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Evolved distance measures for circular intuitionistic fuzzy sets and their exploitation in the technique for order preference by similarity to ideal solutions

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Abstract

Circular intuitionistic fuzzy (C-IF) sets are an up-and-coming tool for enforcing indistinct and imprecise information in variable and convoluted decision-making situations. C-IF sets, as opposed to typical intuitionistic fuzzy sets, are better suited for identifying the evaluation data with uncertainty in intricate realistic decision situations. The architecture of the technique for order preference by similarity to ideal solutions (TOPSIS) provides powerful evaluation tools to aid decision-making in intuitionistic fuzzy conditions. To address appraisal issues associated with decision analysis involving extremely convoluted information, this paper propounds a novel C-IF TOPSIS approach in the context of C-IF uncertainty. This research makes three significant contributions. First, based on the three- and four-term operating rules, this research introduces C-IF Minkowski distance measures, which are new generalized representations of distance metrics applicable to C-IF values and C-IF sets. Such general C-IF distance metrics can alleviate the constraints of established C-IF distance measures, provide usage resiliency through parameter settings, and broaden the applicability of metric analysis. Second, unlike existing C-IF TOPSIS methods, this research fully utilizes C-IF information characteristics and extends the core structure of the classic TOPSIS to C-IF contexts. With the newly developed C-IF Minkowski metrics, this study faithfully demonstrates the trade-off evaluation and compromise decision rules in the TOPSIS framework. Third, this research builds on the core strengths of the pioneered C-IF Minkowski distance measures to create innovative C-IF TOPSIS techniques utilizing four different combinations, including displaced and fixed anchoring frameworks, as well as three- and four-term representations. Such a refined C-IF TOPSIS methodology can assist decision-makers in proactively addressing increasingly sophisticated decision-making problems in practical settings. Finally, this research employs two innovative prioritization algorithms to address a site selection issue of large-scale epidemic hospitals to illustrate the superior capabilities of the C-IF TOPSIS methodology over some current related approaches.

Keywords Circular intuitionistic fuzzy (C-IF) set · Technique for order preference by similarity to ideal solutions (TOPSIS) · C-IF Minkowski distance measure · Anchoring framework · Site selection

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1 Introduction

Multiple criteria decision analysis (MCDA) is generally delineated as an analytical process for making the most suitable choice with the highest relative dominance out of a group of candidate alternatives characterized by multiple performance criteria (Jaini 2023; Kaya et al. 2022; Shen et al. 2022). To tackle and manage real-world decision issues, numerous MCDA models, approaches, and techniques have been created and applied in a variety of application domains (Garg and Rani 2022; Rani and Garg 2022; Reig-Mullor et al. 2022; Tiwari and Gupta 2022). Many MCDA systems employ a step-by-step evaluation process that entails gathering pertinent information, identifying potential alternatives, and assessing workable solutions to aid decision-makers in making more deliberate and meaningful decisions (Chen 2022a, b; Tsao and Chen 2022). In particular, the technique for order preference by similarity to ideal solutions (TOPSIS), which originated from Hwang and Yoon (1981), is a useful prioritization method in important fields of management analysis and decision theory (Alshammari et al. 2022; Yang et al. 2022). TOPSIS exploits a compensatory aggregation approach that is predicated on the geometric distances between performance ratings associated with candidate alternatives and ideal/anti-ideal alternatives (composed of the best/worst performance ratings in terms of each criterion) to assess the relative merits of candidate alternatives (Guan 2022; Yang et al. 2022; Zhang et al. 2022).

The theory of circular intuitionistic fuzzy (C-IF) sets, propounded by Atanassov (2020), is a forward-looking generalization of the intuitionistic fuzzy (IF) theory. In contrast to the IF conformation, an element in the C-IF set is represented by an adjustable circle with a radius parameter, with the center of the circle consisting of membership (i.e., validity) and nonmembership (i.e., non-validity) (Atanassov and Marinov 2021; Çakır et al. 2021). As an illustration in a two-dimensional space, the C-IF conformation is delineated as a high-order uncertain set in which elements in a given finite universe of discourse enjoy the degrees of membership and nonmembership, surrounded by a circle of radius parameter in such a manner that the degrees of membership plus nonmembership do not exceed 1 within this circle (Boltürk and Kahraman 2022; Kahraman and Alkan 2021; Otay and Kahraman 2021). C-IF sets can be efficaciously utilized in MCDA domains because their distinct characteristics can significantly manipulate convolutedly ambiguous information and assist existing MCDA models in obtaining more accurate outcomes, such as a C-IF vlsekriterijumska optimizacija i kompromisno resenje (VIKOR) to treat a waste disposal location selection issue (Kahraman and Otay 2022), an integrated C-IF analytic hierarchy process (AHP) with VIKOR for a multiple expert supplier evaluation problem (Otay and Kahraman 2021), a C-IF decision-making approach through the medium of new defuzzification functions for selecting a health tourism center (Çakır and Taş 2021), a defuzzification and score function-based C-IF decision method for a landfill site selection issue (Çakır et al. 2021), an assessment technique predicated on IF and C-IF sets for evaluating human capital and research indices (Imanov and Aliyev 2021), a present-worth-analysis method grounded in interval-valued IF and C-IF sets for analyzing a water treatment device task (Boltürk and Kahraman 2022), an integrated C-IF AHP approach for assessing remote work adaptation in the midst of the Coronavirus Disease-2019 (COVID-19) pandemic (Çakır and Taş 2022), and an assessment procedure involving C-IF information for selecting industrial symbiotic enterprises (Çakır et al. 2022).

The TOPSIS methodology can serve as a powerful prioritization technique to aid decision support for MCDA matters in the context of C-IF sets. With this in mind, Kahraman and Alkan (2021) devised a new C-IF TOPSIS approach based on vague membership functions and utilized it to a supplier selection problem. In the same vein, Alkan and Kahraman (2022b) developed an akin C-IF TOPSIS framework and delivered an implementation procedure to address epidemiological hospital site selection using pessimistic and optimistic decision matrices. Even though Kahraman and Alkan (2021) and Alkan and Kahraman (2022b) proffered the latest C-IF TOPSIS methods, the application of the core TOPSIS framework to C-IF scenarios still has some limitations and research gaps. First, apart from the two studies mentioned above, there is currently little research that extends the classic TOPSIS architecture within the C-IF decision environment for developing a compromising MCDA method involving C-IF uncertainty. Furthermore, the two current studies deviated significantly from the core architecture of the classic TOPSIS because of many modifications. As a result, how to extend the classic TOPSIS into C-IF backgrounds so that the development can faithfully demonstrate the TOPSIS compromise decision-making spirit has become a significant area of study, which constructs the first motivation.

Furthermore, in the C-IF TOPSIS methods proffered by Kahraman and Alkan (2021) and Alkan and Kahraman (2022b), the C-IF information (including criterion weights and performance ratings of alternatives judged against the criterion) was converted into IF values for use in the calculations of their developed C-IF TOPSIS procedures. By way of illustration, Kahraman and Alkan (2021) and Alkan and Kahraman (2022b) built individual IF decision matrices and IF criterion weights to generate an aggregated IF decision matrix and aggregated IF weights, respectively. The C-IF decision matrix and C-IF weights were then generated using the maximum radius lengths. However, after putting in a lot of time and effort to create the C-IF decision matrix, it was immediately transformed into two IF decision matrices, pessimistic and optimistic. The C-IF decision matrix was specifically transformed into pessimistic and optimistic decision matrices, with the data in these two decision matrices being ordinary IF values. The relative closeness coefficients based on the pessimistic and optimistic matrices were then calculated to generate composite ratio scores for ranking the alternatives. The C-IF data were constructed in terms of IF linguistic information by Kahraman and Alkan (2021) and Alkan and Kahraman (2022b), but they were in turn converted to IF values. Membership and nonmembership, plus or minus the radius parameter of C-IF values, constitute the IF values embedded in the pessimistic and optimistic decision matrices. The C-IF set containing higher-order fuzzy information is transformed into a general IF set using this simplified pessimistic and optimistic estimation procedure. However, the original goal of using C-IF sets to clarify complex uncertainties is lost in this process. The manipulation process reduces the specificity of C-IF information, reducing flexibility and adaptability when dealing with C-IF uncertainty. Conflicting data processing procedures in the two existing C-IF TOPSIS methods create another research limitation and gap, which serves as the foundation for the second motivation.

Aside from that, the distances between target alternatives and ideal/anti-ideal choices are the core concept of TOPSIS, as this is the primary measure of relative closeness coefficients (Alkan and Kahraman 2022a; Han et al. 2022; Sadabadi et al. 2022). Because of this, in C-IF uncertain circumstances, distance measurement is the most important key element in the development of extended TOPSIS. Distance is critical in almost all application problems related to discriminating decision information and comparing performance (Alkan and Kahraman 2022a; Deng and Chen 2022; Szmidt 2014), especially in uncertain circumstances such as the center of gravity distance for fuzzy numbers (Das et al. 2019), the distance and similarity measures for IF sets (Garg and Kumar 2018; Garg and Rani 2021), the exponential distance for interval-valued IF sets (Garg and Kumar 2020), and the distance measures for type 2 IF sets (Garg and Singh 2020; Singh and Garg 2017). Distance metrics, in particular, are not only widely used in decision theory and analysis, but are also crucial in a variety of important domains such as database management, machine learning, knowledge discovery and data mining, information theory, logistics, computer-aided manufacturing, and control theory (Szmidt 2014; Talukdar and Dutta 2021; Tiwari and Gupta 2022). However, the distance measures defined in IF or type-2 IF sets cannot be applied or extended to C-IF environments. The main reason for this is that previous distance measures did not consider intuitionistic ambiguity data for circular structures, nor did they consider the radius of the circular ambiguity range. Accordingly, establishing a metric space is an essential and critical issue in C-IF contexts. Metric space is a fundamental concept in mathematics; its metric is a distance function that can delineate the measure of separation between any two C-IF information within the C-IF environments. As a result, Atanassov and Marinov (2021) propounded four new distance functions for C-IF sets consisting of the three-term based Hamming distance derived from IF Hamming distance, the four-term based Hamming distance stemmed from Szmidt and Kacprzyk's (2000) form of IF Hamming distance, the three-term based Euclidean distance derived from IF Euclidean metric, and the four-term based Euclidean distance stemmed from Szmidt and Kacprzyk's (2000) form of IF Euclidean distance. The four new C-IF distance metrics are easy to use, but they have some limitations. To begin with, Atanassov and Marinov's C-IF distance formulas are only applicable when the radii of the C-IF values in a single C-IF set must be the same length. The advantages of applying the C-IF theory to non-homogeneous uncertainty are diminished by this supposition. Moreover, this supposition restricts the generality and flexibility of C-IF decision information and reduces the effectiveness of applying C-IF theory to real-world issues. The applicability of the C-IF distance formula proposed by Atanassov and Marinov (2021) is therefore severely constrained under such an assumption. Second, the Manhattan metric model and the Euclidean metric model serve as the foundation for the four new distance formulas. However, the Chebyshev metric model, which is commonly used in practical problems, is not covered in the existing literature. Furthermore, generalized forms of C-IF distance measures, such as Minkowski distance (which can be thought of as a generalization of Manhattan distance and Euclidean distance), are not studied in the current C-IF literature. The aforementioned constraints have resulted in research gaps that must be filled as soon as possible, forming the third motivation.

Given the research gaps and motivations, the overall goal of this study is to develop advanced C-IF Minkowski distance measures and to deliver an evolved C-IF TOPSIS methodology to manage the MCDA problem in the presence of C-IF uncertainty. Measuring the distance between C-IF values or C-IF sets is critical for quantifying the separation between C-IF uncertain data and distinguishing C-IF information. To manipulate indistinct and imperfect information in the C-IF TOPSIS procedure, decision-makers would have to employ appropriate C-IF distance metrics to clarify the information content and take befitting measurements to process and differentiate the performance information. The C-IF distance formulas proposed by Atanassov and Marinov (2021) do not account for the fact that elements in the C-IF set may have different radii. These assumptions restrict the applicability of their C-IF distance formulations. Further to that, Atanassov and Marinov's distance formulas only exploited the Manhattan and Euclidean distance models; the Chebyshev distance model, which is commonly used in practical problems, was not discussed. This research establishes two general C-IF distance metrics that cover the existing four C-IF distance metrics, to address the limitations of the existing C-IF distance metrics and increment their flexibility. Based on three- and four-term approaches, this research illustrates generalized representations of two types of C-IF distance measures, dubbed C-IF Minkowski distance measures. The four-term representation takes into account the C-IF set's radius, membership, nonmembership, and hesitation components; additionally, the three-term representation indicates that the radius, membership, and nonmembership components are considered. Moreover, this study explores certain significant properties of these two evolved C-IF Minkowski distance measures. This study constructs an easy-to-operate C-IF TOPSIS technique based on four basic architectures, including four combinations of displaced and fixed anchoring frameworks and using three- and four-term-based C-IF Minkowski metrics, to advance a C-IF TOPSIS methodology for treating the MCDA issue by incorporating the C-IF Minkowski distance measures. The anchoring mechanism includes displaced (or fixed) ideal and anti-ideal C-IF characteristics that can be employed as anchor points in the C-IF TOPSIS process to formulate subsequent compromise indicators. The displaced (or fixed) anchoring mechanism consists of identifying displaced (or fixed) ideal ratings and displaced (or fixed) anti-ideal ratings, constructing displaced (or fixed) ideal and anti-ideal C-IF characteristics, and determining relative closeness coefficients using C-IF Minkowski distance measures with the three- or four-term representation frame. By comparing the relative closeness coefficients, the best compromise solution to support intelligent decision analysis in complex and uncertain environments can be determined. Finally, this study employs the evolved C-IF TOPSIS methodology to investigate the assessment and selection of appropriate sites for large-scale epidemic hospitals. This research conducts a comparative analysis and explores the applied results in comparison to other approaches to validate the applicability, robustness, and flexibility of the current C-IF TOPSIS technique.

The remainder of this paper is structured as follows. Section 2 characterizes some fundamental expressions of C-IF sets. Section 3 presents two evolved C-IF Minkowski distance measures based on three- and four-term approaches to differentiating C-IF information. Section 4 devotes an efficacious C-IF TOPSIS methodology to multiple criteria analysis for decision support. Section 5 explores a realistic application for epidemic hospital siting to illustrate the proposed executive procedure, and comparative research and discussions are conducted to substantiate the strengths of the initiated methodology. Section 6 presents conclusive outcomes as well as promising future research questions.

2 Elementary definitions of C-IF sets

The C-IF set concept generalizes the IF set, which uses a circle to express the object's uncertainty, with the center of the circle consisting of membership and nonmembership degrees. Preliminary definitions of IF sets and C-IFS are presented briefly in this section.

Definition 1 (*Atanassov* 1986) Let Θ be a symbol for a finite universe of discourse. An IF set ϑ is defined in Θ and explained in the following format:

$$\vartheta = \left\{ \langle \theta, m_\vartheta(\theta), n_\vartheta(\theta) \rangle | \theta \in \Theta \right\} \tag{1}$$

where the degree of membership (i.e., validity) $m_\vartheta(\theta) : \Theta \to [0, 1]$, the degree of nonmembership (i.e., non-validity) $n_\vartheta(\theta) : \Theta \to [0, 1]$, and $m_\vartheta(\theta) + n_\vartheta(\theta) \leq 1$ for each element $\theta \in \Theta$ associated with ϑ. The degree of hesitancy (i.e., indeterminacy) corresponding to the IF value $(m_\vartheta(\theta), n_\vartheta(\theta))$ is generated as $h_\vartheta(\theta) = 1 - m_\vartheta(\theta) - n_\vartheta(\theta)$.

Definition 2 (*Atanassov* 2020*; Atanassov and Marinov* 2021) Allow $\mathcal{L}^* = \left\{ \langle \zeta, \zeta' \rangle \middle| \zeta, \zeta' \in [0,1] \text{and} \zeta + \zeta' \leq 1 \right\}$ to be an indication of an L-fuzzy set. A C-IF set $\mathcal{C}$ that has the following format in a given finite universe of discourse Θ:

$$\mathcal{C} = \left\{ \langle \theta, m_{\mathcal{C}}(\theta), n_{\mathcal{C}}(\theta); r_{\mathcal{C}}(\theta) \rangle | \theta \in \Theta \right\} = \left\{ \langle \theta, O_r\left(m_{\mathcal{C}}(\theta), n_{\mathcal{C}}(\theta) \right) \rangle | \theta \in \Theta \right\} \tag{2}$$

where the degree of membership (i.e., validity) $m_{\mathcal{C}}(\theta) : \Theta \to [0,1]$, the degree of non-membership (i.e., non-validity) $n_{\mathcal{C}}(\theta) : \Theta \to [0,1]$, and $m_{\mathcal{C}}(\theta) + n_{\mathcal{C}}(\theta) \leq 1$ for each $\theta \in \Theta$ belonging to $\mathcal{C}$. Moreover, the function O_r externalizes a circle, whose radius is $r_{\mathcal{C}}(\theta) : \Theta \to [0, \sqrt{2}]$ and whose center is $(m_{\mathcal{C}}(\theta), n_{\mathcal{C}}(\theta))$; the expression is as follows:

$$\begin{aligned} O_r\left(m_{\mathcal{C}}(\theta), n_{\mathcal{C}}(\theta) \right) &= \left\{ \zeta, \zeta' \middle| \zeta, \zeta' \in [0,1] \text{ and } \sqrt{\left(m_{\mathcal{C}}(\theta) - \zeta \right)^2 + \left(n_{\mathcal{C}}(\theta) - \zeta' \right)^2} \leq r_{\mathcal{C}}(\theta) \right\} \cap \mathcal{L}^* \\ &= \left\{ \zeta, \zeta' \middle| \zeta, \zeta' \in [0,1], \ \sqrt{\left(m_{\mathcal{C}}(\theta) - \zeta \right)^2 + \left(n_{\mathcal{C}}(\theta) - \zeta' \right)^2} \leq r_{\mathcal{C}}(\theta), \text{ and } \zeta + \zeta' \leq 1 \right\} \end{aligned} \tag{3}$$

The degree of hesitancy (i.e., indeterminacy) for the pair $(m_{\mathcal{C}}(\theta), n_{\mathcal{C}}(\theta))$ is produced in this fashion:

$$h_{\mathcal{C}}(\theta) = 1 - m_{\mathcal{C}}(\theta) - n_{\mathcal{C}}(\theta). \tag{4}$$

Definition 3 (*Atanassov* 2020*; Boltürk and Kahraman* 2022) Let $\varsigma_\iota = (m_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_\iota}(\theta); r_{\mathcal{C}_\iota}(\theta))$ and $\varsigma_{\iota'} = (m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota'}}(\theta); r_{\mathcal{C}_{\iota'}}(\theta))$ signify two C-IF values embedded in the C-IF sets $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$, respectively. Fundamental operational laws of intersection, union, addition, and multiplication involving the min and max types, as well as multiplication by and power of a scalar $\varpi \geq 0$, are stated in this order:

$$\varsigma_\iota \cap_{\min} \varsigma_{\iota'} = \left(\min\left\{ m_{\mathcal{C}_\iota}(\theta), m_{\mathcal{C}_{\iota'}}(\theta) \right\}, \max\left\{ n_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_{\iota'}}(\theta) \right\}; \min\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{5}$$

$$\varsigma_\iota \cap_{\max} \varsigma_{\iota'} = \left(\min\left\{ m_{\mathcal{C}_\iota}(\theta), m_{\mathcal{C}_{\iota'}}(\theta) \right\}, \max\left\{ n_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_{\iota'}}(\theta) \right\}; \max\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{6}$$

$$\varsigma_\iota \cup_{\min} \varsigma_{\iota'} = \left(\max\left\{ m_{\mathcal{C}_\iota}(\theta), m_{\mathcal{C}_{\iota'}}(\theta) \right\}, \min\left\{ n_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_{\iota'}}(\theta) \right\}; \min\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{7}$$

$$\varsigma_\iota \cup_{\max} \varsigma_{\iota'} = \left(\max\left\{ m_{\mathcal{C}_\iota}(\theta), m_{\mathcal{C}_{\iota'}}(\theta) \right\}, \min\left\{ n_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_{\iota'}}(\theta) \right\}; \max\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{8}$$

$$\varsigma_\iota \oplus_{\min} \varsigma_{\iota'} = \left(m_{\mathcal{C}_\iota}(\theta) + m_{\mathcal{C}_{\iota'}}(\theta) - m_{\mathcal{C}_\iota}(\theta) \cdot m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_\iota}(\theta) \cdot n_{\mathcal{C}_{\iota'}}(\theta); \min\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{9}$$

$$\varsigma_\iota \oplus_{\max} \varsigma_{\iota'} = \left(m_{\mathcal{C}_\iota}(\theta) + m_{\mathcal{C}_{\iota'}}(\theta) - m_{\mathcal{C}_\iota}(\theta) \cdot m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_\iota}(\theta) \cdot n_{\mathcal{C}_{\iota'}}(\theta); \max\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{10}$$

$$\varsigma_\iota \otimes_{\min} \varsigma_{\iota'} = \left(m_{\mathcal{C}_\iota}(\theta) \cdot m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_\iota}(\theta) + n_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_\iota}(\theta) \cdot n_{\mathcal{C}_{\iota'}}(\theta); \min\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{11}$$

$$\varsigma_\iota \otimes_{\max} \varsigma_{\iota'} = \left(m_{\mathcal{C}_\iota}(\theta) \cdot m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_\iota}(\theta) + n_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_\iota}(\theta) \cdot n_{\mathcal{C}_{\iota'}}(\theta); \max\left\{ r_{\mathcal{C}_\iota}(\theta), r_{\mathcal{C}_{\iota'}}(\theta) \right\} \right) \tag{12}$$

$$\varpi \odot \varsigma_\iota = \left(1-(1-m_{\mathcal{C}_\iota}(\theta))^{\varpi}, (n_{\mathcal{C}_\iota}(\theta))^{\varpi}; r_{\mathcal{C}_\iota}(\theta)\right) \tag{13}$$

$$(\varsigma_\iota)^{\varpi} = \left((m_{\mathcal{C}_\iota}(\theta))^{\varpi}, 1-(1-n_{\mathcal{C}_\iota}(\theta))^{\varpi}; r_{\mathcal{C}_\iota}(\theta)\right) \tag{14}$$

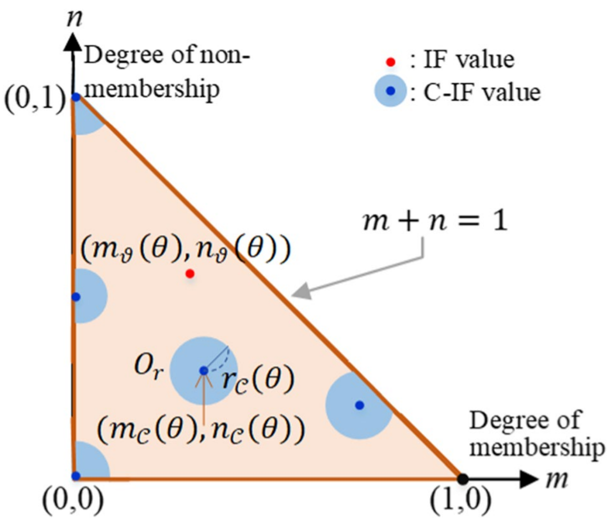
The C-IF set serves as a comprehensive framework for the IF set, the geometric representation of which is sketched in Fig. 1. The graph depicts the distribution space of IF and C-IF values for a triangle in the first quadrant with vertices (0, 0), (1, 0), and (0, 1). The IF value is explained by the point in the triangle with coordinates $(m_\vartheta(\theta), n_\vartheta(\theta))$, while the C-IF value is explained by the circle O_r with center $(m_\mathcal{C}(\theta), n_\mathcal{C}(\theta))$ and radius $r_\mathcal{C}(\theta)$. As defined in Eq. (3), $O_r(m_\mathcal{C}(\theta), n_\mathcal{C}(\theta))$ must satisfy the constraints in $\mathcal{L}^*$, so the circle O_r can take five different forms, as portrayed in the graph. In the event that $r_\mathcal{C}(\theta) = 0$ for all $\theta \in \Theta$, the C-IF set $\mathcal{C}$ composed of C-IF values will degenerate into an IF set; that is, $\mathcal{C} = \left\{\langle \theta, O_0(m_\mathcal{C}(\theta), n_\mathcal{C}(\theta))\rangle | \theta \in \Theta\right\} = \left\{\langle \theta, m_\vartheta(\theta), n_\vartheta(\theta)\rangle | \theta \in \Theta\right\} = \vartheta$.

Definition 4 (*Atanassov and Marinov* 2021) Let *card*(Θ) signify the cardinality of a given finite universe of discourse Θ. Consider two C-IF sets defined in Θ, and assume that $r_{\mathcal{C}_\iota}(\theta) = r_{\mathcal{C}_\iota}$ and $r_{\mathcal{C}_{\iota'}}(\theta) = r_{\mathcal{C}_{\iota'}}$ for all $\theta \in \Theta$, i.e., $\mathcal{C}_\iota = \left\{\langle \theta, m_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_\iota}(\theta); r_{\mathcal{C}_\iota}\rangle | \theta \in \Theta\right\}$ and $\mathcal{C}_{\iota'} = \left\{\langle \theta, m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota'}}(\theta); r_{\mathcal{C}_{\iota'}}\rangle | \theta \in \Theta\right\}$. The Manhattan (so-called Hamming) distance measures between two C-IF sets $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$ based on the three- and four-term approaches are defined in the following order:

$$\mathfrak{D}^M_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \frac{1}{2}\left(\frac{\left|r_{\mathcal{C}_\iota} - r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}} + \frac{1}{2\cdot card(\Theta)}\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\right) \tag{15}$$

$$\mathfrak{D}^M_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \frac{1}{2}\left(\frac{\left|r_{\mathcal{C}_\iota} - r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}} + \frac{1}{2\cdot card(\Theta)}\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\right) \tag{16}$$

Fig. 1 Geometrical representation of a C-IF set



Furthermore, the Euclidean distance measures between $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$ based on the three- and four-term approaches are described in the following order:

$$\mathfrak{D}^E_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \frac{1}{2}\left[\frac{\left|r_{\mathcal{C}_\iota} - r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}} + \left(\frac{1}{2 \cdot card(\Theta)}\sum_{\theta\in\Theta}\left((m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta))^2 + (n_{\mathcal{C}_\iota}(\theta) \ - n_{\mathcal{C}_{\iota'}}(\theta))^2\right)\right)^{\frac{1}{2}}\right] \tag{17}$$

$$\mathfrak{D}^E_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \frac{1}{2}\left[\frac{\left|r_{\mathcal{C}_\iota} - r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}} + \left(\frac{1}{2 \cdot card(\Theta)}\sum_{\theta\in\Theta}\left((m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta))^2 + (n_{\mathcal{C}_\iota}(\theta) \ - n_{\mathcal{C}_{\iota'}}(\theta))^2 + (h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta))^2\right)\right)^{\frac{1}{2}}\right] \tag{18}$$

3 Evolved C-IF Minkowski distance measures

This section will look into new generalized representations of distance metrics that are suitable to the context of C-IF sets. A metric (topological) space is treated as a fundamental space satisfying certain axioms (Das et al. 2019; Garg and Singh 2020; Singh and Garg 2017). Measuring the distance between C-IF uncertain sets is crucial for quantifying the degree of separation between C-IF sets and differentiating C-IF information. Furthermore, the C-IF distance metric can furnish establish more structure than the general topological space. To successfully manage imperfect or uncertain information, decision-makers need to exploit a suitable model to elucidate the information content, and appropriate measures to process and distinguish the information. Among these measures, the notion of distance metric epitomizes a leading role (Garg and Kumar 2018, 2020; Garg and Rani 2021). Atanassov and Marinov (2021) established four fresh distance measures applicable to the C-IF environment; however, they assumed that the radii associated with the two C-IF sets used to calculate the distance are the same. Specifically, Atanassov and Marinov established the special assumption that all elements within a specific C-IF set have the same radius, a narrow assumption that limits the applicability of the four new distance measures. In addition, their distance formulas do not take into account an all-encompassing distance measurement of the general type. This section will propose two general C-IF distance metrics, covering Atanassov and Marinov's four C-IF distance measures, to loosen the restrictions on the current C-IF distance measures and increase their resilience.

