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Modal context restriction for multiagent BDI logics

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Abstract
We present and discuss a novel language restriction for modal logics for multiagent systems, called modal context restriction, that reduces the complexity of the satisfiability problem from EXPTIME complete to NPTIME complete. We focus on BDI multimodal logics that contain fix-point modalities like common beliefs and mutual intentions together with realism and introspection axioms. We show how this combination of modalities and axioms affects complexity of the satisfiability problem and how it can be reduced by restricting the modal context of formulas.

Keywords Modal context restriction · Multiagent theories · BDI · Modal logic · Satisfiability

1 Introduction

Checking the satisfiability of formulas (Garey and Johnson 1990), along with checking whether formulas are satisfied in a given model (i.e. model checking Emerson and Clarke 1980), is an important computational task associated with formalisms used for specifying software systems (Huth and Ryan 2004). In the case of multiagent systems (Wooldridge 2009; Shoham and Leyton-Brown 2008), the satisfiability problem underlies the following two tasks, appearing during system development. The first is the task of specification verification. Checking whether there exists a system that satisfies the given specification is essentially the satisfiability problem. The second task is related to implementation of individual agents (Shoham 1993). Often such implementations are based on logical formalisms and execution of programs of agents involves reasoning tasks related to that formalism. These tasks are based on checking the satisfiability of formulas.

The problem with checking the satisfiability of formulas of modal logics for multiagent systems is its high computational complexity (Halpern and Moses 1992). Such logics usually combine fix-point modalities, like common beliefs and mutual intentions, with axioms interconnecting different types of modalities, like introspection axioms and realism axioms (Levesque et al. 1990; Rao and Georgeff 1991; Wooldridge 2000; Dunin-Kęplicz and Verbrugge 2010). Richness of the formalisms leads to high complexity of

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computational tasks such as checking the satisfiability. One way of addressing this problem is restricting the language of the formalism (Halpern 1995). Knowing possible language restrictions and the associated complexity of computational tasks can help designers of programming languages or the graphical languages for systems design to enforce the chosen restrictions and ensure lower cost of computational tasks like reasoning or checking system validity. A classic example of this approach is the Horn fragment of first order logic that is adopted in logic programming languages, like Prolog.

The rest of the paper is organized as follows. We start with discussing the related literature and the state of the art in Sect. 2. Then we introduce the logical framework in Sect. 3. In Sect. 4 we present the general idea of modal context restriction together with two restrictions for logic TeamLog.We discuss the restrictions and provide examples in Sect. 5. Section 6 contains the complexity results. Section 7 contains the conclusions.

2 Related literature

The main focus of this paper is the complexity of the satisfiability problem for modal BDI logics. Modal logics of agency based on BDI model (Bratman 1987) are formalisms used to specify individual agents in terms of their beliefs, goals/desires and intentions. In the context of multi-agent systems this formalism is extended with fix-point modalities for common beliefs and collective (or joint) intentions and commitments. These extensions where first introduced in the seminal work of Cohen, Levesque et al. (1990). This was followed by a number of well known early formalism including KARO (van Linder et al. 1994, 1998; Meyer et al. 1999; van der Hoek et al. 1999; Aldewereld et al. 2004), $\mathcal{LORA}$ (Wooldridge 2000), and TeamLog (Dunin-Kęplicz and Verbrugge 1996, 2002, 2004, 2010). More recent works on BDI logics and related formalisms focus on extending the formalism and for obtaining better formalisation of concepts like desires, intentions, or coalitions. Dubois et al. (2017) consider the problem of desires revision and propose a formalism for reasoning about desires based on possibilistic logic. Wobcke (2015) proposed an agent dynamic logic (ADL) that allows for reasoning about intentions and action. (Bauters et al. 2014a, b, 2017) develop a BDI formalism that allows for modelling and reasoning under uncertainty. Ågotnes and Alechina (2018) study axiomatisation and complexity of epistemic coalition logic, a formalism combining modalities for expressing knowledge and common knowledge with the modality expressing that a group of agents is effective to make a formula true. Lorini and Sartor (2016) propose a logic for reasoning about social influence based on Belnap et al. (2001) logic of 'seeing to it that', STIT. The BDI model of agency remains an actively studied and used model up to this day. Most recent applications include multiagent system organizations (Keogh and Sonenberg 2020), supply chain quality inspection (Yan et al. 2020), and traffic simulations (Rüb and Dunin-Kęplicz 2020).

A common characteristic of multi-agent BDI logics is adopting, along with standard modal systems K_n, KD_n or $KD45_n$, mixed axioms that interrelate modalities representing different aspects of agent description. Well-known examples of such axioms are the *realism axioms* (Cohen and Levesque 1990; Rao and Georgeff 1998; Wooldridge 2000) and the *introspection axioms* (Rao and Georgeff 1991; Dunin-Kęplicz and Verbrugge 2002, 2004). In the case of basic BDI logics for a single agent (without the temporal or dynamic component), addition of these axioms does not change the complexity of the satisfiability problem, and they all remain PSPACE complete (Rao and Georgeff 1998; Dziubiński et al. 2007). In the multiagent case such logics are extended by lifting individual modalities representing beliefs, goals or intentions to the group level by introducing fixpoint modalities representing common beliefs, mutual goals or mutual intentions (Levesque et al. 1990; Wooldridge 2000; Aldewereld et al. 2004; Dunin-Kęplicz and Verbrugge 2010; Ågotnes and Alechina 2018). Adding such modalities leads to EXPTIME hard satisfiability problem (Halpern and Moses 1992), and presence of mixed axioms does not affect this result (Dziubiński et al. 2007).

One of the ways of dealing with high complexity of logical formalisms is restricting their language, so that the complexity of the satisfiability problem is reduced.[1] Restricting modal depth of formulas by a constant may lead to NPTIME complete satisfiability problem, while combining this with restricting the number of propositional symbols leads to linear time solvability of the problem. This, however, is with a constant that depends exponentially on the number of these symbols (Halpern 1995). In the case of modal logics with fixpoint modalities the restrictions mentioned above are not that promising, as the satisfiability problem remains EXPTIME hard, even when modal depth of formulas is bounded by 2 (Halpern and Moses 1992; Dziubiński et al. 2007). Motivated by these results, Dziubiński (2013) proposed a new kind of language restriction called *modal context restriction* and applied restrictions of this kind to standard systems of multimodal logics (generated by different combinations of axioms **K**, **T**, **D**, **4** and **5**) enriched with fixpoint modalities. This leads to PSPACE completeness and, when combined with modal depth restriction, to NPTIME completeness of the satisfiability problem.

Another type of restrictions considered in the literature are sub-propositional fragments of modal logics. In Nguyen (2005) it is shown that the Horn fragment of some of modal systems has NPTIME complete satisfiability problem, which becomes PTIME complete when combined with modal depth restriction. More recently, Bresolin et al. (2016) and (Bresolin et al. 2018) considered the Horn and the Krom fragments of modal basic logics **K**, **T**, **K4**, and **S4**, also in combination with allowing box or diamond operators in positive literals only. They show that all these fragments are PTIME solvable. Wałęga (2019) considered the core fragment (i.e. an intersection of the Horn and Krom fragments) and showed that if only box operators in positive literals are allowed, the fragment is NL-complete. In Bauland et al. (2006) restrictions on propositional operators used in formulas are considered. To study different sets of propositional operators used in formulas, Post lattice (Post 1941), which has been successfully used to classify the complexity problems for propositional calculus, is used. It is shown that in the case of basic normal modal system K there is a trichotomy: depending on the boolean operators used the satisfiability problem is either PSPACE complete, coNPTIME complete or PTIME solvable). In the case of normal modal system KD there is a dichotomy: the satisfiability problem is either PSPACE complete or PTIME solvable. Almost complete characterization was also obtained for modal systems T, S4 and S5. Similar approach was also applied to LTL in Bauland et al. (2009) and to CTL* and CTL in Meier et al. (2008).

In this paper, we build on the idea of modal context restrictions from Dziubiński (2013) and present modal context restrictions for BDI logics with two types of mixed axioms, realism axioms and introspection axioms, interrelating modalities of different basic multimodal logics. Presence of mixed axioms results in a richer setup which makes the problem of designing the right modal context restriction much more difficult than in Dziubiński (2013) and results in more complex restrictions. In particular, we propose two novel modal context restrictions, called $\mathbf{R}_1$ and $\mathbf{R}_2$. We show that the basic modal context restriction, $\mathbf{R}_1$, leads to PSPACE completeness of the satisfiability problem. However,

[1] In many cases the term reduced is used under the assumption that NPTIME, PSPACE and EXPTIME are different and increasing in this order.

existence of the introspection axioms results in PSPACE hardness of the problem, even if modal depth of formulas is bounded by 2. An interesting feature of the considered logic is the fact that models satisfying the formulas may be exponentially deep (with respect to their size). For this reason a standard tableau based algorithm has to be extended so that it uses polynomial space to decide the satisfiability. Two of the restrictions proposed in the paper, R_2 and $\mathrm{R}_1(c)$, lead to NPTIME solvability of the satisfiability problem, when combined with modal depth restriction. As working formalism we chose TeamLog (Dunin-Kęplicz and Verbrugge 2010), a well-known BDI formalism designed to formalise teamwork in multiagent systems.

3 Logical framework

TeamLog, developed by Dunin-Kęplicz and Verbrugge in a series of papers (Dunin-Kęplicz and Verbrugge 1996, 2002, 2003, 2004) and a book (Dunin-Kęplicz and Verbrugge 2010), is a logical framework proposed to formalize individual and group aspects of BDI systems. The full TeamLog is a very reach formalism allowing for expressing and reasoning about various aspects of individual agents and multiagent systems relevant to cooperative problem solving. Moreover, it is intended to be suitable for possible enrichments that could be designed for chosen classes of multiagent systems. In this paper we focus on the core TeamLog (which will be simply called TeamLog) presented below. For the full framework see (Dunin-Kęplicz and Verbrugge 2010).

TeamLog is a propositional multimodal logic that introduces five sets of modal operators based on a finite and non-empty set of agents $\mathcal{A}$. Three sets of operators, $\Omega^{\mathrm{B}} = \{[\mathrm{B}]_j : j \in \mathcal{A}\}$, $\Omega^{\mathrm{G}} = \{[\mathrm{G}]_j : j \in \mathcal{A}\}$ and $\Omega^{\mathrm{I}} = \{[\mathrm{I}]_j : j \in \mathcal{A}\}$, are used for representing beliefs, goals and intentions, respectively, of individual agents. We call them *individual modalities*, for short, and the set of these modalities is denoted by Ω^{ind}. Additional two sets of operators, $\Omega^{\mathrm{B}^+} = \{[\mathrm{B}]^+_G : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$ and $\Omega^{\mathrm{I}^+} = \{[\mathrm{I}]^+_G : G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}$ are used for representing common beliefs and mutual intentions of groups of agents. We call them *fixpoint modalities*. The set of all modal operators of TeamLog is denoted by Ω^{T}.

The language of TeamLog, denoted by $\mathcal{L}^{\mathrm{T}}$, is based on a countable set of propositional variables $\mathcal{P}$ and on a set of modal operators Ω^{T}. It is a minimal set of formulas satisfying the following properties

- $\mathcal{P} \subseteq \mathcal{L}^{\mathrm{T}}$,
- If $\varphi \in \mathcal{L}^{\mathrm{T}}$, then $\neg\varphi \in \mathcal{L}^{\mathrm{T}}$,
- If $\varphi_1 \in \mathcal{L}^{\mathrm{T}}$ and $\varphi_2 \in \mathcal{L}^{\mathrm{T}}$, then $\varphi_1 \wedge \varphi_2 \in \mathcal{L}^{\mathrm{T}}$,
- If $\varphi \in \mathcal{L}^{\mathrm{T}}$ and $\Box \in \Omega^{\mathrm{T}}$, then $\Box\varphi \in \mathcal{L}^{\mathrm{T}}$.

We will also use the following standard abbreviations for propositional constants and operators:

- $\top \stackrel{\mathrm{def}}{=} p \wedge \neg p$, where $p \in \mathcal{P}$,
- $\top \stackrel{\mathrm{def}}{=} \neg\bot$,
- $\varphi_1 \vee \varphi_2 \stackrel{\mathrm{def}}{=} \neg(\neg\varphi_1 \wedge \neg\varphi_2)$,
- $\varphi_1 \rightarrow \varphi_2 \stackrel{\mathrm{def}}{=} \neg(\varphi_1 \wedge \neg\varphi_2)$,
- $\varphi_1 \leftrightarrow \varphi_2 \stackrel{\mathrm{def}}{=} (\varphi_2 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$,

as well as the following abbreviations for general beliefs and general intentions of a non-empty group of agents G

- $[\mathrm{B}]_G\varphi \overset{\mathrm{def}}{=} \bigwedge_{j\in G}[\mathrm{B}]_j\varphi$,
- $[\mathrm{I}]_G\varphi \overset{\mathrm{def}}{=} \bigwedge_{j\in G}[\mathrm{I}]_j\varphi$.

Given a finite set of formulas Φ, we will use $\bigwedge\Phi$ to denote the conjunction of all formulas in the set, and $\bigvee\Phi$ to denote the disjunction of all formulas in the set. We will also use the conventions that $\bigwedge\emptyset = \top$ and $\bigvee\emptyset = \bot$.

Throughout the paper we will refer to the notion of *single negation*. Given a formula φ,

$$\sim\varphi = \begin{cases} \psi, & \text{if } \varphi = \neg\psi \text{ for some formula } \psi, \\ \neg\varphi, & \text{otherwise.} \end{cases}$$

A set of formulas Φ is *closed under single negation* iff for all $\varphi \in \Phi$, it holds that $\sim\varphi \in \Phi$. Given a set of formulas Φ we will use $\neg\Phi$ to denote the smallest set containing Φ and closed under single negation.

