5], [(s_6, 0.5, 0.4); 0.6]>$	$[(s_5, 0.7, 0.3); 0.8]$	$[(s_4, 0.4, 0.4); 0.6]$

5.1.1 The $\phi_{\chi LWA}$ and $\phi_{\chi LWG}$ operator method

We take the $\phi_{\chi LWA}$ operator as an example to solve this problem.

Step 1. Three DMs give their evaluations based on the LTS:

$$S = \{s_0 = none,\ s_1 = very\ low,\ s_2 = low,\ s_3 = medium,\\ s_4 = high,\ s_5 = very\ high,\ s_6 = extremely\ high\}.$$

The corresponding ambiguity and credibility of the evaluation are also given. The above evaluation information is expressed as IPLVs. The intuitionistic probabilistic linguistic decision matrix of the three DMs is shown in Table 1. By integrating the evaluations from three DMs and solving the approximation degree, we obtain a group decision matrix in the form of χLS. After normalizing the group decision matrix and ranking the χLVs in χLSs in descending order, we obtain Table 2.

Step 2. Use the χLWA operator(Eq. (44)) to aggregate the attribute values of the alternatives $x_\alpha(\alpha = 1, 2, 3)$ to obtain $L_\alpha\chi(x)(\alpha = 1, 2, 3)$.

$$L_1\chi(x)=\left\{\begin{array}{c}\left\langle\begin{array}{c}\{[(s_5,0.66,0.21);0.65];\begin{pmatrix}<(s_{3.63},0.59,0.21);0.56>,\\<(s_5,0.70,0.21);0.67>\end{pmatrix}\},\\\{[(s_{1.6},0.5,0.4);0.18];\begin{pmatrix}<(s_{1.6},0.5,0.4);0.16>,\\<(s_{1.87},0.5,0.27);0.19>\end{pmatrix}\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_{2.9},0.70,0.23);0.55];\begin{pmatrix}<(s_{2.75},0.60,0.21);0.51>,\\<(s_{3.78},0.72,0.22);0.62>\end{pmatrix}\},\\\{[(s_{1.6},0.60,0.20);0.20];\begin{pmatrix}<(s_{1.6},0.50,0.27);0.18>,\\<(s_{1.87},0.60,0.20);0.20>\end{pmatrix}\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_3,0.55,0.21);0.53];\begin{pmatrix}<(s_{2.8},0.52,0.20);0.49>,\\<(s_{3.9},0.66,0.22);0.61>\end{pmatrix}\},\\\{[(s_{2.4},0.40,0.20);0.15];\begin{pmatrix}<(s_{1.87},0.45,0.30);0.15>,\\<(s_{2.4},0.50,0.20);0.18>\end{pmatrix}\}\end{array}\right\rangle\end{array}\right\}.$$

$$L_2\chi(x)=\left\{\begin{array}{c}\left\langle\begin{array}{c}\{[(s_{5.3},0.68,0.10);0.60];\begin{pmatrix}<(s_{3.9},0.66,0.16);0.52>,\\<(s_{5.5},0.72,0.12);0.75>\end{pmatrix}\},\\\{[(s_{2.4},0.40,0.10);0.18];\begin{pmatrix}<(s_2,0.47,0.27);0.18>,\\<(s_{2.4},0.40,0.10);0.19>\end{pmatrix}\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_{3.6},0.57,0.12);0.73];\begin{pmatrix}<(s_{3.1},0.57,0.13);0.59>,\\<(s_{4.7},0.67,0.13);0.80>\end{pmatrix}\},\\\{[(s_{1.6},0.60,0.40);0.20];\begin{pmatrix}<(s_{1.6},0.50,0.35);0.19>,\\<(s_2,0.50,0.25);0.20>\end{pmatrix}\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_{3.6},0.70,0.23);0.68];\begin{pmatrix}<(s_3,0.65,0.20);0.54>,\\<(s_{4.7},0.70,0.15);0.77>\end{pmatrix}\},\\\{[(s_2,0.40,0.30);0.20];\begin{pmatrix}<(s_{1.8},0.40,0.30);0.19>,\\<(s_{2.2},0.47,0.27);0.20>\end{pmatrix}\}\end{array}\right\rangle\end{array}\right\}.$$

$$L_3\chi(x)=\left\{\begin{array}{c}\left\langle\begin{array}{c}\{[(s_{4.3},0.73,0.20);0.73];\begin{pmatrix}<(s_{3.45},0.61,0.24);0.64>,\\<(s_{4.45},0.69,0.16);0.75>\end{pmatrix}\},\\\{[(s_2,0.80,0.10);0.13];\begin{pmatrix}<(s_{1.8},0.67,0.27);0.13>,\\<(s_{2.2},0.80,0.10);0.15>\end{pmatrix}\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_{3.8},0.51,0.38);0.70];\begin{pmatrix}<(s_{3.2},0.51,0.39);0.63>,\\<(s_{4.2},0.59,0.32);0.74>\end{pmatrix}\},\\\{[(s_{1.6},0.70,0.30);0.18];\begin{pmatrix}<(s_{1.6},0.60,0.35);0.15>,\\<(s_2,0.75,0.20);0.18>\end{pmatrix}\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_{2.7},0.56,0.20);0.58];\begin{pmatrix}<(s_{2.5},0.57,0.30);0.57>,\\<(s_{3.8},0.65,0.19);0.68>\end{pmatrix}\},\\\{[(s_{2.4},0.50,0.40);0.15];\begin{pmatrix}<(s_2,0.50,0.40);0.14>,\\<(s_{2.4},0.67,0.27);0.16>\end{pmatrix}\}\end{array}\right\rangle\end{array}\right\}.$$