This study considers two types of generalized representations of C-IF distance measures (see Fig. 2), based on the three- and four-term approaches, known as C-IF Minkowski distance measures. To be more specific, the four-term representation takes into account the radius, membership, nonmembership, and hesitation components that characterize C-IF sets. The three-term representation takes into account only the radius, membership, and nonmembership components. This study also discusses the key properties of these two new Minkowski metrics.

As previously stated, Atanassov and Marinov (2021) unfolded four new distance metrics for quantifying the degree of separation between C-IF sets. The four new C-IF distance metrics are very useful and easy to operate, but they do have some limitations. For starters, Atanassov and Marinov's C-IF distance formulas ignore the fact that elements in a C-IF set may have different radii. As shown in the formulas of Definition 4, the radius of all elements in the C-IF set $\mathcal{C}_\iota$ is equal to $r_{\mathcal{C}_\iota}$ (i.e., $r_{\mathcal{C}_\iota}(\theta) = r_{\mathcal{C}_\iota}$ for each $\theta \in \Theta$); similarly, the radius of all elements in the C-IF set $\mathcal{C}_{\iota'}$ is equal to $r_{\mathcal{C}_{\iota'}}$ (i.e., $r_{\mathcal{C}_{\iota'}}(\theta) = r_{\mathcal{C}_{\iota'}}$ for each $\theta \in \Theta$). Such assumptions are so strong that the applicability of Definition 4 is severely limited.

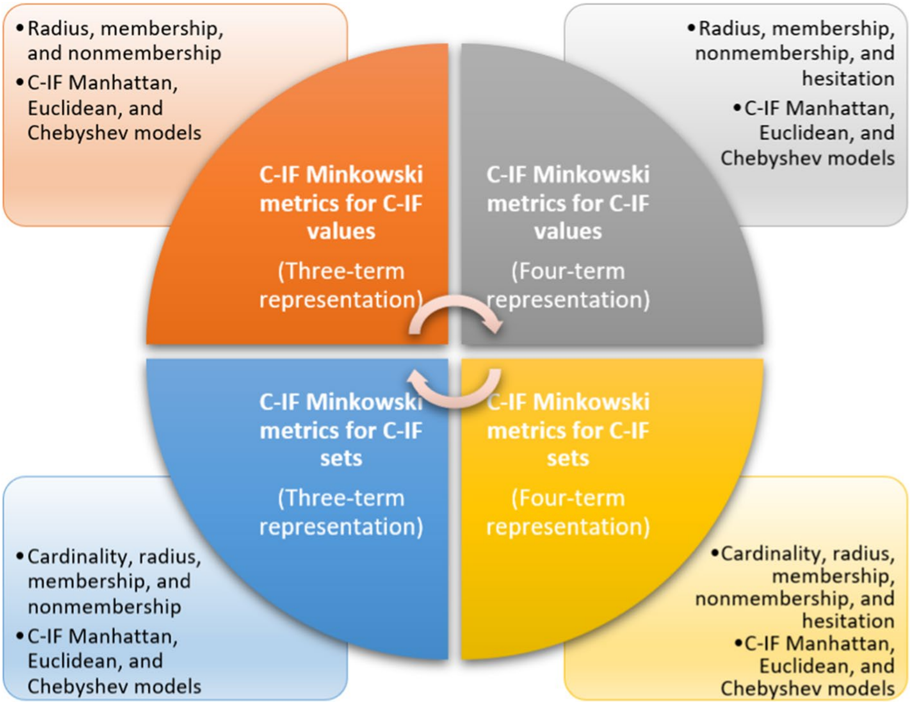


Fig. 2 Proposed generalized representations of C-IF distance measures

Second, the four distance formulas are the Manhattan distance measures using the three- and four-term approaches, and the Euclidean distance measures using the three- and four-term approaches. The Chebyshev distance metrics, which are commonly used in practical problems, are, however, not discussed in the existing literature. To relax the constraints of the current C-IF distance measures and extend their resiliency, this study will put forward two general C-IF distance metrics, covering the foregoing four C-IF distance measures in Definition 4, to provide the metric resiliency and broaden the applicability of decision analysis. In the following, this study exploits the three- and four-term representations to present the two C-IF Minkowski metrics for C-IF values, revealed in the first subsection below. Following this, the second subsection delineates the two C-IF Minkowski metrics for C-IF sets based on the three- and four-term representations.

3.1 C-IF Minkowski metrics for C-IF values

Definition 5 Let a positive integer β indicate the metric parameter in C-IF settings, where $\beta \in Z^{+}$. Consider any two C-IF values $\varsigma_{\iota} = (m_{C_{\iota}}(\theta), n_{C_{\iota}}(\theta); r_{C_{\iota}}(\theta))$ and $\varsigma_{\iota'} = (m_{C_{\iota'}}(\theta), n_{C_{\iota'}}(\theta); r_{C_{\iota'}}(\theta))$. The C-IF Minkowski distance measures separating ς_{ι} and $\varsigma_{\iota'}$ based on the three- and four-term approaches are elucidated in this order:

$$\mathcal{D}^{\beta}_{(3)}(\varsigma_{\iota}, \varsigma_{\iota'}) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2}}\left|r_{C_{\iota}}(\theta) - r_{C_{\iota'}}(\theta)\right| + \left[\frac{1}{2}\left(\left|m_{C_{\iota}}(\theta) - m_{C_{\iota'}}(\theta)\right|^{\beta} + \left|n_{C_{\iota}}(\theta) - n_{C_{\iota'}}(\theta)\right|^{\beta}\right)\right]^{\frac{1}{\beta}} \right\} \quad (19)$$

$$\mathcal{D}^{\beta}_{(4)}(\varsigma_\iota,\varsigma_{\iota'})=\frac{1}{2}\left\{\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\left[\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|^{\beta}+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|^{\beta}\quad+\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|^{\beta}\right)\right]^{\frac{1}{\beta}}\right\} \tag{20}$$

Remark 1 In most cases, the C-IF Minkowski distance measures $\mathcal{D}^{\beta}_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$ and $\mathcal{D}^{\beta}_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$ can be utilized with a metric parameter β of 1 or 2, which are equivalent to the C-IF Manhattan and C-IF Euclidean distances between the two C-IF values ς_ι and $\varsigma_{\iota'}$, respectively. Using the three- and four-term representations, the C-IF Manhattan distance measures are portrayed this wise:

$$\mathcal{D}^{1}_{(3)}\left(\varsigma_\iota,\varsigma_{\iota'}\right)=\frac{1}{2}\left[\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\right] \tag{21}$$

$$\begin{aligned}\mathcal{D}^{1}_{(4)}\left(\varsigma_\iota,\varsigma_{\iota'}\right)=\frac{1}{2}\Bigg[&\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\frac{1}{2}\Big(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|\\&+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\Big)\Bigg]\end{aligned} \tag{22}$$

Moreover, the C-IF Euclidean distance measures are exhibited in this fashion:

$$\mathcal{D}^{2}_{(3)}\left(\varsigma_\iota,\varsigma_{\iota'}\right)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\sqrt{\frac{1}{2}\left[\left(m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right)^2+\left(n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right)^2\right]}\right)(23) \tag{23}$$

$$\begin{aligned}&\mathcal{D}^{2}_{(4)}\left(\varsigma_\iota,\varsigma_{\iota'}\right)\\&=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\sqrt{\frac{1}{2}\left[\left(m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right)^2+\left(n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right)^2+\left(h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right)^2\right]}\right)\end{aligned} \tag{24}$$

In the limiting situation of β reaching infinity, the C-IF Minkowski distance measures correspond to the C-IF Chebyshev distance measures between ς_ι and $\varsigma_{\iota'}$, as demonstrated:

$$\mathcal{D}^{\infty}_{(3)}\left(\varsigma_\iota,\varsigma_{\iota'}\right)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\max\left\{\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}\right) \tag{25}$$

$$\begin{aligned}&\mathcal{D}^{\infty}_{(4)}\left(\varsigma_\iota,\varsigma_{\iota'}\right)\\&=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|+\max\left\{\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|,\ \left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}\right)\end{aligned} \tag{26}$$

where $\mathcal{D}^{\infty}_{(3)}$ and $\mathcal{D}^{\infty}_{(4)}$ indicate $\lim_{\beta\to\infty}\mathcal{D}^{\beta}_{(3)}$ and $\lim_{\beta\to\infty}\mathcal{D}^{\beta}_{(4)}$, respectively.$\mathcal{D}^{\beta}_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$

Example 1 Consider two C-IF values ς_ι =(0.37, 0.13; 1) and $\varsigma_{\iota'}$ =(0.51, 0.24; $\sqrt{2}$). Following the three-term approach, the C-IF Manhattan distance $\mathcal{D}^{1}_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$, C-IF Euclidean distance $\mathcal{D}^{2}_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$, and C-IF Chebyshev distance $\mathcal{D}^{\infty}_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$ can be determined using Eqs. (21), (23), and (25), respectively. Based on the four-term approach, the three

Table 1 Computational illustration of the C-IF distances in Example 1

Approach	β	Demonstration of the calculation process
Three-term	1	$\mathcal{D}^{1}_{(3)}(\varsigma_\iota, \varsigma_{\iota'}) = (1/2)[(1/\sqrt{2})\left\|1-\sqrt{2}\right\| + (1/2)(\|0.37-0.51\| + \|0.13-0.24\|)] = 0.2089$
	2	$\mathcal{D}^{2}_{(3)}(\varsigma_\iota, \varsigma_{\iota'}) = (1/2)\{(1/\sqrt{2})\left\|1-\sqrt{2}\right\| + \sqrt{(1/2)[(0.37-0.51)^2 + (0.13-0.24)^2]}\} = 0.2094$
	∞	$\mathcal{D}^{\infty}_{(3)}(\varsigma_\iota, \varsigma_{\iota'}) = (1/2)[(1/\sqrt{2})\left\|1-\sqrt{2}\right\| + \max\{\|0.37-0.51\|, \|0.13-0.24\|\}] = 0.2164$
Four-term	1	$\mathcal{D}^{1}_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) = (1/2)[(1/\sqrt{2})\left\|1-\sqrt{2}\right\| + (1/2)(\|0.37-0.51\| + \|0.13-0.24\| + \|0.50-0.25\|)] = 0.2714$
	2	$\mathcal{D}^{2}_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) = \left(\frac{1}{2}\right)\left\{\left(\frac{1}{\sqrt{2}}\right)\left\|1-\sqrt{2}\right\| + \sqrt{\left(\frac{1}{2}\right)[(0.37-0.51)^2 + (0.13-0.24)^2 + (0.50-0.25)^2]}\right\} = 0.2550$
	∞	$\mathcal{D}^{\infty}_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) = (1/2)[(1/\sqrt{2})\left\|1-\sqrt{2}\right\| + \max\{\|0.37-0.51\|, \|0.13-0.24\|, \|0.50-0.25\|\}] = 0.2714$

C-IF distances $\mathcal{D}^1_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$, $\mathcal{D}^2_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$, and $\mathcal{D}^\infty_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$ are derived using Eqs. (22), (24), and (26), respectively. The relevant calculation process and results are depicted in Table 1. As revealed in the table, the C-IF distances determined by the three- and four-term approaches fulfill the relationships $\mathcal{D}^\infty_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) > \mathcal{D}^2_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) > \mathcal{D}^1_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$ and $\mathcal{D}^\infty_{(4)}(\varsigma_\iota,\varsigma_{\iota'}) = \mathcal{D}^1_{(4)}(\varsigma_\iota,\varsigma_{\iota'}) > \mathcal{D}^2_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$, respectively. Regardless of β=1, β=2, or $\beta \to \infty$, the C-IF distances generated by the four-term approach are greater than those yielded by the three-term approach, specifically, $\mathcal{D}^1_{(4)}(\varsigma_\iota,\varsigma_{\iota'}) > \mathcal{D}^1_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$, $\mathcal{D}^2_{(4)}(\varsigma_\iota,\varsigma_{\iota'}) > \mathcal{D}^2_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$, and $\mathcal{D}^\infty_{(4)}(\varsigma_\iota,\varsigma_{\iota'}) > \mathcal{D}^\infty_{(3)}(\varsigma_\iota,\varsigma_{\iota'})$. More exploration of the relationship between these distances will be discussed systematically in the following theorems.

Theorem 1 *Based on the three-term representation, the C-IF Minkowski distance measure $\mathcal{D}^\beta_{(3)}$ for any three* C-*IF values ς_ι, $\varsigma_{\iota'}$, and $\varsigma_{\iota\varepsilon}$ fulfills the following properties*:

(T1.1) Non-negativity $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) \geq 0$ for all $\beta \in Z^+$;

(T1.2) Boundedness $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) \leq 1$ for all $\beta \in Z^+$;

(T1.3) Reflexivity $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_\iota) = 0$ for all $\beta \in Z^+$;

(T1.4) Symmetry $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) = \mathcal{D}^\beta_{(3)}(\varsigma_{\iota'},\varsigma_\iota)$ for all $\beta \in Z^+$;

(T1.5) Separability $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) = 0$ for all $\beta \in Z^+$ if and only if $\varsigma_\iota = \varsigma_{\iota'}$;

(T1.6) Triangle inequality $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) \leq \mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota\varepsilon}) + \mathcal{D}^\beta_{(3)}(\varsigma_{\iota\varepsilon},\varsigma_{\iota'})$ when $\beta = 1$ and $\beta \to \infty$.

Proof (T1.1) Because of the absolute value properties, the non-negative property can be deduced immediately.

(T1.2) As stated in Definition 2, $r_\mathcal{C}(\theta) : \Theta \to [0,\sqrt{2}]$ and $m_\mathcal{C}(\theta), n_\mathcal{C}(\theta) : \Theta \to [0,1]$ are known. These axiomatic conditions lead to the conclusion that $|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)| \leq \sqrt{2}$, $|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)| \leq 1$, and $|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)| \leq 1$. It is a direct result of the inequality $|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)|^\beta + |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)|^\beta \leq 2$. As a result, it concludes that $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) \leq (1/2)\{(1/\sqrt{2})\cdot\sqrt{2} + [(1/2)\cdot 2]^{(1/\beta)}\} = 1$ for all $\beta \in Z^+$.

(T1.3) and (T1.4) are trivially correct.

(T1.5) If $\varsigma_\iota = \varsigma_{\iota'}$, one can obtain $m_{\mathcal{C}_\iota}(\theta) = m_{\mathcal{C}_{\iota'}}(\theta)$,$n_{\mathcal{C}_\iota}(\theta) = n_{\mathcal{C}_{\iota'}}(\theta)$, and $r_{\mathcal{C}_\iota}(\theta) = r_{\mathcal{C}_{\iota'}}(\theta)$, and therefore,$\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) = 0$ for all $\beta \in Z^+$. Conversely, if $\mathcal{D}^\beta_{(3)}(\varsigma_\iota,\varsigma_{\iota'}) = 0$ for all $\beta \in Z^+$, the three equations $|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)| = 0$, $|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)| = 0$, and $|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)| = 0$ should be satisfied. As a result, one can directly conclude that $(m_{\mathcal{C}_\iota}(\theta), n_{\mathcal{C}_\iota}(\theta); r_{\mathcal{C}_\iota}(\theta)) = (m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota'}}(\theta); r_{\mathcal{C}_{\iota'}}(\theta))$. The sufficiency and necessity for separability are confirmed, confirming the validity of (T1.5).

(T1.6) Concerning the three radii $r_{\mathcal{C}_\iota}(\theta)$, $r_{\mathcal{C}_{\iota'}}(\theta)$, and $r_{\mathcal{C}_{\iota\varepsilon}}(\theta)$ associated with the C-IF values ς_ι, $\varsigma_{\iota'}$, and $\varsigma_{\iota\varepsilon}$, respectively, the following inequality can be generated:

$$\left|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)\right| = \left|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota\varepsilon}}(\theta) + r_{\mathcal{C}_{\iota\varepsilon}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)\right| \leq \left|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right| + \left|r_{\mathcal{C}_{\iota\varepsilon}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)\right|$$

It immediately follows that $(1/\sqrt{2}) \cdot |r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)| \le (1/\sqrt{2}) \cdot |r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota\varepsilon}}(\theta)| + (1/\sqrt{2}) \cdot |r_{\mathcal{C}_{\iota\varepsilon}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)|$. Similarly, the following two inequalities emerge:

$$\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| = \left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota\varepsilon}}(\theta) + m_{\mathcal{C}_{\iota\varepsilon}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| \le \left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right| + \left|m_{\mathcal{C}_{\iota\varepsilon}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| \tag{27}$$

$$\left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| = \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota\varepsilon}}(\theta) + n_{\mathcal{C}_{\iota\varepsilon}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| \le \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right| + \left|n_{\mathcal{C}_{\iota\varepsilon}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| \tag{28}$$

The following outcomes can be yielded by fusing both sides of the inequalities in Eqs. (27) and (28):

$$\begin{aligned}
&\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\\
&\quad\le \frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota''}}(\theta)\right| + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota''}}(\theta)\right|\right)\\
&\quad+ \frac{1}{2}\left(\left|m_{\mathcal{C}_{\iota''}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_{\iota''}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right)
\end{aligned}$$

Based on the aforementioned outcomes, the validity of the triangle inequality when $\beta = 1$, i.e., $\mathcal{D}^1_{(3)}(\varsigma_\iota, \varsigma_{\iota'}) \le \mathcal{D}^1_{(3)}(\varsigma_\iota, \varsigma_{\iota\varepsilon}) + \mathcal{D}^1_{(3)}(\varsigma_{\iota\varepsilon}, \varsigma_{\iota'})$, can be confirmed. Following that, by taking the maximum operation of both sides of the inequalities in Eqs. (27) and (28), it gives substance to:

$$\begin{aligned}
&\max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}\\
&\quad\le \max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota''}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota''}}(\theta)\right|\right\}\\
&\quad+ \max\left\{\left|m_{\mathcal{C}_{\iota''}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_{\iota''}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}
\end{aligned}$$

Based on this, the authenticity of the triangle inequality when $\beta \to \infty$, i.e., $\mathcal{D}^\infty_{(3)}(\varsigma_\iota, \varsigma_{\iota'}) \le \mathcal{D}^\infty_{(3)}(\varsigma_\iota, \varsigma_{\iota\varepsilon}) + \mathcal{D}^\infty_{(3)}(\varsigma_{\iota\varepsilon}, \varsigma_{\iota'})$, can be confirmed, completing the proof.

Theorem 2 *Based on the four-term representation, the C-IF Minkowski distance measure $\mathcal{D}^\beta_{(4)}$ for any three C-IF values ς_ι, $\varsigma_{\iota'}$, and $\varsigma_{\iota\varepsilon}$ fulfills the following properties*:

(T2.1) Non-negativity	$\mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) \ge 0$ for all $\beta \in Z^+$;
(T2.2) Boundedness	$\mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) \le 1$ for all $\beta \in Z^+$;
(T2.3) Reflexivity	$\mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_\iota) = 0$ for all $\beta \in Z^+$;
(T2.4) Symmetry	$\mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) = \mathcal{D}^\beta_{(4)}(\varsigma_{\iota'}, \varsigma_\iota)$ for all $\beta \in Z^+$;
(T2.5) Separability	$\mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) = 0$ for all $\beta \in Z^+$ if and only if $\varsigma_\iota = \varsigma_{\iota'}$;
(T2.6) Triangle inequality	$\mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_{\iota'}) \le \mathcal{D}^\beta_{(4)}(\varsigma_\iota, \varsigma_{\iota\varepsilon}) + \mathcal{D}^\beta_{(4)}(\varsigma_{\iota\varepsilon}, \varsigma_{\iota'})$ when $\beta = 1$ and $\beta \to \infty$.

Proof The proofs of (T2.1)–(T2.6) resemble those of (T1.1)–(T1.6), respectively.

Following Theorems 1 and 2, the C-IF Minkowski distance measures $\mathcal{D}_{(3)}^{\beta}$ and $\mathcal{D}_{(4)}^{\beta}$ are semi-metrics for C-IF values because they fulfill the essential conditions of reflexivity revealed in (T1.3) and (T2.3), symmetry in (T1.4) and (T2.4), and separability in (T1.5) and (T2.5). Furthermore, the C-IF Manhattan distance measures $\mathcal{D}_{(3)}^{1}$ and $\mathcal{D}_{(4)}^{1}$, as well as the C-IF Chebyshev distance measures $\mathcal{D}_{(3)}^{\infty}$ and $\mathcal{D}_{(4)}^{\infty}$, are well-defined metrics for C-IF information because they satisfy the essential conditions of reflexivity exhibited in (T1.3) and (T2.3), symmetry in (T1.4) and (T2.4), separability in (T1.5) and (T2.5), and triangle inequalities in (T1.6) and (T2.6).

Theorem 3 *The following properties are valid for any two C-IF values ς_ι and $\varsigma_{\iota'}$:*

(T3.1) $\mathcal{D}_{(3)}^{\beta}(\varsigma_\iota, \varsigma_{\iota'}) \leq \mathcal{D}_{(4)}^{\beta}(\varsigma_\iota, \varsigma_{\iota'})$ for all $\beta \in Z^+$;

(T3.2) $\mathcal{D}_{(3)}^{\beta}\left(\varsigma_\iota, \varsigma_{\iota'}\right) = \mathcal{D}_{(4)}^{\beta}(\varsigma_\iota, \varsigma_{\iota'})$ if $h_{\mathcal{C}_\iota}(\theta) = h_{\mathcal{C}_{\iota'}}(\theta)$;

(T3.3) $\mathcal{D}_{(3)}^{\beta}(\varsigma_\iota, \varsigma_{\iota'}) \leq \mathcal{D}_{(3)}^{\infty}(\varsigma_\iota, \varsigma_{\iota'})$ for all $\beta \in Z^+$;

(T3.4) $\mathcal{D}_{(4)}^{\beta}(\varsigma_\iota, \varsigma_{\iota'}) \leq \mathcal{D}_{(4)}^{\infty}(\varsigma_\iota, \varsigma_{\iota'})$ for all $\beta \in Z^+$.

Proof (T3.1) and (T3.2) are trivially correct.

(T3.3) First, consider the case where $\beta = 1$. It is apparent that $|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)| \leq \max\{|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)|, |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)|\}$ and $|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)| \leq \max\{|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)|, |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)|\}$. By aggregating both sides of the two inequalities, it can be generated that $|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)| + |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)| \leq 2 \cdot \max\{|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)|, |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)|\}$. As a consequence, one can obtain $(1/2) \cdot (|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)| + |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)|) \leq \max\{|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)|, |n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)|\}$ Accordingly, it draws the inference that $\mathcal{D}_{(3)}^{1}(\varsigma_\iota, \varsigma_{\iota'}) \leq \mathcal{D}_{(3)}^{\infty}(\varsigma_\iota, \varsigma_{\iota'})$. Furthermore, as a result of the following outcome:

$$
\begin{aligned}
&\left[\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|^{\beta} + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|^{\beta}\right)\right] \\
&\leq \left[\left(\frac{1}{2}\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \frac{1}{2}\left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right)^{\beta}\right]^{\frac{1}{\beta}} \\
&= \frac{1}{2}\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \frac{1}{2}\left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| \\
&\leq \max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\},
\end{aligned}
$$

which leads to the conclusion $\mathcal{D}_{(3)}^{\beta}(\varsigma_\iota, \varsigma_{\iota'}) \leq \mathcal{D}_{(3)}^{\infty}(\varsigma_\iota, \varsigma_{\iota'})$.