We will use $|\varphi|$ to denote the *length of a formula*[2] and $\mathrm{dep}(\varphi)$ to denote the *modal depth* of a formula. Both notions have standard meaning and we omit the definition here. Given a finite set of formulas Φ,

$$\mathrm{dep}(\Phi) = \begin{cases} 0, & \text{if } \Phi = \emptyset \\ \max\{\mathrm{dep}(\varphi) : \varphi \in \Phi\}, & \text{otherwise.} \end{cases}$$

3.1 Deduction system

TeamLog combines axiom systems $\mathrm{KD45}_n$, associated with modal operators from Ω^{B} representing beliefs, K_n, associated with modal operators from Ω^{G} representing goals, and KD_n, associated with modal operators from Ω^{I} representing intentions. Additionally, axioms interrelating different modalities, called *mixed axioms*, as well as axioms related to fixpoint modalities from Ω^{B^+} and Ω^{I^+}, representing common beliefs and mutual intentions, are introduced. All these are presented below.

For each of the modal operators $\Box \in \Omega^{\mathrm{ind}}$, the following axioms and deduction rules of the basic modal system K are adopted:

P	All instances of propositional tautologies	
K	$\Box\varphi \wedge \Box(\varphi \to \psi) \to \Box\psi$	
MP	From φ and $\varphi \to \psi$ infer ψ	(Modus ponens)
GEN	From φ infer $\Box\varphi$	(Generalization).

Additionally, depending on which of the sets Ω^{B}, Ω^{G} or Ω^{I}, $\Box$ belongs to, a subset of the following axioms is adopted:

D	$\neg\Box\bot$
4	$\Box\varphi \to \Box\Box\varphi$
5	$\neg\Box\varphi \to \Box\neg\Box\varphi$.

[2] Given a set X we will also use $|X|$ to denote the cardinality of X. The notation is standard for both length of a formula and cardinality of a set. It will be clear from the context what is meant by particular usages.

The intended meaning of a formula $[\mathrm{B}]_j\varphi$ is that *agent j believes that* φ and axioms and inference rules of the standard doxastic modal logic (c.f. Meyer and van der Hoek 1995; Fagin et al. 2003) forming the $\mathrm{KD45}_n$ system are adopted for modal operators from Ω^{B}. The interpretations of the axioms are in this case as follows: **K** (distribution of beliefs), **D** (consistency of beliefs), **4** (positive introspection) and **5** (negative introspection).

The intended meaning of a formula $[\mathrm{G}]_j\varphi$ is that *agent j has goal* φ and axioms and inference rules of the system K_n are adopted for modal operators from Ω^{G}. Axiom **K** is interpreted as distribution of goals. Note that goals can be inconsistent.

The intended meaning of a formula $[\mathrm{I}]_j\varphi$ is that *agent j intends* φ and axioms and inference rules of the system KD_n are adopted for modal operators from Ω^{I}. The interpretations of the axioms are in this case as follows: **K** (distribution of intentions) and **D** (consistency of intentions).

Mixed axioms, interrelating different modalities, are as follows:

BG4	$[\mathrm{G}]_j\varphi \rightarrow [\mathrm{B}]_j[\mathrm{G}]_j\varphi$	(Positive introspection of goals)
BG5	$\neg[\mathrm{G}]_j\varphi \rightarrow [\mathrm{B}]_j\neg[\mathrm{G}]_j\varphi$	(Negative introspection of goals)
BI4	$[\mathrm{I}]_j\varphi \rightarrow [\mathrm{B}]_j[\mathrm{I}]_j\varphi$	(Positive introspection of intentions)
BI5	$\neg[\mathrm{I}]_j\varphi \rightarrow [\mathrm{B}]_j\neg[\mathrm{I}]_j\varphi$	(Negative introspection of intentions)
IG	$[\mathrm{I}]_j\varphi \rightarrow [\mathrm{G}]_j\varphi$	(Goals and intentions compatibility).

Axioms **BG4** and **BG5** correspond to positive and negative introspection of goals: an agent is aware of the goals it has and of the goals it does not have. Analogous axioms, **BI4** and **BI5**, are adopted for intentions. Axiom **IG** corresponds to goals and intentions compatibility: intentions of an agent are a subset of its goals.

The axioms of positive and negative introspection of goals and intentions were also adopted in the first version of the formalism of Rao and Georgeff (1991). The axiom of goals and intentions compatibility is discussed in Rao and Georgeff (1998) as one of the realism axioms, extending the realism axiom of Cohen and Levesque (1990) for beliefs and goals compatibility to compatibility of goals and intentions. It was also adopted by Wooldridge in his $\mathcal{LORA}$ formalism (Wooldridge 2000).

For fixpoint modalities $[O]^+_G \in \Omega^{\mathrm{B}^+} \cup \Omega^{\mathrm{I}^+}$ the following axiom and a rule of inference are adopted:

C	$[O]^+_G\varphi \leftrightarrow [O]_G(\varphi \wedge [O]^+_G\varphi)$	
RC	From $\varphi \rightarrow [O]_G(\psi \wedge \varphi)$ infer $\varphi \rightarrow [O]^+_G\psi$	(Induction).

The intended meaning of a formula $[\mathrm{B}]^+_G\varphi$ is that *group G has a common belief that* φ. This notion is defined in terms of *general beliefs*, $[\mathrm{B}]_G\varphi$, meaning that every agent in G believes that φ. Thus there is a common belief that φ in group G if and only if every agent in G believes that φ, every agent in G believes that every agent in G believes that φ, etc., ad infinitum.

The intended meaning of a formula $[\mathrm{I}]^+_G\varphi$ is that *group G has a mutual intention that* φ. This notion is defined in terms of *general intentions*, $[\mathrm{I}]_G\varphi$, meaning that every agent in G intends that φ, in analogous way to how common beliefs are defined.

Fixpoint modalities such as common beliefs or mutual intentions are widely used in formalism for multiagent systems such as the formalism of Levesque et al. (1990), $\mathcal{LORA}$

of Wooldridge (2000) and the extension of KARO by Aldewereld, Hoek and Meyer and Bratman (1987).

As we wrote above, operators of general beliefs and general intentions, $[\mathrm{B}]_G$ and $[\mathrm{I}]_G$, are defined in terms of individual beliefs and individual intentions and a formula with these operators can be translated to a formula without them in linear time, which increases the size of the formula be a linear factor $\leq |\mathcal{A}|$. For that reason we will omit them from now on, as they are not relevant for the complexity issues considered in this paper.

3.2 Semantics

Formulas from $\mathcal{L}^{\mathrm{T}}$ are interpreted in Kripke models with accessibility relations corresponding to modalities from Ω^{T}. Since accessibility relations corresponding to operators from Ω^{B^+} and Ω^{I^+} can be defined in terms of accessibility relations corresponding to individual modalities Ω^{ind}, the definition of Kripke models is based on relations corresponding to these operators only.

Definition 1 (Kripke frame) A *Kripke frame* is a tuple $\mathcal{F} = \left(W, \left\{O_j : [O]_j \in \Omega^{\mathrm{ind}}\right\}\right)$, where

- $W \neq \emptyset$ is the set of possible worlds.
- For all $[O]_j \in \Omega^{\mathrm{ind}}$, $O_j \subseteq W \times W$. Each relation O_j stands for the accessibility relation corresponding to the operator $[O]_j$.

Definition 2 (Kripke model) A *Kripke model* is a pair $\mathcal{M} = (\mathcal{F}, Val)$, where $\mathcal{F}$ is a Kripke frame and

- $Val : \mathcal{P} \times W \to \{0, 1\}$ is a valuation function that assigns the truth values to atomic propositions in worlds.

Given a binary relation $R \subseteq W \times W$ and $w \in W$ we will use $R(w)$ to denote the set of worlds accessible from w, that is $R(w) = \{v \in W : (w, v) \in R\}$. Moreover, we will use R^+ to denote the transitive closure of R. Additionally, given a family of relations $\{R_j : j \in \mathcal{A}\}$ and a set of agents $G \subseteq \mathcal{A}$, relation $R_G = \bigcup_{j \in G} R_j$. The relation corresponding to a modal operator $[O]_G^+ \in \Omega^{\mathrm{T}}$ is O_G^+.

Definition 3 (Satisfaction) Let $\mathcal{M}$ be a Kripke model, w be a world in $\mathcal{M}$ and $\varphi \in \mathcal{L}^{\mathrm{T}}$ be a formula. The notion of φ being *satisfied* (or being *true* or *holding*) in $\mathcal{M}$ at w is defined inductively as follows:

$$
\begin{array}{lll}
(\mathcal{M}, w) \vDash p & \text{iff} & Val(p, w) = 1, \\
(\mathcal{M}, w) \vDash \neg\varphi & \text{iff} & (\mathcal{M}, w) \nvDash \varphi, \\
(\mathcal{M}, w) \vDash \varphi_1 \wedge \varphi_2 & \text{iff} & (\mathcal{M}, w) \vDash \varphi_1 \text{ and } (\mathcal{M}, w) \vDash \varphi_2, \\
(\mathcal{M}, w) \vDash [O]_j \varphi & \text{iff} & (\mathcal{M}, v) \vDash \varphi, \text{ for all } v \in O_j(w), \\
(\mathcal{M}, w) \vDash [O]_G^+ \varphi & \text{iff} & (\mathcal{M}, v) \vDash \varphi, \text{ for all } v \in O_G^+(w).
\end{array}
$$

Let $\varphi \in \mathcal{L}^{\mathrm{T}}$ be a formula. We say that φ is *valid* in a Kripke model $\mathcal{M}$ if for every world w in $\mathcal{M}$, $(\mathcal{M}, w) \vDash \varphi$. We denote this fact by $\mathcal{M} \vDash \varphi$. We say that φ is *satisfiable in* $\mathcal{M}$ if there exists a world w in $\mathcal{M}$ such that $(\mathcal{M}, w) \vDash \varphi$. Let $\mathcal{C}$ be a class of Kripke models. We say that φ is *valid in* $\mathcal{C}$ if $\mathcal{M} \vDash \varphi$, for every $\mathcal{M} \in \mathcal{C}$. We denote this fact by $\mathcal{C} \vDash \varphi$. We say that φ is *satisfiable in* $\mathcal{C}$ if there exists $\mathcal{M} \in \mathcal{C}$ such that φ is satisfiable in $\mathcal{M}$.

Axioms of modal systems K_n, KD_n and $\mathrm{KD45}_n$, as far as they do not hold on all frames like **K**, correspond to well-known structural properties on Kripke frames, in the sense that they hold on all frames having certain structural properties (c.f. van Benthem 1984). Axiom **D** (adopted for $[\mathrm{B}]_j$ and $[\mathrm{I}]_j$) corresponds to seriality of O_j (where $O \in \{\mathrm{B}, \mathrm{I}\}$)

$$\forall s (O_j(s) \neq \varnothing),$$

axiom **4** (adopted for $[\mathrm{B}]_j$) corresponds to transitivity of B_j

$$\forall s, t (t \in \mathrm{B}_j(s) \rightarrow \mathrm{B}_j(t) \subseteq \mathrm{B}_j(s)),$$

and axiom **5** (adopted for $[\mathrm{B}]_j$) corresponds to Euclidity of B_j

$$\forall s, t (t \in \mathrm{B}_j(s) \rightarrow \mathrm{B}_j(s) \subseteq \mathrm{B}_j(t)).$$

Similarly, the mixed axioms correspond to certain properties of Kripke frames. Axioms of positive introspection, **B** ***O*4** with $O \in \{\mathrm{G}, \mathrm{I}\}$, correspond to the following property

$$\forall s, t (t \in \mathrm{B}_j(s) \rightarrow O_j(t) \subseteq O_j(s)).$$

We will call this property *generalized transitivity*. Axioms of negative introspection, **B** ***O*5**, with $O \in \{\mathrm{G}, \mathrm{I}\}$, correspond to the property

$$\forall s, t (t \in \mathrm{B}_j(s) \rightarrow O_j(s) \subseteq O_j(t)).$$

We will call this property *generalized Euclidity*. Finally, axiom **IG** corresponds to the property

$$\mathrm{G}_j \subseteq \mathrm{I}_j.$$

Proofs of these correspondences are given in Dunin-Kęplicz and Verbrugge (2004). Also, they follow directly from Sahlqvist theorem (c.f. Blackburn et al. 2002, for example). The class of all Kripke frames with accessibility relations satisfying the properties above will be called TeamLog *frames*. Analogously TeamLog *models* are defined. We will say that φ is TeamLog *provable*, denoted by $\vdash_T \varphi$, if there exists a proof of φ that includes axioms from TeamLog. The deduction system of TeamLog is sound and complete with respect to the class of TeamLog models, as was shown in Dunin-Kęplicz and Verbrugge (2002).

Theorem 1 *Let $\mathcal{T}$ be the class of* TeamLog *models. Then for any $\varphi \in \mathcal{L}^{\mathrm{T}}$*

$$\mathcal{T} \vDash \varphi \text{ iff } \vdash_T \varphi$$

4 Modal context restriction

In this section we introduce a family of language restrictions for modal logics called modal context restrictions. The idea was already presented in Dziubiński (2013). After introducing the concept, we propose two restrictions of this kind, called $\mathbf{R}_1$ and $\mathbf{R}_2$, as well as a refinement of one of them, called $\mathbf{R}_{1(c)}$, which will be studied in the remaining part of the paper.

We start by defining the notion of modal context restriction for a general language of multimodal logic.[3] First we need a notion of modal context of a formula within a formula. Consider a case of a formula appearing only once as a subformula of some given formula. This subformula is in a scope of a sequence (possibly empty) of modal operators. This sequence is the modal context of the subformula in the given formula. In general a formula may appear more than once as a subformula of another formula. In this case there is a set of sequence of modal operators, one for each appearance of the subformula. To get an intuition of this concept, assume a modal language over the set of two modal operators $\Omega = \{\Box_0, \Box_1\}$. Every formula has an expression-tree representation associated with it. Leafs of such a tree are labelled with propositional symbols and internal nodes are labelled with operators. Each subtree of the expression-tree corresponds to a subformula of the formula. For example, a formula

$$\varphi \equiv \Box_1(p \wedge \Box_0(q \vee \Box_1 p))$$

is represented by a tree in Fig. 1.

Expression-tree representation is unique (up to isomorphism). Modal context of a subformula of a formula is the set of sequences of modal operators on the paths from the root of the expression-tree representing the formula to the roots of expression-trees representing the subformula. In the example above, modal context of q in φ is $\{\Box_1\Box_0\}$, modal context of p in φ is $\{\Box_1, \Box_1\Box_0\Box_1\}$, modal context of $p \wedge \Box_0(q \vee \Box_1 p)$ in φ is $\{\Box_1\}$, and modal context of φ in φ is the empty sequence, ε. If a formula is not a subformula of the other formula, then its modal context is $\emptyset$.