Step 3. Construct a possibility degree matrix $P = (p_{\alpha k})_{3\times 3}$.

$$P = \begin{pmatrix} 0.5 & 0.27 & 0.44 \\ 0.73 & 0.5 & 0.65 \\ 0.56 & 0.35 & 0.5 \end{pmatrix}.$$

Step 4. Rank the cities $x_\alpha(\alpha = 1,2,3)$ with the values of $p_\alpha(\alpha = 1,2,3)$ in descending order and select the most risky city.

$$p_1 \approx 1.21, p_2 \approx 1.87, p_3 \approx 1.42. \quad \because p_2 > p_3 > p_1, \therefore x_2 \succ x_3 \succ x_1.$$

Therefore, the most risky city is x_2.

5.1.2 The χLVIKOR method

Next, we use the χLVIKOR method of χLSs to calculate this numerical example again.

Step 1. Construct a normalized χ-linguistic decision matrix with all χLVs are in descending order, donated as Table 2.

Step 2. Determine the best solution $L\chi(x)^* = \{L_1\chi(x)^*, L_2\chi(x)^*, L_3\chi(x)^*\}$ and the worst solution $L\chi(x)^\# = \{L_1\chi(x)^\#, L_2\chi(x)^\#, L_3\chi(x)^\#\}$ with Definitions 35 and 36.

$$L_1\chi(x)^* = \left\{ \begin{array}{c} \left\langle \begin{array}{l} \{[(s_5, 0.8, 0.1); 0.9]; \begin{pmatrix} <(s_{3.33}, 0.6, 0.3); 0.63>, \\ <(s_5, 0.63, 0.3); 0.9> \end{pmatrix}\}, \\ \{[(s_6, 0.8, 0.1); 0.7]; \begin{pmatrix} <(s_{4.5}, 0.67, 0.27); 0.5>, \\ <(s_{4.67}, 0.5, 0.27); 0.75> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \{[(s_6, 0.7, 0.1); 0.9]; \begin{pmatrix} <(s_2, 0.6, 0.3); 0.5>, \\ <(s_{3.33}, 0.63, 0.3); 0.75> \end{pmatrix}\}, \\ \{[(s_4, 0.6, 0.2); 0.8]; \begin{pmatrix} <(s_4, 0.6, 0.35); 0.6>, \\ <(s_{4.67}, 0.6, 0.2); 0.8> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{l} \{[(s_3, 0.7, 0.2); 0.7]; \begin{pmatrix} <(s_3, 0.6, 0.15); 0.55>, \\ <(s_{4.67}, 0.75, 0.13); 0.75> \end{pmatrix}\}, \\ \{[(s_6, 0.5, 0.4); 0.8]; \begin{pmatrix} <(s_5, 0.5, 0.4); 0.55>, \\ <(s_{5.5}, 0.47, 0.27); 0.8> \end{pmatrix}\} \end{array} \right\rangle \end{array} \right\}.$$

$$L_2\chi(x)^* = \left\{ \begin{array}{c} \left\langle \begin{array}{c} \{[(s_5, 0.6, 0.2); 0.8]; \begin{pmatrix} <(s_{4.5}, 0.57, 0.3); 0.77>, \\ <(s_{5.5}, 0.6, 0.2); 0.8> \end{pmatrix}\}, \\ \{[(s_0, 0, 0); 0]; \begin{pmatrix} <(s_0, 0, 0); 0>, \\ <(s_0, 0, 0); 0> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{[(s_6, 0.8, 0.2); 0.7]; \begin{pmatrix} <(s_5, 0.5, 0.5); 0.7>, \\ <(s_6, 0.5, 0.5); 0.77> \end{pmatrix}\}, \\ \{[(s_0, 0, 0); 0]; \begin{pmatrix} <(s_0, 0, 0); 0>, \\ <(s_0, 0, 0); 0> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{[(s_4, 0.5, 0.1); 0.8]; \begin{pmatrix} <(s_4, 0.57, 0.3); 0.77>, \\ <(s_5, 0.6, 0.2); 0.8> \end{pmatrix}\}, \\ \{[(s_0, 0, 0); 0]; \begin{pmatrix} <(s_0, 0, 0); 0>, \\ <(s_0, 0, 0); 0> \end{pmatrix}\} \end{array} \right\rangle \end{array} \right\}.$$