(T3.4) Remember, $m_{\mathcal{C}_\iota}(\theta) + n_{\mathcal{C}_\iota}(\theta) + h_{\mathcal{C}_\iota}(\theta) = 1$, and $m_{\mathcal{C}_{\iota'}}(\theta) + n_{\mathcal{C}_{\iota'}}(\theta) + h_{\mathcal{C}_{\iota'}}(\theta) = 1$. Concerning the comparisons between $m_{\mathcal{C}_\iota}(\theta)$ and $m_{\mathcal{C}_{\iota'}}(\theta)$, $n_{\mathcal{C}_\iota}(\theta)$ and $n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_\iota}(\theta)$ and $h_{\mathcal{C}_{\iota'}}(\theta)$, the contrast outcomes have six situations, consisting of (i) $m_{\mathcal{C}_\iota}(\theta) \geq m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_\iota}(\theta) \geq n_{\mathcal{C}_{\iota'}}(\theta), h_{\mathcal{C}_\iota}(\theta) \leq h_{\mathcal{C}_{\iota'}}(\theta)$, (ii) $m_{\mathcal{C}_\iota}(\theta) \geq m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_\iota}(\theta) \leq n_{\mathcal{C}_{\iota'}}(\theta), h_{\mathcal{C}_\iota}(\theta) \geq h_{\mathcal{C}_{\iota'}}(\theta)$, (iii)

$m_{\mathcal{C}_{\iota}}(\theta) \le m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \ge n_{\mathcal{C}_{\iota'}}(\theta), h_{\mathcal{C}_{\iota}}(\theta) \ge h_{\mathcal{C}_{\iota'}}(\theta),$ (iv)
$m_{\mathcal{C}_{\iota}}(\theta) \ge m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \le n_{\mathcal{C}_{\iota'}}(\theta), h_{\mathcal{C}_{\iota}}(\theta) \le h_{\mathcal{C}_{\iota'}}(\theta),$ (v)
$m_{\mathcal{C}_{\iota}}(\theta) \le m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \ge n_{\mathcal{C}_{\iota'}}(\theta), h_{\mathcal{C}_{\iota}}(\theta) \le h_{\mathcal{C}_{\iota'}}(\theta),$ and (vi)
$m_{\mathcal{C}_{\iota}}(\theta) \le m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \le n_{\mathcal{C}_{\iota'}}(\theta), h_{\mathcal{C}_{\iota}}(\theta) \ge h_{\mathcal{C}_{\iota'}}(\theta)$. In Case (i), the presuppositions of $m_{\mathcal{C}_{\iota}}(\theta) \ge m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \ge n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_{\iota}}(\theta) \le h_{\mathcal{C}_{\iota'}}(\theta)$ give substance to the following results:

$$
\begin{aligned}
&\frac{1}{2}\left(\left|m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\\
&\quad = \frac{1}{2}\left(m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) + n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) - h_{\mathcal{C}_{\iota}}(\theta) + h_{\mathcal{C}_{\iota'}}(\theta)\right)\\
&\quad = \frac{1}{2}\left[m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) + n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) - \left(1 - m_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota}}(\theta)\right) + \left(1 - m_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right)\right]\\
&\quad = \frac{1}{2}\left(2 \cdot m_{\mathcal{C}_{\iota}}(\theta) - 2 \cdot m_{\mathcal{C}_{\iota'}}(\theta) + 2 \cdot n_{\mathcal{C}_{\iota}}(\theta) - 2 \cdot n_{\mathcal{C}_{\iota'}}(\theta)\right)\\
&\quad = m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) + n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) = -h_{\mathcal{C}_{\iota}}(\theta) + h_{\mathcal{C}_{\iota'}}(\theta) = \left|h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|.
\end{aligned}
$$

In Case (ii), the presuppositions of $m_{\mathcal{C}_{\iota}}(\theta) \ge m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \le n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_{\iota}}(\theta) \ge h_{\mathcal{C}_{\iota'}}(\theta)$ demonstrate the truth of the following outcomes:

$$
\begin{aligned}
&\frac{1}{2}\left(\left|m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\\
&\quad = \frac{1}{2}\left[m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_{\iota}}(\theta) + n_{\mathcal{C}_{\iota'}}(\theta) + \left(1 - m_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota}}(\theta)\right) - \left(1 - m_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right)\right]\\
&\quad = \frac{1}{2}\left(-2 \cdot n_{\mathcal{C}_{\iota}}(\theta) + 2 \cdot n_{\mathcal{C}_{\iota'}}(\theta)\right) = -n_{\mathcal{C}_{\iota}}(\theta) + n_{\mathcal{C}_{\iota'}}(\theta) = \left|n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|.
\end{aligned}
$$

In Case (iii), the presuppositions of $m_{\mathcal{C}_{\iota}}(\theta) \le m_{\mathcal{C}_{\iota'}}(\theta)$, $n_{\mathcal{C}_{\iota}}(\theta) \ge n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_{\iota}}(\theta) \ge h_{\mathcal{C}_{\iota'}}(\theta)$ produce the following outcomes:

$$
\begin{aligned}
&\frac{1}{2}\left(\left|m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\\
&\quad = \frac{1}{2}\left[-m_{\mathcal{C}_{\iota}}(\theta) + m_{\mathcal{C}_{\iota'}}(\theta) + n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) + \left(1 - m_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota}}(\theta)\right) - \left(1 - m_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right)\right]\\
&\quad = \frac{1}{2}\left(-2 \cdot m_{\mathcal{C}_{\iota}}(\theta) + 2 \cdot m_{\mathcal{C}_{\iota'}}(\theta)\right) = -m_{\mathcal{C}_{\iota}}(\theta) + m_{\mathcal{C}_{\iota'}}(\theta) = \left|m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|.
\end{aligned}
$$

In Case (iv), the presuppositions of $m_{\mathcal{C}_{\iota}}(\theta) \ge m_{\mathcal{C}_{\iota'}}(\theta)$, $n_{\mathcal{C}_{\iota}}(\theta) \le n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_{\iota}}(\theta) \le h_{\mathcal{C}_{\iota'}}(\theta)$ generate the following outcomes:

$$
\begin{aligned}
&\frac{1}{2}\left(\left|m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right| + \left|h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)\\
&\quad = \frac{1}{2}\left[m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_{\iota}}(\theta) + n_{\mathcal{C}_{\iota'}}(\theta) - \left(1 - m_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota}}(\theta)\right) + \left(1 - m_{\mathcal{C}_{\iota'}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right)\right]\\
&\quad = \frac{1}{2}\left(2 \cdot m_{\mathcal{C}_{\iota}}(\theta) - 2 \cdot m_{\mathcal{C}_{\iota'}}(\theta)\right) = m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) = \left|m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|.
\end{aligned}
$$

In Case (v), the presuppositions of $m_{\mathcal{C}_{\iota}}(\theta) \le m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota}}(\theta) \ge n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_{\iota}}(\theta) \le h_{\mathcal{C}_{\iota'}}(\theta)$ yield the following outcomes:

$$\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)$$
$$=\frac{1}{2}\left[-m_{\mathcal{C}_\iota}(\theta)+m_{\mathcal{C}_{\iota'}}(\theta)+n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)-\left(1-m_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_\iota}(\theta)\right)+\left(1-m_{\mathcal{C}_{\iota'}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right)\right]$$
$$=\frac{1}{2}\left(2\cdot n_{\mathcal{C}_\iota}(\theta)-2\cdot n_{\mathcal{C}_{\iota'}}(\theta)\right)=n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)=\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|.$$

In Case (vi), the presuppositions of $m_{\mathcal{C}_\iota}(\theta)\le m_{\mathcal{C}_{\iota'}}(\theta)$, $n_{\mathcal{C}_\iota}(\theta)\le n_{\mathcal{C}_{\iota'}}(\theta)$, and $h_{\mathcal{C}_\iota}(\theta)\ge h_{\mathcal{C}_{\iota'}}(\theta)$ the produce following outcomes:

$$\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)$$
$$=\frac{1}{2}\left[-m_{\mathcal{C}_\iota}(\theta)+m_{\mathcal{C}_{\iota'}}(\theta)-n_{\mathcal{C}_\iota}(\theta)+n_{\mathcal{C}_{\iota'}}(\theta)+\left(1-m_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_\iota}(\theta)\right)-\left(1-m_{\mathcal{C}_{\iota'}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right)\right]$$
$$=\frac{1}{2}\left(-2\cdot m_{\mathcal{C}_\iota}(\theta)+2\cdot m_{\mathcal{C}_{\iota'}}(\theta)-2\cdot n_{\mathcal{C}_\iota}(\theta)+2\cdot n_{\mathcal{C}_{\iota'}}(\theta)\right)$$
$$=-m_{\mathcal{C}_\iota}(\theta)+m_{\mathcal{C}_{\iota'}}(\theta)-n_{\mathcal{C}_\iota}(\theta)+n_{\mathcal{C}_{\iota'}}(\theta)=\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|.$$

The following inequality can be deduced by summarizing the results obtained under the above presuppositions in in Cases (i) – (vi):

$$\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)$$
$$\le\max\left\{\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}$$

As a result, $\mathcal{D}^1_{(4)}(\varsigma_\iota,\varsigma_{\iota'})\le\mathcal{D}^\infty_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$. Furthermore, the following outcome is received:

$$\left[\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta+\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta+\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta\right)\right]^{\frac{1}{\beta}}$$
$$\le\left[\left(\frac{1}{2}\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\frac{1}{2}\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|+\frac{1}{2}\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right)^\beta\right]^{\frac{1}{\beta}}$$
$$=\frac{1}{2}\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\frac{1}{2}\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|+\frac{1}{2}\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|$$
$$\le\max\left\{\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|h_{\mathcal{C}_\iota}(\theta)-h_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}.$$

As a result, it confirms the truth of $\mathcal{D}^\beta_{(4)}(\varsigma_\iota,\varsigma_{\iota'})\le\mathcal{D}^\infty_{(4)}(\varsigma_\iota,\varsigma_{\iota'})$. This concludes the proof.

3.2 C-IF Minkowski metrics for C-IF sets

In addition to the C-IF Minkowski distance measures between C-IF values described in the previous subsection, this subsection further introduces new distance metrics for C-IF sets based on the above distance measures.

Definition 6 Given a finite universe of discourse Θ (with the cardinality $card(\Theta)$) and a metric parameter $\beta\in Z^+$, the C-IF Minkowski distance measures separating the C-IF sets $\mathcal{C}_\iota=\left\{\langle\theta,m_{\mathcal{C}_\iota}(\theta),n_{\mathcal{C}_\iota}(\theta);r_{\mathcal{C}_\iota}(\theta)\rangle|\theta\in\Theta\right\}$ (involving hesitancy $h_{\mathcal{C}_\iota}(\theta)$) and

$\mathcal{C}_{\iota'} = \left\{ \langle \theta, m_{\mathcal{C}_{\iota'}}(\theta), n_{\mathcal{C}_{\iota'}}(\theta); r_{\mathcal{C}_{\iota'}}(\theta) \rangle | \theta \in \Theta \right\}$ (involving hesitancy $h_{\mathcal{C}_{\iota'}}(\theta)$) based on the three- and four-term approaches are expounded in this order:xx

$$\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}_{\iota}, \mathcal{C}_{\iota'}\right) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2} \cdot card(\Theta)} \sum_{\theta \in \Theta} \left| r_{\mathcal{C}_{\iota}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta) \right| + \left[\frac{1}{2 \cdot card(\Theta)} \sum_{\theta \in \Theta} \left(\left| m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) \right|^{\beta} + \left| n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) \right|^{\beta} \right) \right]^{\frac{1}{\beta}} \right\} \tag{29}$$

$$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}_{\iota}, \mathcal{C}_{\iota'}) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2} \cdot card(\Theta)} \sum_{\theta \in \Theta} \left| r_{\mathcal{C}_{\iota}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta) \right| + \left[\frac{1}{2 \cdot card(\Theta)} \sum_{\theta \in \Theta} \left(\left| m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) \right|^{\beta} + \left| n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) \right|^{\beta} + \left| h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta) \right|^{\beta} \right) \right]^{\frac{1}{\beta}} \right\}. \tag{30}$$

Remark 2 The C-IF Manhattan distance measures between two C-IF sets $\mathcal{C}_{\iota}$ and $\mathcal{C}_{\iota'}$ are depicted as follows using the three- and four-term representations:

$$\mathfrak{D}^{1}_{(3)}(\mathcal{C}_{\iota}, \mathcal{C}_{\iota'}) = \frac{1}{2}\left[\frac{1}{\sqrt{2} \cdot card(\Theta)} \sum_{\theta \in \Theta} \left| r_{\mathcal{C}_{\iota}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta) \right| + \frac{1}{2 \cdot card(\Theta)} \sum_{\theta \in \Theta} \left(\left| m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) \right| + \left| n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) \right| \right) \right] \tag{31}$$

$$\mathfrak{D}^{1}_{(4)}(\mathcal{C}_{\iota}, \mathcal{C}_{\iota'}) = \frac{1}{2}\left[\frac{1}{\sqrt{2} \cdot card(\Theta)} \sum_{\theta \in \Theta} \left| r_{\mathcal{C}_{\iota}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta) \right| + \frac{1}{2 \cdot card(\Theta)} \sum_{\theta \in \Theta} \left(\left| m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta) \right| + \left| n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta) \right| + \left| h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta) \right| \right) \right] \tag{32}$$

The C-IF Euclidean distance measures between $\mathcal{C}_{\iota}$ and $\mathcal{C}_{\iota'}$ are drawn as follows:

$$\mathfrak{D}^{2}_{(3)}(\mathcal{C}_{\iota}, \mathcal{C}_{\iota'}) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2} \cdot card(\Theta)} \sum_{\theta \in \Theta} \left| r_{\mathcal{C}_{\iota}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta) \right| + \left[\frac{1}{2 \cdot card(\Theta)} \sum_{\theta \in \Theta} \left((m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta))^{2} + (n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta))^{2} \right) \right]^{\frac{1}{2}} \right\} \tag{33}$$

$$\mathfrak{D}^{2}_{(4)}(\mathcal{C}_{\iota}, \mathcal{C}_{\iota'}) = \frac{1}{2}\left\{ \frac{1}{\sqrt{2} \cdot card(\Theta)} \sum_{\theta \in \Theta} \left| r_{\mathcal{C}_{\iota}}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta) \right| + \left[\frac{1}{2 \cdot card(\Theta)} \sum_{\theta \in \Theta} \left((m_{\mathcal{C}_{\iota}}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta))^{2} + (n_{\mathcal{C}_{\iota}}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta))^{2} + (h_{\mathcal{C}_{\iota}}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta))^{2} \right) \right]^{\frac{1}{2}} \right\} \tag{34}$$

The C-IF Chebyshev distance measures between $\mathcal{C}_{\iota}$ and $\mathcal{C}_{\iota'}$ are depicted like this:

Table 2 Data and computation results of the C-IF distances in Example 2

θ_i	Data embedded in the C-IF set $\mathcal{C}_\iota$				Data embedded in the C-IF set $\mathcal{C}_{\iota'}$			
	$m_{\mathcal{C}_\iota}(\theta_i)$	$n_{\mathcal{C}_\iota}(\theta_i)$	$r_{\mathcal{C}_\iota}(\theta_i)$	$h_{\mathcal{C}_\iota}(\theta_i)$	$m_{\mathcal{C}_{\iota'}}(\theta_i)$	$n_{\mathcal{C}_{\iota'}}(\theta_i)$	$r_{\mathcal{C}_{\iota'}}(\theta_i)$	$h_{\mathcal{C}_{\iota'}}(\theta_i)$
θ_1	0.16	0.12	0.07	0.72	0.13	0.23	0.08	0.64
θ_2	0.28	0.60	0.06	0.12	0.28	0.48	0.10	0.24
θ_3	0.56	0.26	0.09	0.18	0.52	0.13	0.11	0.35

C-IF distance based on the three-term approach			C-IF distance based on the four-term approach		
$\mathfrak{D}^1_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$	$\mathfrak{D}^2_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$	$\mathfrak{D}^\infty_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$	$\mathfrak{D}^1_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$	$\mathfrak{D}^2_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$	$\mathfrak{D}^\infty_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$
0.0441	0.0520	0.0682	0.0749	0.0714	0.0749

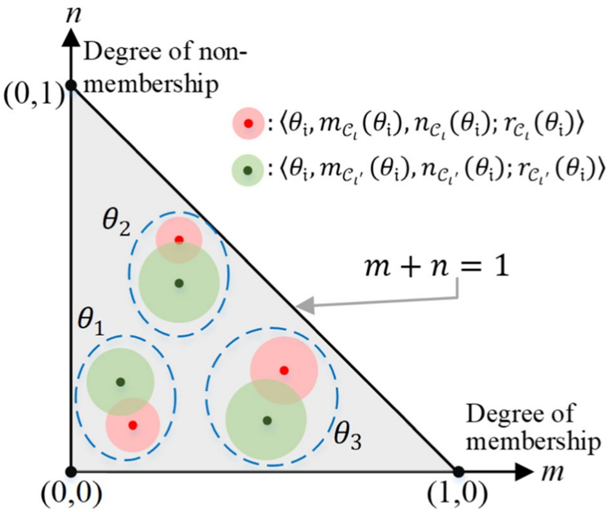


Fig. 3 Geometrical representation of the C-IF sets $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$

$$\mathfrak{D}^\infty_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \frac{1}{2}\left[\frac{1}{\sqrt{2}\cdot card(\Theta)}\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)\right| + \frac{1}{card(\Theta)}\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|,\ \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}\right] \tag{35}$$

$$\mathfrak{D}^\infty_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \frac{1}{2}\left[\frac{1}{\sqrt{2}\cdot card(\Theta)}\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_\iota}(\theta) - r_{\mathcal{C}_{\iota'}}(\theta)\right| + \frac{1}{card(\Theta)}\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|,\ \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}\right] \tag{36}$$

where $\mathfrak{D}^{\infty}_{(3)}$ and $\mathfrak{D}^{\infty}_{(4)}$ indicate $\lim_{\beta\to\infty}\mathfrak{D}^{\beta}_{(3)}$ and $\lim_{\beta\to\infty}\mathfrak{D}^{\beta}_{(4)}$, respectively.

Example 2 Given a finite universe of discourse $\Theta = \{\theta_1, \theta_2, \theta_3\}$, consider two C-IF sets $\mathcal{C}_\iota = \{\langle\theta_i, m_{\mathcal{C}_\iota}(\theta_i), n_{\mathcal{C}_\iota}(\theta_i); r_{\mathcal{C}_\iota}(\theta_i)\rangle | \theta_i \in \Theta\}$ (involving the degree of hesitancy $h_{\mathcal{C}_\iota}(\theta_i)$ for $\theta_i \in \Theta$) and $\mathcal{C}_{\iota'} = \{\langle\theta_i, m_{\mathcal{C}_{\iota'}}(\theta_i), n_{\mathcal{C}_{\iota'}}(\theta_i); r_{\mathcal{C}_{\iota'}}(\theta_i)\rangle | \theta_i \in \Theta\}$ (involving the degree of hesitancy $h_{\mathcal{C}_{\iota'}}(\theta_i)$ for $\theta_i \in \Theta$). The detailed data embedded in $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$ are shown in the upper part of Table 2. More precisely, the geometrical representation of $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$ is portrayed in Fig. 3.

The cardinality of Θ is given as $card(\Theta) = 3$ in this example. The C-IF Manhattan, Euclidean, and Chebyshev distances between $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$ are derived using the three-term approach as follows:

$$\begin{aligned}\mathfrak{D}^{1}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) &= \frac{1}{2}\Bigg[\frac{1}{\sqrt{2}\cdot 3}(|0.07-0.08| + |0.06-0.10| + |0.09-0.11|) + \frac{1}{2\cdot 3}(|0.16-0.13| \\ &+ |0.12-0.23| + |0.28-0.28| + |0.60-0.48| + |0.56-0.52| + |0.26-0.13|)\Bigg] \\ &= 0.0441\end{aligned}$$

$$\begin{aligned}\mathfrak{D}^{2}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) &= \frac{1}{2}\Bigg\{\Bigg\{\frac{1}{\sqrt{2}\cdot 3}(|0.07-0.08| + |0.06-0.10| + |0.09-0.11|) \\ &+ \Bigg[\frac{1}{2\cdot 3}((0.16-0.13)^2 + (0.12-0.23)^2 + (0.28-0.28)^2 \\ &+ (0.60-0.48)^2 + (0.56-0.52)^2 + (0.26-0.13)^2\frac{1}{2} = 0.0520\end{aligned}$$

$$\begin{aligned}\mathfrak{D}^{\infty}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) &= \frac{1}{2}\Bigg[\frac{1}{\sqrt{2}\cdot 3}(|0.07-0.08| + |0.06-0.10| + |0.09-0.11|) \\ &+ \frac{1}{3}(\max\{|0.16-0.13|, |0.12-0.23|\} + \max\{|0.28-0.28|, \\ &|0.60-0.48|\} | + \max\{|0.56-0.52|, 0.26-0.13|\})] = 0.0682\end{aligned}$$

Similarly, using the four-term approach, the C-IF Manhattan, Euclidean, and Chebyshev distances between $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$ are as follows: $\mathfrak{D}^{1}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = 0.0749$, $\mathfrak{D}^{2}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = 0.0714$, and $\mathfrak{D}^{\infty}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = 0.0749$. The C-IF distances calculated above are listed in the lower part of Table 2. As shown in the table, the C-IF Manhattan, Euclidean, and Chebyshev distances produced by the three-term approach are smaller than those rendered by the four-term approach, i.e., $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) \leq \mathfrak{D}^{\beta}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$ for β=1, β=2, and $\beta \to \infty$. Additionally, it can be observed that $\mathfrak{D}^{1}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) < \mathfrak{D}^{2}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) < \mathfrak{D}^{\infty}_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$ and $\mathfrak{D}^{2}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) < \mathfrak{D}^{1}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) = \mathfrak{D}^{\infty}_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$ in this example. The following three theorems will systematically investigate the essential properties of these C-IF distances and their relationship.

Theorem 4 *The C-IF Minkowski distance measure* $\mathfrak{D}^{\beta}_{(3)}$ *for any three C-IF sets* $\mathcal{C}_\iota$, $\mathcal{C}_{\iota'}$, *and* $\mathcal{C}_{\iota\varepsilon}$ *fulfills the following properties based on the three-term representation:*

(T4.1) Non-negativity $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})\geq 0$ for all $\beta\in Z^{+}$;

(T4.2) Boundedness $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})\leq 1$ for all $\beta\in Z^{+}$;

(T4.3) Reflexivity $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota})=0$ for all $\beta\in Z^{+}$;

(T4.4) Symmetry $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})=\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota'},\mathcal{C}_{\iota})$ for all $\beta\in Z^{+}$;

(T4.5) Separability $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})=0$ for all $\beta\in Z^{+}$ if and only if $\mathcal{C}_{\iota}=\mathcal{C}_{\iota'}$;

(T4.6) Triangle inequality $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})\leq\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota\varepsilon})+\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota\varepsilon},\mathcal{C}_{\iota'})$ when $\beta=1$ and $\beta\to\infty$.

Proof (T4.1) The non-negative property is demonstrated by a non-negative cardinality, i.e., $card(\Theta)\geq 0$, and the non-negative property in (T1.1).

(T4.2) The cardinality $card(\Theta)$ is a measure of the number of elements contained in Θ. By reason of $|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)|\leq\sqrt{2}$ and $|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)|^{\beta}+|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)|^{\beta}\leq 2$ (for $\beta\in Z^{+}$), it is recognized that $\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|\leq\sqrt{2}\cdot card(\Theta)$ and $\sum_{\theta\in\Theta}|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)|^{\beta}$ $+|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)|^{\beta}\leq 2\cdot card(\Theta)$. Consequently, for all $\beta\in Z^{+}$, it is acquired that $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})\leq(1/2)\{[1/(\sqrt{2}\cdot card(\Theta))]\cdot(\sqrt{2}\cdot card(\Theta))+[(1/(2\cdot card(\Theta)))\cdot(2\cdot card(\Theta))]^{(1/\beta)}\}=1$.

(T4.3) and (T4.4) are trivially correct.

(T4.5) In accordance with Definition 2, $\mathcal{C}_{\iota}=\{\langle\theta,m_{\mathcal{C}_{\iota}}(\theta),n_{\mathcal{C}_{\iota}}(\theta);r_{\mathcal{C}_{\iota}}(\theta)\rangle|\theta\in\Theta\}$ and $\mathcal{C}_{\iota'}=\{\langle\theta,m_{\mathcal{C}_{\iota'}}(\theta),n_{\mathcal{C}_{\iota'}}(\theta);r_{\mathcal{C}_{\iota'}}(\theta)\rangle|\theta\in\Theta\}$. The given condition $\mathcal{C}_{\iota}=\mathcal{C}_{\iota'}$ gives substance to $m_{\mathcal{C}_{\iota}}(\theta)=m_{\mathcal{C}_{\iota'}}(\theta)$,$n_{\mathcal{C}_{\iota}}(\theta)=n_{\mathcal{C}_{\iota'}}(\theta)$, and $r_{\mathcal{C}_{\iota}}(\theta)=r_{\mathcal{C}_{\iota'}}(\theta)$ for all $\theta\in\Theta$, thereby $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})=0$ for all $\beta\in Z^{+}$. By contrast, the condition $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}_{\iota},\mathcal{C}_{\iota'})=0$ for all $\beta\in Z^{+}$ establishes evidence of $|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)|=0$, $|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)|=0$, and $|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)|=0$ for each $\theta\in\Theta$. As a result, the correctness of $\{\langle\theta,m_{\mathcal{C}_{\iota}}(\theta),n_{\mathcal{C}_{\iota}}(\theta);r_{\mathcal{C}_{\iota}}(\theta)\rangle|\theta\in\Theta\}=\{\langle\theta,m_{\mathcal{C}_{\iota'}}(\theta),n_{\mathcal{C}_{\iota'}}(\theta);r_{\mathcal{C}_{\iota'}}(\theta)\rangle|\theta\in\Theta\}$ can be supported, corroborating the sufficiency and necessity for the separability property.

(T4.6) The following inequalities are fulfilled for each $\theta\in\Theta$, as attested by the property in (T1.6): $|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)|\leq|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota\varepsilon}}(\theta)|+|r_{\mathcal{C}_{\iota\varepsilon}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)|$, $|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)|\leq|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota\varepsilon}}(\theta)|+|m_{\mathcal{C}_{\iota\varepsilon}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)|$, and $|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)|\leq|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota\varepsilon}}(\theta)|+|n_{\mathcal{C}_{\iota\varepsilon}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)|$. It is the direct result of the following inequalities:

$$\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|\leq\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_{\iota}}(\theta)-r_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right|+\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_{\iota\varepsilon}}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|,$$

$$\begin{aligned}\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|\right)&\leq\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_{\iota}}(\theta)-m_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right|+\left|n_{\mathcal{C}_{\iota}}(\theta)-n_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right|\right)\\&+\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_{\iota\varepsilon}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|+\left|n_{\mathcal{C}_{\iota\varepsilon}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|\right),\end{aligned}$$

$$\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}\le\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_\iota}(\theta)-m_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right|,\ \left|n_{\mathcal{C}_\iota}(\theta)-n_{\mathcal{C}_{\iota\varepsilon}}(\theta)\right|\right\}$$
$$+\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_{\iota\varepsilon}}(\theta)-m_{\mathcal{C}_{\iota'}}(\theta)\right|,\left|n_{\mathcal{C}_{\iota\varepsilon}}(\theta)-n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}.$$

In light of the previously obtained results and the formulas in Remark 2, the correctness of the triangle inequalities $\mathfrak{D}^1_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le\mathfrak{D}^1_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota\varepsilon})+\mathfrak{D}^1_{(3)}(\mathcal{C}_{\iota\varepsilon},\mathcal{C}_{\iota'})$ and $\mathfrak{D}^\infty_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le\mathfrak{D}^\infty_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota\varepsilon})+\mathfrak{D}^\infty_{(3)}(\mathcal{C}_{\iota\varepsilon},\mathcal{C}_{\iota'})$ when $\beta=1$ and $\beta\to\infty$, respectively, can be corroborated, completing the proof.

Theorem 5 *The C-IF Minkowski distance measure $\mathfrak{D}^\beta_{(4)}$ for any three C-IF sets $\mathcal{C}_\iota$, $\mathcal{C}_{\iota'}$, and $\mathcal{C}_{\iota\varepsilon}$ fulfills the following properties based on the four-term representation:*

(T5.1) Non-negativity $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\ge 0$ for all $\beta\in Z^+$;

(T5.2) Boundedness: $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le 1$ for all $\beta\in Z^+$;

(T5.3) Reflexivity $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_\iota)=0$ for all $\beta\in Z^+$;

(T5.4) Symmetry $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})=\mathfrak{D}^\beta_{(4)}(\mathcal{C}_{\iota'},\mathcal{C}_\iota)$ for all $\beta\in Z^+$;

(T5.5) Separability $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})=0$ for all $\beta\in Z^+$ if and only if $\mathcal{C}_\iota=\mathcal{C}_{\iota'}$;

(T5.6) Triangle inequality $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota\varepsilon})+\mathfrak{D}^\beta_{(4)}(\mathcal{C}_{\iota\varepsilon},\mathcal{C}_{\iota'})$ when $\beta=1$ and $\beta\to\infty$.

Proof The proving processes of (T5.1) – (T5.6) are akin to those of (T4.1) – (T4.6), respectively.

According to Theorems 4 and 5, the C-IF Minkowski distance measures $\mathfrak{D}^\beta_{(3)}$ and $\mathfrak{D}^\beta_{(4)}$ are semi-metrics of the C-IF sets because they satisfy the reflexivity in (T4.3) and (T5.3), symmetry in (T4.4) and (T5.4), and separability in (T4.5) and (T5.5). Furthermore, the C-IF Manhattan distance measures $\mathfrak{D}^1_{(3)}$ and $\mathfrak{D}^1_{(4)}$, as well as the C-IF Chebyshev distance measures $\mathfrak{D}^\infty_{(3)}$ and $\mathfrak{D}^\infty_{(4)}$, are well-defined measures of C-IF sets because they satisfy the reflexivity in (T4.3) and (T5.3), symmetry in (T4.4) and (T5.4), separability in (T4.5) and (T5.5), and triangle inequalities in (T4.6) and (T5.6).