Formally, let $\mathcal{L}[\mathcal{P}, \Omega]$ be a multimodal language over a set of propositional variables $\mathcal{P}$ and a set of (unary) modal operators Ω. Given $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$, let

$$\mathrm{Sub}(\varphi) = \{\psi : \psi \text{ is a subformula of } \varphi\}$$

be the set of all subformulas of φ.

Definition 4 (Modal context of a formula within a formula) Let $\{\varphi, \xi\} \in \mathcal{L}[\mathcal{P}, \Omega]$. The *modal context of formula* ξ *within formula* φ is a set of finite sequences over Ω, $\mathrm{cont}(\xi, \varphi) \subseteq \Omega^*$, defined inductively as follows:

- $\mathrm{cont}(\xi, \varphi) = \emptyset$, if $\xi \notin \mathrm{Sub}(\varphi)$,
- $\mathrm{cont}(\varphi, \varphi) = \{\varepsilon\}$,
- $\mathrm{cont}(\xi, \neg\psi) = \mathrm{cont}(\xi, \psi)$, if $\xi \neq \neg\psi$,
- $\mathrm{cont}(\xi, \psi_1 \wedge \psi_2) = \mathrm{cont}(\xi, \psi_1) \cup \mathrm{cont}(\xi, \psi_2)$, if $\xi \neq \psi_1 \wedge \psi_2$,
- $\mathrm{cont}(\xi, \Box\psi) = \Box \cdot \mathrm{cont}(\xi, \psi)$, if $\xi \neq \Box\psi$ and $\Box \in \Omega$,

[3] The definition of modal context restriction is given for a propositional modal logic with unary modal operators. It can be extended to the case of modal operators of arbitrary arities and to FOL modal logic in a natural way.

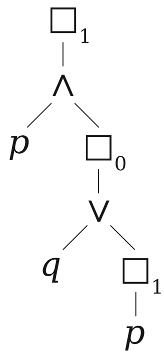

Fig. 1 Expression tree for formula $\Box_1(p \wedge \Box_0(q \vee \Box_1 p)$

where $\Box \cdot S = \{\Box \cdot s : s \in S\}$, for $\Box \in \Omega$ and $S \subseteq \Omega^*$.

Definition 5 (Modal context of a formula) *Modal context of a formula* $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$, $\mathrm{cont}(\varphi)$, is the sum of modal contexts of all its subformulas, that is

$$\mathrm{cont}(\varphi) = \bigcup_{\xi \in \mathrm{Sub}(\varphi)} \mathrm{cont}(\xi, \varphi).$$

Having defined modal context of a formula, we are ready to define the notion of modal context restriction. The restriction is the set of sequences of modal operators that are allowed in modal contexts of formulas.

Definition 6 (Modal context restriction) A *modal context restriction* is a set of sequences over Ω, $R \subseteq \Omega^*$, constraining possible modal contexts of subformulas within formulas. We say that $\varphi \in \mathcal{L}[\mathcal{P}, \Omega]$ *satisfies a modal context restriction* $R \subseteq \Omega^*$ iff $\mathrm{cont}(\varphi) \subseteq R$.

To see an example of modal context restriction, consider the multi-modal language over the set of modal operators $\Omega = \{\Box_0, \Box_1\}$. Suppose that we would like to allow only the formulas were modal operators (if present) are alternating in modal contexts of subformulas. Such a restriction can be defined by a regular expression $((\varepsilon \cup \Box_1) \cdot (\Box_0\Box_1)^*) \cup ((\varepsilon \cup \Box_0) \cdot (\Box_1\Box_0)^*)$. Alternatively, we could define it by specifying that subsequences $\Box_1\Box_1$ and $\Box_0\Box_0$ are forbidden: $\Omega^* \setminus (\Omega^* \cdot (\Box_1\Box_1 \cup \Box_0\Box_0) \cdot \Omega^*)$ (that is, as a complement of the language containing all the sequences that we want to forbid).

Notice that modal context restriction is a generalisation of restricting modal depth of formulas by a constant c. Indeed, we could define it as a set of sequences of modal operators of length at most c.

4.1 Restricting modal context of TeamLog

In this paper we study two modal context restrictions for the language of TeamLog that lead to PSPACE completeness of the satisfiability problem. The restrictions are presented below.

The first of the restrictions, $\mathbf{R}_1$, is motivated by the formula used to show that the satisfiability problem for basic multimodal logics with fixpoint modalities is EXPTIME hard, even if depth of formulas is bounded by 2 (c.f. Dziubiński et al. 2007). The modal context of this formula contains sequences of the form $[O]^+_G[O]_j$ with $j \in G$. Restriction $\mathbf{R}_1$ forbids such sequences in modal contexts of formulas. It forbids also sequences $[O]^+_G[O]^+_H$ with $G \cap H \neq \emptyset$. This is motivated by the fact that in TeamLog a formula $[O]^+_G[O]^+_H\psi$ is equivalent to $[O]^+_G \bigwedge_{j \in H}([O]_j\psi \wedge [O]_j[O]^+_H\psi)$, the modal context of which contains the sequence $[O]^+_G[O]_j$ that we want to forbid. In the definition below these forbidden sequences are grouped in the sets $S_O(G)$. Additionally the restriction forbids subsequences $[\mathrm{I}]^+_G[\mathrm{B}]_j[\mathrm{I}]_j$ with $j \in G$. This is motivated by the fact that, due to mixed axioms **BI4** and **BI5**, the formula $[\mathrm{I}]^+_G[\mathrm{B}]_j[\mathrm{I}]_j\varphi$ is equivalent to $[\mathrm{I}]^+_G[\mathrm{I}]_j\varphi$, which is the sequence that we initially wanted to forbid. For similar reasons sequences $[\mathrm{I}]^+_G[\mathrm{B}]^+_H[\mathrm{I}]_j$, $[\mathrm{I}]^+_G[\mathrm{B}]_j[\mathrm{I}]^+_F$ and $[\mathrm{I}]^+_G[\mathrm{B}]^+_H[\mathrm{I}]^+_F$ are forbidden for if $j \in G \cap H$ or $j \in G \cap F$ or $F \cap G \cap H \neq \emptyset$, respectively. In the definition below these forbidden sequences are grouped in the sets $S_{\mathrm{IB}(G)}$.

Definition 7 (Restriction $\mathbf{R}_1$) Let

$$\mathbf{R_1} = \Omega^* \setminus \left(\Omega^* \cdot \left[\bigcup_{G \in \mathrm{P}(\mathcal{A}) \setminus \{\emptyset\}} (S_{\mathrm{I}}(G) \cup S_{\mathrm{IB}}(G)) \cup \bigcup_{G \in \mathrm{P}(\mathcal{A}), |G| \geq 2} S_{\mathrm{B}}(G)\right] \cdot \Omega^*\right),$$

where

$$S_{\mathrm{IB}}(G) = \bigcup_{j \in G} [\mathrm{I}]^+_G \cdot \left\{[\mathrm{B}]_j, [\mathrm{B}]^+_{\{j\}}\right\}^* \cdot T_{\mathrm{B}}(\{j\}) \cdot T_{\mathrm{I}}(\{j\}), \text{ and}$$

$$S_O(G) = [O]^+_G \cdot T_O(G),$$

$$T_O(G) = \{[O]_j : j \in G\} \cup \{[O]^+_H : H \in \mathrm{P}(\mathcal{A}), H \cap G \neq \emptyset\},$$

for $O \in \{\mathrm{B}, \mathrm{I}\}$. The set of formulas in $\mathcal{L}^{\mathrm{T}}$ satisfying restriction $\mathbf{R}_1$ will be denoted by $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$.

Roughly speaking, restriction $\mathbf{R}_1$ forbids any operator $[O]_j$ or $[O]^+_H$, with $O \in \{\mathrm{B}, \mathrm{I}\}$ in scope of operator $[O]^+_G$, if $j \in G$ or $G \cap H \neq \emptyset$. The following formulas satisfy restriction $\mathbf{R}_1$:

$$[\mathrm{B}]^+_{\{1,2\}}[\mathrm{I}]^+_{\{1,2\}}p,$$

Indeed, the modal context of the first formula is $\left\{\varepsilon, [\mathrm{B}]^+_{\{1,2\}}, [\mathrm{B}]^+_{\{1,2\}}[\mathrm{I}]^+_{\{1,2\}}\right\}$ and none of these sequences contains any of the forbidden ones. The modal context of the second formula is $\left\{\varepsilon, [\mathrm{B}]^+_{\{1,2\}}, [\mathrm{B}]^+_{\{1,2\}}[\mathrm{G}]_1\right\}$ and, again, none of these sequences contains any of the forbidden ones. The following formulas violate restriction $\mathbf{R}_1$:

$$[\mathrm{I}]^+_{\{1,2\}}[\mathrm{B}]_1[\mathrm{I}]_1 q,$$

The modal context of the first one is $\left\{\varepsilon, [\mathrm{I}]^+_{\{1,2\}}, [\mathrm{I}]^+_{\{1,2\}}[\mathrm{B}]_1, [\mathrm{I}]^+_{\{1,2\}}[\mathrm{B}]_1[\mathrm{I}]_1\right\}$, and it contains a forbidden sequence $[\mathrm{I}]^+_{\{1,2\}}[\mathrm{B}]_1[\mathrm{I}]_1$. The modal context of the second one is $\left\{\varepsilon, [\mathrm{B}]^+_{\{1,2\}}, [\mathrm{B}]^+_{\{1,2\}}[\mathrm{B}]_1\right\}$, and it contains a forbidden sequence $[\mathrm{B}]^+_{\{1,2\}}[\mathrm{B}]_1$.

As we show in Sect. 6 (Proposition 1), the satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ is PSPACE hard, even if modal depth of formulas is bounded by 2. This result is shown using a formula which contains, in its modal context, sequences of the form $[\mathrm{B}]^{+}_{G}[\mathrm{G}]_j$ and $[\mathrm{B}]^{+}_{G}[\mathrm{I}]_j$, with $j \in G$. Motivated by that, we study another modal context restriction, $\mathbf{R}_2$, which refines $\mathbf{R}_1$ by forbidding such sequences. Additionally, for the reasons similar to those explained in the case of restriction $\mathbf{R}_1$, subsequences forbidden by restriction $\mathbf{R}_2$ contain sequences of the form $[\mathrm{B}]^{+}_{G}[\mathrm{I}]^{+}_{H}$ with $G \cap H \neq \emptyset$. Restriction $\mathbf{R}_2$, when combined with restricting modal depth of formulas by a constant, makes the TeamLog satisfiability problem NPTIME solvable.

Definition 8 (Restriction $\mathbf{R}_2$) Let

$$\mathbf{R}_2 = \Omega^* \setminus \left(\Omega^* \cdot \left[\bigcup_{G \in \mathrm{P}(\mathcal{A}) \setminus \{\emptyset\}} (S_{\mathrm{I}}(G) \cup S_{\mathrm{IB}}(G)) \cup \bigcup_{G \in \mathrm{P}(\mathcal{A}), |G| \geq 2} \tilde{S}_{\mathrm{B}}(G) \right] \cdot \Omega^* \right),$$

where

$$\tilde{S}_{\mathrm{B}}(G) = [\mathrm{B}]^{+}_{G} \cdot \left(\{[\mathrm{G}]_j : j \in G\} \cup \bigcup_{O \in \{\mathrm{B}, \mathrm{I}\}} T_O(G) \right)$$

and S_{IB}, S_{I} and T_O, for $O \in \{\mathrm{B}, \mathrm{I}\}$, are defined like in the case of restriction $\mathbf{R}_1$. The set of formulas in $\mathcal{L}^{\mathrm{T}}$ satisfying restriction $\mathbf{R}_2$ will be denoted by $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$.

Restriction $\mathbf{R}_2$ extends $\mathbf{R}_1$ by forbidding any operator $[O]_j$ or $[O]^{+}_{H}$, with $O \in \{\mathrm{G}, \mathrm{I}\}$, in the context of $[\mathrm{B}]^{+}_{G}$, if $j \in G$ or $H \cap G \neq \emptyset$. Thus any formula $\varphi \in \mathcal{L}^{\mathrm{T}}$ satisfying restriction $\mathbf{R}_2$, satisfies restriction $\mathbf{R}_1$ as well, that is $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2 \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}}$. Notice that if $|\mathcal{A}| = 1$, then $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2 = \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}}$.

The following formulas satisfy restriction $\mathbf{R}_2$.

$$[\mathrm{I}]^{+}_{\{1,2\}}[\mathrm{B}]_1 p,$$

Indeed, the modal context of the first formula is $\left\{\varepsilon, [\mathrm{I}]^{+}_{\{1,2\}}, [\mathrm{I}]^{+}_{\{1,2\}}[\mathrm{B}]_1\right\}$ and none of these sequences contains any of the forbidden ones. The modal context of the second formula is $\left\{\varepsilon, [\mathrm{B}]^{+}_{\{1,2\}}, [\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{B}]_3\right\}$ and, again, none of these sequences contains any of the forbidden ones.

The following formulas violate restriction $\mathbf{R}_2$

$$[\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{I}]^{+}_{\{1,2\}} p,$$

Modal context of the first formula is $\left\{\varepsilon, [\mathrm{B}]^{+}_{\{1,2\}}, [\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{I}]^{+}_{\{1,2\}}\right\}$, and it contains a forbidden sequence $[\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{I}]^{+}_{\{1,2\}}$. Modal context of the second formula is $\left\{\varepsilon, [\mathrm{B}]^{+}_{\{1,2\}}, [\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{G}]_1\right\}$, and it contains a forbidden sequence $[\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{G}]_1$. Notice that the modal contexts above do not contain sequences forbidden by restriction $\mathbf{R}_1$, and so the formulas belong to $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$.