$$L_3\chi(x)^* = \left\{ \begin{array}{c} \left\langle \begin{array}{c} \{[(s_6, 0.8, 0.1); 0.7]; \begin{pmatrix} <(s_3, 0.6, 0.27); 0.6>, \\ <(s_4, 0.8, 0.2); 0.7> \end{pmatrix}\}, \\ \{[(s_0, 0, 0); 0]; \begin{pmatrix} <(s_0, 0, 0); 0>, \\ <(s_0, 0, 0); 0> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{[(s_4, 0.8, 0.1); 0.9]; \begin{pmatrix} <(s_3, 0.4, 0.4); 0.55>, \\ <(s_4, 0.6, 0.27); 0.65> \end{pmatrix}\}, \\ \{[(s_0, 0, 0); 0]; \begin{pmatrix} <(s_0, 0, 0); 0>, \\ <(s_0, 0, 0); 0> \end{pmatrix}\} \end{array} \right\rangle, \\ \left\langle \begin{array}{c} \{[(s_4, 0.8, 0.2); 0.8]; \begin{pmatrix} <(s_1, 0.5, 0.3); 0.5>, \\ <(s_3, 0.7, 0.2); 0.6> \end{pmatrix}\}, \\ \{[(s_0, 0, 0); 0]; \begin{pmatrix} <(s_0, 0, 0); 0>, \\ <(s_0, 0, 0); 0> \end{pmatrix}\} \end{array} \right\rangle \end{array} \right\}.$$

$$L_1\chi(x)^{\#}=\left\{\begin{array}{c}\left\langle\begin{array}{c}\{[(s_4,0.6,0.3);0.7];\left(\begin{array}{c}<(s_3,0.67,0.17);0.6>,\\<(s_4,0.7,0.1);0.8>\end{array}\right)\},\\\{[(s_5,0.5,0.4);0.5];\left(\begin{array}{c}<(s_{4.5},0.67,0.27);0.5>,\\<(s_{5.5},0.8,0.1);0.6>\end{array}\right)\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_2,0.7,0.1);0.4];\left(\begin{array}{c}<(s_{4.67},0.5,0.1);0.4>,\\<(s_6,0.67,0.13);0.63>\end{array}\right)\},\\\{[(s_4,0.6,0.4);0.7];\left(\begin{array}{c}<(s_4,0.5,0.35);0.77>,\\<(s_5,0.5,0.25);0.8>\end{array}\right)\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_3,0.6,0.3);0.5];\left(\begin{array}{c}<(s_{2.5},0.63,0.3);0.4>,\\<(s_4,0.7,0.3);0.63>\end{array}\right)\},\\\{[(s_4,0.4,0.3);0.6];\left(\begin{array}{c}<(s_{4.67},0.45,0.3);0.6>,\\<(s_6,0.5,0.2);0.7>\end{array}\right)\}\end{array}\right\rangle\end{array}\right\}.$$

$$L_2\chi(x)^{\#}=\left\{\begin{array}{c}\left\langle\begin{array}{c}\{[(s_5,0.6,0.2);0.7];\left(\begin{array}{c}<(s_{3.67},0.5,0.2);0.6>,\\<(s_5,0.7,0.2);0.7>\end{array}\right)\},\\\{[(s_0,0,0);0];\left(\begin{array}{c}<(s_0,0,0);0>,\\<(s_0,0,0);0>\end{array}\right)\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_3,0.8,0.2);0.6];\left(\begin{array}{c}<(s_3,0.6,0.2);0.55>,\\<(s_{3.67},0.8,0.2);0.65>\end{array}\right)\},\\\{[(s_0,0,0);0];\left(\begin{array}{c}<(s_0,0,0);0>,\\<(s_0,0,0);0>\end{array}\right)\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_4,0.6,0.3);0.4];\left(\begin{array}{c}<(s_3,0.6,0.3);0.4>,\\<(s_{4.5},0.57,0.2);0.63>\end{array}\right)\},\\\{[(s_0,0,0);0];\left(\begin{array}{c}<(s_0,0,0);0>,\\<(s_0,0,0);0>\end{array}\right)\}\end{array}\right\rangle\end{array}\right\}.$$

$$L_3\chi(x)^{\#}=\left\{\begin{array}{c}\left\langle\begin{array}{c}\{[(s_6,0.7,0.1);0.4];\left(\begin{array}{c}<(s_4,0.75,0.15);0.4>,\\<(s_6,0.77,0.13);0.7>\end{array}\right)\},\\\{[(s_0,0,0);0];\left(\begin{array}{c}<(s_0,0,0);0>,\\<(s_0,0,0);0>\end{array}\right)\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_4,0.7,0.2);0.5];\left(\begin{array}{c}<(s_{3.5},0.6,0.15);0.5>,\\<(s_{4.5},0.75,0.15);0.53>\end{array}\right)\},\\\{[(s_0,0,0);0];\left(\begin{array}{c}<(s_0,0,0);0>,\\<(s_0,0,0);0>\end{array}\right)\}\end{array}\right\rangle,\\\left\langle\begin{array}{c}\{[(s_4,0.8,0.2);0.8];\left(\begin{array}{c}<(s_3,0.75,0.15);0.6>,\\<(s_5,0.77,0.13);0.85>\end{array}\right)\},\\\{[(s_0,0,0);0];\left(\begin{array}{c}<(s_0,0,0);0>,\\<(s_0,0,0);0>\end{array}\right)\}\end{array}\right\rangle\end{array}\right\}.$$