Theorem 6 *The following properties are valid for any two C-IF sets $\mathcal{C}_\iota$ and $\mathcal{C}_{\iota'}$:*

(T6.1) $\mathfrak{D}^\beta_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$ for all $\beta\in Z^+$;

(T6.2) $\mathfrak{D}^\beta_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})=\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$ if $h_{\mathcal{C}_\iota}(\theta)=h_{\mathcal{C}_{\iota'}}(\theta)$ for all $\theta\in\Theta$;

(T6.3) $\mathfrak{D}^\beta_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le\mathfrak{D}^\infty_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$ for all $\beta\in Z^+$;

(T6.4) $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})\le\mathfrak{D}^\infty_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$ for all $\beta\in Z^+$.

Proof (T6.1) and (T6.2) are trivially correct.

(T6.3) Based on the proving process in (T3.3), the following inequality holds true for $\theta \in \Theta$:

$$\left[\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta\right)\right]^{\frac{1}{\beta}} \leq \max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}.$$

When both sides of the inequality are averaged across all $\theta \in \Theta$, the following result can be deduced:

$$\left[\frac{1}{2 \cdot card(\Theta)}\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta\right)\right]^{\frac{1}{\beta}}$$

$$\leq \frac{1}{card(\Theta)}\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}.$$

It directly draws the inference of $\mathfrak{D}^\beta_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) \leq \mathfrak{D}^\infty_{(3)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$ for all $\beta \in Z^+$.

(T6.4) The following inequality holds true for $\theta \in \Theta$, as supported by the proving process in (T3.4):

$$\left[\frac{1}{2}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta + \left|h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta\right)\right]^{\frac{1}{\beta}}$$

$$\leq \max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}.$$

When both sides of the inequality are averaged across all $\theta \in \Theta$, the following result can be deduced:

$$\left[\frac{1}{2 \cdot card(\Theta)}\sum_{\theta\in\Theta}\left(\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta + \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta + \left|h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|^\beta\right)\right]^{\frac{1}{\beta}}$$

$$\leq \frac{1}{card(\Theta)}\sum_{\theta\in\Theta}\max\left\{\left|m_{\mathcal{C}_\iota}(\theta) - m_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|n_{\mathcal{C}_\iota}(\theta) - n_{\mathcal{C}_{\iota'}}(\theta)\right|, \left|h_{\mathcal{C}_\iota}(\theta) - h_{\mathcal{C}_{\iota'}}(\theta)\right|\right\}.$$

It comes to the conclusion of $\mathfrak{D}^\beta_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'}) \leq \mathfrak{D}^\infty_{(4)}(\mathcal{C}_\iota, \mathcal{C}_{\iota'})$ for all $\beta \in Z^+$.

As previously stated, Atanassov and Marinov (2021) constructed four new C-IF distance metrics. These four new distance measures, however, have significant limitations. Atanassov and Marinov's C-IF distance formulas, as shown in Definition 4, do not account for the fact that the elements in a C-IF set have different radii. The distance measures advanced by Atanassov and Marinov (2021) are special cases of the evolved C-IF Minkowski metrics, according to the C-IF Minkowski metrics established in this study, if the radius of all elements in each C-IF set $\mathcal{C}_\iota$ is assumed to be equal to $r_{\mathcal{C}_\iota}(\theta)$. Specifically, assume that $r_{\mathcal{C}_\iota}(\theta) = r_{\mathcal{C}_\iota}$ and $r_{\mathcal{C}_{\iota'}}(\theta) = r_{\mathcal{C}_{\iota'}}$ for all $\theta \in \Theta$. One has:

$$\frac{1}{\sqrt{2}\cdot card(\Theta)}\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_\iota}(\theta)-r_{\mathcal{C}_{\iota'}}(\theta)\right|=\frac{\sum_{\theta\in\Theta}\left|r_{\mathcal{C}_\iota}-r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}\cdot card(\Theta)}=\frac{card(\Theta)\cdot\left|r_{\mathcal{C}_\iota}-r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}\cdot card(\Theta)}=\frac{\left|r_{\mathcal{C}_\iota}-r_{\mathcal{C}_{\iota'}}\right|}{\sqrt{2}}.$$

Therefore, $\mathfrak{D}^{M}_{(3)}\left(\mathcal{C}_\iota,\mathcal{C}_{\iota'}\right)=\mathfrak{D}^{1}_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$, $\mathfrak{D}^{M}_{(4)}\left(\mathcal{C}_\iota,\mathcal{C}_{\iota'}\right)=\mathfrak{D}^{1}_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$, $\mathfrak{D}^{E}_{(3)}\left(\mathcal{C}_\iota,\mathcal{C}_{\iota'}\right)=\mathfrak{D}^{2}_{(3)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$, and $\mathfrak{D}^{E}_{(4)}\left(\mathcal{C}_\iota,\mathcal{C}_{\iota'}\right)=\mathfrak{D}^{2}_{(4)}(\mathcal{C}_\iota,\mathcal{C}_{\iota'})$. The C-IF Manhattan and Euclidean distance measures for C-IF sets revealed in Remark 2 can cover the four C-IF distance metrics in Definition 4. More importantly, two evolved general distance metrics based on three- and four-term representations, namely the C-IF Minkowski metrics for C-IF values in Definition 5 and the C-IF Minkowski metrics for C-IF sets in Definition 6, can improve the limitations of existing C-IF distance metrics and expand their flexibility and applicability.

4 Evolved C-IF TOPSIS methodology

This section is dedicated to the development of an evolved C-IF TOPSIS methodology predicated on the C-IF Minkowski metrics to tackle MCDA problems with complex and uncertain decision information as a reference for intelligent decision assistance. As previously stated, Alkan and Kahraman (2022b) and Kahraman and Alkan (2021) developed the C-IF TOPSIS methods. However, they did not fully follow the core architecture of classic TOPSIS but made substantial modifications. Moreover, they converted C-IF information (i.e., criteria weights and performance ratings of alternatives judging by criteria) into IF information for processing in the calculation process of their C-IF TOPSIS procedures. Specifically, the C-IF decision matrix was transformed into pessimistic and optimistic decision matrices, and the data in these two decision matrices are ordinary IF values. This operating procedure may lose the specificity of C-IF information and reduce its flexibility to deal with uncertainty. The question of how to extend the classic TOPSIS architecture to the C-IF environment in such a way that the development can faithfully demonstrate the spirit of TOPSIS's compromise approach has become critical. As a result, this section is dedicated to creating a new C-IF TOPSIS methodology based on the C-IF Minkowski metric within the TOPSIS core architecture. The C-IF TOPSIS technique proposed in this section is based on four fundamental structures, each of which contains four combinations of the displaced and fixed anchoring frames, as well as the C-IF Minkowski metrics using three- and four-term approaches. Through such a methodological evolution, the theoretical framework of the prestigious TOPSIS can be extended to intricate C-IF environments, thereby expanding the application scope of TOPSIS.

4.1 Proposed C-IF TOPSIS approaches

In this study, the MCDA issue is investigated by evaluating a limited number of candidate alternatives across multiple performance criteria in C-IF uncertain circumstances. Let $\mathcal{A}=\left\{a_1,a_2,\cdots,a_{\mathfrak{A}}\right\}$ signify a collection of the $\mathfrak{A}(\geq 2)$ candidate alternatives; similarly, let $\mathcal{P}=\left\{p_1,p_2,\cdots,p_{\mathfrak{P}}\right\}$ signify a collection of the $\mathfrak{P}(\geq 2)$ performance criteria, wherein $\mathfrak{A}$ and $\mathfrak{C}$ are positive integers. Furthermore, the collection $\mathcal{P}$ is split into two sub-collections, one for the collection $\mathcal{P}_{\mathrm{I}}$ of beneficial criteria (to be maximized) and the other for the collection $\mathcal{P}_{\mathrm{II}}$ of non-beneficial criteria (to be minimized), where $\mathcal{P}_{\mathrm{I}}\cap\mathcal{P}_{\mathrm{II}}=\varnothing$ and $\mathcal{P}_{\mathrm{I}}\cup\mathcal{P}_{\mathrm{II}}=\mathcal{P}$.

The C-IF performance rating of a candidate alternative a_i $(i = 1, 2, \cdots, \mathfrak{A})$ judging by a performance criterion p_j $(j = 1, 2, \cdots, \mathfrak{P})$ is expressed as $\varsigma_{ij} = (m_{ij}, n_{ij}; r_{ij})$, in which the degree of hesitancy is derived as $h_{ij} = 1 - m_{ij} - n_{ij}$. The C-IF characteristic $\mathcal{C}_i$ and its embedded function O_r are expounded for each candidate alternative $a_i \in \mathcal{A}$ as follows:

$$\mathcal{C}_i = \left\{ \langle p_j, m_{ij}, n_{ij}; r_{ij} \rangle \middle| p_j \in \mathcal{P} \right\} = \left\{ \langle p_j, O_r\left(m_{ij}, n_{ij}\right) \rangle \middle| p_j \in \mathcal{P} \right\} \tag{37}$$

and $O_r(m_{ij}, n_{ij}) = \{\langle \zeta, \zeta' \rangle | \zeta, \zeta' \in [0, 1], \sqrt{(m_{ij} - \zeta)^2 + (n_{ij} - \zeta')^2} \le r_{ij}, \text{and} \zeta + \zeta' \le 1\}$. The C-IF importance weight of a performance criterion p_j $(j = 1, 2, \cdots, \mathfrak{P})$ is explicated as $\mathcal{w}_j = (w_j, \omega_j; \gamma_j)$, in which the degree of hesitancy is produced as $\langle_{ij} = 1 - w_j - \omega_j$.

Remember that the intersection, union, addition, and multiplication operations described in Definition 3 involve the min types (i.e., $\cap_{\min}$, $\cup_{\min}$, $\oplus_{\min}$, and $\otimes_{\min}$) as well as the max types (i.e., $\cap_{\max}$, $\cup_{\max}$, $\oplus_{\max}$, and $\otimes_{\max}$). Herein, these operations using the min and max types are determined by the minimum radius and maximum radius, respectively. In conformity with Kahraman and Alkan (2021), the operations with the min and max types will deliver the outcomes involving minimum and maximum uncertainty levels, respectively. To be more specific, a smaller radius indicates less ambiguity for the operation result of the C-IF paired information, while a larger radius indicates greater ambiguity. This study intends to exploit the multiplication operation with the max type to generate weighted performance ratings in this regard.

The weighted performance rating ς_{ij}^{w} of a_i judging by p_j is generated using the multiplication operation $\otimes_{\max}$ in Definition 3, i.e., $\varsigma_{ij}^{w} = \mathcal{w}_j \otimes_{\max} \varsigma_{ij}$, as demonstrated below:

$$\varsigma_{ij}^{w} = \left(m_{ij}^{w}, n_{ij}^{w}; r_{ij}^{w} \right) = \left(w_j \cdot m_{ij}, \omega_j + n_{ij} - \omega_j \cdot n_{ij}; \max\left\{ \gamma_j, r_{ij} \right\} \right) \tag{38}$$

where the degree of hesitancy is derived as $h_{ij}^{w} = 1 - m_{ij}^{w} - n_{ij}^{w}$. By collecting the weighted performance rating ς_{ij}^{w} over the $\mathfrak{P}$ criteria, the C-IF characteristic $\mathcal{C}_i^{w}$ for each a_i is elucidated as:

$$\mathcal{C}_i^{w} = \left\{ \langle p_j, m_{ij}^{w}, n_{ij}^{w}; r_{ij}^{w} \rangle \middle| p_j \in \mathcal{P} \right\} = \left\{ \langle p_j, O_r\left(m_{ij}^{w}, n_{ij}^{w} \right) \rangle \middle| p_j \in \mathcal{P} \right\} \tag{39}$$

in which $O_r(m_{ij}^{w}, n_{ij}^{w}) = \{\langle \zeta, \zeta' \rangle | \zeta, \zeta' \in [0, 1], \sqrt{(m_{ij}^{w} - \zeta)^2 + (n_{ij}^{w} - \zeta')^2} \le r_{ij}^{w}, \text{and} \zeta + \zeta' \le 1\}$.

The C-IF TOPSIS methodology, which is scheduled to be developed in this study, is based on the aforementioned C-IF Minkowski metrics using three- and four-term approaches, as well as the displaced and fixed anchoring frameworks. The anchoring mechanism consists of displaced and fixed ideal and anti-ideal C-IF characteristics that can serve as anchoring points in the C-IF TOPSIS procedure to formulate the subsequent compromise indices. The displaced anchoring framework, in particular, can be elicited from the intersection and union operations using the min and max types. To capture the most uncertainty, the operators $\cap_{\max}$ and $\cup_{\max}$ are utilized to identify the displaced ideal and anti-ideal ratings, as well as C-IF characteristics. The displaced ideal rating ς_{*j}^{w} can be yielded by performing the union operation $\varsigma_{1j}^{w} \cup_{\max} \varsigma_{2j}^{w} \cup_{\max} \cdots \cup_{\max} \varsigma_{\mathfrak{A}j}^{w}$ and the intersection operation $\varsigma_{1j}^{w} \cap_{\max} \varsigma_{2j}^{w} \cap_{\max} \cdots \cap_{\max} \varsigma_{\mathfrak{A}j}^{w}$ if $p_j \in \mathcal{P}_{\mathrm{I}}$ and $p_j \in \mathcal{P}_{\mathrm{II}}$, respectively. The displaced anti-ideal rating $\varsigma_{\neg j}^{w}$, on the other hand, can be produced by performing the intersection operation $\varsigma_{1j}^{w} \cap_{\max} \varsigma_{2j}^{w} \cap_{\max} \cdots \cap_{\max} \varsigma_{\mathfrak{A}j}^{w}$ and the union operation $\varsigma_{1j}^{w} \cup_{\max} \varsigma_{2j}^{w} \cup_{\max} \cdots \cup_{\max} \varsigma_{\mathfrak{A}j}^{w}$ if $p_j \in \mathcal{P}_{\mathrm{I}}$

and $p_j \in \mathcal{P}_{\text{II}}$, respectively. More specifically, the displaced ideal and anti-ideal ratings are derived as follows:

$$\varsigma_{*j}^{w} = \left(m_{*j}^{w}, n_{*j}^{w}; r_{*j}^{w}\right) = \begin{cases} \left(\max_{i=1}^{\mathfrak{A}} m_{ij}^{w}, \min_{i=1}^{\mathfrak{A}} n_{ij}^{w}; \max_{i=1}^{\mathfrak{A}} r_{ij}^{w}\right) & \text{if} p_j \in \mathcal{P}_{\text{I}}, \\ \left(\min_{i=1}^{\mathfrak{A}} m_{ij}^{w}, \max_{i=1}^{\mathfrak{A}} n_{ij}^{w}; \max_{i=1}^{\mathfrak{A}} r_{ij}^{w}\right) & \text{if} p_j \in \mathcal{P}_{\text{II}}; \end{cases} \tag{40}$$

$$\varsigma_{\neg j}^{w} = \left(m_{\neg j}^{w}, n_{\neg j}^{w}; r_{\neg j}^{w}\right) = \begin{cases} \left(\min_{i=1}^{\mathfrak{A}} m_{ij}^{w}, \max_{i=1}^{\mathfrak{A}} n_{ij}^{w}; \max_{i=1}^{\mathfrak{A}} r_{ij}^{w}\right) & \text{if} p_j \in \mathcal{P}_{\text{I}}, \\ \left(\max_{i=1}^{\mathfrak{A}} m_{ij}^{w}, \min_{i=1}^{\mathfrak{A}} n_{ij}^{w}; \max_{i=1}^{\mathfrak{A}} r_{ij}^{w}\right) & \text{if} p_j \in \mathcal{P}_{\text{II}}, \end{cases} \tag{41}$$

where $h_{*j}^{w} = 1 - m_{*j}^{w} - n_{*j}^{w}$ and $h_{\neg j}^{w} = 1 - m_{\neg j}^{w} - n_{\neg j}^{w}$ are the degrees of hesitancy. Furthermore, the displaced ideal and anti-ideal C-IF characteristics are delineated in this wise:

$$\mathcal{C}_{*}^{w} = \left\{ \langle p_j, m_{*j}^{w}, n_{*j}^{w}; r_{*j}^{w} \rangle \middle| p_j \in \mathcal{P} \right\} = \left\{ \langle p_j, O_r\left(m_{*j}^{w}, n_{*j}^{w}\right) \rangle \middle| p_j \in \mathcal{P} \right\} \tag{42}$$

$$\mathcal{C}_{\neg}^{w} = \left\{ \langle p_j, m_{\neg j}^{w}, n_{\neg j}^{w}; r_{\neg j}^{w} \rangle \middle| p_j \in \mathcal{P} \right\} = \left\{ \langle p_j, O_r\left(m_{\neg j}^{w}, n_{\neg j}^{w}\right) \rangle \middle| p_j \in \mathcal{P} \right\} \tag{43}$$

where $O_r(m_{*j}^{w}, n_{*j}^{w}) = \{\langle \zeta, \zeta' \rangle | \zeta, \zeta' \in [0,1], \sqrt{(m_{*j}^{w} - \zeta)^2 + (n_{*j}^{w} - \zeta')^2} \leq r_{*j}^{w}, \text{and} \zeta + \zeta' \leq 1\}$ and $O_r(m_{\neg j}^{w}, n_{\neg j}^{w}) = \{\langle \zeta, \zeta' \rangle | \zeta, \zeta' \in [0,1], \sqrt{(m_{\neg j}^{w} - \zeta)^2 + (n_{\neg j}^{w} - \zeta')^2} \leq r_{\neg j}^{w}, \text{and} \zeta + \zeta' \leq 1\}$.

The relative closeness of the C-IF characteristic $\mathcal{C}_i^{w}$ to the ideal and anti-ideal C-IF characteristics $\mathcal{C}_{*}^{w}$ and $\mathcal{C}_{\neg}^{w}$ can be delineated in the displaced anchoring framework along these lines: one is derived from the C-IF Minkowski distance measure using the three-term approach, and the other is established using the four-term approach. The relative closeness coefficients $\mathfrak{R}_{(3)}^{\beta *}(a_i)$ and $\mathfrak{R}_{(4)}^{\beta *}(a_i)$ are derived as:

$$\mathfrak{R}_{(3)}^{\beta *}\left(a_i\right) = \frac{\mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_i^{w}, \mathcal{C}_{\neg}^{w}\right)}{\mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_i^{w}, \mathcal{C}_{*}^{w}\right) + \mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_i^{w}, \mathcal{C}_{\neg}^{w}\right)} \tag{44}$$

$$\mathfrak{R}_{(4)}^{\beta *}\left(a_i\right) = \frac{\mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_i^{w}, \mathcal{C}_{\neg}^{w}\right)}{\mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_i^{w}, \mathcal{C}_{*}^{w}\right) + \mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_i^{w}, \mathcal{C}_{\neg}^{w}\right)}. \tag{45}$$

Remark 3 The relative closeness coefficients $\mathfrak{R}_{(3)}^{\beta *}(a_i)$ and $\mathfrak{R}_{(4)}^{\beta *}(a_i)$ have the following characteristics:

(R3.1) $0 \leq \mathfrak{R}_{(3)}^{\beta *}\left(a_i\right) \leq 1$ and $0 \leq \mathfrak{R}_{(4)}^{\beta *}\left(a_i\right) \leq 1$ for all $\beta \in Z^{+}$;
(R3.2) $\mathfrak{R}_{(3)}^{\beta *}\left(a_i\right) = 0$ and $\mathfrak{R}_{(4)}^{\beta *}\left(a_i\right) = 0$ for all $\beta \in Z^{+}$ if and only if $\mathcal{C}_i^{w} = \mathcal{C}_{\neg}^{w}$;
(R3.3) $\mathfrak{R}_{(3)}^{\beta *}\left(a_i\right) = 1$ and $\mathfrak{R}_{(4)}^{\beta *}\left(a_i\right) = 1$ for all $\beta \in Z^{+}$ if and only if $\mathcal{C}_i^{w} = \mathcal{C}_{*}^{w}$.

Proof (R3.1) For all $\beta \in Z^{+}$, it is recognized that $0 \leq \mathfrak{R}_{(3)}^{\beta *}\left(a_i\right) \leq 1$ and $0 \leq \mathfrak{R}_{(4)}^{\beta *}\left(a_i\right) \leq 1$ as a consequence of the non-negativity properties in (T4.1) and (T5.1) and the boundedness properties in (T4.2) and (T5.2) (i.e., $0 \leq \mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right), \mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right), \mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right), \mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) \leq 1$).

(R3.2) The preconditions $\mathfrak{R}^{\beta*}_{(3)}\left(a_i\right) = 0$ and $\mathfrak{R}^{\beta*}_{(4)}\left(a_i\right) = 0$ draw the inferences of zero numerators in Eqs. (44) and (45), respectively; that is, $\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) = 0$ and $\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) = 0$. As attested by the separability property in (T4.5) and (T5.5), it is received that $\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) = 0$ and $\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) = 0$, respectively, if and only if $\mathcal{C}^{w}_{i} = \mathcal{C}^{w}_{\neg}$. From these explicit discussions, it deduces that $\mathfrak{R}^{\beta*}_{(3)}\left(a_i\right) = \mathfrak{R}^{\beta*}_{(4)}\left(a_i\right) = 0$ for all $\beta \in Z^{+}$ if and only if $\mathcal{C}^{w}_{i} = \mathcal{C}^{w}_{\neg}$.

(R3.3) Concerning all $\beta \in Z^{+}$, the given conditions $\mathfrak{R}^{\beta*}_{(3)}\left(a_i\right) = 1$ and $\mathfrak{R}^{\beta*}_{(4)}\left(a_i\right) = 1$ indicate that $\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) = \mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right) + \mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right)$ and $\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right) = \mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right) + \mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{\neg}\right)$, respectively. Accordingly, the two equalities give rise to $\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right) = 0$ and $\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right) = 0$. Based on the separability property in (T4.5) and (T5.5), the sufficient and necessary condition of $\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right) = 0$ and $\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{*}\right) = 0$, respectively, is $\mathcal{C}^{w}_{i} = \mathcal{C}^{w}_{*}$. It concludes that $\mathfrak{R}^{\beta*}_{(3)}\left(a_i\right) = 1$ and $\mathfrak{R}^{\beta*}_{(4)}\left(a_i\right) = 1$ for all $\beta \in Z^{+}$ if and only if $\mathcal{C}^{w}_{i} = \mathcal{C}^{w}_{*}$.

In the displaced anchoring framework, all of the $\mathfrak{A}$ candidate alternatives are sorted in diminishing order of their relative closeness coefficients $\mathfrak{R}^{\beta*}_{(3)}(a_i)$ or $\mathfrak{R}^{\beta*}_{(4)}(a_i)$. This method yields the best compromise collections containing the candidates with the highest relative closeness coefficients:

$$\mathcal{A}^{\beta*}_{(3)} = \left\{ a_i \middle| \max_{i=1}^{\mathfrak{A}} \mathfrak{R}^{\beta*}_{(3)}(a_i), a_i \in \mathcal{A} \right\}, \tag{46}$$

$$\mathcal{A}^{\beta*}_{(4)} = \left\{ a_i \middle| \max_{i=1}^{\mathfrak{A}} \mathfrak{R}^{\beta*}_{(4)}(a_i), a_i \in \mathcal{A} \right\} \tag{47}$$

On the flip side, the fixed anchoring framework can be established on the fixed ideal and anti-ideal ratings as well as C-IF characteristics. Concerning a beneficial criterion $p_j \in \mathcal{P}_{\mathrm{I}}$, the fixed ideal rating ς^{w}_{+j} is rendered by multiplying the C-IF importance weight w_j by the highest performance $(1, 0;0)$. Furthermore, the fixed anti-ideal rating ς^{w}_{-j} can be yielded by multiplying w_j by the lowest performance $(0, 1;0)$. In the case of a non-beneficial criterion $p_j \in \mathcal{P}_{\mathrm{II}}$, the fixed ideal rating ς^{w}_{+j} is obtained by multiplying w_j by the highest performance $(0, 1;0)$; additionally, the fixed anti-ideal rating ς^{w}_{-j} is derived by multiplying w_j by the lowest performance $(1, 0;0)$. To be more specific, the fixed ideal and anti-ideal ratings are identified in this fashion:

$$\varsigma^{w}_{+j} = \left(m^{w}_{+j}, n^{w}_{+j}; r^{w}_{+j}\right) = \begin{cases} (w_j, \omega_j;\gamma_j)\otimes_{\max}(1, 0;0) = \left(w_j, \omega_j;\gamma_j\right) & \text{if} p_j \in \mathcal{P}_{\mathrm{I}}, \\ (w_j, \omega_j;\gamma_j)\otimes_{\max}(0, 1;0) = \left(0, 1;\gamma_j\right) & \text{if} p_j \in \mathcal{P}_{\mathrm{II}}; \end{cases} \tag{48}$$

$$\varsigma^{w}_{-j} = \left(m^{w}_{-j}, n^{w}_{-j}; r^{w}_{-j}\right) = \begin{cases} (w_j, \omega_j;\gamma_j)\otimes_{\max}(0, 1;0) = \left(0, 1;\gamma_j\right) & \text{if} p_j \in \mathcal{P}_{\mathrm{I}}, \\ (w_j, \omega_j;\gamma_j)\otimes_{\max}(1, 0;0) = \left(w_j, \omega_j;\gamma_j\right) & \text{if} p_j \in \mathcal{P}_{\mathrm{II}}; \end{cases} \tag{49}$$

where $h^{w}_{+j} = 1 - m^{w}_{+j} - n^{w}_{+j}$ and $h^{w}_{-j} = 1 - m^{w}_{-j} - n^{w}_{-j}$ are the degrees of hesitancy. The fixed ideal and anti-ideal C-IF characteristics are portrayed in the following manner:

$$\mathcal{C}^{w}_{+} = \left\{ \langle p_j, m^{w}_{+j}, n^{w}_{+j}; r^{w}_{+j} \rangle \middle| p_j \in \mathcal{P} \right\} = \left\{ \langle p_j, O_r\left(m^{w}_{+j}, n^{w}_{+j} \right) \rangle \middle| p_j \in \mathcal{P} \right\} \tag{50}$$

$$\mathcal{C}^{w}_{-} = \left\{ \langle p_j, m^{w}_{-j}, n^{w}_{-j}; r^{w}_{-j} \rangle \middle| p_j \in \mathcal{P} \right\} = \left\{ \langle p_j, O_r\left(m^{w}_{-j}, n^{w}_{-j} \right) \rangle \middle| p_j \in \mathcal{P} \right\} \tag{51}$$

in which $O_r(m^{w}_{+j}, n^{w}_{+j}) = \{\langle \zeta, \zeta' \rangle | \zeta, \zeta' \in [0,1], \sqrt{(m^{w}_{+j} - \zeta)^2 + (n^{w}_{+j} - \zeta')^2} \le r^{w}_{+j}, \text{and} \zeta + \zeta' \le 1\}$ and $O_r(m^{w}_{-j}, n^{w}_{-j}) = \{\langle \zeta, \zeta' \rangle | \zeta, \zeta' \in [0,1], \sqrt{(m^{w}_{-j} - \zeta)^2 + (n^{w}_{-j} - \zeta')^2} \le r^{w}_{-j}, \text{and} \zeta + \zeta' \le 1\}$.