4.2 Restriction $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$

As we show in Sect. 5, restriction $\mathbf{R}_2$ is too strong, as it forbids formulas such as collective intentions, important for specifying cooperating teams of agents (Dunin-Kęplicz and Verbrugge 2010). Therefore we consider a refinement of restriction $\mathbf{R}_1$ that, when combined with restricting modal depth, makes the [TeamLog satisfiability problem NPTIME solvable.

The restriction is motivated by the formula used in proof of Proposition 1. One of the conjuncts in this formula is a formula of the form $[\mathrm{B}]^{+}_{G}\psi$ where ψ is a propositional formula built of $2N$ atoms of the form $[\mathrm{I}]_j\xi$ (alternatively $[\mathrm{G}]_j\xi$), with $j \in G$. These atoms are used to implement a counter that can enforce a path of length $\mathcal{O}(2^N)$ in the model for the formula. To prevent such a construction, restriction $\mathbf{R}_{1(c)}$ bounds, by a constant c, the number of subformulas of the form $[\mathrm{I}]_j\xi$, $[\mathrm{G}]_j\xi$ and $[\mathrm{I}]^{+}_{H}\xi$, within the direct context of modal operators $[\mathrm{B}]^{+}_{G}$ with $j \in G$ or $H \cap G \neq \emptyset$. That is, whenever a formula from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ has a subformula which violates modal context restriction $\mathbf{R}_2$, then this formula must satisfy this additional restriction.

To define the restriction, we need to define the set of subformulas of a formula taken with respect to propositional operators only. Let $\mathrm{PT}(\varphi)$ be defined inductively as follows:

1. $\mathrm{PT}(p) = \{p\}$, where $p \in \mathcal{P}$,
2. $\mathrm{PT}(\neg\psi) = \{\neg\psi\} \cup \mathrm{PT}(\psi)$,
3. $\mathrm{PT}(\psi_1 \wedge \psi_2) = \mathrm{PT}(\psi_1) \cup \mathrm{PT}(\psi_2)$.
4. $\mathrm{PT}(\Box\psi) = \{\Box\psi\}$, where $\Box \in \Omega^{\mathrm{T}}$.

Given a formula $\Box\varphi$, the set $\mathrm{PT}(\varphi)$ contains the subformulas in the direct context of a modal operator $\Box$. So for example in the case of $\Box(p \wedge \neg q \vee \Box(\Box p \wedge r) \vee \Box r)$, $\mathrm{PT}(p \wedge \neg q \vee \Box(\Box p \wedge r) \vee \Box r) = \{p, q, \neg q, \Box(\Box p \wedge r), \Box r\}$ contains the subformulas in the direct context of the most external operator $\Box$.

The restriction is defined as follows.

Definition 9 (Restriction $\mathbf{R}_{1(c)}$) Let $c \geq 0$. A formula φ satisfies the restriction $\mathbf{R}_{1(c)}$ if $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ and either $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ or one of the following holds:

- φ is of the form $\neg\psi$ and ψ satisfies restriction $\mathbf{R}_{1(c)}$,
- φ is of the form $\psi_1 \wedge \psi_2$ and ψ_1 and ψ_2 satisfy restriction $\mathbf{R}_{1(c)}$,
- φ is of the form $[O]_j\psi$, with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$, and ψ satisfies restriction $\mathbf{R}_{1(c)}$,
- φ is of the form $[\mathrm{I}]^{+}_{G}\psi$, with $G \in \mathrm{P}(\mathcal{A}) \setminus \{\emptyset\}$, and ψ satisfies restriction $\mathbf{R}_{1(c)}$,
- φ is of the form $[\mathrm{B}]^{+}_{G}\psi$, with $G \in \mathrm{P}(\mathcal{A}) \setminus \{\emptyset\}$, ψ satisfies restriction $\mathbf{R}_{1(c)}$ and $\left|\{[O]_j\xi : [O]_j\xi \in \neg\mathrm{PT}(\psi) \text{ and } j \in G\} \cup \{[\mathrm{I}]^{+}_{H}\xi : [\mathrm{I}]^{+}_{H}\xi \in \neg\mathrm{PT}(\psi) \text{ and } H \cap G \neq \emptyset\}\right| \leq c$.

The set of formulas in $\mathcal{L}^{\mathrm{T}}$ satisfying restriction $\mathbf{R}_{1(c)}$ will be denoted by $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$.

The following formulas satisfy restriction $\mathbf{R}_{1(1)}$.

$$[\mathrm{B}]^{+}_{\{1,2\}}[\mathrm{I}]^{+}_{\{1,2\}}p,$$

Firstly, both the formulas satisfy restriction $\mathbf{R}_1$. The set of formulas in the direct context of $[\mathrm{B}]^+_{\{1,2\}}$ in the first formula, $\mathrm{PT}([\mathrm{I}]^+_{\{1,2\}}p) = \{[\mathrm{I}]^+_{\{1,2\}}p\}$ and since this set contains only one formula, restriction $\mathbf{R}_{1(1)}$ is satisfied. The set of formulas in the direct context of $[\mathrm{B}]^+_{\{1,2\}}$ in the second formula, $\mathrm{PT}([\mathrm{I}]_2 p \vee q) = \{[\mathrm{I}]_2 p, q\}$ and since this set contains only one formula of the form $[\mathrm{I}]_j\xi$, $[\mathrm{G}]_j\xi$ or $[\mathrm{I}]^+_H\xi$ with $j \in \{1,2\}$ or $H \cap \{1,2\} \neq \varnothing$, namely $[\mathrm{I}]_2 p$, so restriction $\mathbf{R}_{1(1)}$ is satisfied as well.

The following formulas violate restriction $\mathbf{R}_{1(1)}$ and satisfy restriction $\mathbf{R}_{1(2)}$

$$[\mathrm{B}]^+_{\{1,2\}}\big([\mathrm{I}]_1 p \wedge [\mathrm{I}]_2 q\big),$$

Both the formulas satisfy restriction $\mathbf{R}_1$. The set of formulas in the direct context of $[\mathrm{B}]^+_{\{1,2\}}$ in the first formula, $\mathrm{PT}([\mathrm{I}]_1 p \wedge [\mathrm{I}]_2 q) = \{[\mathrm{I}]_1 p, [\mathrm{I}]_2 q)$ and since this set contains two formulas of the form $[\mathrm{I}]_j\xi$ with $j \in \{1,2\}$, so restriction $\mathbf{R}_{1(c)}$ with $c = 1$ is violated, but it is satisfied with $c = 2$. The set of formulas in the direct context of $[\mathrm{B}]^+_{\{1,2\}}$ in the second formula, $\mathrm{PT}([\mathrm{G}]_2 p \vee [\mathrm{I}]^+_{\{2,3\}}q) = \{[\mathrm{G}]_2 p, [\mathrm{I}]^+_{\{2,3\}}q\}$ and since this set contains two formulas, $[\mathrm{G}]_j p$ with $j \in \{1,2\}$, and $[\mathrm{I}]^+_{\{H\}}q$ with $H \cap \{1,2\} \neq \varnothing$, so restriction $\mathbf{R}_{1(c)}$ with $c = 1$ is violated, but it is satisfied with $c = 2$.

5 Discussion

Let us start the discussion of the two restrictions with formulas specifying beliefs of groups of agents. When interpreted in the context of BDI agents, $\mathbf{R}_1$ can be seen as forbidding common introspection of beliefs within a group of agents. In other words, it forbids any formula of the form $[\mathrm{B}]^+_G\varphi$ where φ contains, within the scope of propositional operators, any formulas referring to beliefs of agents from G. For example the following formula specifies that group G commonly beliefs that some agent j believes that φ holds

$$[\mathrm{B}]^+_G[\mathrm{B}]_j\varphi$$

If $j \notin G$ (and if φ satisfies $\mathbf{R}_1$), then this formula satisfies restriction $\mathbf{R}_1$. However, if $j \in G$, then the formula does not satisfy the restriction. Similarly, the following formula specifies that group G commonly believes that some other group H commonly believes that φ holds

$$[\mathrm{B}]^+_G[\mathrm{B}]^+_H\varphi$$

If $G \cap H = \varnothing$ (and if φ satisfies $\mathbf{R}_1$), then this formula satisfies restriction $\mathbf{R}_1$ and it violates it if $G \cap H \neq \varnothing$. The second case could be seen as a consequence of the first one, given that the formula $[\mathrm{B}]^+_G[\mathrm{B}]^+_H\varphi \leftrightarrow [\mathrm{B}]^+_G\Big([\mathrm{B}]^+_H\varphi \wedge [\mathrm{B}]_j\varphi\Big)$ is provable in TeamLog. Summarizing, as long as the objects of common beliefs of a group of agents are ‘external’ for that group, i.e. do not concern beliefs of the agents in the group, restriction $\mathbf{R}_1$ is not violated.

Let us look at restriction $\mathbf{R}_2$ now. It forbids, in addition to what is forbidden by $\mathbf{R}_1$, common introspection of goals and intentions within a group of agents. Hence the following formula, specifying that group G of agents commonly believes that agent j has intention φ:

$$[\mathrm{B}]^+_G[\mathrm{I}]_j\varphi$$

is allowed by $\mathbf{R}_2$ if $j \notin G$ (and φ satisfies $\mathbf{R}_2$) and is forbidden otherwise. Similarly with the formula $[\mathrm{B}]^+_G[\mathrm{G}]_j\varphi$, specifying that group G of agents has common belief that agent j has goal φ, as well as with the formula

$$[\mathrm{B}]^+_G[\mathrm{I}]^+_H\varphi,$$

specifying that group G of agents has common belief that group H of agents has mutual intention φ. In this case the formula satisfies condition $\mathbf{R}_2$ as long as $G \cap H = \emptyset$ (and φ satisfies $\mathbf{R}_2$) and violates it otherwise.

Summarizing, similarly like in the case of restriction $\mathbf{R}_1$, restriction $\mathbf{R}_2$ is not violated as long as the objects of common beliefs of a group of agents are 'external' for that group, where external means this time that it does not concern any informational or motivational attitudes of the agents in the group.

Interpretation of the restrictions in the case of mutual intentions of groups of agents is similar to that of common beliefs. There is one addition, however. Both restrictions, $\mathbf{R}_1$ and $\mathbf{R}_2$, forbid formulas of the form

$$[\mathrm{I}]^+_G[\mathrm{B}]_j[\mathrm{I}]_j\varphi,$$

with $j \in G$, specifying that group G of agents has mutual intention that agent j believes that it has intention φ. This is needed because, due to awareness axioms, formulas of this form are equivalent to formulas of the form $[\mathrm{I}]^+_G[\mathrm{I}]_j\varphi$ (which are forbidden like the analogous ones for common beliefs). For similar reasons formulas $[\mathrm{I}]^+_G[\mathrm{B}]^+_H[\mathrm{I}]_j\varphi$, $[\mathrm{I}]^+_G[\mathrm{B}]_j[\mathrm{I}]^+_F\varphi$ and $[\mathrm{I}]^+_G[\mathrm{B}]^+_H[\mathrm{I}]^+_F\varphi$ are forbidden by $\mathbf{R}_1$ and $\mathbf{R}_2$, if $j \in G \cap H$ or $j \in G \cap F$ or $F \cap G \cap H \neq \emptyset$, respectively.

Now let us turn to formulas specifying important properties of multiagent systems. For example one of the fundamental notions underlying teamwork is that of collective intention, defined as follows (Dunin-Kęplicz and Verbrugge 2010):

$$\text{C- INT}_G(\varphi) \equiv [\mathrm{I}]^+_G\varphi \wedge [\mathrm{B}]^+_G[\mathrm{I}]^+_G\varphi$$

does not satisfy $\mathbf{R}_2$, while it satisfies $\mathbf{R}_1$ (as long as it is satisfied by formulas φ and $[\mathrm{I}]^+_G\varphi$).

Another notion fundamental for specifying teamwork is (bilateral) social commitments between agents in a team. For example social commitment of agent i towards agent j with respect to some action α is defined as follows (Dunin-Kęplicz and Verbrugge 2010):[4]

$$\text{COMM}(i,j,\alpha) \equiv [\mathrm{I}]_j\alpha \wedge [\mathrm{G}]_j done(i,\alpha) \wedge [\mathrm{B}]^+_{\{i,j\}}\Big([\mathrm{I}]_j done(i,\alpha) \wedge [\mathrm{G}]_j done(i,\alpha)\Big).$$

This formula does not satisfy $\mathbf{R}_2$, because of presence of formulas $[\mathrm{I}]_j done(i,a)$ and $[\mathrm{G}]_j done(i,a)$ in the direct context of operator $[\mathrm{B}]^+_{\{i,j\}}$. However, it satisfies restriction $\mathbf{R}_1$. Notice also that the formula satisfies $\mathbf{R}_1(c)$ with $c = 2$.

The third important notion is collective commitment. Several variants of it can be defined, corresponding to different strength of motivational and informational interdependencies within a team of agents. In this case even restriction $\mathbf{R}_1$ can be too strong. Consider, for example, the strongest two forms of collective commitment, robust and strong collective commitment:

[4] Proposition $done(i,a)$ is true iff agent i successfully executed a (simple or complex) action a. Proposition $constitutes(P,\varphi)$ is true iff P is a plan for achieving φ. Both these propositions can be defined formally in richer version of TeamLog which are out of the scope in this paper.

$$\begin{aligned}\text{R-COMM}_{G,P}(\varphi) \equiv{} & \text{C-INT}_G(\varphi) \wedge \mathit{constitutes}(P,\varphi) \wedge [\mathrm{B}]^+_G \mathit{constitutes}(P,\varphi)\\ & \wedge \bigwedge_{\alpha\in P}\bigvee_{i,j\in G}[\mathrm{B}]^+_G \text{COMM}(i,j,\alpha)\end{aligned}$$

$$\begin{aligned}\text{S-COMM}_{G,P}(\varphi) \equiv{} & \text{C-INT}_G(\varphi) \wedge \mathit{constitutes}(P,\varphi) \wedge [\mathrm{B}]^+_G \mathit{constitutes}(P,\varphi)\\ & \wedge [\mathrm{B}]^+_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G}\text{COMM}(i,j,\alpha)\right)\end{aligned}$$

In both cases the last component, expressing team awareness about the distribution of social commitments within the group involves common beliefs within the group about beliefs of agents from this group (which are contained in the definition of $\text{COMM}(i,j,\alpha)$). Hence, both definitions of commitments do not satisfy the restriction $\mathbf{R}_1$. The formula defining the robust commitment is not a problem, because it can be replaced by an equivalent formula that satisfies restriction $\mathbf{R}_1$. Let *responsibility of agent i towards agent j with respect to action* α be defined as follows:

$$\text{RESP}(i,j,\alpha) \equiv [\mathrm{I}]_j\alpha \wedge [\mathrm{G}]_j \mathit{done}(i,a)$$

Notice that social commitment of one agent towards another is responsibility plus awareness about this responsibility. Consider now the following definition of robust commitment:

$$\begin{aligned}\text{R-COMM}'_{G,P}(\varphi) \equiv{} & \text{C-INT}_G(\varphi) \wedge \mathit{constitutes}(P,\varphi) \wedge [\mathrm{B}]^+_G \mathit{constitutes}(P,\varphi)\\ & \wedge \bigwedge_{\alpha\in P}\bigvee_{i,j\in G}[\mathrm{B}]^+_G \text{RESP}(i,j,\alpha)\end{aligned}$$

It can be easily seen that $\text{R-COMM}_{G,P}(\varphi)$ is equivalent to $\text{R-COMM}'_{G,P}(\varphi)$, as the formula $[\mathrm{B}]^+_G\left(\psi \wedge [\mathrm{B}]_j\psi\right) \leftrightarrow [\mathrm{B}]^+_G\psi$ is provable in TeamLog for any ψ, $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$ and $j \in G$.