Table 3 The normalized χ-linguistic decision matrix of the group

	a_1	a_2	a_3
x_1	$\left\{ \left\langle \{[(s_5,0.6,0.3);0.9];(<(s_{3.33},0.6,0.3);0.63>,<(s_5,0.63,0.3);0.9>)\}, \{[(s_4,0.5,0.4);0.7];(<(s_4,0.5,0.4);0.65>,<(s_{4.67},0.5,0.27);0.75>)\} \right\rangle, \left\langle \{[(s_2,0.6,0.3);0.6];(<(s_2,0.6,0.3);0.5>,<(s_{3.33},0.63,0.3);0.75>)\}, \{[(s_4,0.6,0.2);0.8];(<(s_4,0.5,0.27);0.7>,<(s_{4.67},0.6,0.2);0.8>)\} \right\rangle, \left\langle \{[(s_3,0.7,0.3);0.4];(<(s_{2.5},0.63,0.3);0.4>,<(s_4,0.7,0.3);0.63>)\}, \{[(s_6,0.4,0.2);0.6];(<(s_{4.67},0.45,0.3);0.6>,<(s_6,0.5,0.2);0.7>)\} \right\rangle \right\}$	$\left\{ \left\langle \{[(s_5,0.6,0.2);0.7];(<(s_{3.67},0.5,0.2);0.6>,<(s_5,0.7,0.2);0.7>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_3,0.8,0.2);0.6];(<(s_3,0.6,0.2);0.55>,<(s_{3.67},0.8,0.2);0.65>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_3,0.4,0.2);0.5];(<(s_3,0.4,0.2);0.5>,<(s_{3.67},0.6,0.2);0.6>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle \right\}$	$\left\{ \left\langle \{[(s_5,0.8,0.1);0.5];(<(s_4,0.67,0.13);0.5>,<(s_5,0.8,0.1);0.53>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_4,0.7,0.2);0.5];(<(s_{3.5},0.6,0.15);0.5>,<(s_{4.5},0.75,0.15);0.53>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_3,0.5,0.1);0.6];(<(s_3,0.5,0.1);0.53>,<(s_4,0.67,0.13);0.6>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle \right\}$
x_2	$\left\{ \left\langle \{[(s_5,0.8,0.1);0.8];(<(s_4,0.67,0.13);0.63>,<(s_5,0.8,0.1);0.8>)\}, \{[(s_6,0.4,0.1);0.7];(<(s_5,0.47,0.27);0.7>,<(s_6,0.4,0.1);0.77>)\} \right\rangle, \left\langle \{[(s_6,0.5,0.1);0.4];(<(s_{4.67},0.5,0.1);0.4>,<(s_6,0.67,0.13);0.63>)\}, \{[(s_4,0.6,0.4);0.8];(<(s_4,0.5,0.35);0.77>,<(s_5,0.5,0.25);0.8>)\} \right\rangle, \left\langle \{[(s_3,0.7,0.2);0.7];(<(s_3,0.6,0.15);0.55>,<(s_{4.67},0.75,0.13);0.75>)\}, \{[(s_5,0.4,0.3);0.8];(<(s_{4.5},0.4,0.3);0.77>,<(s_{5.5},0.47,0.27);0.8>)\} \right\rangle \right\}$	$\left\{ \left\langle \{[(s_5,0.5,0.1);0.8];(<(s_{3.67},0.57,0.2);0.63>,<(s_5,0.55,0.15);0.8>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_2,0.6,0.2);0.7];(<(s_2,0.57,0.2);0.55>,<(s_{3.67},0.55,0.15);0.75>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_4,0.6,0.3);0.4];(<(s_3,0.6,0.3);0.4>,<(s_{4.5},0.57,0.2);0.63>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle \right\}$	$\left\{ \left\langle \{[(s_6,0.7,0.1);0.4];(<(s_4,0.75,0.15);0.4>,<(s_6,0.77,0.13);0.7>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_2,0.8,0.1);0.9];(<(s_2,0.77,0.13);0.7><(s_4,0.8,0.1);0.9>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_4,0.8,0.2);0.8];(<(s_3,0.75,0.15);0.6>,<(s_5,0.77,0.13);0.85>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle \right\}$
x_3	$\left\{ \left\langle \{[(s_4,0.7,0.1);0.7];(<(s_3,0.67,0.17);0.6>,<(s_4,0.7,0.1);0.8>)\}, \{[(s_5,0.8,0.1);0.5];(<(s_{4.5},0.67,0.27);0.5>,<(s_{5.5},0.8,0.1);0.6>)\} \right\rangle, \left\langle \{[(s_2,0.7,0.1);0.9];(<(s_2,0.67,0.17);0.7>,<(s_3,0.7,0.1);0.9>)\}, \{[(s_4,0.7,0.3);0.7];(<(s_4,0.6,0.35);0.6>,<(s_5,0.75,0.2);0.7>)\} \right\rangle, \left\langle \{[(s_3,0.6,0.3);0.5];(<(s_{2.5},0.6,0.3);0.5>,<(s_{3.5},0.67,0.17);0.7>)\}, \{[(s_6,0.5,0.4);0.6];(<(s_5,0.5,0.4);0.55>,<(s_6,0.67,0.27);0.65>)\} \right\rangle \right\}$	$\left\{ \left\langle \{[(s_5,0.7,0.3);0.8];(<(s_{4.5},0.57,0.3);0.77>,<(s_{5.5},0.6,0.2);0.8>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_6,0.5,0.5);0.7];(<(s_5,0.5,0.5);0.7>,<(s_5,0.5,0.5);0.77>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_4,0.5,0.1);0.8];(<(s_4,0.57,0.3);0.77>,<(s_5,0.6,0.2);0.8>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle \right\}$	$\left\{ \left\langle \{[(s_4,0.8,0.2);0.7];(<(s_3,0.6,0.27);0.6>,<(s_4,0.8,0.2);0.7>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_4,0.4,0.4);0.6];(<(s_3,0.4,0.4);0.55>,<(s_4,0.6,0.27);0.65>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle, \left\langle \{[(s_1,0.6,0.2);0.5];(<(s_1,0.5,0.3);0.5>,<(s_3,0.7,0.2);0.6>)\}, \{[(s_0,0,0);0];(<(s_0,0,0);0>,<(s_0,0,0);0>)\} \right\rangle \right\}$