In the fixed anchoring framework, the relative closeness of the C-IF characteristic $\mathcal{C}^{w}_{i}$ to the ideal and anti-ideal C-IF characteristics $\mathcal{C}^{w}_{+}$ and $\mathcal{C}^{w}_{-}$ can be generated predicated on the C-IF Minkowski distance measures using the three- and four-term approaches. The relative closeness coefficients $\mathfrak{R}^{\beta+}_{(3)}(a_i)$ and $\mathfrak{R}^{\beta+}_{(4)}(a_i)$ are rendered in this fashion:

$$\mathfrak{R}^{\beta+}_{(3)}\left(a_i\right) = \frac{\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{-}\right)}{\mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{+}\right) + \mathfrak{D}^{\beta}_{(3)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{-}\right)} \tag{52}$$

$$\mathfrak{R}^{\beta+}_{(4)}\left(a_i\right) = \frac{\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{-}\right)}{\mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{+}\right) + \mathfrak{D}^{\beta}_{(4)}\left(\mathcal{C}^{w}_{i}, \mathcal{C}^{w}_{-}\right)} \tag{53}$$

Remark 4 The relative closeness coefficients $\mathfrak{R}^{\beta+}_{(3)}\left(a_i\right)$ and $\mathfrak{R}^{\beta+}_{(4)}\left(a_i\right)$ have the following characteristics:

(R4.1) $0 \le \mathfrak{R}^{\beta+}_{(3)}\left(a_i\right) \le 1$ and $0 \le \mathfrak{R}^{\beta+}_{(4)}\left(a_i\right) \le 1$ for all $\beta \in Z^{+}$;
(R4.2) $\mathfrak{R}^{\beta+}_{(3)}\left(a_i\right) = 0$ and $\mathfrak{R}^{\beta+}_{(4)}\left(a_i\right) = 0$ for all $\beta \in Z^{+}$ if and only if $\mathcal{C}^{w}_{i} = \mathcal{C}^{w}_{-}$;
(R4.3) $\mathfrak{R}^{\beta+}_{(3)}\left(a_i\right) = 1$ and $\mathfrak{R}^{\beta+}_{(4)}\left(a_i\right) = 1$ for all $\beta \in Z^{+}$ if and only if $\mathcal{C}^{w}_{i} = \mathcal{C}^{w}_{+}$.

Proof The proofs of (R4.1)–(R4.3) are as analogous to those of (R3.1)–(R3.3), respectively.

Finally, all of the $\mathfrak{A}$ candidate alternatives are sorted in decreasing order of their relative closeness coefficients $\mathfrak{R}^{\beta+}_{(3)}(a_i)$ or $\mathfrak{R}^{\beta+}_{(4)}(a_i)$. The best compromise collections with the highest relative closeness coefficients in the fixed anchoring framework can be generated by

$$\mathcal{A}^{\beta+}_{(3)} = \left\{ a_i \middle| \max\nolimits_{i=1}^{\mathfrak{A}} \mathfrak{R}^{\beta+}_{(3)}(a_i), a_i \in \mathcal{A} \right\} \tag{54}$$

$$\mathcal{A}^{\beta+}_{(4)} = \left\{ a_i \middle| \max\nolimits_{i=1}^{\mathfrak{A}} \mathfrak{R}^{\beta+}_{(4)}(a_i), a_i \in \mathcal{A} \right\} \tag{55}$$

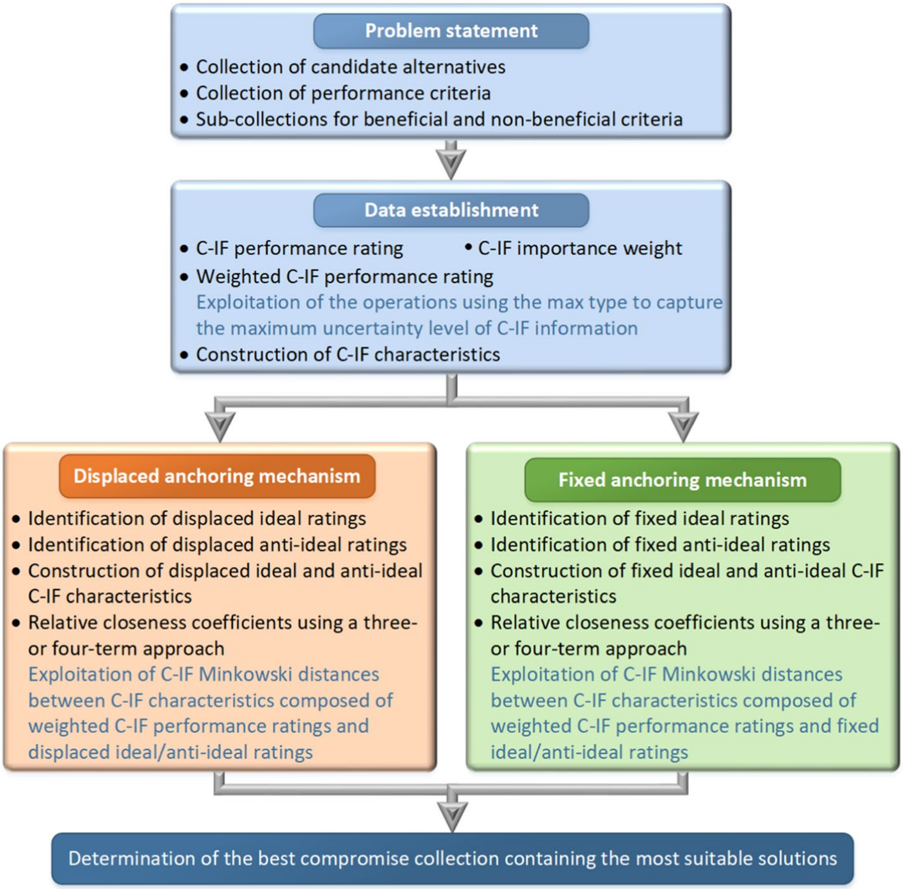


Fig. 4 Schematic representation of the C-IF TOPSIS methodology

4.2 Proposed C-IF TOPSIS algorithms

The evolved C-IF TOPSIS methodology employs four fundamental architectures to tackle MCDA issues in complex and uncertain situations. As stated in the C-IF TOPSIS technique in the previous subsection, the four basic architectures are: (1) a displaced anchoring frame based on the three-term representation, (2) a displaced anchoring frame based on the four-term representation, (3) a fixed anchoring frame based on the three-term representation, and (4) a fixed anchoring frame based on the four-term representation. The schematic representation of the C-IF TOPSIS methodology is depicted in Fig. 4. The C-IF TOPSIS procedure is divided into two phases: the foundation phase and the critical core technology phase. The foundation phase consists of two tasks: problem statement and data establishment. The problem statement, for example, includes two collections of candidate alternatives and performance criteria (divided into beneficial and non-beneficial criteria). The data establishment includes the ascertainment of C-IF performance ratings and C-IF importance weights in order to calculate weighted C-IF performance ratings and their corresponding C-IF characteristics. Following that, the critical core technology phase involves either a displaced anchoring mechanism or a fixed anchoring mechanism. The displaced anchoring mechanism encompasses the recognition of displaced ideal and displaced anti-ideal ratings, the construction of displaced ideal/anti-ideal C-IF characteristics, and the ascertainment of relative closeness coefficients using the C-IF Minkowski distance measures with a three- or four-term representation frame. Similarly, the fixed anchoring mechanism encompasses the recognition of fixed ideal and anti-ideal ratings, the construction of fixed ideal/anti-ideal C-IF characteristics, and the ascertainment of relative closeness coefficients using a three- or four-term approach. The best compromise collection containing the most suitable solutions can be determined by comparing relative closeness coefficients to support intelligent decision analysis under complicated uncertain scenarios.

The evolved C-IF TOPSIS methodology is implemented using the subsequent algorithmic steps:

Step 1 Compile a collection of candidate alternatives $\mathcal{A} = \{a_1, a_2, \cdots, a_{\mathfrak{A}}\}$. Make a collection of performance criteria $\mathcal{P} = \{p_1, p_2, \cdots, p_{\mathfrak{P}}\}$ and divide it into two sub-collections: $\mathcal{P}_{\mathrm{I}}$, which contains beneficial criteria, and $\mathcal{P}_{\mathrm{II}}$, which contains non-beneficial criteria.

Step 2 Utilize data collection tools such as semantic assessments to construct the C-IF importance weight $\varpi_j = (w_j, \omega_j; \gamma_j)$ for all $p_j \in \mathcal{P}$ and the C-IF performance rating $\varsigma_{ij} = (m_{ij}, n_{ij}; r_{ij})$ of $a_i \in \mathcal{A}$ judging by each p_j.

Step 3 Combine ς_{ij} with ϖ_j to calculate the weighted performance rating $\varsigma_{ij}^{\varpi} = \left(m_{ij}^{\varpi}, n_{ij}^{\varpi}; r_{ij}^{\varpi}\right)$ using Eq. (38), which yields the corresponding hesitancy h_{ij}^{ϖ}. Employ Eq. (39) to form the C-IF characteristic $\mathcal{C}_i^{\varpi}$ for each $a_i \in \mathcal{A}$

Steps 4−7 Take either a displaced anchoring mechanism or a fixed anchoring mechanism.

Prioritization algorithm based on the displaced anchoring mechanism:

Step 4 Identify the displaced ideal rating $\varsigma_{*j}^{\varpi} = \left(m_{*j}^{\varpi}, n_{*j}^{\varpi}; r_{*j}^{\varpi}\right)$ (involving hesitancy h_{*j}^{ϖ}) using Eq. (40). Establish the displaced ideal C-IF characteristic $\mathcal{C}_*^{\varpi}$ using Eq. (42).

Step 5 Produce the displaced anti-ideal rating $\varsigma_{\neg j}^{\varpi} = \left(m_{\neg j}^{\varpi}, n_{\neg j}^{\varpi}; r_{\neg j}^{\varpi}\right)$ (involving hesitancy $h_{\neg j}^{\varpi}$) using Eq. (41). Build the displaced anti-ideal C-IF characteristic $\mathcal{C}_{\neg}^{\varpi}$ using Eq. (43).

Step 6 Assign a metric parameter $\beta \in Z^+$. Calculate the C-IF Minkowski distances $\mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_i^{\varpi}, \mathcal{C}_*^{\varpi}\right)$ and $\mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_i^{\varpi}, \mathcal{C}_{\neg}^{\varpi}\right)$ using Eq. (29), or alternately $\mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_i^{\varpi}, \mathcal{C}_*^{\varpi}\right)$ and $\mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_i^{\varpi}, \mathcal{C}_{\neg}^{\varpi}\right)$ using Eq. (30).

Step 7 Derive the relative closeness coefficient $\mathfrak{R}_{(3)}^{\beta*}(a_i)$ (or $\mathfrak{R}_{(4)}^{\beta*}(a_i)$) using Eq. (44) (or Eq. (45)) for each $a_i \in \mathcal{A}$. Compose the best compromise collection $\mathcal{A}_{(3)}^{\beta*}$ (or $\mathcal{A}_{(4)}^{\beta*}$) using Eq. (46) (or Eq. (47)) to obtain the most suitable solution(s).

Prioritization algorithm based on the fixed anchoring mechanism:

Step 4′ Identify the fixed ideal rating $\varsigma_{+j}^{\varpi} = \left(m_{+j}^{\varpi}, n_{+j}^{\varpi}; r_{+j}^{\varpi}\right)$ (involving hesitancy h_{+j}^{ϖ}) using Eq. (48). Establish the fixed ideal C-IF characteristic $\mathcal{C}_+^{\varpi}$ using Eq. (50).

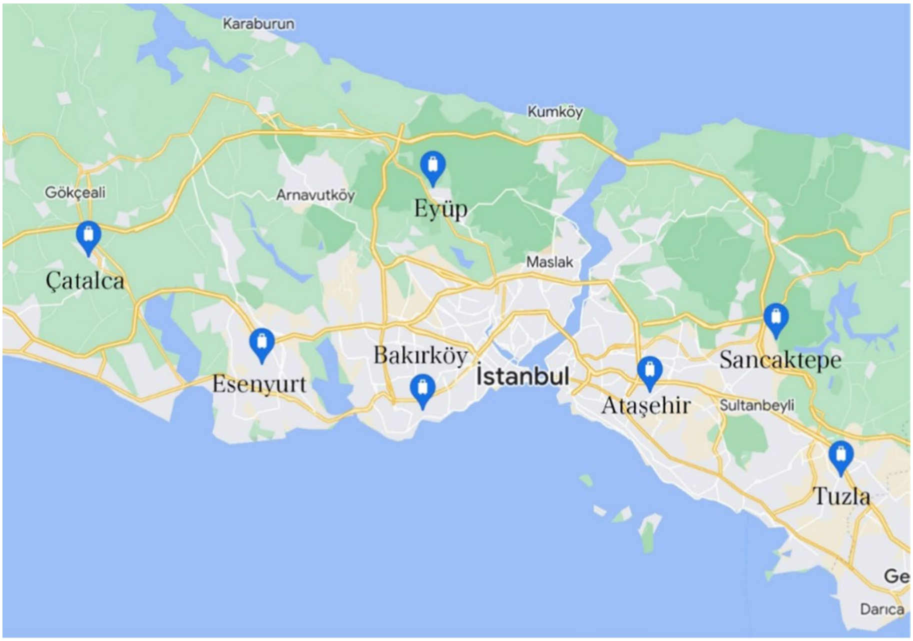


Fig. 5 Candidate locations for the Istanbul Epidemic Hospital

Step 5′ Produce the fixed anti-ideal rating $\varsigma_{-j}^{w} = \left(m_{-j}^{w}, n_{-j}^{w}; r_{-j}^{w}\right)$ (involving hesitancy h_{-j}^{w}) using Eq. (49). Build the fixed anti-ideal C-IF characteristic $\mathcal{C}_{-}^{w}$ using Eq. (51).

Step 6′ Assign a metric parameter $\beta \in Z^{+}$. Calculate the C-IF Minkowski distances $\mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_{i}^{w}, \mathcal{C}_{+}^{w}\right)$ and $\mathfrak{D}_{(3)}^{\beta}\left(\mathcal{C}_{i}^{w}, \mathcal{C}_{-}^{w}\right)$ using Eq. (29), or alternately $\mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_{i}^{w}, \mathcal{C}_{+}^{w}\right)$ and $\mathfrak{D}_{(4)}^{\beta}\left(\mathcal{C}_{i}^{w}, \mathcal{C}_{-}^{w}\right)$ using Eq. (30).

Step 7′ Derive the relative closeness coefficient $\mathfrak{R}_{(3)}^{\beta+}\left(a_{i}\right)$ (or $\mathfrak{R}_{(4)}^{\beta+}\left(a_{i}\right)$) using Eq. (52) (or Eq. (53)) for each $a_{i} \in \mathcal{A}$. Compose the best compromise collection $\mathcal{A}_{(3)}^{\beta+}$ (or $\mathcal{A}_{(4)}^{\beta+}$) using Eq. (54) (or Eq. (55)) to obtain the most suitable solution(s).

5 Model application and comparative study

This section applies the evolved C-IF TOPSIS methodology to investigate a site selection issue at an epidemic hospital in Istanbul, Turkey. This question was first posed by Alkan and Kahraman (2022b), who examined a real-world MCDA task to assess and select suitable sites for large-scale epidemic hospitals for Istanbul authorities in the post-COVID-19 era. Furthermore, this section will perform sensitivity analyses and comparative studies with other relevant fuzzy TOPSIS methods to confirm the robustness, adaptability, and utility of the current C-IF TOPSIS techniques.

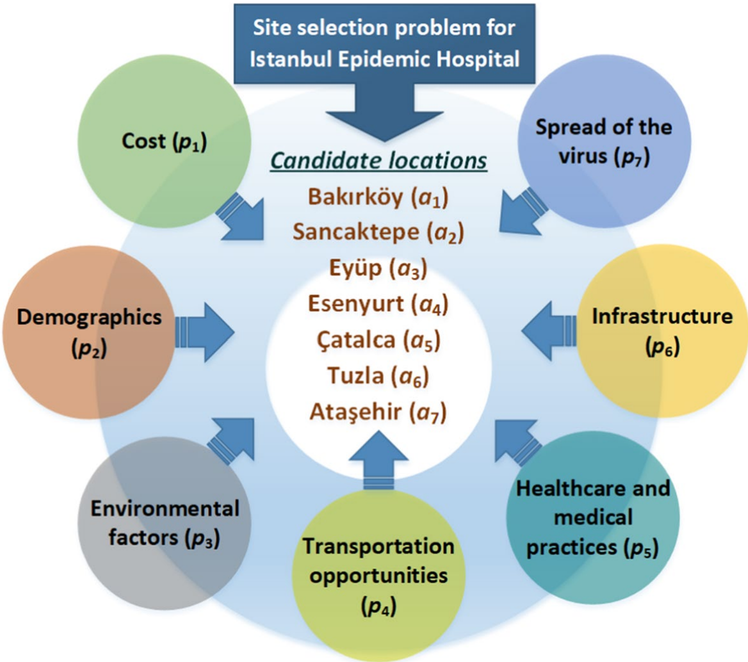


Fig. 6 Brief description of epidemic hospital location issues

Table 3 Data on C-IF performance ratings and C-IF importance weights

a_i	$\varsigma_{i1}=(m_{i1},n_{i1};r_{i1})$	$\varsigma_{i2}=(m_{i2},n_{i2};r_{i2})$	$\varsigma_{i3}=(m_{i3},n_{i3};r_{i3})$	$\varsigma_{i4}=(m_{i4},n_{i4};r_{i4})$
a_1	(0.667, 0.283; 0.094)	(0.567, 0.383; 0.094)	(0.333, 0.617; 0.094)	(0.800, 0.167; 0.130)
a_2	(0.333, 0.617; 0.094)	(0.767, 0.183; 0.094)	(0.533, 0.417; 0.094)	(0.500, 0.450; 0.000)
a_3	(0.533, 0.417; 0.094)	(0.733, 0.217; 0.094)	(0.367, 0.583; 0.094)	(0.633, 0.317; 0.094)
a_4	(0.267, 0.683; 0.094)	(0.833, 0.133; 0.075)	(0.267, 0.683; 0.094)	(0.433, 0.517; 0.094)
a_5	(0.433, 0.517; 0.094)	(0.333, 0.617; 0.094)	(0.667, 0.283; 0.094)	(0.367, 0.583; 0.094)
a_6	(0.500, 0.450; 0.000)	(0.567, 0.383; 0.094)	(0.467, 0.483; 0.094)	(0.367, 0.583; 0.094)
a_7	(0.667, 0.283; 0.094)	(0.733, 0.217; 0.094)	(0.467, 0.483; 0.094)	(0.633, 0.317; 0.094)

	$\varsigma_{i5}=(m_{i5},n_{i5};r_{i5})$	$\varsigma_{i6}=(m_{i6},n_{i6};r_{i6})$	$\varsigma_{i7}=(m_{i7},n_{i7};r_{i7})$	p_j	$\varpi_j=(w_j,\omega_j;\gamma_j)$
a_1	(0.633, 0.317; 0.094)	(0.367, 0.583; 0.094)	(0.767, 0.183; 0.094)	p_1	(0.533, 0.417; 0.094)
a_2	(0.467, 0.483; 0.094)	(0.600, 0.350; 0.141)	(0.667, 0.283; 0.094)	p_2	(0.767, 0.183; 0.094)
a_3	(0.633, 0.317; 0.094)	(0.633, 0.317; 0.094)	(0.667, 0.283; 0.094)	p_3	(0.667, 0.283; 0.094)
a_4	(0.533, 0.417; 0.094)	(0.133, 0.850; 0.120)	(0.867, 0.117; 0.075)	p_4	(0.733, 0.217; 0.094)
a_5	(0.400, 0.550; 0.141)	(0.867, 0.117; 0.075)	(0.167, 0.800; 0.120)	p_5	(0.367, 0.583; 0.094)
a_6	(0.400, 0.550; 0.141)	(0.567, 0.383; 0.094)	(0.333, 0.617; 0.094)	p_6	(0.267, 0.683; 0.094)
a_7	(0.433, 0.517; 0.094)	(0.267, 0.683; 0.094)	(0.733, 0.217; 0.094)	p_7	(0.833, 0.133; 0.075)

Table 4 Outcomes of weighted performance ratings and their corresponding degrees of hesitancy

a_i	$\varsigma_{i1}^{w} = \left(m_{i1}^{w}, n_{i1}^{w}; r_{i1}^{w}\right)$	h_{i1}^{w}	$\varsigma_{i2}^{w} = \left(m_{i2}^{w}, n_{i2}^{w}; r_{i2}^{w}\right)$	h_{i2}^{w}	$\varsigma_{i3}^{w} = \left(m_{i3}^{w}, n_{i3}^{w}; r_{i3}^{w}\right)$	h_{i3}^{w}
a_1	(0.3555, 0.5820; 0.094)	0.0625	(0.4349, 0.4959; 0.094)	0.0692	(0.2221, 0.7254; 0.094)	0.0525
a_2	(0.1775, 0.7767; 0.094)	0.0458	(0.5883, 0.3325; 0.094)	0.0792	(0.3555, 0.5820; 0.094)	0.0625
a_3	(0.2841, 0.6601; 0.094)	0.0558	(0.5622, 0.3603; 0.094)	0.0775	(0.2448, 0.7010; 0.094)	0.0542
a_4	(0.1423, 0.8152; 0.094)	0.0425	(0.6389, 0.2917; 0.094)	0.0694	(0.1781, 0.7727; 0.094)	0.0492
a_5	(0.2308, 0.7184; 0.094)	0.0508	(0.2554, 0.6871; 0.094)	0.0575	(0.4449, 0.4859; 0.094)	0.0692
a_6	(0.2665, 0.6794; 0.094)	0.0542	(0.4349, 0.4959; 0.094)	0.0692	(0.3115, 0.6293; 0.094)	0.0592
a_7	(0.3555, 0.5820; 0.094)	0.0625	(0.5622, 0.3603; 0.094)	0.0775	(0.3115, 0.6293; 0.094)	0.0592
	$\varsigma_{i4}^{w} = \left(m_{i4}^{w}, n_{i4}^{w}; r_{i41}^{w}\right)$	h_{i4}^{w}	$\varsigma_{i5}^{w} = \left(m_{i5}^{w}, n_{i5}^{w}; r_{i5}^{w}\right)$	h_{i5}^{w}	$\varsigma_{i6}^{w} = \left(m_{i6}^{w}, n_{i6}^{w}; r_{i6}^{w}\right)$	h_{i6}^{w}
a_1	(0.5864, 0.3478; 0.130)	0.0658	(0.2323, 0.7152; 0.094)	0.0525	(0.0980, 0.8678; 0.094)	0.0342
a_2	(0.3665, 0.5694; 0.094)	0.0641	(0.1714, 0.7844; 0.094)	0.0442	(0.1602, 0.7940; 0.141)	0.0459
a_3	(0.4640, 0.4652; 0.094)	0.0708	(0.2323, 0.7152; 0.094)	0.0525	(0.1690, 0.7835; 0.094)	0.0475
a_4	(0.3174, 0.6218; 0.094)	0.0608	(0.1956, 0.7569; 0.094)	0.0475	(0.0355, 0.9525; 0.120)	0.0120
a_5	(0.2690, 0.6735; 0.094)	0.0575	(0.1468, 0.8124; 0.141)	0.0408	(0.2315, 0.7201; 0.094)	0.0484
a_6	(0.2690, 0.6735; 0.094)	0.0575	(0.1468, 0.8124; 0.141)	0.0408	(0.1514, 0.8044; 0.094)	0.0442
a_7	(0.4640, 0.4652; 0.094)	0.0708	(0.1589, 0.7986; 0.094)	0.0425	(0.0713, 0.8995; 0.094)	0.0292
	$\varsigma_{i7}^{w} = \left(m_{i7}^{w}, n_{i7}^{w}; r_{i7}^{w}\right)$	h_{i7}^{w}	The C-IF characteristic $\mathcal{C}_i^{w} = \left\{ \langle p_j, m_{ij}^{w}, n_{ij}^{w}; r_{ij}^{w} \rangle \middle| p_j \in \mathcal{P} \right\}$			
a_1	(0.6389, 0.2917; 0.094)	0.0694	$\mathcal{C}_1^{w} = \{\langle p_1, m_{11}^{w}, n_{11}^{w}; r_{11}^{w}\rangle, \langle p_2, m_{12}^{w}, n_{12}^{w}; r_{12}^{w}\rangle, \cdots, \langle p_7, m_{17}^{w}, n_{17}^{w}; r_{17}^{w}\rangle\}$			
a_2	(0.5556, 0.3784; 0.094)	0.0660	$\mathcal{C}_2^{w} = \{\langle p_1, m_{21}^{w}, n_{21}^{w}; r_{21}^{w}\rangle, \langle p_2, m_{22}^{w}, n_{22}^{w}; r_{22}^{w}\rangle, \cdots, \langle p_7, m_{27}^{w}, n_{27}^{w}; r_{27}^{w}\rangle\}$			
a_3	(0.5556, 0.3784; 0.094)	0.0660	$\mathcal{C}_3^{w} = \{\langle p_1, m_{31}^{w}, n_{31}^{w}; r_{31}^{w}\rangle, \langle p_2, m_{32}^{w}, n_{32}^{w}; r_{32}^{w}\rangle, \cdots, \langle p_7, m_{37}^{w}, n_{37}^{w}; r_{37}^{w}\rangle\}$			
a_4	(0.7222, 0.2344; 0.075)	0.0434	$\mathcal{C}_4^{w} = \{\langle p_1, m_{41}^{w}, n_{41}^{w}; r_{41}^{w}\rangle, \langle p_2, m_{42}^{w}, n_{42}^{w}; r_{42}^{w}\rangle, \cdots, \langle p_7, m_{47}^{w}, n_{47}^{w}; r_{47}^{w}\rangle\}$			
a_5	(0.1391, 0.8266; 0.120)	0.0343	$\mathcal{C}_5^{w} = \{\langle p_1, m_{51}^{w}, n_{51}^{w}; r_{51}^{w}\rangle, \langle p_2, m_{52}^{w}, n_{52}^{w}; r_{52}^{w}\rangle, \cdots, \langle p_7, m_{57}^{w}, n_{57}^{w}; r_{57}^{w}\rangle\}$			
a_6	(0.2774, 0.6679; 0.094)	0.0547	$\mathcal{C}_6^{w} = \{\langle p_1, m_{61}^{w}, n_{61}^{w}; r_{61}^{w}\rangle, \langle p_2, m_{62}^{w}, n_{62}^{w}; r_{62}^{w}\rangle, \cdots, \langle p_7, m_{67}^{w}, n_{67}^{w}; r_{67}^{w}\rangle\}$			
a_7	(0.6106, 0.3211; 0.094)	0.0683	$\mathcal{C}_7^{w} = \{\langle p_1, m_{71}^{w}, n_{71}^{w}; r_{71}^{w}\rangle, \langle p_2, m_{72}^{w}, n_{72}^{w}; r_{72}^{w}\rangle, \cdots, \langle p_7, m_{77}^{w}, n_{77}^{w}; r_{77}^{w}\rangle\}$			

5.1 Application to epidemic hospital site selection

Emerging infectious diseases have spread rapidly in humans in recent years. COVID-19, for example, which appeared in December 2019, is caused by a new type of coronavirus. The COVID-19 outbreak is rapidly spreading worldwide. The majority of infected patients can recover and regain their health. However, a small number of infected patients may develop respiratory distress syndrome, severe pneumonia, shock, multiple organ failure, or even death in severe cases. Various emerging infectious diseases adhere to their transmission pathways and protection principles. However, there are still differences in the characteristics of various pathogens. As a result, in the face of emerging major infectious diseases with unknown causes, health authorities and epidemic prevention experts in various countries must review and revise countermeasures by collecting epidemic development and relevant latest information on a continuous basis. In the post-COVID-19 era, many countries are focusing on establishing a comprehensive medical network for epidemic prevention and control, as well as establishing dedicated epidemic hospitals. The establishment of large-scale epidemic hospitals can improve the grading and treatment of mild and severe cases, patient triage, care subdivision, visitor control, and environmental management. Setting up

Table 5 Displaced ideal and anti-ideal ratings and their corresponding degrees of hesitancy

p_j	$\varsigma^{w}_{*j} = \left(m^{w}_{*j}, n^{w}_{*j}; r^{w}_{*j}\right)$	h^{w}_{*j}	$\varsigma^{w}_{\neg j} = \left(m^{w}_{\neg j}, n^{w}_{\neg j}; r^{w}_{\neg j}\right)$	$h^{w}_{\neg j}$
p_1	(0.1423, 0.8152; 0.0940)	0.0425	(0.3555, 0.5820; 0.0940)	0.0625
p_2	(0.6389, 0.2917; 0.0940)	0.0694	(0.2554, 0.6871; 0.0940)	0.0575
p_3	(0.4449, 0.4859; 0.0940)	0.0692	(0.1781, 0.7727; 0.0940)	0.0492
p_4	(0.5864, 0.3478; 0.1300)	0.0658	(0.2690, 0.6735; 0.1300)	0.0575
p_5	(0.2323, 0.7152; 0.1410)	0.0525	(0.1468, 0.8124; 0.1410)	0.0408
p_6	(0.2315, 0.7201; 0.1410)	0.0484	(0.0355, 0.9525; 0.1410)	0.0120
p_7	(0.7222, 0.2344; 0.1200)	0.0434	(0.1391, 0.8266; 0.1200)	0.0343

epidemic hospitals allows infected people to be treated while keeping the health care system running normally.