In the case of strong commitment one could deal with the problem by lowering the level of awareness about the distribution of the social commitments, replacing subformula $[\mathrm{B}]^+_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G}\text{COMM}(i,j,\alpha)\right)$ with $[\mathrm{B}]_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G}\text{COMM}(i,j,\alpha)\right)$.[5] Another possibility is to consider an alternative definition of bilateral commitment, where social commitments are replaced by bilateral responsibilities

$$\begin{aligned}\text{S-COMM}'_{G,P}(\varphi) \leftrightarrow{} & \text{C-INT}_G(\varphi) \wedge \mathit{constitutes}(P,\varphi) \wedge [\mathrm{B}]^+_G \mathit{constitutes}(P,\varphi)\\ & \wedge [\mathrm{B}]^+_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G}\text{RESP}(i,j,\alpha)\right)\\ & \wedge \bigwedge_{\alpha\in P}\bigwedge_{i,j\in G}\text{RESP}(i,j,\alpha) \rightarrow [\mathrm{B}]^+_{\{i,j\}}(\text{RESP}(i,j,\alpha))\end{aligned}$$

[5] This could be strengthened by finitely nesting the operators of general beliefs: $[\mathrm{B}]^m_G\left(\bigwedge_{\alpha\in P}\bigvee_{i,j\in G}\text{COMM}(i,j,\alpha)\right)$, where $[\mathrm{B}]^m_G$ stands for $[\mathrm{B}]_G$ repeated m times. See (Dunin-Kęplicz and Verbrugge 2010) for the discussion of this form weakening the awareness components of multiagent systems specifications.

In this version of the notion of strong commitment the group is fully aware of bilateral responsibilities within the group with regard to the actions of the plan P for achieving the goal φ, but the awareness about existence of bilateral awareness about these responsibilities is not required.

As the examples above show, restriction $\mathbf{R}_1$ is sufficiently weak to allow for expressing the key properties of cooperating teams of agents, such as collective intentions and social (bilateral commitments). What the restriction forbids, are the formulas where groups of agents have common beliefs about beliefs of agents from the group or have mutual intentions with regards to intentions of agents from the group. Essentially, the objects of common beliefs must be 'external' with respect to the group. In particular, they may be beliefs of agents from outside the group about the beliefs of agents in the group. Similarly in the case of mutual intentions. This restriction may be a problem when some kinds of collective commitments are concerned. The problem there is the awareness part, where common beliefs about social (bilateral) commitments within a group are expressed. It can be overcome by restating the formula expressing the collective commitment (like in the case of robust commitment). If this is impossible (like in the case of social commitment), alternative forms of collective commitments, where social commitments are replaced with responsibilities, can be considered. In these variants, the group as a whole holds a common belief about all the bilateral responsibilities, but it does not hold common belief about the bilateral awareness about these responsibilities. We would like to note that other, weaker, forms of collective commitments, like the team commitment and the distributed commitment (c.f. Dunin-Kęplicz and Verbrugge 2010) are expressible with restriction $\mathbf{R}_1$.

Restriction $\mathbf{R}_2$ is too strong to allow for expressing even collective intentions. However it is still of use. Methodologies of agent oriented modelling and design, like the one proposed by Kinny et al. (1996); Kinny and Georgeff (1997); Kinny (1998), often divide the process of agents modelling by separating construction of belief model, goal model and plan model. Similarly, in specification of multiagent systems using formalisms like TeamLog, separate parts could be distinguished, where purely informational and purely motivational aspects of individual agents and groups of agents are specified and parts where interrelations between these parts are specified. In such cases restriction $\mathbf{R}_2$ can be applied to the purely informational or purely motivational parts, while restriction $\mathbf{R}_1$ can be applied to the mixed parts.

In the discussion above we restricted attention to TeamLog formalism. However, similar formulas, where agents in a group hold common beliefs about intentions or goals of group members appear also in other theories of teamwork. See for example (Levesque et al. 1990; Wooldridge and Jennings 1999; Aldewereld et al. 2004).

6 Complexity of the satisfiability problem

In this section we study the complexity of checking the satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ and $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$. We focus on presenting the algorithms and general ideas. For that reason we moved most of the proofs to the Appendix.

For checking the satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ we will use the method based on pre-tableau construction presented in Halpern and Moses (1992). However, adopting a similar algorithm for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ would not work. This is because, as we show below, formulas of $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ may require an exponentially deep model with respect to the size of input formula, while all the algorithms based on the pre-tableau method perform a depth first search constructing sequences of nodes that constitute the tree-like structure of the pre-tableau for a given input.

Proposition 1 *Let* $|\mathcal{A}| \geq 2$*. Then there exists a satisfiable formula* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ *such that any* TeamLog *model* $\mathcal{M}$ *in which it is satisfied contains a sequence of pairwise different worlds of length exponential with respect to* $|\varphi|$.

Proposition 1 is shown by writing a formula that implements a binary counter. That is, the formula enforces a path in its model, where formulas of the form $[\mathrm{I}]_j\xi_1, \ldots, [\mathrm{I}]_{\xi_N}$ (alternatively $[\mathrm{G}]_j\xi_1, \ldots, [\mathrm{G}]_{\xi_N}$) have different valuations, implementing a counter over N bits.

6.1 Complexity of $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$

Algorithm 1 for checking satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$ presented in this section is based on the standard tableau method for modal logics. Our presentation follows (Halpern and Moses 1992) in the general methodology, which can be summarized as consisting of the following steps:

1. Define the notion of modal tableau for the logic in question. A modal tableau is a Kripke frame with worlds labelled with sets of formulas and accessibility relations satisfying additional properties associated with axioms generating the logic considered.
2. Show that any formula of the logic is satisfiable iff there is a tableau for it.
3. Give an algorithm for checking the satisfiability of a formula. The algorithm constructs a tree-like structure called a pre-tableau which forms a basis for the tableau for the formula.
4. Show that the algorithm has a termination property and is valid.
5. Analyse the computational complexity of the algorithm.

In steps 1 and 2 we follow (Dziubiński et al. 2007) where modal tableau for TeamLog is defined. The main difficulty are steps 3 and 4, particularly the termination property. In step 3 we extend the algorithm from (Dziubiński et al. 2007) so that fixpoint modalities can be dealt with. For step 4 we find the property of sets of formulas processed by the algorithm that 'decreases' during execution of the algorithm. In these steps we extend the ideas used in Dziubiński (2013) for standard systems of multimodal logics enriched with fixpoint modalities.

The notion of modal tableau is based on the notion of model graph, which we define below.

Definition 10 (Model graph) A *model graph* $\mathcal{T}$ is a tuple $\mathcal{T} = \left(W, \left\{O_j : [O]_j \in \Omega^{\mathrm{ind}}\right\}, L\right)$, where W and O_j are defined like in a Kripke frame and L is a *labelling function* associating with each state $w \in W$ a set $L(w)$ of formulas.

The modal tableau for TeamLog is a model graph with labelling sets of formulas satisfying additional properties, which we define below. Firstly, all labels of states are closed propositional tableaux. We define these notions below.

Definition 11 (Closed set of formulas) A set of formulas Φ is *closed* if it satisfies the following condition, for all $G \subseteq \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{I}\}$:

Cl If $[O]_G^+\varphi \in \Phi$, then $\{[O]_j[O]_G^+\varphi, [O]_j\varphi : j \in G\} \subseteq \Phi$,

Given a formula φ, we will use $\mathrm{Cl}(\varphi)$ to denote the smallest closed set of formulas containing φ. Similarly, given a set of formulas Φ we will use $\mathrm{Cl}(\Phi)$ to denote the smallest closed set of formulas having Φ as a subset.

Definition 12 (Propositional tableau) A *propositional tableau* is a set $\mathcal{T}$ of formulas such that $\mathcal{T}$ is not trivially inconsistent[6] and:

1. If $\neg\neg\psi \in \mathcal{T}$ then $\psi \in \mathcal{T}$.
2. If $\varphi \wedge \psi \in \mathcal{T}$ then $\varphi \in \mathcal{T}$ and $\psi \in \mathcal{T}$.
3. If $\neg(\varphi \wedge \psi) \in \mathcal{T}$ then either $\{\sim\varphi, \psi\} \subseteq T$ or $\{\varphi, \sim\psi\} \subseteq \mathcal{T}$ or $\{\sim\varphi, \sim\psi\} \subseteq \mathcal{T}$.

A *propositional tableau for a formula* φ is a minimal propositional tableau $\mathcal{T}$ such that $\varphi \in \mathcal{T}$. It is easy to see that every closed propositional tableau for φ is a maximal consistent subset of $\neg\mathrm{Cl}(\mathrm{PT}(\varphi))$.[7] Notice that, by definition, a propositional tableau cannot be trivially inconsistent.

Algorithm 1: DecideSatisfiability

Input: a formula φ
Output: a decision whether φ is satisfiable or not

```
/* Pre-tableau construction                                          */
Construct a pre-tableau consisting of single node root, with L(root) = {φ} and all successor relations being
  empty;
repeat
    Let Z be the set of all leaves of the pre-tableau with labelling sets that are not trivially inconsistent;
    if there is n ∈ Z such that n is not a state and ψ ∈ L(n) is a witness to that then
        FormState(n,ψ);
    else if there is s ∈ Z then
        foreach ψ ∈ L(s) do
            CreateSuccessorsB(s,ψ);
            CreateSuccessorsG(s,ψ);
            CreateSuccessorsI(s,ψ);
until no change occurred;
/* Marking nodes and deciding satisfiability                          */
repeat
    MarkNodes;
until no new node marked;
if root is marked sat then
    return sat;
else
    return unsat;
```

A modal tableau for TeamLog, called a TeamLog tableau is defined as follows.

Definition 13 (TeamLog tableau) A *modal tableau* is a model graph $\mathcal{T} = \left(W, \left\{O_j : [O]_j \in \Omega^{\mathrm{ind}}\right\}, L\right)$ such that for all $w \in W$, $L(w)$ is a closed propositional tableau.

[6] A set of formulas Φ is *trivially inconsistent* if there is a formula $\varphi \in \Phi$ such that $\neg\varphi \in \Phi$.

[7] Recall that $\neg\mathrm{Cl}(\mathrm{PT}(\varphi))$ is the closure of $\mathrm{Cl}(\mathrm{PT}(\varphi))$ with respect to single negation.

Moreover, for any $[O]_j \in \Omega^{\mathrm{T}}$ and any $[O]_G^+ \in \Omega^{\mathrm{T}}$ the following conditions are satisfied, for all $w \in W$:

T1 If $[O]_j\varphi \in L(w)$ and $v \in O_j(w)$, then $\varphi \in L(v)$. If $[O]_G^+\varphi \in L(w)$ and $v \in O_G^+(w)$, then $\varphi \in L(v)$.

T2 If $\neg[O]_j\varphi \in L(w)$, then there exists $v \in O_j(w)$ such that $\sim\varphi \in L(v)$. If $\neg[O]_G^+\varphi \in L(w)$, then there exists $v \in O_G^+(w)$ such that $\sim\varphi \in L(v)$.

The following conditions are satisfied if $[O]_j \in \Omega^{\mathrm{T}}$ is associated with additional axioms from **D** – 5 (c.f. Halpern and Moses 1992):

- If $[O]_j$ is associated with axiom **D**, then the following condition is satisfied, for any $w \in W$:

  TD If $[O]_j\varphi \in L(w)$, then either $\varphi \in L(w)$ or $O_j(w) \neq \emptyset$.

- If $[O]_j$ is associated with axiom 4, then the following condition is satisfied, for any $w \in W$:

  T4 If $v \in O_j(w)$ and $[O]_j\varphi \in L(w)$, then $[O]_j\varphi \in L(v)$.

- If $[O]_j$ is associated with axiom 5, then the following condition is satisfied, for any $w \in W$:

  T5 If $v \in O_j(w)$ and $[O]_j\varphi \in L(v)$, then $[O]_j\varphi \in L(w)$.

The following additional conditions, associated with axioms **B** ***O***4, **B** ***O***5 (with $O \in \{\mathrm{G}, \mathrm{I}\}$) and **IG** are satisfied for all $j \in \mathcal{A}$ and $w \in W$:

TB ***O***4 If $v \in \mathrm{B}_j(w)$ and $[O]_j\varphi \in L(w)$, then $[O]_j\varphi \in L(v)$.
TB ***O***5 If $v \in \mathrm{B}_j(w)$ and $[O]_j\varphi \in L(v)$, then $[O]_j\varphi \in L(w)$.
TIG If $v \in \mathrm{G}_j(w)$ and $[\mathrm{I}]_j\varphi \in L(w)$, then $\varphi \in L(v)$.