Table 4 Three ranking results

Values	Ranking
$\mathcal{S}_\alpha$	$x_3 \succ x_2 \succ x_1$
$\mathcal{R}_\alpha$	$x_3 \succ x_2 \succ x_1$
$\mathcal{Q}_\alpha$	$x_3 \succ x_2 \succ x_1$

Table 5 Ranking results of different A, B, and C based on the χLWA and χLWG operator

$(A, B, C)^T$	TheχLWA oparator		The χLWG operator	
	$(p_1, p_2, p_3)^T$	Ranking results	$(p_1, p_2, p_3)^T$	Ranking results
$(1, 0, 0)^T$	$(1.37, 1.72, 1.40)^T$	$x_2 \succ x_3 \succ x_1$	$(1.55, 1.66, 1.29)^T$	$x_2 \succ x_1 \succ x_3$
$(0.5, 0.5, 0)^T$	$(1.36, 1.91, 1.26)^T$	$x_2 \succ x_1 \succ x_3$	$(1.49, 1.69, 1.32)^T$	$x_2 \succ x_1 \succ x_3$
$(0.5, 0.3, 0.2)^T$	$(1.30, 1.82, 1.39)^T$	$x_2 \succ x_3 \succ x_1$	$(1.48, 1.68, 1.34)^T$	$x_2 \succ x_1 \succ x_3$
$(0.5, 0, 0.5)^T$	$(1.37, 1.73, 1.40)^T$	$x_2 \succ x_3 \succ x_1$	$(1.55, 1.66, 1.29)^T$	$x_2 \succ x_1 \succ x_3$

Step 3. Calculate values $\mathcal{S}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{Q}_\alpha(\alpha = 1, 2, 3)$ with Eqs.(50-52).

$$\mathcal{S}_1 = 17.29,\ \mathcal{S}_2 = 2.40,\ \mathcal{S}_3 = 1.02.$$
$$\mathcal{R}_1 = 15.87,\ \mathcal{R}_2 = 1.40,\ \mathcal{R}_3 = 0.68.$$
$$\mathcal{Q}_1 = 8.41,\ \mathcal{Q}_2 = 0.72,\ \mathcal{Q}_3 = 0.32.$$

Step 4. Obtain three ranking results by values $\mathcal{S}_\alpha$, $\mathcal{R}_\alpha$, and $\mathcal{Q}_\alpha(\alpha = 1, 2, \ldots, t)$, and show them in Table 3.

Step 5. Determine the compromise solution (set) by conditions 1 and 2.

In the ranking obtained by $\mathcal{Q}$, x_3 and x_2 are ranked first and second. For $\frac{1}{t-1} = 0.5$ and

$$\mathcal{Q}_2 - \mathcal{Q}_3 = 0.4 < 0.5,$$

it does not satisfy condition 1. x_3 is also the best alternative by $\mathcal{S}$ and $\mathcal{R}$. Therefore, it satisfy condition 2.

$$\mathcal{Q}_1 - \mathcal{Q}_3 = 8.09 > 0.5.$$

Therefore, $\{x_3, x_2\}$ is the compromise solution set. x_3 and x_2 are the most risky cities. And x_3 has more risky than x_2.

5.2 Analysis and discussion

In the previous section, the $\phi_{\chi LWA}$ (or $\phi_{\chi LWG}$) operator and the χLVIKOR method are used to calculate the numerical example, and the obtained most risky city are different. This is caused by the different characteristics of the two methods. The former is easy to calculate but has the problem of information loss, and the latter effectively reduce information loss but is complicated in calculation. DMs can choose methods and results based on specific issues and needs.