According to Alkan and Kahraman (2022b), the COVID-19 outbreak spread quickly during the pandemic in several cities in Turkey, particularly Istanbul, where large numbers of infected patients accumulated. The Istanbul authorities have decided to establish a large-scale epidemic hospital in seven candidate locations, consisting of Bakırköy (a_1), Sancaktepe (a_2), Eyüp (a_3), Esenyurt (a_4), Çatalca (a_5), Tuzla (a_6), and Ataşehir (a_7), to prevent the spread of COVID-19 or other infectious diseases in the future and to allow infected patients to receive proper treatment. Figure 5 depicts a Google map of the seven candidate locations in Istanbul.

Authorities anticipate that the epidemic hospital will prioritize and treat patients with infectious diseases. Aside from housing COVID-19 patients, the epidemic hospital also prevents other epidemics from spreading to other patients or visitors within the hospital. Furthermore, the epidemic hospital will actively and fully cooperate with the policy of the infectious disease prevention and treatment network, as well as the inspection of facilities, equipment, protective equipment, and work quality in isolation wards, as determined by the competent authority. The Istanbul authorities used seven criteria to evaluate the optimal location of the Istanbul Epidemic Hospital in the problem described by Alkan and Kahraman (2022b), consisting of cost (p_1), demographics (p_2), environmental factors (p_3), transportation opportunities (p_4), healthcare and medical practices (p_5), infrastructure (p_6), and spread of the virus (p_7). Only p_1 is a non-beneficial criterion among the above, while the other six (i.e., p_2–p_7) are beneficial. Figure 6 depicts a high-level overview of the site selection problem for the Istanbul Epidemic Hospital under consideration.

This study attempts to employ the C-IF TOPSIS algorithm to deal with the site selection issue of the Istanbul Epidemic Hospital to scrutinize the workableness and suitability of the initiated methodology in realistic problems. In Step 1, the collections of candidate alternatives (i.e., seven locations) and performance criteria are represented as $\mathcal{A} = \left\{a_1, a_2, \cdots, a_7\right\}$ ($\mathfrak{A} = 7$) and $\mathcal{P} = \left\{p_1, p_2, \cdots, p_7\right\}$ ($\mathfrak{P} = 7$), respectively, based on Fig. 6. Furthermore, $\mathcal{P}$ is divided into two sub-collections: $\mathcal{P}_{\mathrm{I}} = \left\{p_2, p_3, \cdots, p_7\right\}$, which contains beneficial criteria, and $\mathcal{P}_{\mathrm{II}} = \left\{p_1\right\}$, which contains non-beneficial criteria.

In Step 2, as pointed out in the research by Alkan and Kahraman (2022b), the expert group has given semantic assessments of the significance or importance of criteria and the performance of a candidate location judging by each criterion, based on individual professional experience and judgment. After pooling the opinions offered by the expert group, Alkan and Kahraman transformed these semantic assessments into the C-IF values. The data of the C-IF performance rating ς_{ij} and the C-IF importance weight ϖ_j are revealed in Table 3.

In Step 3, the weighted performance rating ς_{ij}^{ϖ} and its corresponding degree of hesitancy h_{ij}^{ϖ} were calculated by combining ς_{ij} with ϖ_j. Furthermore, the C-IF characteristic $\mathcal{C}_i^{\varpi}$ can be constituted by collecting the obtained ς_{ij}^{ϖ} across the seven performance criteria. Table 4 displays the results of the related ascertainment.

Following that, the prioritization algorithm of the C-IF TOPSIS methodology is run. Steps 4−7 of the displaced anchoring mechanism or Steps 4′−7′ of the fixed anchoring mechanism can be employed by decision-makers or analysts. This study first demonstrates the implementation process of Steps 4−7. In Step 4, this study produced the displaced ideal rating ς_{*j}^{ϖ} using Eq. (40) and the corresponding degree of hesitancy h_{*j}^{ϖ}, and the outcomes are listed in the left section of Table 5. Taking $p_1 \in \mathcal{P}_{\mathrm{II}}$ and $p_7 \in \mathcal{P}_{\mathrm{I}}$ as examples, the ratings ς_{*1}^{ϖ} and ς_{*7}^{ϖ} were calculated as follows:

$$\varsigma_{*1}^{\varpi} = \left(m_{*1}^{\varpi}, n_{*1}^{\varpi}; r_{*1}^{\varpi}\right) = \left(\min_{i=1}^{7} m_{i1}^{\varpi}, \max_{i=1}^{7} n_{i1}^{\varpi}; \max_{i=1}^{7} r_{i1}^{\varpi}\right)$$

$$= (\min\{0.3555, 0.1775, 0.2841, 0.1423, 0.2308, 0.2665, 0.3555\}, \max\{0.5820, 0.7767, 0.6601,$$

$$0.8152, 0.7184, 0.6794, 0.5820\}; \max\{0.094, 0.094, 0.094, 0.094, 0.094, 0.094, 0.094\})$$

$$= (0.1423, 0.8152, 0.094),$$

$$\varsigma_{*7}^{\varpi} = \left(m_{*7}^{\varpi}, n_{*7}^{\varpi}; r_{*7}^{\varpi}\right) = \left(\max_{i=1}^{7} m_{i7}^{\varpi}, \min_{i=1}^{7} n_{i7}^{\varpi}; \max_{i=1}^{7} r_{i7}^{\varpi}\right)$$

$$= (\max\{0.6389, 0.5556, 0.5556, 0.7222, 0.1391, 0.2774, 0.6106\}, \min\{0.2917, 0.3784, 0.3784,$$

$$0.2344, 0.8266, 0.6679, 0.3211\}; \max\{0.094, 0.094, 0.094, 0.075, 0.120, 0.094, 0.094\})$$

$$= (0.7222, 0.2344, 0.120).$$

Furthermore, their respective hesitancy degrees were calculated as $h_{*1}^{\varpi} = 1 - 0.1423 - 0.8152 = 0.0425$ and $h_{*7}^{\varpi} = 1 - 0.7222 - 0.2344 = 0.0434$. In a similar vein, based on Step 5, this study generated the displaced anti-ideal rating $\varsigma_{\neg j}^{\varpi}$ using Eq. (41) and its corresponding degree of hesitancy $h_{\neg j}^{\varpi}$; the outcomes are sketched in the right section of Table 5. The displaced ideal C-IF characteristic $\mathcal{C}_*^{\varpi}$ and the displaced anti-ideal C-IF characteristic $\mathcal{C}_\neg^{\varpi}$ can be formed by gathering the obtained ς_{*j}^{ϖ} and $\varsigma_{\neg j}^{\varpi}$, respectively, across all $p_j \in \mathcal{P}$. Specifically, $\mathcal{C}_*^{\varpi} = \left\{\langle p_1, m_{*1}^{\varpi}, n_{*1}^{\varpi}; r_{*1}^{\varpi}\rangle, \langle p_2, m_{*2}^{\varpi}, n_{*2}^{\varpi}; r_{*2}^{\varpi}\rangle, \cdots, \langle p_7, m_{*7}^{\varpi}, n_{*7}^{\varpi}; r_{*7}^{\varpi}\rangle\right\}$ and $\mathcal{C}_\neg^{\varpi} = \{\langle p_1, m_{\neg 1}^{\varpi}, n_{\neg 1}^{\varpi}; r_{\neg 1}^{\varpi}\rangle, \langle p_2, m_{\neg 2}^{\varpi}, n_{\neg 2}^{\varpi}; r_{\neg 2}^{\varpi}\rangle, \cdots, \langle p_7, m_{\neg 7}^{\varpi}, n_{\neg 7}^{\varpi}; r_{\neg 7}^{\varpi}\rangle\}$.

The conventional TOPSIS method measures the degrees of separation between candidate alternatives and ideal/anti-ideal solutions using geometric distances in the Euclidean plane. As a result, in the demonstrative case, this study assumes $\beta = 2$. The C-IF Euclidean distance will be exploited to measure the degree of separation between the C-IF characteristic $\mathcal{C}_i^{\varpi}$ and the displaced ideal C-IF characteristic $\mathcal{C}_*^{\varpi}$, as well as the degree of separation between $\mathcal{C}_i^{\varpi}$ and the displaced anti-ideal C-IF characteristic $\mathcal{C}_\neg^{\varpi}$.

Table 6 Fixed ideal and anti-ideal ratings and their corresponding degrees of hesitancy

p_j	$\varsigma_{+j}^{w} = \left(m_{+j}^{w}, n_{+j}^{w}; r_{+j}^{w}\right)$	h_{+j}^{w}	$\varsigma_{-j}^{w} = \left(m_{-j}^{w}, n_{-j}^{w}; r_{-j}^{w}\right)$	h_{-j}^{w}
p_1	(0.0000, 1.0000; 0.0940)	0.0000	(0.5330, 0.4170; 0.0940)	0.0500
p_2	(0.7670, 0.1830; 0.0940)	0.0500	(0.0000, 1.0000; 0.0940)	0.0000
p_3	(0.6670, 0.2830; 0.0940)	0.0500	(0.0000, 1.0000; 0.0940)	0.0000
p_4	(0.7330, 0.2170; 0.0940)	0.0500	(0.0000, 1.0000; 0.0940)	0.0000
p_5	(0.3670, 0.5830; 0.0940)	0.0500	(0.0000, 1.0000; 0.0940)	0.0000
p_6	(0.2670, 0.6830; 0.0940)	0.0500	(0.0000, 1.0000; 0.0940)	0.0000
p_7	(0.8330, 0.1330; 0.0750)	0.0340	(0.0000, 1.0000; 0.0750)	0.0000

It is worth noting that the cardinality of the collection $\mathcal{P}$ is given as $card(\mathcal{P}) = 7$. In Step 6, based on the three-term representation in Eq. (33), the following C-IF Euclidean distances were rendered, including $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_1^{w}, \mathcal{C}_*^{w}\right) = 0.0840$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_2^{w}, \mathcal{C}_*^{w}\right) = 0.0636$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_3^{w}, \mathcal{C}_*^{w}\right) = 0.0714$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_4^{w}, \mathcal{C}_*^{w}\right) = 0.0917$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_5^{w}, \mathcal{C}_*^{w}\right) = 0.1527$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_6^{w}, \mathcal{C}_*^{w}\right) = 0.1234$, and $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_7^{w}, \mathcal{C}_*^{w}\right) = 0.0770$ with relevance to the displaced ideal C-IF characteristic $\mathcal{C}_*^{w}$; and $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_1^{w}, \mathcal{C}_\neg^{w}\right) = 0.1288$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_2^{w}, \mathcal{C}_\neg^{w}\right) = 0.1258$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_3^{w}, \mathcal{C}_\neg^{w}\right) = 0.1229$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_4^{w}, \mathcal{C}_\neg^{w}\right) = 0.1480$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_5^{w}, \mathcal{C}_\neg^{w}\right) = 0.0749$, $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_6^{w}, \mathcal{C}_\neg^{w}\right) = 0.0659$, and $\mathfrak{D}_{(3)}^{2}\left(\mathcal{C}_7^{w}, \mathcal{C}_\neg^{w}\right) = 0.1276$ in relation to the displaced anti-ideal C-IF characteristic $\mathcal{C}_\neg^{w}$. Furthermore, based on the four-term representation in Eq. (34), the following C-IF Euclidean distances were generated, including $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_1^{w}, \mathcal{C}_*^{w}\right) = 0.0842$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_2^{w}, \mathcal{C}_*^{w}\right) = 0.0637$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_3^{w}, \mathcal{C}_*^{w}\right) = 0.0716$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_4^{w}, \mathcal{C}_*^{w}\right) = 0.0918$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_5^{w}, \mathcal{C}_*^{w}\right) = 0.1527$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_6^{w}, \mathcal{C}_*^{w}\right) = 0.1235$, and $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_7^{w}, \mathcal{C}_*^{w}\right) = 0.0772$ with relevance to $\mathcal{C}_*^{w}$; and $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_1^{w}, \mathcal{C}_\neg^{w}\right) = 0.1289$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_2^{w}, \mathcal{C}_\neg^{w}\right) = 0.1261$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_3^{w}, \mathcal{C}_\neg^{w}\right) = 0.1231$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_4^{w}, \mathcal{C}_\neg^{w}\right) = 0.1480$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_5^{w}, \mathcal{C}_\neg^{w}\right) = 0.0751$, $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_6^{w}, \mathcal{C}_\neg^{w}\right) = 0.0661$, and $\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_7^{w}, \mathcal{C}_\neg^{w}\right) = 0.1278$ in connection with $\mathcal{C}_\neg^{w}$. Considering the measure $\mathfrak{D}_{(4)}^{2}$ between $\mathcal{C}_4^{w}$ and $\mathcal{C}_\neg^{w}$, one has:

$$
\begin{aligned}
\mathfrak{D}_{(4)}^{2}\left(\mathcal{C}_4^{w}, \mathcal{C}_\neg^{w}\right) = \frac{1}{2}\Bigg\{ & \frac{1}{\sqrt{2}\cdot 7}(|0.094-0.094| + |0.094-0.094| + |0.094-0.094| + |0.094-0.130| \\
& +|0.094-0.141| + |0.120-0.141| + |0.075-0.120|) \\
& +\left[\frac{1}{2\cdot 3}\left(|0.1423-0.3555|^2 + |0.6389-0.2554|^2 + |0.1781-0.1781|^2 + |0.3174-0.2690|^2\right.\right. \\
& +|0.1956-0.1468|^2 + |0.0355-0.0355|^2 + |0.7222-0.1391|^2 + |0.8152-0.5820|^2 \\
& +|0.2917-0.6871|^2 + |0.7727-0.7727|^2 + |0.6218-0.6735|^2 + |0.7569-0.8124|^2 \\
& +|0.9525-0.9525|^2 + |0.2344-0.8266|^2 + |0.0425-0.0625|^2 + |0.0694-0.0575|^2
\end{aligned}
$$

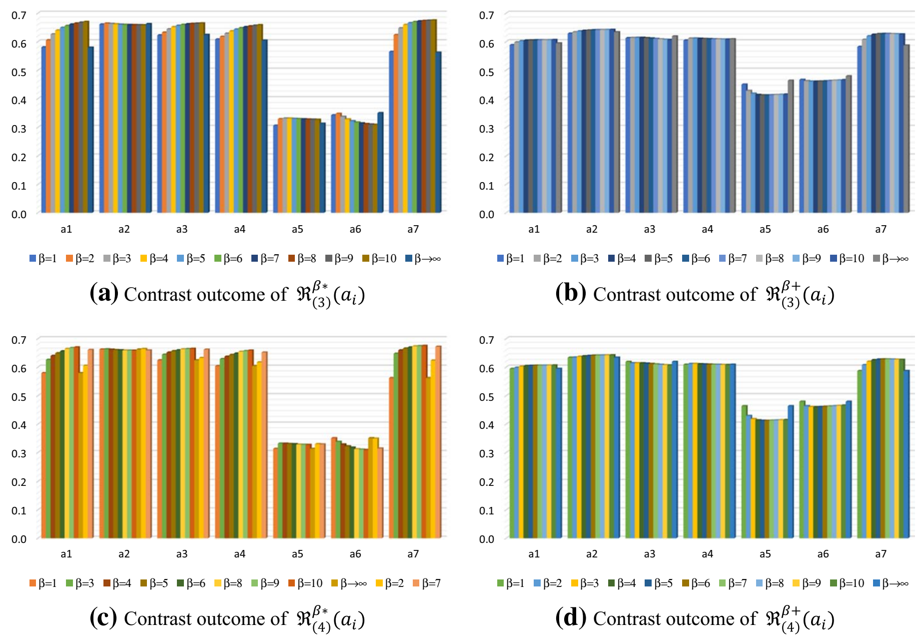
$$+|0.0492-0.0492|^2+|0.0608-0.0575|^2+|0.0475-0.0408|^2+|0.0120-0.0120|^2$$

$$+|0.0434-0.0343|^2)\big]^{\frac{1}{2}}\Big\}=0.1480.$$

Continue using $\beta=2$ as an example to show how the relative closeness coefficients $\mathfrak{R}^{2*}_{(3)}(a_i)$ and $\mathfrak{R}^{2*}_{(4)}(a_i)$ are calculated in Step 7. Using the three-term approach, Eq. (44) yields the following: $\mathfrak{R}^{2*}_{(3)}(a_1)=\mathfrak{D}^2_{(3)}(\mathcal{C}^w_1,\mathcal{C}^w_\neg)/\left(\mathfrak{D}^2_{(3)}(\mathcal{C}^w_1,\mathcal{C}^w_*)+\mathfrak{D}^2_{(3)}(\mathcal{C}^w_1,\mathcal{C}^w_\neg)\right)=0.1288/(0.0840+0.1288)=0.6053$, $\mathfrak{R}^{2*}_{(3)}(a_2)=0.6642$, $\mathfrak{R}^{2*}_{(3)}(a_3)=0.6324$, $\mathfrak{R}^{2*}_{(3)}(a_4)=0.6175$, $\mathfrak{R}^{2*}_{(3)}(a_5)=0.3291$, $\mathfrak{R}^{2*}_{(3)}(a_6)=0.3481$, and $\mathfrak{R}^{2*}_{(3)}(a_7)=0.6236$. Since $\mathfrak{R}^{2*}_{(3)}(a_2)>\mathfrak{R}^{2*}_{(3)}(a_3)>\mathfrak{R}^{2*}_{(3)}(a_7)>\mathfrak{R}^{2*}_{(3)}(a_4)>\mathfrak{R}^{2*}_{(3)}(a_1)>\mathfrak{R}^{2*}_{(3)}(a_6)>\mathfrak{R}^{2*}_{(3)}(a_5)$, the priority order of the seven candidate locations is $a_2\succ a_3\succ a_7\succ a_4\succ a_1\succ a_6\succ a_5$, which is consistent with the sorting result solved by Alkan and Kahraman (2022b). Based on Eq. (46), the best compromise collection is $\mathcal{A}^{2*}_{(3)}=\left\{a_i\middle|\max_{i=1}^7\mathfrak{R}^{2*}_{(3)}(a_i),a_i\in\mathcal{A}\right\}=\{a_2\}$. Besides this, based on Eq. (45) using the four-term approach, one has $\mathfrak{R}^{2*}_{(4)}(a_1)=0.6051$, $\mathfrak{R}^{2*}_{(4)}(a_2)=0.6642$, $\mathfrak{R}^{2*}_{(4)}(a_3)=0.6324$, $\mathfrak{R}^{2*}_{(4)}(a_4)=0.6171$, $\mathfrak{R}^{2*}_{(4)}(a_5)=0.3298$, $\mathfrak{R}^{2*}_{(4)}(a_6)=0.3489$, and $\mathfrak{R}^{2*}_{(4)}(a_7)=0.6232$. In view of $\mathfrak{R}^{2*}_{(4)}(a_2)>\mathfrak{R}^{2*}_{(4)}(a_3)>\mathfrak{R}^{2*}_{(4)}(a_7)>\mathfrak{R}^{2*}_{(4)}(a_4)>\mathfrak{R}^{2*}_{(4)}(a_1)>\mathfrak{R}^{2*}_{(4)}(a_6)>\mathfrak{R}^{2*}_{(4)}(a_5)$, the priority order of the candidate locations is $a_2\succ a_3\succ a_7\succ a_4\succ a_1\succ a_6\succ a_5$, which is also concordant with the outcome in Alkan and Kahraman (2022b). The best compromise collection, according to Eq. (47), is $\mathcal{A}^{2*}_{(4)}=\left\{a_i\middle|\max_{i=1}^7\mathfrak{R}^{2*}_{(4)}(a_i),a_i\in\mathcal{A}\right\}=\{a_2\}$. Using the three- and four-term approaches in the displaced anchoring mechanism, the C-IF TOPSIS ranking results based on the C-IF Euclidean distance measure, as well as the best compromise collection and the most suitable solution, are the same. In other words, $\mathcal{A}^{2*}_{(3)}=\mathcal{A}^{2*}_{(4)}=\{a_2\}$, and the candidate location Sancaktepe (a_2) is the most suitable solution.

Next, this study explicates the calculation process of Steps $4'-7'$ using the fixed anchoring mechanism. In Step $4'$, this study produced the fixed ideal rating ς^w_{+j} using Eq. (48) and the corresponding degree of hesitancy h^w_{+j}, and the outcomes are manifested in the left section of Table 6. Using p_1 and p_7 as an illustration again, the ratings ς^w_{+1} and ς^w_{+7} were rendered in this fashion: $\varsigma^w_{+1}=\left(m^w_{+1},n^w_{+1};r^w_{+1}\right)=(0.533,0.417;0.094)\otimes_{\max}(0,1;0)=(0,1;0.094)$, and $\varsigma^w_{+7}=(m^w_{+7},n^w_{+7};\ r^w_{+7})=(0.833,0.133;0.075)\otimes_{\max}(1,0;0)=(0.833,0.133;0.075)$. Additionally, the corresponding hesitancy degrees were derived as $h^w_{+1}=1-0-1=0$ and $h^w_{+7}=1-0.833-0.133=0.034$. In Step $5'$, this study identified the fixed anti-ideal rating ς^w_{-j} using Eq. (49) and its corresponding degree of hesitancy h^w_{-j}; the outcomes are portrayed in the right section of Table 6. The fixed ideal C-IF characteristic $\mathcal{C}^w_+$ and the fixed anti-ideal C-IF characteristic $\mathcal{C}^w_-$ can be constructed by $\mathcal{C}^w_+=\left\{\langle p_1,m^w_{+1},n^w_{+1};r^w_{+1}\rangle,\langle p_2,m^w_{+2},n^w_{+2};r^w_{+2}\rangle,\cdots,\langle p_7,m^w_{+7},n^w_{+7};r^w_{+7}\rangle\right\}$ and $\mathcal{C}^w_-=\{\langle p_1,m^w_{-1},n^w_{-1};r^w_{-1}\rangle,\langle p_2,m^w_{-2},n^w_{-2};r^w_{-2}\rangle,\cdots,\langle p_7,m^w_{-7},n^w_{-7};r^w_{-7}\rangle\}$, respectively.

This study still requires $\beta=2$ (i.e., C-IF Euclidean distance measure) to provide an example of the execution procedure under the fixed anchoring mechanism. In Step $6'$, the following C-IF Euclidean distances were calculated using the



(a) Contrast outcome of $\mathfrak{R}^{\beta*}_{(3)}(a_i)$ **(b)** Contrast outcome of $\mathfrak{R}^{\beta+}_{(3)}(a_i)$

(c) Contrast outcome of $\mathfrak{R}^{\beta*}_{(4)}(a_i)$ **(d)** Contrast outcome of $\mathfrak{R}^{\beta+}_{(4)}(a_i)$

Fig. 7 The juxtaposition of relative closeness coefficients in various scenarios

three-term representation in Eq. (33): $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{+}\right)=0.1431$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{2},\mathcal{C}^{w}_{+}\right)=0.1244$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{3},\mathcal{C}^{w}_{+}\right)=0.1311$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{4},\mathcal{C}^{w}_{+}\right)=0.1419$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{5},\mathcal{C}^{w}_{+}\right)=0.2041$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{6},\mathcal{C}^{w}_{+}\right)=0.1814$, and $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{7},\mathcal{C}^{w}_{+}\right)=0.1356$ relating to the fixed ideal C-IF characteristic $\mathcal{C}^{w}_{+}$; and $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{-}\right)=0.2127$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{2},\mathcal{C}^{w}_{-}\right)=0.2149$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{3},\mathcal{C}^{w}_{-}\right)=0.2081$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{4},\mathcal{C}^{w}_{-}\right)=0.2232$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{5},\mathcal{C}^{w}_{-}\right)=0.1525$, $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{6},\mathcal{C}^{w}_{-}\right)=0.1558$, and $\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{7},\mathcal{C}^{w}_{-}\right)=0.2095$ with respect to the fixed anti-ideal C-IF characteristic $\mathcal{C}^{w}_{-}$. On the other hand, using the four-term representation in Eq. (34), the following C-IF Euclidean distances were calculated: $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{+}\right)=0.1435$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{2},\mathcal{C}^{w}_{+}\right)=0.1247$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{3},\mathcal{C}^{w}_{+}\right)=0.1315$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{4},\mathcal{C}^{w}_{+}\right)=0.1421$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{5},\mathcal{C}^{w}_{+}\right)=0.2043$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{6},\mathcal{C}^{w}_{+}\right)=0.1816$, and $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{7},\mathcal{C}^{w}_{+}\right)=0.1360$ in relation to $\mathcal{C}^{w}_{+}$; and $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{-}\right)=0.2136$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{2},\mathcal{C}^{w}_{-}\right)=0.2159$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{3},\mathcal{C}^{w}_{-}\right)=0.2091$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{4},\mathcal{C}^{w}_{-}\right)=0.2238$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{5},\mathcal{C}^{w}_{-}\right)=0.1535$, $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{6},\mathcal{C}^{w}_{-}\right)=0.1569$, and $\mathfrak{D}^2_{(4)}\left(\mathcal{C}^{w}_{7},\mathcal{C}^{w}_{-}\right)=0.2104$ with relevance to $\mathcal{C}^{w}_{-}$.