A TeamLog *tableau* is a modal tableau satisfying conditions **T**1 and **T**2 (for all $[O]_j$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$), condition **TD** (for all $[O]_j$ with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j \in \mathcal{A}$), conditions **T**4 and **T**5 (for all $[\mathrm{B}]_j$ with $j \in \mathcal{A}$) and conditions **TBG**4, **TBG**5, **TBI**4, **TBI**5 and **TIG**. Given a formula φ, we say that $\mathcal{T}$ is a *tableau for* φ if there exists a state $w \in W$ such that $\varphi \in L(w)$.

The following proposition links existence of TeamLog tableau for a formula with its satisfiability.

Proposition 2 *A formula $\varphi \in \mathcal{L}^{\mathrm{T}}$ is satisfiable iff there is a* TeamLog *tableau for φ.*

6.1.1 Algorithm for $\mathcal{L}^{\mathrm{T}}_{\mathrm{R}_2}$

Algorithm 1 tries to construct a pre-tableau – a tree-like structure that forms the basis for a TeamLog tableau for an input formula φ. A pre-tableau consists of nodes connected with a successor relation. Each node can have zero or more successors and each of them has zero or one predecessor. There is at most one node in the pre-tableau that has no predecessors and it is called the root. Each node is labelled with a set of formulas. The root is labelled with the set containing the input formula only. The nodes of a pre-tableau can be divided into two groups: internal nodes and states. Successors of states correspond to accessibility relations and are created for formulas of the form $\neg[O]_j\psi$ in the labels of states and, in the case of modal operators with which axiom **D** is associated, for formulas of the form $[O]_j\psi$ in the labels of states. To construct a TeamLog tableau based on a given pre-tableau, a subset of states of the pre-tableau is selected and the accessibility relations are constructed on the basis of successor relations for states. Labels of states must be closed propositional tableaux satisfying additional requirements given below. Internal nodes correspond to subsequent steps of constructing labels of states. For the more detailed explanation of the notion of pre-tableaux see (Halpern and Moses 1992).

Now we turn to defining additional properties that will be satisfied by states of pre-tableaux constructed by Algorithm 1. Firstly, the notion of $[O]^+$-expanded set of formulas, for $O \in \{\mathrm{B}, \mathrm{I}\}$, related to fixpoint modalities is needed.

Definition 14 ($[O]^+$-expanded set of formulas) A set of formulas $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ is $[O]^+$-*expanded*, with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $G \subseteq \mathcal{A}$, if the following condition is satisfied:

CE If $\neg[O]^+_G\psi \in \Phi$, then for all $j \in G$, $\{[O]_j\psi, [O]_j[O]^+_G\psi\} \subseteq \neg\Phi$ and there exists $j \in G$ such that either $\neg[O]_j\psi \in \Phi$ or $\neg[O]_j[O]^+_G\psi \in \Phi$.

Secondly, the notion of [B]-expanded set of formulas is needed

Definition 15 ([B]-expanded tableau) A [B]-*expanded tableau* is a $[\mathrm{B}]^+$-expanded and $[\mathrm{I}]^+$-expanded closed propositional tableau $\mathcal{T}$ such that for all $j \in \mathcal{A}$:

1. If $[\mathrm{B}]_j\varphi \in \neg\mathcal{T}$ and $[O]_j\psi \in \neg\mathrm{PT}(\varphi)$, then $[O]_j\psi \in \neg\mathcal{T}$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$.
2. If $[\mathrm{B}]_j\varphi \in \neg\mathcal{T}$ and $[O]^+_G\psi \in \neg\mathrm{PT}(\varphi)$ with $j \in G$, then $[O]_j\psi \in \neg\mathcal{T}$ and $[O]_j[O]^+_G\psi \in \neg\mathcal{T}$, where $O \in \{\mathrm{B}, \mathrm{I}\}$.

A [B]-*expanded tableau for a formula* φ is a minimal [B]-expanded tableau $\mathcal{T}$ such that $\varphi \in \mathcal{T}$. Given a formula φ we will use $\mathrm{BT}(\varphi)$ to denote the union of all [B]-expanded tableaux for φ. Notice that any [B]-expanded tableau is a maximal consistent subset of $\neg\mathrm{BT}(\varphi)$. Nodes of a pre-tableau constructed for an input formula φ are labelled with subsets of $\neg\mathrm{Cl}(\varphi)$. States are nodes with labels being [B]-expanded tableaux. This is required because of axioms **5**, **BG5** and **BI5**, associated with operators $[\mathrm{B}]_j$, and is needed to prevent construction of too long paths in the pre-tableaux.[8]

Algorithm 1 consists of two stages: the stage of pre-tableau construction and the stage of marking nodes. In the first stage the algorithm attempts to construct a pre-tableau based on the input formula. This stage consists of two general steps: the step of state construction

[8] Algorithm given in Halpern and Moses (1992) for modal logic $\mathrm{KD45}_n$ and in Dziubiński et al. (2007) for the fragment of TeamLog without fixpoint modalities deal with the problem of long paths in the case of formulas of the form $[\mathrm{B}]_j\psi$ in a different way, by requiring that states are fully expanded modal tableaux, which contain all the subformulas (possibly negated) of ψ. This approach would be too strong in our case and is not necessary. It suffices to require that labels of state are [B]-expanded tableaux.

and the step of state successors creation. In the step of state construction labels of internal nodes are properly extended with new formulas, resulting in new successor nodes, until a node which is a state is obtained. This step is described in Procedure 2.

Procedure 2: FormState

Input: a node n and a formula ψ

if *ψ is of the form* $\neg\neg\xi$ **then**
  Create a successor m of n and set $L(m) := L(n) \cup \{\xi\}$;

else if *ψ is of the form* $\xi_1 \wedge \xi_2$ **then**
  Create a successor m of n and set $L(m) := L(n) \cup \{\xi_1, \xi_2\}$;

else if *ψ is of the form* $\neg(\xi_1 \wedge \xi_2)$ **then**
  Create three successors m_1, m_2 and m_3 of n and set $L(m_1) := L(n) \cup \{\sim\xi_1, \xi_2\}$,
  $L(m_2) := L(n) \cup \{\xi_1, \sim\xi_2\}$ and $L(m_3) := L(n) \cup \{\sim\xi_1, \sim\xi_2\}$;

else if *ψ is of the form* $[\mathrm{B}]_j\xi$ *or of the form* $\neg[\mathrm{B}]_j\xi$ **then**
  if *there is* $[O]_j\zeta \in \neg\mathrm{PT}(\xi)$ *such that* $[O]_j\zeta \notin \neg L(n)$ **then**
    Create two successors m_1 and m_2 of n and set $L(m_1) := L(n) \cup \{[O]_j\zeta\}$ and
    $L(m_2) := L(n) \cup \{\neg[O]_j\zeta\}$;

  else if *there is* $[O]_G^+\zeta \in \neg\mathrm{PT}(\xi)$ *with* $j \in G$ *such that either* $[O]_j\zeta \notin \neg L(n)$ *or* $[O]_j[O]_G^+\zeta \notin \neg L(n)$ **then**
    Create four successors m_1, m_2, m_3 and m_4 of n and set $L(m_1) := L(n) \cup \{[O]_j\zeta, [O]_j[O]_G^+\zeta\}$,
    $L(m_2) := L(n) \cup \{[O]_j\zeta, \neg[O]_j[O]_G^+\zeta\}$, $L(m_3) := L(n) \cup \{\neg[O]_j\zeta, [O]_j[O]_G^+\zeta\}$,
    $L(m_4) := L(n) \cup \{\neg[O]_j\zeta, \neg[O]_j[O]_G^+\zeta\}$;

else if *ψ is of the form* $\neg[O]_G^+\xi$ **then**
  foreach $(H_1, H_2) \in \mathrm{P}(G) \times \mathrm{P}(G)$ *such that* $H_1 \cup H_2 \neq \varnothing$ **do**
    Create a successor m of n and set $L(m) :=$
    $L(n) \cup \bigcup_{j \in H_1} \{\neg[O]_j\xi\} \cup \bigcup_{j \in G\setminus H_1} \{[O]_j\xi\} \cup \bigcup_{j \in H_2} \{\neg[O]_j[O]_G^+\xi\} \cup \bigcup_{j \in G\setminus H_2} \{[O]_j[O]_G^+\xi\}$;

else if *ψ is of the form* $[O]_G^+\xi$ **then**
  Create a successor m of n and set $L(m) := L(n) \cup \bigcup_{j \in G} \{[O]_j\xi, [O]_j[O]_G^+\xi\}$;

The step of state successors creation is described in three Procedures 3, 4 and 5, creating state successors associated with operators $[\mathrm{B}]_j$, $[\mathrm{G}]_j$ and $[\mathrm{I}]_j$, respectively. We will call these successors O_j-successors, for $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. A node which is an O_j-successor for some $j \in \mathcal{A}$ will be called an O-successor. Additionally, a successor created for a formula ξ will be called a ξ-successor. Similar notions can be defined for subsequent states on paths of the pre-tableau. State t is called a O_j-Successor of state s if t is a descendant of s in the pre-tableau, there are no states between s and t and t is an O_j-successor. The relations of O-successor and ξ-Successor are defined analogously. In the presentation we will also refer to similar notions of -predecessors and -Predecessors. The following set of formulas are used in the procedures to define labels of the newly created successors of a state ($O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, $j \in \mathcal{A}$ and Φ is a set of formulas)

$$\Phi^{\neg[O]_j}(\psi) = \{\sim\psi\} \cup \Phi^{[O]_j}.$$

The definition of set $\Phi^{[O]_j}$ depends on the axioms associated with $[O]$:

$$\Phi^{[\mathrm{I}]_j} = \Phi/[\mathrm{I}]_j, \quad \Phi^{[\mathrm{G}]_j} = (\Phi/[\mathrm{G}]_j) \cup \Phi^{[\mathrm{I}]_j}, \quad \Phi^{[\mathrm{B}]_j} = (\Phi/[\mathrm{B}]_j) \cup (\Phi \sqcap j)$$

and

$$\begin{aligned}
\Phi/[O]_j &= \{\psi : [O]_j\psi \in \Phi\},\\
\Phi \sqcap [O]_j &= \{[O]_j\psi : [O]_j\psi \in \Phi\},\\
\Phi \sqcap \neg[O]_j &= \{\neg[O]_j\psi : \neg[O]_j\psi \in \Phi\},\\
\Phi \sqcap j &= (\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap [\mathrm{G}]_j) \cup (\Phi \sqcap [\mathrm{I}]_j) \cup\\
&\quad (\Phi \sqcap \neg[\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{G}]_j) \cup (\Phi \sqcap \neg[\mathrm{I}]_j).
\end{aligned}$$

Given a state s and its label $L(s)$, we will write $L^{[O]_j}(s)$ to denote $(L(s))^{[O]_j}$ and $L^{\neg[O]_j}(s,\psi)$ to denote $(L(s))^{\neg[O]_j}(\psi)$.

Procedure 3: CreateSuccessorsB

Input: a state s and a formula $\psi \in L(s)$

if *ψ is of the form* $\neg[\mathrm{B}]_j\xi$ **then**
- **if** *there is an* B_j*-Predecessor t of s such that* $\neg[\mathrm{B}]_j\xi \in L(t)$ *and* $L^{\neg[\mathrm{B}]j}(t,\xi) = L^{\neg[\mathrm{B}]j}(s,\xi)$ **then**
  - **if** $\xi = [\mathrm{B}]^+_G\zeta$ *with* $j \in G$ *and s is a* $\neg[\mathrm{B}]_j\xi$*-Successor of t* **then**
    - For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$, where n is an B_j-successor of t on the path from t to s;
- **else if** $\xi = [\mathrm{B}]^+_G\zeta$ *with* $j \in G$ *and there is a* $\neg[\mathrm{B}]^+_G\zeta$*-Ancestor t of s such that its* $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\zeta$*-successor n is on the path from t to s and* $L^{[\mathrm{B}]j}(s) = L^{[\mathrm{B}]j}(t)$ **then**
  - For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$;
- **else** Create an B_j-Successor v of s with $L(v) = L^{\neg[\mathrm{B}]j}(s,\xi)$;
- ;

else if *ψ is of the form* $[\mathrm{B}]_j\xi$ *and there are no formulas of the form* $\neg[\mathrm{B}]_j\zeta \in L(s)$ **then**
- If there is no B_j-Predecessor t of s such that $[\mathrm{B}]_j\xi \in L(t)$ and $L^{[\mathrm{B}]j}(t) = L^{[\mathrm{B}]j}(s)$ and $L^{[\mathrm{B}]j}(s) \not\subseteq L(s)$, then create an B_j-successor u of s with $L(u) = L^{[\mathrm{B}]j}(s)$;

Procedure 4: CreateSuccessorsG

Input: a state s and a formula $\psi \in L(s)$

if *ψ is of the form* $\neg[\mathrm{G}]_j\xi$ **then**
- If there is no B_j-predecessor state t of s such that $L^{\neg[\mathrm{G}]j}(t,\xi) = L^{\neg[\mathrm{G}]j}(s,\xi)$, then create an G_j-successor u of s with $L(u) = L^{\neg[\mathrm{G}]j}(s,\xi)$;

Procedure 5: CreateSuccessorsI

Input: a state s and a formula $\psi \in L(s)$

if *ψ is of the form* $\neg[\mathrm{I}]_j\xi$ **then**
- **if** *there is no* B_j*-predecessor state t of s such that* $\neg[\mathrm{I}]_j\xi \in L(t)$ *and* $L^{\neg[\mathrm{I}]j}(t,\xi) = L^{\neg[\mathrm{I}]j}(s,\xi)$ **then**
  - **if** $\xi = [\mathrm{I}]^+_G\zeta$ *with* $j \in G$ *and there is a* $\neg[\mathrm{I}]^+_G\zeta$*-Ancestor t of s such that its* $\neg[\mathrm{I}]_j[\mathrm{I}]^+_G\zeta$*-successor n is on the path from t to s and* $L^{[\mathrm{I}]j}(s) = L^{[\mathrm{I}]j}(t)$ **then**
    - For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$;
  - **else** Create an I_j-successor v of s with $L(v) = L^{\neg[\mathrm{I}]j}(s,\xi)$;
  - ;

else if *ψ is of the form* $[\mathrm{I}]_j\xi$ *and there are no formulas of the form* $\neg[\mathrm{I}]_j\zeta \in L(s)$ **then**
- If there is no B_j-Predecessor t of s such that $[\mathrm{I}]_j\xi \in L(t)$ and $L^{[\mathrm{I}]j}(t) = L^{[\mathrm{I}]j}(s)$ and $L^{[\mathrm{I}]j}(s) \not\subseteq L(s)$, then create an I_j-successor u of s with $L(u) = L^{[\mathrm{I}]j}(s)$;

Procedures 3, 4 and 5 are based on the algorithms described in Halpern and Moses (1992) which need to be considerably extended to address axioms of TeamLog. Firstly, they are affected by mixed axioms (see Dziubiński et al. 2007 for the discussion of these aspects). Secondly they need to deal with existence of fixpoint modalities, to avoid construction of too long or infinite paths. When creating $\neg[O]_j[O]^+_G\xi$-successor (with $O \in \{\mathrm{B}, \mathrm{I}\}$ and $j \in G$) the sets $L^{[O]_j}(\cdot)$ for all $[O]^+_G\psi$-Ancestors are checked. A state t is called a $[O]^+_G\xi$-Ancestor of state s if t is an ancestor of s and for every state u on the path from t to s such that $u \neq s$, there exists $j \in G$ such that u is a $\neg[O]_j[O]^+_G\xi$-Successor. If the label of a potential O_j-successor, with $O \in \{\mathrm{B}, \mathrm{I}\}$, of a state s is equal to the label of a successor node n of some $[O]^+_G\psi$-Ancestor of s which is on the path from t to s, then construction of the successor of s is blocked by n. This is illustrated in Fig. 2.