Next, we compare and discussion from three aspects: the influence of changes in the values of A, B, and C, the influence of changes in the value of λ, and the comparison with other existing methods.

Table 6 Ranking results of different A, B, and C based on the χLVIKOR method

$(A, B, C)^T$	$\mathcal{S}$	Ranking results by $\mathcal{S}$	$\mathcal{R}$	Ranking results by $\mathcal{R}$	$\mathcal{Q}$	Ranking results by $\mathcal{Q}$	The compromise solution (set)
$(0.5, 0, 0.5)^T$	$(5.68, 1.61, 1.17)^T$	$x_3 \succ x_2 \succ x_1$	$(2.63, 1.02, 0.62)^T$	$x_3 \succ x_2 \succ x_1$	$(1.66, 0.41, 0.16)^T$	$x_3 \succ x_2 \succ x_1$	$\{x_3, x_2\}$

Table 7 Ranking results of different λ based on the χLVIKOR method

λ	$\mathcal{S}$	Ranking results by $\mathcal{S}$	$\mathcal{R}$	Ranking results by $\mathcal{R}$	$\mathcal{Q}$	Ranking results by $\mathcal{Q}$	The compromise solution (set)
0.1	$(3.69, 0.83, 0.90)^T$	$x_2 \succ x_3 \succ x_1$	$(2.01, 0.38, 0.52)^T$	$x_2 \succ x_3 \succ x_1$	$(1.39, 0.07, 0.15)^T$	$x_2 \succ x_3 \succ x_1$	$\{x_2, x_3\}$
0.5	$(4.12, 1.01, 1.14)^T$	$x_2 \succ x_3 \succ x_1$	$(2.63, 0.37, 0.69)^T$	$x_2 \succ x_3 \succ x_1$	$(1.74, 0.10, 0.29)^T$	$x_2 \succ x_3 \succ x_1$	$\{x_2, x_3\}$
1	$(5.91, 1.37, 1.25)^T$	$x_3 \succ x_2 \succ x_1$	$(4.46, 0.61, 0.78)^T$	$x_2 \succ x_3 \succ x_1$	$(2.65, 0.24, 0.31)^T$	$x_2 \succ x_3 \succ x_1$	$\{x_2, x_3\}$
2	$(17.29, 2.40, 1.02)^T$	$x_3 \succ x_2 \succ x_1$	$(15.87, 1.40, 0.68)^T$	$x_3 \succ x_2 \succ x_1$	$(8.41, 0.72, 0.32)^T$	$x_3 \succ x_2 \succ x_1$	x_3
5	$(47.48, 4.67, 0.73)^T$	$x_3 \succ x_2 \succ x_1$	$(46.19, 4.06, 0.45)^T$	$x_3 \succ x_2 \succ x_1$	$(23.59, 2.07, 0.22)^T$	$x_3 \succ x_2 \succ x_1$	$\{x_2, x_3\}$
10	$(50.72, 5.06, 0.71)^T$	$x_3 \succ x_2 \succ x_1$	$(49.46, 4.55, 0.40)^T$	$x_3 \succ x_2 \succ x_1$	$(25.23, 2.31, 0.20)^T$	$x_3 \succ x_2 \succ x_1$	x_3

Table 8 Comparison with existing methods

Methods	Dimension	Ambiguity information	Probabilistic/ Credibility information	Data sources for decision information	Ranking results
ILS(this paper)	$n(n \geq 1)$	Completely known	Completely known	Partly given, partly calculated	$x_2 \succ x_3 \succ x_1$
HFLTS Rodriguez et al. (2012)	1	N/A	N/A	Given	$x_2 \succ x_3 \succ x_1$
PLTS Pang et al. (2016)	1	N/A	Partly known	Given	$x_2 \succ x_1 \succ x_3$
Intuitionistic linguistic fuzzy set Wang and Li (2010)	1	Completely known	N/A	Given	N/A
Z-linguistic set Xian et al. (2019)	1	N/A	Completely known	Given	N/A
intuitionistic Z-linguistic set Xian et al. (2019, 2022)	1	Completely known	Completely known	Given	N/A
Fuzzy rough set Dubois and Prade (1990)	1	N/A	N/A	Calculated	N/A
PDHL-VIKOR Gou et al. (2021)	2	N/A	Partly known	Given	$x_2 \succ x_1 \succ x_3$

5.2.1 The influence of the values *A*, *B*, and *C* on the ranking results

In different decision making problems, the importance of the ambiguity, possibility, and approximation of χLSs is correspondingly different. To distinguish the difference in importance among the above three, the variable parameter values A, B, and C are added when defining $r(l^k\chi(x))$, $\vartheta(l^k\chi(x))$, $\varphi(l^k\chi(x))$, and the distance between two χLSs or χLVs.