In Step 7′, using the three-term approach and Eq. (52), one can calculate the relative closeness coefficients in this fashion: $\mathfrak{R}^{2+}_{(3)}\left(a_1\right)=\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{-}\right)/(\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{+}\right)+\mathfrak{D}^2_{(3)}\left(\mathcal{C}^{w}_{1},\mathcal{C}^{w}_{-}\right))=0.2127/(0.1431+0.2127)=0.5978$, $\mathfrak{R}^{2+}_{(3)}\left(a_2\right)=0.6335$, $\mathfrak{R}^{2+}_{(3)}\left(a_3\right)=0.6134$, $\mathfrak{R}^{2+}_{(3)}\left(a_4\right)=0.6114$, $\mathfrak{R}^{2+}_{(3)}\left(a_5\right)=0.4276$, $\mathfrak{R}^{2+}_{(3)}\left(a_6\right)=0.4620$, and $\mathfrak{R}^{2+}_{(3)}\left(a_7\right)=0.6071$. Since $\mathfrak{R}^{2+}_{(3)}\left(a_2\right)>\mathfrak{R}^{2+}_{(3)}\left(a_3\right)>\mathfrak{R}^{2+}_{(3)}\left(a_4\right)>\mathfrak{R}^{2+}_{(3)}\left(a_7\right)>\mathfrak{R}^{2+}_{(3)}\left(a_1\right)>\mathfrak{R}^{2+}_{(3)}\left(a_6\right)>\mathfrak{R}^{2+}_{(3)}\left(a_5\right)$, the priority order of the candidate locations is $a_2\succ a_3\succ a_4\succ a_7\succ a_1\succ a_6\succ a_5$. In line with

Eq. (54), the best compromise collection is $\mathcal{A}^{2+}_{(3)} = \left\{ a_i \middle| \max_{i=1}^{7} \mathfrak{R}^{2+}_{(3)}(a_i), a_i \in \mathcal{A} \right\} = \{a_2\}$. On the flip side, based on Eq. (53) using the four-term approach, the following relative closeness coefficients were derived: $\mathfrak{R}^{2+}_{(4)}(a_1) = 0.5982$, $\mathfrak{R}^{2+}_{(4)}(a_2) = 0.6339$, $\mathfrak{R}^{2+}_{(4)}(a_3) = 0.6139$, $\mathfrak{R}^{2+}_{(4)}(a_4) = 0.6116$, $\mathfrak{R}^{2+}_{(4)}(a_5) = 0.4291$, $\mathfrak{R}^{2+}_{(4)}(a_6) = 0.4634$, and $\mathfrak{R}^{2+}_{(4)}(a_7) = 0.6074$, which follows that $\mathfrak{R}^{2+}_{(4)}(a_2) > \mathfrak{R}^{2+}_{(4)}(a_3) > \mathfrak{R}^{2+}_{(4)}(a_4) > \mathfrak{R}^{2+}_{(4)}(a_7) > \mathfrak{R}^{2+}_{(4)}(a_1) > \mathfrak{R}^{2+}_{(4)}(a_6) > \mathfrak{R}^{2+}_{(4)}(a_5)$. As a result, the candidate locations are prioritized as $a_2 \succ a_3 \succ a_4 \succ a_7 \succ a_1 \succ a_6 \succ a_5$; additionally, the best compromise collection is identified as $\mathcal{A}^{2+}_{(4)} = \left\{ a_i \middle| \max_{i=1}^{7} \mathfrak{R}^{2+}_{(4)}(a_i), a_i \in \mathcal{A} \right\} = \{a_2\}$ using Eq. (55). The C-IF TOPSIS ranking results, as well as the best compromise collection and the most suitable solution, are the same using the three- and four-term approaches in the fixed anchoring mechanism based on the C-IF Euclidean distance model. The priority ranking $a_2 \succ a_3 \succ a_4 \succ a_7 \succ a_1 \succ a_6 \succ a_5$ obtained from the C-IF TOPSIS prioritization procedure in the fixed anchoring mechanism is comparable to the sorting result obtained by Alkan and Kahraman (2022b). The only difference is the outranking relationship between a_4 and a_7. Even so, a_2 is still the most suitable solution because $\mathcal{A}^{2+}_{(3)} = \mathcal{A}^{2+}_{(4)} = \{a_2\}$.

The application of the epidemic hospital site selection case demonstrates that the evolved C-IF TOPSIS methodology propounded in this work is both practicable and effective. The operability and ease of execution of the computational process are also very high, as demonstrated by its operation. Furthermore, by utilizing the C-IF Minkowski distances, the current techniques provide flexible and diverse options, such as a displaced anchoring mechanism based on a three-term approach, a displaced anchoring mechanism based on a four-term approach, a fixed anchoring mechanism based on a three-term approach, and a fixed anchoring mechanism based on a four-term approach.

5.2 Comparative research and discussions

This subsection performs four comparative studies to examine the validity and explore the merits of the evolved C-IF TOPSIS methodology. First, this subsection compares and thoroughly investigates the comparative evaluation of relative closeness coefficients under displaced and fixed anchoring mechanisms. Following that, this subsection discusses and focuses on how the relative closeness coefficient varies at each candidate position level under different anchoring mechanisms and parameter settings. Concerning the Manhattan, Euclidean, and Chebyshev distance models, which are frequently used in practice, this subsection investigates additional comparisons of the applied results yielded by the C-IF Manhattan, Euclidean, and Chebyshev distance measures based on three- and four-term representations. Finally, intending to address the epidemic hospital site selection case within IF or other relevant fuzzy environments, this subsection compares and analyzes the application outcomes of the evolved methodology with other TOPSIS-related methods (such as other C-IF TOPSIS or IF TOPSIS approaches) and has reviewed the strengths of the current C-IF TOPSIS technique in this study.

To begin, the C-IF Minkowski distances and relative closeness coefficients were calculated under different metric parameter settings for the first and second comparative studies. The metric parameter was set using $\beta = 1, 2, \cdots, 10$ and $\beta \to \infty$. The main comparison outcomes of the C-IF Minkowski distances and relative closeness coefficients are documented in the Appendix, including the detailed results based on the displaced and fixed anchoring mechanisms in Tables 10 and 11, respectively, for reference herein. Table 11 reveals

Fig. 8 Variation of relative closeness coefficients for individual candidate location levels

the results of $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{*})$, $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{\neg})$, and $\mathfrak{R}^{\beta *}_{(3)}(a_i)$ using the three-term approach, as well as $\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{*})$, $\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{\neg})$, and $\mathfrak{R}^{\beta *}_{(4)}(a_i)$ using the four-term approach. Moreover, Table 11 displays the results of $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{+})$, $\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{-})$, and $\mathfrak{R}^{\beta +}_{(3)}(a_i)$ using the three-term approach, as well as $\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{+})$, $\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^{w}_{i},\mathcal{C}^{w}_{-})$, and $\mathfrak{R}^{\beta +}_{(4)}(a_i)$ using the four-term approach.

Table 7 Summary of the priority ranking orders under various settings

	The metric parameter β (three-term approach)											The metric parameter β (four-term approach)										
	1	2	3	4	5	6	7	8	9	10	∞	1	2	3	4	5	6	7	8	9	10	∞
	Priority ranking order based on $\mathfrak{R}_{(3)}^{\beta *}(a_i)$											Priority ranking order based on $\mathfrak{R}_{(4)}^{\beta *}(a_i)$										
a_1	4	5	5	4	4	4	3	2	2	2	4	4	5	5	4	4	4	3	2	2	2	4
a_2	1	1	1	1	2	3	4	4	4	5	1	1	1	1	1	2	3	4	4	4	5	1
a_3	2	2	3	3	3	2	2	3	3	3	2	2	2	3	3	3	2	2	3	3	3	2
a_4	3	4	4	5	5	5	5	5	5	4	3	3	4	4	5	5	5	5	5	5	4	3
a_5	7	7	7	6	6	6	6	6	6	6	7	7	7	7	6	6	6	6	6	6	6	7
a_6	6	6	6	7	7	7	7	7	7	7	6	6	6	6	7	7	7	7	7	7	7	6
a_7	5	3	2	2	1	1	1	1	1	1	5	5	3	2	2	1	1	1	1	1	1	5
	Priority ranking order based on $\mathfrak{R}_{(3)}^{\beta +}(a_i)$											Priority ranking order based on $\mathfrak{R}_{(4)}^{\beta +}(a_i)$										
a_1	4	5	5	5	5	5	5	5	5	5	4	4	5	5	5	5	5	5	5	5	5	4
a_2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a_3	2	2	3	3	3	3	3	3	4	4	2	2	2	3	3	3	3	3	3	4	4	2
a_4	3	3	4	4	4	4	4	4	3	3	3	3	3	4	4	4	4	4	4	3	3	3
a_5	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
a_6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
a_7	5	4	2	2	2	2	2	2	2	2	5	5	4	2	2	2	2	2	2	2	2	5

Remarks about the backdrop color:

Grey: $a_2 \succ a_3 \succ a_4 \succ a_1 \succ a_7 \succ a_6 \succ a_5$

Purple: $a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_6 \succ a_5$

Jasmine: $a_2 \succ a_7 \succ a_3 \succ a_4 \succ a_1 \succ a_6 \succ a_5$

Fluorescent green: $a_2 \succ a_7 \succ a_3 \succ a_1 \succ a_4 \succ a_5 \succ a_6$

Light red: $a_7 \succ a_2 \succ a_3 \succ a_1 \succ a_4 \succ a_5 \succ a_6$

Aqua blue: $a_7 \succ a_3 \succ a_2 \succ a_1 \succ a_4 \succ a_5 \succ a_6$

Orange brown: $a_7 \succ a_3 \succ a_1 \succ a_2 \succ a_4 \succ a_5 \succ a_6$

Light green: $a_7 \succ a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5 \succ a_6$

Fluorescent yellow: $a_7 \succ a_1 \succ a_3 \succ a_4 \succ a_2 \succ a_5 \succ a_6$

Pink: $a_2 \succ a_3 \succ a_4 \succ a_7 \succ a_1 \succ a_6 \succ a_5$

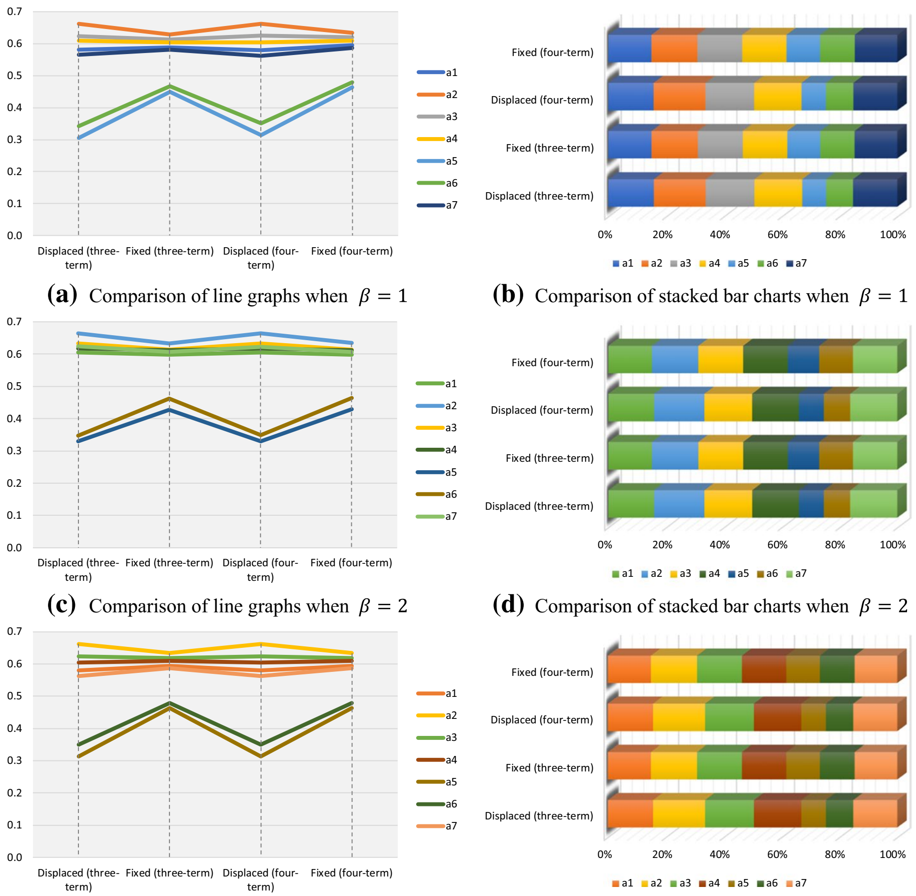
Dutch blue: $a_2 \succ a_7 \succ a_4 \succ a_3 \succ a_1 \succ a_6 \succ a_5$

Table 8 Prioritization results based on the C-IF Manhattan, Euclidean, and Chebyshev distances

a_i	Displaced anchoring mechanism (three-term approach)			Fixed anchoring mechanism (three-term approach)		
	$\mathfrak{R}^{1*}_{(3)}(a_i)$(Rank)	$\mathfrak{R}^{2*}_{(3)}(a_i)$(Rank)	$\mathfrak{R}^{\infty*}_{(3)}(a_i)$(Rank)	$\mathfrak{R}^{1+}_{(3)}(a_i)$(Rank)	$\mathfrak{R}^{2+}_{(3)}(a_i)$(Rank)	$\mathfrak{R}^{\infty+}_{(3)}(a_i)$(Rank)
a_1	0.5804 (4th)	0.6053 (5th)	0.5791 (4th)	0.5881 (4th)	0.5978 (5th)	0.5941 (4th)
a_2	0.6608 (1st)	0.6642 (1st)	0.6622 (1st)	0.6285 (1st)	0.6335 (1st)	0.6334 (1st)
a_3	0.6227 (2nd)	0.6324 (2nd)	0.6240 (2nd)	0.6126 (2nd)	0.6134 (2nd)	0.6187 (2nd)
a_4	0.6082 (3rd)	0.6175 (4th)	0.6039 (3rd)	0.6044 (3rd)	0.6114 (3rd)	0.6088 (3rd)
a_5	0.3062 (7th)	0.3291 (7th)	0.3132 (7th)	0.4497 (7th)	0.4276 (7th)	0.4633 (7th)
a_6	0.3429 (6th)	0.3481 (6th)	0.3503 (6th)	0.4664 (6th)	0.4620 (6th)	0.4790 (6th)
a_7	0.5647 (5th)	0.6236 (3rd)	0.5620 (5th)	0.5817 (5th)	0.6071 (4th)	0.5865 (5th)
	Displaced anchoring mechanism (four-term approach)			Fixed anchoring mechanism (four-term approach)		
	$\mathfrak{R}^{1*}_{(4)}(a_i)$(Rank)	$\mathfrak{R}^{2*}_{(4)}(a_i)$(Rank)	$\mathfrak{R}^{\infty*}_{(4)}(a_i)$(Rank)	$\mathfrak{R}^{1+}_{(4)}(a_i)$(Rank)	$\mathfrak{R}^{2+}_{(4)}(a_i)$(Rank)	$\mathfrak{R}^{\infty+}_{(4)}(a_i)$(Rank)
a_1	0.5791 (4th)	0.6051 (5th)	0.5791 (4th)	0.5941 (4th)	0.5982 (5th)	0.5941 (4th)
a_2	0.6622 (1st)	0.6642 (1st)	0.6622 (1st)	0.6334 (1st)	0.6339 (1st)	0.6334 (1st)
a_3	0.6240 (2nd)	0.6324 (2nd)	0.6240 (2nd)	0.6187 (2nd)	0.6139 (2nd)	0.6187 (2nd)
a_4	0.6039 (3rd)	0.6171 (4th)	0.6039 (3rd)	0.6088 (3rd)	0.6116 (3rd)	0.6088 (3rd)
a_5	0.3132 (7th)	0.3298 (7th)	0.3132 (7th)	0.4633 (7th)	0.4291 (7th)	0.4633 (7th)
a_6	0.3503 (6th)	0.3489 (6th)	0.3503 (6th)	0.4790 (6th)	0.4634 (6th)	0.4790 (6th)
a_7	0.5620 (5th)	0.6232 (3rd)	0.5620 (5th)	0.5865 (5th)	0.6074 (4th)	0.5865 (5th)

The first comparative study's goal is to thoroughly examine the juxtaposition situations of relative closeness coefficients under the displaced and fixed anchoring mechanisms. Figure 7 shows the comparison outcomes based on the three- and four-term representations for the same site selection case of the Istanbul Epidemic Hospital; additionally, these figures sketch the relative closeness coefficients being placed close together to see the contrasting effects in various settings of the metric parameter β. When $\beta = 1, 2, \cdots, 10$ and $\beta \to \infty$, the comparison and distribution of relative closeness coefficients are exhibited in this figure, which includes the juxtaposition of $\mathfrak{R}^{\beta*}_{(3)}(a_i)$ based on the three-term approach under the displaced anchoring mechanism in Fig. 7a, the juxtaposition of $\mathfrak{R}^{\beta+}_{(3)}(a_i)$ based on the three-term approach under the fixed anchoring mechanism in Fig. 7b, the juxtaposition of $\mathfrak{R}^{\beta*}_{(4)}(a_i)$ based on the four-term approach under the displaced anchoring mechanism in Fig. 7c, and the juxtaposition of $\mathfrak{R}^{\beta+}_{(4)}(a_i)$ based on the four-term approach under the fixed anchoring mechanism in Fig. 7d.

Overall, the relative closeness coefficients of the candidate locations a_1, a_2, a_3, a_4, and a_7 are greater than those of a_5 and a_6 in all discussed scenarios. This means that a_1, a_2, a_3, a_4, and a_7 are far superior to a_5 and a_6. Furthermore, in Fig. 7a and c, the comparative advantages of the five better candidate locations and the two worst locations are clearly visible, but such gaps are narrowed in Fig. 7b and d. It is recognized from Fig. 7b and d that the relative closeness coefficients $\mathfrak{R}^{\beta+}_{(3)}(a_i)$ and $\mathfrak{R}^{\beta+}_{(4)}(a_i)$ do not vary much depending on the metric parameter setting values. This demonstrates that, with the fixed anchoring mechanism, the relative closeness coefficients yielded by the C-IF TOPSIS procedure are relatively stable across a range of β values. In contrast, as displayed in Fig. 7a and c, the relative closeness coefficients $\mathfrak{R}^{\beta*}_{(3)}(a_i)$ and $\mathfrak{R}^{\beta*}_{(4)}(a_i)$ vary relatively greatly in different metric parameter



(a) Comparison of line graphs when $\beta = 1$ **(b)** Comparison of stacked bar charts when $\beta = 1$

(c) Comparison of line graphs when $\beta = 2$ **(d)** Comparison of stacked bar charts when $\beta = 2$

(e) Comparison of line graphs when $\beta \to \infty$ **(f)** Comparison of stacked bar charts when $\beta \to \infty$

Fig. 9 Contrast results based on the C-IF Manhattan, Euclidean, and Chebyshev distances

settings, particularly in Fig. 7c. This indicates that the relative closeness coefficients generated by the C-IF TOPSIS procedure are moderately affected by different β values under the displaced anchoring mechanism. In a nutshell, the relative closeness coefficients based on the C-IF Minkowski distance measure vary slightly under different β values and the displaced anchoring mechanism but are more stable under the fixed anchoring mechanism.

The second comparative study focuses on the variation of relative closeness coefficients for each candidate location level under different anchoring mechanisms based on the C-IF Minkowski distance measures in the epidemic hospital site selection case using three- and four-term approaches. Figure 8 depicts the relative comparison results in different scenarios concerning individual candidate location levels; that is, the outcomes corresponding to the candidate locations $a_1, a_2, \cdots, a_7$ are portrayed in Fig. 8a–g, respectively. The cases of $\beta = 1$, $\beta = 2$, and $\beta \to \infty$ represent commonly used distance models, namely the C-IF Manhattan, Euclidean, and Chebyshev metrics, and are highlighted as line graphs in the figure. Group histograms are used to represent other β-value cases.

The priority ranking outcomes of the seven candidate locations are displayed in Table 7 following the decreasing order of the relative closeness coefficients produced by the two C-IF TOPSIS prioritization algorithms. This table compares the priority ranks of each candidate location sequentially under various metric parameter settings, containing the priority ranks using the three- and four-term approaches with displaced and fixed anchoring mechanisms.

Based on the previous juxtaposition and contrasting effects in Fig. 8 and Table 7, it is clear that under the fixed anchoring mechanism, the candidate location a_2 enjoys the highest relative closeness coefficients among all the settings of the metric parameter β, implying that a_2 is a stable and consistent most suitable solution. In contrast, the candidate locations a_2 and a_7 have the highest relative closeness coefficients under the displaced anchoring mechanism in the settings of $\beta = 1, 2, 3, 4$ as well as $\beta \to \infty$ and $\beta = 5, 6, \cdots, 10$, respectively, so a_2 and a_7 are tied as the most suitable solutions with relative stability. More specifically, based on the three-term approach under the displaced anchoring mechanism, the following priority rankings of the seven candidate locations were generated from Table 7: $a_2 \succ a_3 \succ a_4 \succ a_1 \succ a_7 \succ a_6 \succ a_5$ when $\beta = 1$ and $\beta \to \infty$, $a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_6 \succ a_5$ when $\beta = 2$, $a_2 \succ a_7 \succ a_3 \succ a_4 \succ a_1 \succ a_6 \succ a_5$ when $\beta = 3$, $a_2 \succ a_7 \succ a_3 \succ a_1 \succ a_4 \succ a_5 \succ a_6$ when $\beta = 4$, $a_7 \succ a_2 \succ a_3 \succ a_1 \succ a_4 \succ a_5 \succ a_6$ when $\beta = 5$, $a_7 \succ a_3 \succ a_2 \succ a_1 \succ a_4 \succ a_5 \succ a_6$ when $\beta = 6$, $a_7 \succ a_3 \succ a_1 \succ a_2 \succ a_4 \succ a_5 \succ a_6$ when $\beta = 7$, $a_7 \succ a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5 \succ a_6$ when $\beta = 8$ and $\beta = 9$, and $a_7 \succ a_1 \succ a_3 \succ a_4 \succ a_2 \succ a_5 \succ a_6$ when $\beta = 10$. Except in the case of $\beta \to \infty$, the relative advantage of a_2 decreases gradually as the metric parameter β increases. When $\beta = 10$, the priority of a_2 has dropped to fifth. In contrast, the priority of a_7 is slightly more pronounced, as it ranks 1st, 2nd, and 3rd when $\beta = 5, 6, \cdots, 10$, $\beta = 3, 4$, and $\beta = 2$, respectively. The same observation can be found in the C-IF TOPSIS solution results using the four-term approach with the displaced anchoring mechanism.

On the flip side, based on the three-term approach under the fixed anchoring mechanism, the following priority rankings of candidate locations were rendered from Table 7: $a_2 \succ a_3 \succ a_4 \succ a_1 \succ a_7 \succ a_6 \succ a_5$ when $\beta = 1$ and $\beta \to \infty$, $a_2 \succ a_3 \succ a_4 \succ a_7 \succ a_1 \succ a_6 \succ a_5$ when $\beta = 2$, $a_2 \succ a_7 \succ a_3 \succ a_4 \succ a_1 \succ a_6 \succ a_5$ when $\beta = 3, 4, \cdots, 8$, and $a_2 \succ a_7 \succ a_4 \succ a_3 \succ a_1 \succ a_6 \succ a_5$ when $\beta = 9$ and $\beta = 10$. The same ranking outcomes can be found in the C-IF TOPSIS solutions based on the four-term approach with the fixed anchoring mechanism. As a result, the candidate position a_7 is the most stable and suitable solution in the fixed anchoring mechanism, with the highest relative closeness coefficients of all β settings. The preceding comparative discussions indicate that when decision-makers require a stable consequence of the most suitable solution, it is best to exploit the fixed anchoring mechanism in the evolved C-IF TOPSIS prioritization algorithm. When decision-makers prefer elastic and changing solution results as references, the displaced anchoring mechanism in the prioritization algorithm is recommended.

Given the generality and utility of the Manhattan, Euclidean, and Chebyshev metrics, the third comparative study expands on the application outcomes produced by the C-IF Manhattan, Euclidean, and Chebyshev distance measures (i.e., $\beta = 1, 2$, and $\beta \to \infty$, respectively, in $\mathfrak{D}^{\beta}_{(3)}$ and $\mathfrak{D}^{\beta}_{(4)}$) using three- and four-term approaches under different anchoring mechanisms. The prioritization results are revealed in Table 8, including the relative closeness coefficients $\mathfrak{R}^{\beta*}_{(3)}(a_i)$, $\mathfrak{R}^{\beta+}_{(3)}\left(a_i\right)$, $\mathfrak{R}^{\beta*}_{(4)}(a_i)$, and $\mathfrak{R}^{\beta+}_{(4)}\left(a_i\right)$, as well as their corresponding priority orders. The outcomes of the relative closeness coefficients calculated using the

Table 9 Comparison results with other fuzzy TOPSIS-related methods

Source of methods	Method	Judgment basis	Priority rankings of candidate locations
TOPSIS-related methods in C-IF, IF, picture fuzzy, and Pythagorean fuzzy settings			
Alkan and Kahraman (2022b)	C-IF TOPSIS method	Composite ratio score	$a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_6 \succ a_5$
Boran et al. (2009)	IF TOPSIS approach	Closeness coefficient	$a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_6 \succ a_5$
Ashraf et al. (2019)	Picture fuzzy TOPSIS approach	Closeness coefficient	$a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_5 \succ a_6$
Akram et al. (2019)	Pythagorean fuzzy TOPSIS approach	Closeness coefficient	$a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_5 \succ a_6$
Judgment basis	**Anchoring frame**	**Metric parameter**	**Priority rankings of candidate locations**
Evolved C-IF TOPSIS methodology in the current paper			
Relative closeness coefficient (Three- and four-term approaches)	Displaced anchoring mechanism	$\beta = 1, \beta \to \infty$	$a_2 \succ a_3 \succ a_4 \succ a_1 \succ a_7 \succ a_6 \succ a_5$
		$\beta = 2$	$a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_6 \succ a_5$
		$\beta = 3$	$a_2 \succ a_7 \succ a_3 \succ a_4 \succ a_1 \succ a_6 \succ a_5$
		$\beta = 4$	$a_2 \succ a_7 \succ a_3 \succ a_1 \succ a_4 \succ a_5 \succ a_6$
		$\beta = 5$	$a_7 \succ a_2 \succ a_3 \succ a_1 \succ a_4 \succ a_5 \succ a_6$
		$\beta = 6$	$a_7 \succ a_3 \succ a_2 \succ a_1 \succ a_4 \succ a_5 \succ a_6$
		$\beta = 7$	$a_7 \succ a_3 \succ a_1 \succ a_2 \succ a_4 \succ a_5 \succ a_6$
		$\beta = 8, 9$	$a_7 \succ a_1 \succ a_3 \succ a_2 \succ a_4 \succ a_5 \succ a_6$
		$\beta = 10$	$a_7 \succ a_1 \succ a_3 \succ a_4 \succ a_2 \succ a_5 \succ a_6$
Relative closeness coefficient (Three- and four-term approaches)	Fixed anchoring Mechanism	$\beta = 1, \beta \to \infty$	$a_2 \succ a_3 \succ a_4 \succ a_1 \succ a_7 \succ a_6 \succ a_5$
		$\beta = 2$	$a_2 \succ a_3 \succ a_4 \succ a_7 \succ a_1 \succ a_6 \succ a_5$
		$\beta = 3, 4, \ldots, 8$	$a_2 \succ a_7 \succ a_3 \succ a_4 \succ a_1 \succ a_6 \succ a_5$
		$\beta = 9, 10$	$a_2 \succ a_7 \succ a_4 \succ a_3 \succ a_1 \succ a_6 \succ a_5$

C-IF Manhattan and Chebyshev distances are similar; additionally, their priority rankings are the same, and both are $a_2 \succ a_3 \succ a_4 \succ a_1 \succ a_7 \succ a_6 \succ a_5$. Based on the C-IF Euclidean distance, the priority rankings judging by the relative closeness coefficients $\mathfrak{R}^{2*}_{(3)}(a_i)$ (as well as $\mathfrak{R}^{2*}_{(4)}(a_i)$) and $\mathfrak{R}^{2+}_{(3)}(a_i)$ (as well as $\mathfrak{R}^{2+}_{(4)}(a_i)$) are $a_2 \succ a_3 \succ a_7 \succ a_4 \succ a_1 \succ a_6 \succ a_5$ and $a_2 \succ a_3 \succ a_4 \succ a_7 \succ a_1 \succ a_6 \succ a_5$, respectively, which are slightly different from the cases of $\beta = 1$ and $\beta \to \infty$. The priority orders in the three candidate locations a_1, a_4, and a_7 differ from the prioritization results in the displaced anchoring mechanism. When $\beta = 1$ and $\beta \to \infty$, the priority orders of a_1, a_4, and a_7 are 4th, 3rd, and 5th, respectively. When $\beta = 2$, however, the priority orders of a_1, a_4, and a_7 are 5th, 4th, and 3rd, respectively. Taking into account the prioritization results in the fixed anchoring mechanism, the priority orders in the two candidate locations a_1 and a_7 differ. When $\beta = 1$ and $\beta \to \infty$, the priority orders of a_1 and a_7 are 4th and 5th, respectively, whereas the priority orders of a_1 and a_7 are 5th and 4th, respectively, when $\beta = 2$.