To decide whether a node containing a formula of the form $\neg[O]^+_G\xi$ is satisfiable, it has to be checked whether an appropriate sequence of states can be constructed, that would indicate that this formula is satisfiable. Since creation of successors for formulas of the form $\neg[O]_j[O]^+_G\xi$, with $j \in G$, may be blocked by some ancestor node, the decision whether such an appropriate sequence of states can be constructed may have to be suspended until the satisfiability of the ancestors is checked. Therefore for each node n there is a set $B(n)$ associated with it and containing weak ancestors[9] that block creation successors of states in the n-subtree of the pre-tableau (c.f. Fig. 2).[10] Whenever a new node n is created by the algorithm, the associated set of nodes $B(n)$ is set to $\varnothing$.

During the stage of marking nodes, nodes of the pre-tableau are marked either `sat`, `unsat`, or `undec`. A node n being marked `undec` indicates that satisfiability of $\bigwedge L(n)$ could not be decided due to existence of a formula of the form $\neg[O]^+_G\psi$ in its label for which an appropriate sequence of states was not constructed yet. We call such formulas unresolved in a given node, as defined below. In the definition we refer to a notion of ξ-descendant. A node is a *ξ-descendant* if it is a ξ-successor or a descendant of a ξ-successor such that there is no state between it and the ξ-successor.

Definition 16 (Unresolved formula) Let n be a node in a pre-tableau and let $\neg[O]^+_G\psi \in L(n)$ with $O \in \{\mathrm{B}, \mathrm{I}\}$. A formula $\neg[O]^+_G\psi$ is *unresolved* in n if one of the following holds:

- n is an internal node and a $\neg[O]_j[O]^+_G\psi$-descendant with $j \in G$, none of its successors is marked `sat`, there exists a successor of n marked `undec` and $B(n) \neq \varnothing$,
- n is a state and a $\neg[O]_j[O]^+_G\psi$-Successor with $j \in G$, $B(n) \neq \varnothing$, none of $\neg[O]_k[O]^+_G\psi$-successors of n, with $k \in G$, is marked `sat` and $[O]_k\psi \in L(n)$, for all $k \in G$.

Notice that if $B(n) = \varnothing$, then a node cannot be marked `undec`. The stage of marking nodes is described in Procedure 6.

We show first that for any input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ the algorithm for checking satisfiability terminates. The usual method of showing that is by showing that the *height*[11] of every node in the constructed pre-tableau is bounded. In the case of standard modal systems (e.g. K_n, KD_n,

[9] A *weak ancestor* of n is either an ancestor of n or n.

[10] The *n-subtree* is a subtree of the pre-tableau with n being its root.

[11] Height of a node is the number of nodes on the path from the node to the root of the pre-tableau.

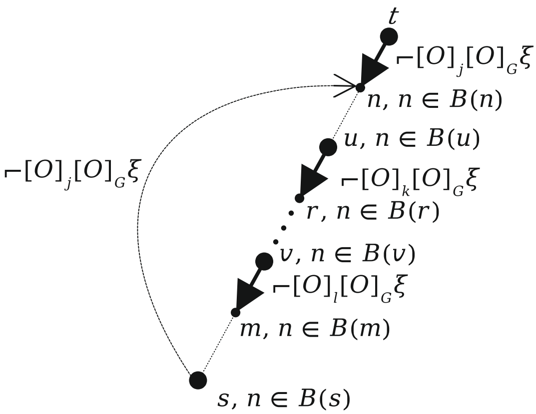

Fig. 2 Creation of $\neg[O]_j[O]^+_G\xi$-successor of s is blocked by its ancestor n which is a $\neg[O]_j[O]^+_\xi$-successor of a $[O]^+_G\xi$-Ancestor t of s. Dotted lines depict sequences of internal nodes (these sequences can be empty, in which case the starting node coincides with the ending state)

$\mathrm{KD45}_n$) this is shown by showing that modal depth of formulas in subsequent states on paths of the pre-tableaux is falling with distance from the root. In the case of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ this approach would not work and we need to find a different parameter of the labels that changes with distance from the root. The main problem here are the formulas of the form $[O]^+_G\psi$ or $\neg[O]^+_G\psi$. This is because if t is an O_j-Successor of s in a pre-tableau constructed by Algorithm 1, then any formula $[O]^+_G\psi \in L(s)$ with $j \in G$ is in $L(t)$ as well. Similarly with a formula of the form $\neg[O]^+_G\psi \in L(s)$, if t is a $\neg[O]_j[O]^+_G\psi$-Successor of s. Moreover, if additionally u is a O_k-Successor of t and $k \in G$, then for any formula $\xi \in \neg\mathrm{BT}(\psi)$, $\xi \in \neg L(t)$ (as it is added during [B]-expanded tableau formation) and $\xi \in \neg L(u)$, as $[O]^+_G\psi \in L(u)$. Thus formulas of the form $[O]^+_G\psi$ may carry over to the label of the O_j-Successor formulas from $\neg\mathrm{BT}(\psi)$. Similarly, they may carry over the formulas $[O]_j[O]^+_G\psi$ and $[O]_j\psi$, that are added to the label during the closed propositional tableau formation.

To analyse the length of sequences of O-Successors in a pre-tableau constructed by Algorithm 1, we need to separate the formulas in labels of states which are carried by some other formulas from those which are not carried by any other formula. We will say that a formula $[O]^+_G\psi$ *carries* a formula ξ if $\xi \in \neg\mathrm{BT}(\psi)$ or $\xi \in \widetilde{\mathrm{Cl}}([O]^+_G\psi)$, where $\widetilde{\mathrm{Cl}}([O]^+_G\psi) = \{[O]_j\psi : j \in G\} \cup \{[O]_j[O]^+_G\psi : j \in G\}$. Similarly, a formula $\neg[O]^+_G\psi$ carries a formula ξ if $\xi \in \neg\mathrm{BT}(\psi)$ or $\xi \in \widetilde{\mathrm{Cl}}(\neg[O]^+_G\psi)$, where $\widetilde{\mathrm{Cl}}(\neg[O]^+_G\psi) = \neg\widetilde{\mathrm{Cl}}([O]^+_G\psi)$. Given a set of formulas Φ and a formula ξ we will say that ξ is *carried by* Φ if there is a formula in Φ which carries it.

First we will consider the carried formulas of the form $[O]^+_H\zeta$ or $\neg[O]^+_H\zeta$. Notice that in this case such a formula is carried by some formula $[O]^+_G\psi$ or $\neg[O]^+_G\psi$ if and only if it is in $\neg\mathrm{PT}(\psi)$. Given a set of formulas Φ, let[12]

$$\mathrm{Gr}(\Phi) = \bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \left(\left(\Phi \sqcap [O]^+\right) \cup \left(\Phi \sqcap \neg[O]^+\right)\right),$$

[12] The name Gr is from ‘group’, as it selects the formulas starting with modalities $[O]^+$ related to properties of groups of agents.

where

$$\Phi/[O]^{+} = \{\psi : [O]^{+}_{G}\psi \in \Phi,\ \text{for some } G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\},$$
$$\Phi \sqcap [O]^{+} = \{[O]^{+}_{G}\psi : [O]^{+}_{G}\psi \in \Phi,\ \text{for some } G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\},$$
$$\Phi \sqcap \neg[O]^{+} = \{\neg[O]^{+}_{G}\psi : \neg[O]^{+}_{G}\psi \in \Phi,\ \text{for some } G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}\}.$$

Let $\mathcal{F}_{\Phi} : \mathrm{P}(\mathcal{L}^{\mathrm{T}}) \to \mathrm{P}(\mathcal{L}^{\mathrm{T}})$ be defined as follows, for $\Psi \subseteq \mathcal{L}^{\mathrm{T}}$,

$$\mathcal{F}_{\Phi}(\Psi) = \mathrm{Gr}(\Phi) \setminus \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg\Psi/[O]^{+}\right),$$

The operator $\mathcal{F}_{\Phi}$, when applied to a set of formulas Ψ, removes from Φ all the formulas from $\mathrm{Gr}(\Phi)$ which are carried by Ψ.

Procedure 6: MarkNodes

```
repeat
    if n is an unmarked state then
        if n has a successor that is marked unsat then
            Mark n unsat;
        else if n does not have an unmarked successor then
            if there is a formula ¬[O]+_G ψ ∈ L(n) which is unresolved in n then
                if B(n) = {n} then
                    Mark n unsat;
                else
                    Mark n undec;
            else
                Mark n sat;
    else if n is an unmarked internal node then
        if L(n) is trivially inconsistent or all successors of n are marked unsat then
            Mark n unsat;
        else if there exists a successor of n marked sat then
            Mark n sat;
        else if n does not have an unmarked successor then
            if there exists a formula ¬[O]+_G ψ ∈ L(n) which is unresolved in n and B(n) = {n} then
                Mark n unsat;
            else
                Mark n undec;
until no new node marked;
```

Given a set of formulas Φ and a formula ψ we will say that ψ *is uncarried in* Φ if $\psi \in \Phi$ and ψ is not carried by Φ. We will be interested in sets of formulas which are *carry-free*, that is Φ such that all the formulas in Φ are uncarried in it. Given $i \in \mathbb{N}$, let

$$F^{(i)}_{\Phi} = \begin{cases} \varnothing, & \text{if } i = 0, \\ \mathcal{F}_{\Phi}\left(F^{(i-1)}_{\Phi}\right), & \text{if } i > 0 \end{cases}$$

and let $F^{(\infty)}_{\Phi} = \lim_{i\to\infty} F^{(i)}_{\Phi}$. As we show below, for any $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ the limit $F^{(\infty)}_{\Phi}$ exists.

Lemma 1 *For any* $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$, $F_{\Phi}^{(\infty)}$ *exists.*

Let $\widehat{\mathrm{Gr}}(\Phi) = F_{\Phi}^{(\infty)}$, that is

$$\widehat{\mathrm{Gr}}(\Phi) = \mathrm{Gr}(\Phi) \setminus \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg\widehat{\mathrm{Gr}}(\Phi)/[O]^{+}\right).$$

The set $\widehat{\mathrm{Gr}}(\Phi)$ it the maximal carry-free subset of $\mathrm{Gr}(\Phi)$ containing all the formulas which are uncarried in $\mathrm{Gr}(\Phi)$.

Now we turn to the uncarried formulas which are not in $\mathrm{Gr}(\Phi)$ but can 'carry' other formulas to successor labels. These are those formulas in Φ which are (possibly negated) formulas of the form $[O]_j\xi$. More precisely, we will be interested in those of such formulas which are not carried by $\widehat{\mathrm{Gr}}(\Phi)$ nor are elements of $\widetilde{\mathrm{Cl}}(\Phi)$, where $\widetilde{\mathrm{Cl}}(\Phi) = \bigcup_{\psi\in\mathrm{Gr}(\Phi)} \widetilde{\mathrm{Cl}}(\psi)$. The set of such formulas is[13]

$$\mathrm{Ind}(\Phi) = \left(\bigcup_{j\in\mathcal{A}} \Phi \sqcap j\right) \setminus \left(\widetilde{\mathrm{Cl}}(\Phi) \cup \neg\mathrm{BT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg\widehat{\mathrm{Gr}}(\Phi)/[O]^{+}\right)\right).$$

Although the modal depth of $\widehat{\mathrm{Gr}}(\cdot)$ of labels may not change between O-Successors in the sequence of states in a pre-tableau constructed by Algorithm 1, there is one more parameter of these sets that will change. Given a formula ψ and $O \in \{\mathrm{B},\mathrm{I}\}$, we define a set[14]

$$\mathrm{ag}(\psi,[O]^{+}) = \begin{cases} G, & \text{if } \psi \text{ is of the form } [O]_G^{+}\xi \text{ or } \neg[O]_G^{+}\xi,\\ \mathcal{A} \cup \{\omega\}, & \text{otherwise,}\end{cases}$$

where $\omega \notin \mathcal{A}$. Given a set of formulas $\Phi \neq \emptyset$ and $O \in \{\mathrm{B},\mathrm{I}\}$ we define

$$\mathrm{ag}(\Phi,[O]^{+}) = \begin{cases} \bigcap_{\psi\in\Phi} \mathrm{ag}(\psi,[O]^{+}), & \text{if } \Phi \neq \emptyset,\\ \mathcal{A} \cup \{\omega\}, & \text{otherwise.}\end{cases}$$

Notice that $\omega \in \mathrm{ag}(\Phi,[O]^{+})$ implies that there are no formulas of the form $[O]_G^{+}\xi$ nor $\neg[O]_G^{+}\xi$ in Φ. Also, when formulas are removed from Φ, then $\mathrm{ag}(\Phi,[O]^{+})$ either remains unchanged or increases.