In order to explore the influence of the values A, B, and C on the decision results, we take different values and calculate them with the methods based on the χLWA operator and the χLWG operator, respectively. The ranking results are shown in Table 4. Obviously, in the ILWA operator, when $(A, B, C)^T$ takes $(0.5, 0.5, 0)^T$, the ranking result is $x_2 \succ x_1 \succ x_3$, which is different from the ranking result $x_2 \succ x_3 \succ x_1$ obtained with other values. In the χLWG operator, when $(A, B, C)^T$ takes $(1/3, 1/3, 1/3)^T$, the ranking result is $x_3 \succ x_2 \succ x_1$, which is different from the ranking result $x_2 \succ x_1 \succ x_3$ obtained with other values. Table 5 shows the ranking results under different values A, B, and C obtained by the χLVIKOR method. It can be seen that the obtained compromise solutions (sets) are not exactly the same.

These shows that the values A, B, and C have an impact on the ranking results. According to the different needs of the decision problems, they can be given different values, which makes the decision more flexible and changeable.

5.2.2 The influence of the value λ on the ranking results

The value λ is used to define the distance measurement, which can express the risk attitude of DMs and give DMs more choices.

Next, we explore the influence of λ on the ranking results. We use different λ to calculate the numerical example with the χLVIKOR method. The ranking results are shown in Table 6. As shown in the table, when λ takes 0.1, 0.5, 1, and 5, the compromise solution set is $\{x_2, x_3\}$; when λ takes 2 and 10, the compromise solution changes to x_3.

Therefore, the variable parameter λ influence the ranking result. The value of λ can be determined according to the risk attitude of DMs.

5.2.3 Comparative analysis with existing methods

In this section, we compare the χLS proposed in this paper with other similar methods in the concept and ranking results. Comparison methods include HFLTS, PLTS, ILFS, Z-linguistic set, intuitionistic Z-linguistic set and fuzzy rough set. In the calculation, we process the data in the numerical example. Taking the HFLTS as an example, we only consider one dimension and ignore ambiguity, credibility, and approximation. The calculations are all based on operator-based method. We also compared the results with the PDHL-VIKOR[58] method. Table 7 shows the comparison results.

The numerical example in this paper is a MAGDM problem, where the attribute weights are known and DMs' weights are unknown. The weights of DMs are not needed when using χLSs, HFLTSs, and PLTSs to do GDM, but ILFSs, Z-linguistic sets, intuitionistic Z-linguistic set and fuzzy rough sets need (Table 8). Therefore, the numerical example cannot be solved by the last four methods. The ranking obtained by using χLSs is different from the ranking obtained by using PLTSs and PDHL-VIKOR, which is caused by subjective and incomplete representation of the decision information of PLTSs and PDHLs. Although the ranking obtained by using χLSs and HFLTSs are the same, HFLTSs can only represent one-dimensional information, which may cause inaccurate decision making.

Through conceptual comparison, we can find the superiority of the χLS. The χLS represent n-dimensional$(n \geq 1)$ information, and other methods only represent one-dimensional information. Ambiguity and probabilistic(credibility) information are included, but other methods have the problem of missing information. Decision making data is a combination of the DMs' subjective ideas and objective calculations, while most other methods are too subjective, making the decision results lose objectivity.

6 Conclusion

The traditional linguistic representation methods are single in information representation and cannot represent all uncertainty of decision information, which leads to inaccurate decision results. Therefore, we put forward a new concept called χLSs, taking into account the fuzziness, ambiguity, randomness and approximation of language. It effectively solves the problem of information loss and represent multi-dimensional information. The basic theory of χLSs is also proposed. The new χLVIKOR method for a MAGDM problem is proposed. Subsequently, the method is applied to COVID-19 risk assessment in cities. Through comparative analysis with χLWA (χLWG) operator and other linguistic representation methods, we prove the effectiveness and superiority of χLSs.

In the future, the theory of χLSs needs to be improved. In the MAGDM problem based on χLSs, the determination of attribute weights is a problem to be solved. The computational complexity is high, simplifying the operation of χLSs is an optimization direction.

The applications in artificial intelligence, innovative technology and other fields are interesting topics.

Proof of Theorem 2

Proof According to Eq. (34), we have

$$
\begin{aligned}
& p(L_1\chi(x) \geq L_2\chi(x)) + p(L_2\chi(x) \geq L_1\chi(x)) \\
&= \frac{\max\left(0, E(L_1\chi(x)^+) - E(L_2\chi(x)^-)\right) - \max\left(0, E(L_1\chi(x)^-) - E(L_2\chi(x)^+)\right)}{\left(E(L_1\chi(x)^+) - E(L_1\chi(x)^-)\right) + \left(E(L_2\chi(x)^+) - E(L_2\chi(x)^-)\right)} \\
&+ \frac{\max\left(0, E(L_2\chi(x)^+) - E(L_1\chi(x)^-)\right) - \max\left(0, E(L_2\chi(x)^-) - E(L_1\chi(x)^+)\right)}{\left(E(L_2\chi(x)^+) - E(L_2\chi(x)^-)\right) + \left(E(L_1\chi(x)^+) - E(L_1\chi(x)^-)\right)} \\
&= \frac{E(L_1\chi(x)^+) - E(L_2\chi(x)^-) - E(L_1\chi(x)^-) + E(L_2\chi(x)^+) + 0 + 0}{\left(E(L_1\chi(x)^+) - E(L_1\chi(x)^-)\right) + \left(E(L_2\chi(x)^+) - E(L_2\chi(x)^-)\right)} = 1.
\end{aligned}
$$

When $L_1\chi(x) = L_2\chi(x)$, it is obvious that $p(L_1\chi(x) \geq L_2\chi(x)) = p(L_2\chi(x) \geq L_1\chi(x)) = 0.5$.