For further observation, this study plots the comparison results based on the C-IF Manhattan, Euclidean, and Chebyshev distances, as exhibited in Fig. 9, involving the comparison of line graphs in Fig. 9a, c, e and the comparison of stacked bar charts in Fig. 9b, d, f. As can be confirmed from Fig. 9a, c, e, whether based on the C-IF Manhattan distance, Euclidean distance, or Chebyshev distance, the relative closeness coefficient of a_2 performs the best in all four scenarios. On the contrary, the relative closeness coefficients of a_5 and a_6 are always the worst two in the four scenarios, with significant numerical differences with the other five candidate locations. Figure 9c shows that the relative closeness coefficients enjoyed by a_1, a_3, a_4, and a_7 based on the C-IF Euclidean distance metric are very close. The relative benefits of the four candidate locations are fairly close. Although the relative closeness coefficients of a_1, a_3, a_4, and a_7 are very close in Fig. 9a and e, there is still a slight difference in their relative advantages. The stacked bar charts in Fig. 9b, d, f, on the other hand, can display part-to-whole relationships. These 100% stacked bar charts, in particular, can display the relative percentages of the relative closeness coefficients of the seven locations $a_1 - a_7$ in the stacked bar, where the cumulative sum of the stacked bars invariably equals 100%. These stacked bar charts manifest how the proportion changes with different anchoring mechanisms and three- and four-term representations, such as the displaced anchoring mechanism with a three-term representation, the fixed anchoring mechanism with a three-term representation, the displaced anchoring mechanism with a four-term representation, and the fixed anchoring mechanism with a four-term representation.

In the case of the Istanbul Epidemic Hospital site selection, the fourth comparative research examines the rationality, usefulness, and adaptation of the evolved C-IF TOPSIS methodology by performing an overall comparison of the application outcomes generated by the current approach and other TOPSIS-related methods. It should be noted that Alkan and Kahraman (2022b) converted the C-IF performance ratings and C-IF importance weights into IF, picture fuzzy, and Pythagorean fuzzy information for ease of comparison. Furthermore, Alkan and Kahraman (2022b) compared their propounded C-IF TOPSIS method to the IF, picture fuzzy, and Pythagorean fuzzy TOPSIS approaches originally established by Boran et al. (2009), Ashraf et al. (2019), and Akram et al. (2019), respectively. The bases for judging the pros and cons of the candidate locations in this fourth comparative study include the composite ratio scores used in Alkan and Kahraman's C-IF TOPSIS method, the closeness coefficients used in the IF, picture fuzzy, and Pythagorean fuzzy TOPSIS approaches, and the relative closeness coefficients used in the evolved C-IF TOPSIS methodology. Table 9 reveals the comparison outcomes between the evolved C-IF TOPSIS techniques proposed in this study and the preceding fuzzy TOPSIS-related methods.

Based on the comparison outcomes in Table 9, it is concluded that a_5 and a_6 are the two alternatives with the worst overall performance. Among the prioritization results obtained by the four fuzzy TOPSIS-related methods (i.e., Alkan and Kahraman's C-IF TOPSIS method, as well as the IF, picture fuzzy, and Pythagorean fuzzy TOPSIS approaches) and the evolved C-IF TOPSIS methodology (under the displaced anchoring mechanism in the situations of $\beta = 1, 2, 3, 4$ and $\beta \to \infty$, and the fixed anchoring mechanism in the situations of $\beta = 1, 2, \cdots, 10$ and $\beta \to \infty$), a_2 has the best overall performance. In particular, when applying the evolved C-IF TOPSIS methodology to the site selection case of the Istanbul Epidemic Hospital, a_7 emerges as the most suitable solution under the displaced anchoring mechanism in the cases of $\beta = 5, 6, \cdots, 10$. The priority ranking outcomes produced by the four fuzzy TOPSIS-related methods were found to be highly similar. The main distinction occurs only in the outranking relationship between a_5 and a_6. Using Alkan and Kahraman's C-IF TOPSIS and the IF TOPSIS approaches, we obtained $a_6 \succ a_5$; however, using the picture fuzzy and Pythagorean fuzzy TOPSIS approaches, we obtained $a_5 \succ a_6$. Aside from that, the outcomes of the four fuzzy TOPSIS-related methods are nearly identical. Furthermore, their execution procedures were incapable of producing tunable or elastic results in response to changing circumstances. It implies that these methods did not provide a simple and straightforward mechanism for obtaining adaptive outcomes in various scenarios for decision-making purposes. On the contrary, the evolved C-IF TOPSIS methodology can easily and effectively adapt to different scenarios, including the setting of the metric parameter and the designation of the anchoring mechanism. The evolved C-IF TOPSIS methodology can generate flexible prioritization results by adjusting the metric parameter and setting the displaced or fixed anchoring mechanism. In conclusion, the current C-IF TOPSIS techniques can produce stable but still resilient results.

6 Conclusion and future scope

The circumstances confronting the MCDA problem are becoming increasingly complex, and there is much incomplete, indistinct, and inconsistent information in some unpredictable emergencies or the ever-changing social environment. The evolved C-IF TOPSIS methodology can provide decision-makers with recommendations for making the most appropriate choices within complex realistic environments. Specifically, this study makes the following main contributions to address the limitations of the existing literature and overcome the challenges of research gaps: (1) Based on three- and four-term approaches, this study constructed new C-IF Minkowski distance measures, and such general-purpose metrics can relax the constraints of the current C-IF distance metrics, provide flexibility of use through parameter settings, and broaden the applicability of quantitative analysis. (2) This study applied our newly developed C-IF Minkowski distance measures to the development process of the C-IF TOPSIS methodology, which can better determine the separation of incomplete, uncertain, and inconsistent information and delineate trade-off evaluations and compromise decision rules. (3) This study built displaced and fixed anchoring frameworks with three- and four-term representations in order to create an evolved C-IF TOPSIS methodology for dealing with increasingly complex MCDA problems in real-world settings. (4) This study developed two new effective prioritization algorithms that were practically applied to the site selection issue of large epidemic hospitals, demonstrating the applicability of the new techniques and their superior ability when compared to existing approaches through comparative analysis.

However, there may be some limitations to this study. As previously indicated, the C-IF decision matrix was converted into pessimistic and optimistic decision matrices expressed in terms of IF values by Kahraman and Alkan (2021) and Alkan and Kahraman (2022b). They transformed the higher-order fuzzy information included in the C-IF set into a standard IF set with streamlined pessimistic and optimistic estimate processes. While their method may lose sight of the original goal of dealing with complex uncertainties using the C-IF set, the pessimistic and optimistic estimates associated with the C-IF decision setting are indeed issues that are not addressed in the approach presented in this study. If the decision-maker has a particularly pessimistic or optimistic view of the decision-making environment, the results obtained using the suggested C-IF TOPSIS technique must be interpreted with caution. This is the main limitation faced by this study.

Other implementation limitations may exist for the proposed C-IF TOPSIS approach. To be specific, possible constraints include the mechanism for setting the metric parameter, and how to specify appropriate parameter values in various MCDA application libraries. Furthermore, a more thorough analysis of case applicability is necessary since the displaced or fixed anchoring frameworks with three- and four-term representations are relevant to various MCDA case bases. This opens up the possibility of conducting a future study in two directions. The relationship between the metric parameter and the optimal solution outcomes should be taken into consideration when building additional C-IF MCDA models using the suggested C-IF Minkowski distance measures. Moreover, an optimal setting mechanism for the metric parameter should be established to enhance the effectiveness and performance of the C-IF MCDA model. Furthermore, how to apply the evolved C-IF TOPSIS methodology to various practical cases, as well as the correctness and validity analysis of the consequences, warrant further investigation.

Appendix

See Tables 10 and 11

Table 10 Main comparison results based on the displaced anchoring mechanism

a_i	$\mathfrak{D}$ or $\mathfrak{R}$	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$	$\beta=6$	$\beta=7$	$\beta=8$	$\beta=9$	$\beta=10$	$\beta=\infty$
C-IF Minkowski distances and relative closeness coefficients using the three-term approach												
a_1	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_1, \mathcal{C}^w_*)$	0.0682	0.0840	0.0918	0.0967	0.1001	0.1026	0.1046	0.1062	0.1075	0.1086	0.0709
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_1, \mathcal{C}^w_\neg)$	0.0943	0.1288	0.1536	0.1716	0.1851	0.1954	0.2036	0.2103	0.2157	0.2203	0.0976
	$\mathfrak{R}^{\beta*}_{(3)}(a_1)$	0.5804	0.6053	0.6260	0.6397	0.6490	0.6557	0.6606	0.6644	0.6673	0.6697	0.5791
a_2	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_2, \mathcal{C}^w_*)$	0.0547	0.0636	0.0716	0.0780	0.0831	0.0871	0.0902	0.0928	0.0949	0.0967	0.0567
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_2, \mathcal{C}^w_\neg)$	0.1066	0.1258	0.1408	0.1524	0.1615	0.1686	0.1743	0.1790	0.1829	0.1862	0.1111
	$\mathfrak{R}^{\beta*}_{(3)}(a_2)$	0.6608	0.6642	0.6629	0.6614	0.6603	0.6595	0.6590	0.6586	0.6584	0.6582	0.6622
a_3	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_3, \mathcal{C}^w_*)$	0.0627	0.0714	0.0769	0.0811	0.0844	0.0871	0.0894	0.0914	0.0932	0.0947	0.0650
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_3, \mathcal{C}^w_\neg)$	0.1034	0.1229	0.1391	0.1515	0.1612	0.1688	0.1749	0.1799	0.1841	0.1877	0.1078
	$\mathfrak{R}^{\beta*}_{(3)}(a_3)$	0.6227	0.6324	0.6438	0.6515	0.6564	0.6595	0.6616	0.6630	0.6640	0.6647	0.6240
a_4	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_4, \mathcal{C}^w_*)$	0.0648	0.0917	0.1047	0.1124	0.1175	0.1212	0.1240	0.1262	0.1281	0.1296	0.0672
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_4, \mathcal{C}^w_\neg)$	0.1006	0.1480	0.1772	0.1972	0.2118	0.2230	0.2319	0.2390	0.2449	0.2498	0.1024
	$\mathfrak{R}^{\beta*}_{(3)}(a_4)$	0.6082	0.6175	0.6285	0.6369	0.6431	0.6479	0.6515	0.6544	0.6567	0.6585	0.6039
a_5	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_5, \mathcal{C}^w_*)$	0.1101	0.1527	0.1786	0.1965	0.2099	0.2204	0.2289	0.2359	0.2417	0.2466	0.1119
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_5, \mathcal{C}^w_\neg)$	0.0486	0.0749	0.0885	0.0973	0.1035	0.1083	0.1121	0.1151	0.1177	0.1198	0.0510
	$\mathfrak{R}^{\beta*}_{(3)}(a_5)$	0.3062	0.3291	0.3314	0.3311	0.3303	0.3295	0.3287	0.3280	0.3275	0.3271	0.3132
a_6	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_6, \mathcal{C}^w_*)$	0.1060	0.1234	0.1388	0.1512	0.1608	0.1684	0.1746	0.1795	0.1837	0.1872	0.1080
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_6, \mathcal{C}^w_\neg)$	0.0553	0.0659	0.0708	0.0740	0.0764	0.0783	0.0800	0.0814	0.0827	0.0838	0.0583
	$\mathfrak{R}^{\beta*}_{(3)}(a_6)$	0.3429	0.3481	0.3377	0.3286	0.3220	0.3174	0.3143	0.3120	0.3104	0.3092	0.3503
a_7	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_7, \mathcal{C}^w_*)$	0.0723	0.0770	0.0816	0.0857	0.0893	0.0925	0.0952	0.0975	0.0995	0.1013	0.0757
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_7, \mathcal{C}^w_\neg)$	0.0938	0.1276	0.1497	0.1657	0.1778	0.1873	0.1948	0.2009	0.2060	0.2103	0.0972
	$\mathfrak{R}^{\beta*}_{(3)}(a_7)$	0.5647	0.6236	0.6473	0.6591	0.6656	0.6694	0.6718	0.6733	0.6743	0.6750	0.5620

Table 10 (continued)

a_i	$\mathfrak{D}$ or $\mathfrak{R}$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$	$\beta = 8$	$\beta = 9$	$\beta = 10$	$\beta = \infty$
C-IF Minkowski distances and relative closeness coefficients using the four-term approach												
a_1	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_1, \mathcal{C}^w_*)$	0.0709	0.0842	0.0918	0.0967	0.1001	0.1026	0.1046	0.1062	0.1075	0.1086	0.0709
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_1, \mathcal{C}^w_\neg)$	0.0976	0.1289	0.1536	0.1716	0.1851	0.1954	0.2036	0.2103	0.2157	0.2203	0.0976
	$\mathfrak{R}^{\beta*}_{(4)}(a_1)$	0.5791	0.6051	0.6260	0.6397	0.6490	0.6557	0.6606	0.6644	0.6673	0.6697	0.5791
a_2	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_2, \mathcal{C}^w_*)$	0.0567	0.0637	0.0716	0.0780	0.0831	0.0871	0.0902	0.0928	0.0949	0.0967	0.0567
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_2, \mathcal{C}^w_\neg)$	0.1111	0.1261	0.1408	0.1524	0.1615	0.1686	0.1743	0.1790	0.1829	0.1862	0.1111
	$\mathfrak{R}^{\beta*}_{(4)}(a_2)$	0.6622	0.6642	0.6629	0.6614	0.6603	0.6595	0.6590	0.6586	0.6584	0.6582	0.6622
a_3	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_3, \mathcal{C}^w_*)$	0.0650	0.0716	0.0769	0.0811	0.0844	0.0871	0.0894	0.0914	0.0932	0.0947	0.0650
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_3, \mathcal{C}^w_\neg)$	0.1078	0.1231	0.1391	0.1516	0.1612	0.1688	0.1749	0.1799	0.1841	0.1877	0.1078
	$\mathfrak{R}^{\beta*}_{(4)}(a_3)$	0.6240	0.6324	0.6438	0.6515	0.6564	0.6595	0.6616	0.6630	0.6640	0.6647	0.6240
a_4	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_4, \mathcal{C}^w_*)$	0.0672	0.0918	0.1047	0.1124	0.1175	0.1212	0.1240	0.1262	0.1281	0.1296	0.0672
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_4, \mathcal{C}^w_\neg)$	0.1024	0.1480	0.1772	0.1972	0.2118	0.2230	0.2319	0.2390	0.2449	0.2498	0.1024
	$\mathfrak{R}^{\beta*}_{(4)}(a_4)$	0.6039	0.6171	0.6285	0.6369	0.6431	0.6479	0.6515	0.6544	0.6567	0.6585	0.6039
a_5	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_5, \mathcal{C}^w_*)$	0.1119	0.1527	0.1786	0.1965	0.2099	0.2204	0.2289	0.2359	0.2417	0.2466	0.1119
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_5, \mathcal{C}^w_\neg)$	0.0510	0.0751	0.0885	0.0973	0.1035	0.1083	0.1121	0.1151	0.1177	0.1198	0.0510
	$\mathfrak{R}^{\beta*}_{(4)}(a_5)$	0.3132	0.3298	0.3315	0.3311	0.3303	0.3295	0.3287	0.3280	0.3275	0.3271	0.3132
a_6	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_6, \mathcal{C}^w_*)$	0.1080	0.1235	0.1388	0.1512	0.1608	0.1684	0.1746	0.1795	0.1837	0.1872	0.1080
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_6, \mathcal{C}^w_\neg)$	0.0583	0.0661	0.0708	0.0740	0.0764	0.0783	0.0800	0.0814	0.0827	0.0838	0.0583
	$\mathfrak{R}^{\beta*}_{(4)}(a_6)$	0.3503	0.3489	0.3378	0.3286	0.3220	0.3174	0.3143	0.3120	0.3104	0.3092	0.3503
a_7	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_7, \mathcal{C}^w_*)$	0.0757	0.0772	0.0816	0.0857	0.0893	0.0925	0.0952	0.0975	0.0995	0.1013	0.0757
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_7, \mathcal{C}^w_\neg)$	0.0972	0.1278	0.1497	0.1657	0.1778	0.1873	0.1948	0.2009	0.2060	0.2103	0.0972
	$\mathfrak{R}^{\beta*}_{(4)}(a_7)$	0.5620	0.6232	0.6473	0.6591	0.6656	0.6694	0.6718	0.6733	0.6743	0.6750	0.5620

Table 11 Main comparison results based on the fixed anchoring mechanism

a_i	$\mathfrak{D}$ or $\mathfrak{R}$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$	$\beta = 8$	$\beta = 9$	$\beta = 10$	$\beta = \infty$
C-IF Minkowski distances and relative closeness coefficients using the three-term approach												
a_1	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_1, \mathcal{C}^w_+)$	0.1298	0.1431	0.1547	0.1640	0.1712	0.1770	0.1816	0.1854	0.1886	0.1913	0.1353
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_1, \mathcal{C}^w_-)$	0.1853	0.2127	0.2341	0.2501	0.2621	0.2714	0.2788	0.2848	0.2898	0.2941	0.1980
	$\mathfrak{R}^{\beta+}_{(3)}(a_1)$	0.5881	0.5978	0.6022	0.6040	0.6048	0.6053	0.6055	0.6057	0.6058	0.6060	0.5941
a_2	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_2, \mathcal{C}^w_+)$	0.1175	0.1244	0.1305	0.1357	0.1402	0.1441	0.1474	0.1502	0.1527	0.1549	0.1226
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_2, \mathcal{C}^w_-)$	0.1988	0.2149	0.2285	0.2398	0.2491	0.2568	0.2633	0.2688	0.2735	0.2776	0.2118
	$\mathfrak{R}^{\beta+}_{(3)}(a_2)$	0.6285	0.6335	0.6366	0.6386	0.6398	0.6406	0.6411	0.6415	0.6417	0.6418	0.6334
a_3	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_3, \mathcal{C}^w_+)$	0.1207	0.1311	0.1402	0.1479	0.1544	0.1599	0.1647	0.1688	0.1723	0.1754	0.1259
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_3, \mathcal{C}^w_-)$	0.1908	0.2081	0.2230	0.2350	0.2443	0.2517	0.2576	0.2625	0.2665	0.2700	0.2042
	$\mathfrak{R}^{\beta+}_{(3)}(a_3)$	0.6126	0.6134	0.6140	0.6138	0.6128	0.6115	0.6100	0.6086	0.6073	0.6062	0.6187
a_4	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_4, \mathcal{C}^w_+)$	0.1235	0.1419	0.1579	0.1705	0.1802	0.1878	0.1937	0.1986	0.2026	0.2060	0.1279
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_4, \mathcal{C}^w_-)$	0.1887	0.2232	0.2484	0.2673	0.2817	0.2928	0.3015	0.3086	0.3144	0.3192	0.1990
	$\mathfrak{R}^{\beta+}_{(3)}(a_4)$	0.6044	0.6114	0.6114	0.6106	0.6098	0.6093	0.6088	0.6084	0.6081	0.6078	0.6088
a_5	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_5, \mathcal{C}^w_+)$	0.1755	0.2041	0.2252	0.2415	0.2543	0.2647	0.2733	0.2805	0.2866	0.2919	0.1789
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_5, \mathcal{C}^w_-)$	0.1434	0.1525	0.1615	0.1701	0.1782	0.1854	0.1919	0.1975	0.2025	0.2068	0.1544
	$\mathfrak{R}^{\beta+}_{(3)}(a_5)$	0.4497	0.4276	0.4177	0.4134	0.4119	0.4119	0.4124	0.4132	0.4140	0.4147	0.4633
a_6	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_6, \mathcal{C}^w_+)$	0.1687	0.1814	0.1919	0.2007	0.2082	0.2146	0.2200	0.2247	0.2288	0.2323	0.1732
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_6, \mathcal{C}^w_-)$	0.1475	0.1558	0.1636	0.1709	0.1776	0.1837	0.1892	0.1941	0.1985	0.2023	0.1593
	$\mathfrak{R}^{\beta+}_{(3)}(a_6)$	0.4664	0.4620	0.4602	0.4599	0.4604	0.4613	0.4623	0.4635	0.4645	0.4656	0.4790
a_7	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_7, \mathcal{C}^w_+)$	0.1303	0.1356	0.1410	0.1463	0.1513	0.1557	0.1597	0.1632	0.1662	0.1689	0.1368
	$\mathfrak{D}^{\beta}_{(3)}(\mathcal{C}^w_7, \mathcal{C}^w_-)$	0.1812	0.2095	0.2295	0.2438	0.2544	0.2625	0.2690	0.2743	0.2788	0.2826	0.1941
	$\mathfrak{R}^{\beta+}_{(3)}(a_7)$	0.5817	0.6071	0.6194	0.6249	0.6271	0.6276	0.6275	0.6270	0.6265	0.6259	0.5865

Table 11 (continued)

a_i	$\mathfrak{D}$ or $\mathfrak{R}$	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$	$\beta=6$	$\beta=7$	$\beta=8$	$\beta=9$	$\beta=10$	$\beta=\infty$
C-IF Minkowski distances and relative closeness coefficients using the four-term approach												
a_1	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_1,\mathcal{C}^w_+)$	0.1353	0.1435	0.1547	0.1640	0.1712	0.1770	0.1816	0.1854	0.1886	0.1913	0.1353
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_1,\mathcal{C}^w_-)$	0.1980	0.2136	0.2342	0.2501	0.2621	0.2714	0.2788	0.2848	0.2898	0.2941	0.1980
	$\mathfrak{R}^{\beta+}_{(4)}(a_1)$	0.5941	0.5982	0.6022	0.6040	0.6048	0.6053	0.6055	0.6057	0.6058	0.6060	0.5941
a_2	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_2,\mathcal{C}^w_+)$	0.1226	0.1247	0.1305	0.1357	0.1402	0.1441	0.1474	0.1502	0.1527	0.1549	0.1226
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_2,\mathcal{C}^w_-)$	0.2118	0.2159	0.2286	0.2398	0.2491	0.2568	0.2633	0.2688	0.2735	0.2776	0.2118
	$\mathfrak{R}^{\beta+}_{(4)}(a_2)$	0.6334	0.6339	0.6367	0.6386	0.6398	0.6406	0.6411	0.6415	0.6417	0.6418	0.6334
a_3	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_3,\mathcal{C}^w_+)$	0.1259	0.1315	0.1402	0.1479	0.1544	0.1599	0.1647	0.1688	0.1723	0.1754	0.1259
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_3,\mathcal{C}^w_-)$	0.2042	0.2091	0.2231	0.2350	0.2443	0.2517	0.2576	0.2625	0.2665	0.2700	0.2042
	$\mathfrak{R}^{\beta+}_{(4)}(a_3)$	0.6187	0.6139	0.6140	0.6138	0.6128	0.6115	0.6100	0.6086	0.6073	0.6062	0.6187
a_4	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_4,\mathcal{C}^w_+)$	0.1279	0.1421	0.1579	0.1705	0.1802	0.1878	0.1937	0.1986	0.2026	0.2060	0.1279
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_4,\mathcal{C}^w_-)$	0.1990	0.2238	0.2485	0.2673	0.2817	0.2928	0.3015	0.3086	0.3144	0.3192	0.1990
	$\mathfrak{R}^{\beta+}_{(4)}(a_4)$	0.6088	0.6116	0.6115	0.6106	0.6098	0.6093	0.6088	0.6084	0.6081	0.6078	0.6088
a_5	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_5,\mathcal{C}^w_+)$	0.1789	0.2043	0.2252	0.2415	0.2543	0.2647	0.2733	0.2805	0.2866	0.2919	0.1789
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_5,\mathcal{C}^w_-)$	0.1544	0.1535	0.1616	0.1702	0.1782	0.1854	0.1919	0.1975	0.2025	0.2068	0.1544
	$\mathfrak{R}^{\beta+}_{(4)}(a_5)$	0.4633	0.4291	0.4179	0.4134	0.4119	0.4119	0.4124	0.4132	0.4140	0.4147	0.4633
a_6	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_6,\mathcal{C}^w_+)$	0.1732	0.1816	0.1919	0.2007	0.2082	0.2146	0.2200	0.2247	0.2288	0.2323	0.1732
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_6,\mathcal{C}^w_-)$	0.1593	0.1569	0.1638	0.1710	0.1776	0.1837	0.1892	0.1941	0.1985	0.2023	0.1593
	$\mathfrak{R}^{\beta+}_{(4)}(a_6)$	0.4790	0.4634	0.4604	0.4599	0.4604	0.4613	0.4623	0.4635	0.4645	0.4656	0.4790
a_7	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_7,\mathcal{C}^w_+)$	0.1368	0.1360	0.1411	0.1463	0.1513	0.1557	0.1597	0.1632	0.1662	0.1689	0.1368
	$\mathfrak{D}^{\beta}_{(4)}(\mathcal{C}^w_7,\mathcal{C}^w_-)$	0.1941	0.2104	0.2296	0.2438	0.2544	0.2625	0.2690	0.2743	0.2788	0.2826	0.1941

Table 11 (continued)

a_i	$\mathfrak{D}$ or $\mathfrak{R}$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$	$\beta = 8$	$\beta = 9$	$\beta = 10$	$\beta = \infty$
	$\mathfrak{R}_{(4)}^{\beta+}(a_7)$	0.5865	0.6074	0.6195	0.6249	0.6271	0.6276	0.6275	0.6270	0.6265	0.6259	0.5865

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Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The author declare that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Human and animal rights This article does not contain any studies with human participants or animals that were performed by any of the authors.

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