Proof of Theorem 3

Proof (1) $l_1\chi(x) \oplus l_2\chi(x) = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\theta^n(x_1)+\theta^n(x_2)}, \frac{\theta^n(x_1)\mu^n_{\sigma(x_1)}+\theta^n(x_2)\mu^n_{\sigma(x_2)}}{\theta^n(x_1)+\theta^n(x_2)}, \frac{\theta^n(x_1)v^n_{\sigma(x_1)}+\theta^n(x_2)v^n_{\sigma(x_2)}}{\theta^n(x_1)+\theta^n(x_2)}); \frac{p^n_{\rho(x_1)}+p^n_{\rho(x_2)}}{2}]; g^n_{\varsigma(x_1)} \oplus g^n_{\varsigma(x_2)}\} > = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\theta^n(x_2)+\theta^n(x_1)}, \frac{\theta^n(x_2)\mu^n_{\sigma(x_2)}+\theta^n(x_1)\mu^n_{\sigma(x_1)}}{\theta^n(x_2)+\theta^n(x_1)}, \frac{\theta^n(x_2)v^n_{\sigma(x_2)}+\theta^n(x_1)}{} v^n_{\sigma(x_1)}\theta^n(x_2)+\theta^n(x_1)); \frac{p^n_{\rho(x_2)}+p^n_{\rho(x_1)}}{2}]; g^n_{\varsigma(x_2)} \oplus g^n_{\varsigma(x_1)}\} > = l_2\chi(x) \oplus l_1\chi(x)$.
(2) $l_1\chi(x) \otimes l_2\chi(x) = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\theta^n(x_1)\times\theta^n(x_2)}, \mu^n_{\sigma(x_1)}\mu^n_{\sigma(x_2)}, v^n_{\sigma(x_1)}v^n_{\sigma(x_2)}); p^n_{\rho(x_1)} \times p^n_{\rho(x_2)}]; g^n_{\varsigma(x_1)} \otimes g^n_{\varsigma(x_2)}\} > = <[\bigcup_{n=1,2,\ldots,n_1} \{(s_{\theta^n(x_2)\times\theta^n(x_1)}, \mu^n_{\sigma(x_2)}\mu^n_{\sigma(x_1)}, v^n_{\sigma(x_2)}v^n_{\sigma(x_1)}); p^n_{\rho(x_2)} \times p^n_{\rho(x_1)}]; g^n_{\varsigma(x_2)} \otimes g^n_{\varsigma(x_1)}\} > = l_2\chi(x) \otimes l_1\chi(x)$.
The proof process of (3) is similar, omitted here.

Proof of Theorem 5

Proof (1) $d(l_1\chi(x), l_2\chi(x)) \geq 0$ is obvious.
Since $E(f^1), E(f^2), E(p^1), E(p^2), E(g^1), E(g^2) \in [0, 1]$, and $\lambda > 0$, then we have

$$
\left|E(f^1) - E(f^2)\right|^\lambda \leq 1, \left|E(p^1) - E(p^2)\right|^\lambda \leq 1, \left|E(g^1) - E(g^2)\right|^\lambda \leq 1.
$$

For $0 \leq A, B, C \leq 1$ and $A + B + C = 1$, then

$$
\begin{aligned}
& d(l_1\chi(x), l_2\chi(x)) \\
&= \left(A\left|E(f^1) - E(f^2)\right|^\lambda + B\left|E(p^1) - E(p^2)\right|^\lambda + C\left|E(g^1) - E(g^2)\right|^\lambda\right)^{1/\lambda} \leq 1.
\end{aligned}
$$

Therefore, $0 \leq d(l_1\chi(x), l_2\chi(x)) \leq 1$.

(2)
$$\begin{aligned}&d(l_1\chi(x), l_2\chi(x)) = \\ &= 0 \Rightarrow E(f^1) = E(f^2), E(p^1) = E(p^2), E(g^1) = E(g^2) \\ &\Rightarrow r(l_1\chi(x)) = r(l_2\chi(x)), \vartheta(l_1\chi(x)) = \vartheta(l_2\chi(x)), \varphi(l_1\chi(x)) = \varphi(l_2\chi(x)) \\ &\Rightarrow l_1\chi(x) \sim l_2\chi(x).\end{aligned}$$

(3)
$$\begin{aligned}&d(l_1\chi(x), l_2\chi(x)) = \\ &= \left(A\left|E(f^2) - E(f^1)\right|^{\lambda} + B\left|E(p^2) - E(p^1)\right|^{\lambda} + C\left|E(g^2) - E(g^1)\right|^{\lambda} \right)^{1/\lambda}. \\ &= d(l_2\chi(x), l_1\chi(x))\end{aligned}$$

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Declarations

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