When analysing how labels of subsequent states change, we will divide them into subsets (levels) of different modal depth of formulas and then we will look at the sets $\mathrm{ag}(\cdot,[O]^{+})$ at different levels. Given a set of formulas Φ, let $\Phi_d = \{\psi \in \Phi : \mathrm{dep}(\psi) = d\}$. Also, let

$$\mathrm{ag}(\Phi,[O]^{+},d) = \mathrm{ag}(\Phi_d,[O]^{+}).$$

[13] The name Ind comes from 'individual', as the formulas it selects start with modal operators associated with individual properties of agents.

[14] The name $\mathcal{A}$ comes from 'agents', because it relates to the sets of agents associated with group modalities.

Notice that $\mathrm{ag}\big(\Phi, [O]^+, d\big)$ is well defined even for $d > \mathrm{dep}(\Phi)$. Is it simply $\mathcal{A} \cup \{\omega\}$ then. Similarly for levels $d \leq \mathrm{dep}(\Phi)$ at which there are no formulas of the form $[O]^+_G\xi$ nor $\neg[O]^+_G\xi$. Notice also, that $\mathrm{ag}\big(\Phi, [O]^+, 0\big) = \mathcal{A} \cup \{\omega\}$.

The main difficulty in showing the boundaries on the lengths of paths in pre-tableau constructed by Algorithm 1 is in showing that sequences of B-Successors and I-Successors are properly bounded. The key lemmas to this result are Lemma 2 and Lemma 3 proven below. The lemmas give properties of $\widehat{\mathrm{Gr}}(\cdot)$ and $\mathrm{Ind}(\cdot)$ that follow from modal context restrictions $\mathbf{R}_1$ and $\mathbf{R}_2$. In the case of I-successors restriction $\mathbf{R}_1$ is assumed only. This restriction guarantees that if a formula of the form $[O]^+_H\xi$ is carried by a formula $[O]^+_G\psi$, then it must be that $G \cap H = \varnothing$. Similarly, if a formula of the form $[O]_j\xi$ is carried by a formula $[O]^+_G\psi$, then it must be that $j \notin G$. Thus if $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$ and there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\Phi_d$ at levels $d > D$, then any formula of the form $[\mathrm{I}]^+_H\xi \in \neg\Phi$ with modal depth $\geq D$ must be uncarried in Φ. Also, if a formula $[\mathrm{I}]_j\xi$ is in $\neg\widetilde{\mathrm{Cl}}(\Phi)$, then it must be in $\neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(\Phi))$.

Lemma 2 *Let $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ and let $D \geq 0$ and $j \in \mathcal{A}$ be such that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$, for all $d \geq D + 1$. Then the following hold*:

(i) if $[\mathrm{I}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]^+_G\psi) \geq D$, then $[\mathrm{I}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$,

(ii) if $\mathrm{dep}(\mathrm{Ind}(\Phi)) \leq D$ and $[\mathrm{I}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{I}]_j\psi) \geq D + 1$, then $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$.

The analogous lemma for B-Successors differs from the case of I-Successors in two aspects. If t is a B_j-Successor of state s, then, by construction of the algorithm, $L(s) \sqcap j \subseteq L(t)$. For this reason the set $\mathrm{Ind}(\Phi) \sqcap j$ rather than the set $\mathrm{Ind}(\Phi)$ is used. Secondly the lemma has an additional point that requires restriction $\mathbf{R}_2$. The point is crucial for having bounds on the lengths of sequences of B-Successors in the pre-tableau.

Lemma 3 *Let $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ and let $D \geq 0$ and $j \in \mathcal{A}$ be such that $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D + 1$. Then the following hold*:

(i) if $[\mathrm{B}]^+_G\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{B}]^+_G\psi) \geq D$, then $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$,

(ii) if $\mathrm{dep}(\mathrm{Ind}(\Phi) \sqcap j) \leq D$ and $[\mathrm{B}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{B}]_j\psi) \geq D + 1$, then $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$,

(iii) if $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$, $\mathrm{dep}(\mathrm{Ind}(\Phi) \sqcap j) \leq D$, $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\Big)$, for all $d \geq D$ and $[O]_j\psi \in \neg\Phi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $\mathrm{dep}([O]_j\psi) \geq D + 1$, then $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\Big(\widehat{\mathrm{Gr}}(\Phi)\Big)$ and $O = \mathrm{B}$.

In what follows we will concentrate on sequences of B-Successors. The general approach in the case of I-Successors is similar and easier. In proofs, given in the Appendix, we indicate where the differences in proofs for these to cases lie. For the detailed proofs we refer the reader to Dziubiński (2011). Lemma 3 allows us to analyse the origins of formulas in labels of B_j-Successors. The corollary below points out the sources of formulas in the successor state with modal depth not smaller than $\mathrm{dep}(\mathrm{Ind}(\cdot)) \sqcap j$ of the predecessor state. Roughly speaking all such formulas are either added when the label of the successor state is being closed or the formulas are carried by the uncarried formulas from the label of the predecessor state, or are uncarried formulas inherited from the label of the predecessor state. The analogous corollary for I-successors would concern set $\mathrm{Ind}(\cdot)$ instead of $\mathrm{dep}(\mathrm{Ind}(\cdot)) \sqcap j$.

Corollary 1 *Let t be an B_j-Successor of state s in the pre-tableau constructed by Algorithm 1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$, with $D \geq 0$ such that $\mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j) \leq D$, $j \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\Big)$ and $\omega \in \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\Big)$, for all $d \geq D+1$. Then for all $\psi \in L(t)$ with $\mathrm{dep}(\psi) \geq D$ one of the following holds*

(i) $\psi \in L(s) \sqcap j$ or
(ii) $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ or
(iii) there exists $[\mathrm{B}]^+_G \xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $j \in G$ such that $\psi \in \neg\mathrm{BT}(\xi)$ or
(iv) ψ is of the form $[\mathrm{B}]^+_G \eta$ with $j \in G$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$ or
(v) ψ is of the form $\neg[\mathrm{B}]^+_G \eta$ with $j \in G$, $\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G \eta \in L(s)$ and t is a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G \eta$-Successor of s.

We will also need the following auxiliary lemma which will be useful in analysing the lengths of sequences involving B-successors. The lemma extends a similar one used in the analysis of the complexity of TeamLog without fixpoint modalities in Dziubiński et al. (2007).

Lemma 4 *Let t be a B_j-Successor of state s in the pre-tableau constructed by Algorithm 1 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}$. Then the following hold for $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$:*

1. $\neg[O]_j \xi \in L(s)$ iff $\neg[O]_j \xi \in L(t)$.
2. $[O]_j \xi \in L(s)$ iff $[O]_j \xi \in L(t)$.
3. $L^{[O]_j}(s) = L^{[O]_j}(t)$.
4. $L^{\neg[O]_j}(s, \xi) = L^{\neg[O]_j}(t, \xi)$.

We are now ready to state the lemma about the bounds on the length of a sequence of B-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm 1 for some input φ. To assess lengths of sequences of B-Successors with unchanged modal depth of labels, we will show that the sets $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{B}]^+, d\Big)$ and $\mathrm{ag}\Big(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{I}]^+, d\Big)$ must gradually increase proceeding top down, from $d = \mathrm{dep}(\Phi)$ to $d = 1$. For this reason we will need to assess for how long these sets may remain unchanged at different levels. This is expressed by the following properties of states, for $O \in \{\mathrm{B}, \mathrm{I}\}$:

$$\mathbf{P}\ \boldsymbol{O1}\ \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(s)), [O]^+, d\Big) = \mathrm{ag}\Big(\widehat{\mathrm{Gr}}(L(t)), [O]^+, d\Big),$$

We say that the sequence of states $s_0, \ldots, s_m$ satisfies **P *O1*** if for all $0 < k \leq m$, states s_{k-1} and s_k satisfy **P *O1***.

Additional factor that needs to be taken into account is the set of formulas of the form $\neg[O]^+_G\xi$ at different levels of the set $\widehat{\mathrm{Gr}}(\cdot)$. The following property states that this set remains unchanged between two states:

$$\mathbf{P}\ \boldsymbol{O2}\ \left(\widehat{\mathrm{Gr}}(L(s)) \sqcap \neg[O]^+\right)_d = \left(\widehat{\mathrm{Gr}}(L(t)) \sqcap \neg[O]^+\right)_d.$$

We say that the sequence of states $s_0, \ldots, s_m$ satisfies $\mathbf{P}\ \boldsymbol{O2}$, if for all $0 < k \leq m$, states s_{k-1} and s_k satisfy $\mathbf{P}\ \boldsymbol{O2}$.

Lemma 5 *The maximal length of sequence of* B*-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ *is* $\leq \mathcal{O}\left(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\right)$.

Analogous lemma for sequences of I-successors can be shown with restriction $\mathbf{R}_1$ only.

Lemma 6 *The maximal length of a sequence of* I*-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *is* $\leq \mathcal{O}\left(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\right)$.

Now we are ready to prove the lemma on bounds on the height and *state height*[15] of the pre-tableau constructed by Algorithm 1 for an input formula satisfying modal context restriction $\mathbf{R}_2$. The height is bounded by a polynomial depending on $|\varphi|$, while the state height is bounded by a polynomial depending on $\mathrm{dep}(\varphi)$.

Lemma 7 *The state height of the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ *is* $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$ *and its height is* $\leq \mathcal{O}(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$.

Proof For any node n in a pre-tableau constructed by the algorithm $|L(n)| \leq (2|\mathcal{A}| + 1)|\varphi|$, as $L(n) \subseteq \neg\mathrm{Cl}(\varphi)$. Thus the path between any subsequent states s and t can contain at most $(2|\mathcal{A}| + 1)|\varphi| - 1$ internal nodes. Moreover, for any states s and t such that t is a descendant of s it must be that $\mathrm{dep}(L(t)) \leq \mathrm{dep}(L(s))$.

If s and t are states, such that t is an G-Successor of s, then $\mathrm{dep}(L(t)) < \mathrm{dep}(L(s))$. Thus any sequence of states in the pre-tableau can contain at most $\mathrm{dep}(\varphi)$ G-Successors. Also, if s, t and u are states such that t is an B_j-Successor of s and u is an I_k-Successor of t with $j \neq k$, then it holds that $\mathrm{dep}(L(u)) < \mathrm{dep}(L(s))$. By Lemma 4 and construction of the algorithm, if s, t and u are states such that t is an B_j-Successor of s and u is an O_k-Successor of t, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$, then it must hold that $j \neq k$. By Lemma 6, the maximal length of a sequence of I-Successors with the same modal depth of labels in a pre-tableau constructed by the algorithm is $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|})$. Similarly, by Lemma 5, the maximal length of a sequence of B-Successors with the same modal depth of labels in a pre-tableau constructed by the algorithm is $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|})$. Hence any sequence of nodes in the pre-tableau must be of length $\leq \mathcal{O}(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$ and contains $\leq \mathcal{O}(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$ states. $\square$

[15] State height of a node is the number of states on a path from the root to the node in the pre-tableau.

Since the height of the pre-tableau constructed by the algorithm for an input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ is bounded and the number of successor of any state is bounded as well so we have the following proposition as a corollary of Lemma 7.

Proposition 3 *For any input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ Algorithm 1 terminates.*

For soundness and validity, we show that Algorithm 1 is sound and valid for any formula $\varphi \in \mathcal{L}^{\mathrm{T}}$ for which it terminates. Thus as long as we can show that the algorithm terminates for a fragment of $\mathcal{L}^{\mathrm{T}}$, we have a sound and valid method of checking TeamLog satisfiability.

Proposition 4 *Suppose that Algorithm 1 terminates on a formula $\varphi \in \mathcal{L}^{\mathrm{T}}$. Then φ is satisfiable iff Algorithm 1 returns* `sat` *on the input φ.*

By Proposition 3, Algorithm 1 terminates, so we have the following corollary from Proposition 4, stating soundness and validity of the algorithm.

Proposition 5 *A formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ is satisfiable iff Algorithm 1 returns* `sat` *on the input φ.*

By Lemma 7, Algorithm 1 can be run on a Turing machine using space bounded from above by a polynomial depending on $|\varphi|$. Thus the TeamLog satisfiability problem for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ is PSPACE solvable. It is also PSPACE hard as it is PSPACE hard for $\mathcal{L}^{\mathrm{T}}$ without fixpoint modalities (Dziubiński et al. 2007). Thus we have the following theorem.

Theorem 2 *The* TeamLog *satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ is PSPACE complete.*

As Lemma 7 and proof of Proposition 4 suggest, bounding modal depth of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ makes the TeamLog satisfiability problem NPTIME complete.

Theorem 3 *For any fixed k, if modal depth of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ is bounded by k, then the* TeamLog *satisfiability problem for them is NPTIME complete.*

Proof By Lemma 7 and the construction of TeamLog tableau based on the pre-tableau constructed by Algorithm 1 presented in Proposition 4, the size of the tableau for a satisfiable formula j is bounded by $\mathcal{O}\left(((2|\mathcal{A}|+1)|\varphi|)^{\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}}\right)$. Hence, if modal depth of φ is bounded by k, then the size of the tableau is bounded by $\mathcal{O}\left(((2|\mathcal{A}|+1)|\varphi|)^{k^{2|\mathcal{A}|+1}}\right)$. This means that the satisfiability of φ with bounded modal depth can be checked by the following non-deterministic Algorithm 7.

Algorithm 7: DecideSatisfiabilityNonDeterministic

Input: a formula $\varphi \in \mathscr{L}^{\mathrm{T}}_{\mathbf{R_2}}$
Output: a decision whether φ is satisfiable or not

Guess a TEAMLOG tableau $\mathscr{T}$ satisfying φ;
if *$\mathscr{T}$ is a tableau for φ* **then**
  return `satisfiable`;

Since tableau $\mathcal{T}$ constructed by Algorithm 7 is of polynomial size, so checking if it is a tableau for φ can be realized in polynomial time. This shows that satisfiability of φ can be checked in NPTIME. The problem is also NPTIME complete, as the satisfiability problem for propositional logic is NPTIME hard. □

Procedure 8: CreateSuccessorsB2

Input: a state s and a formula $\psi \in L(s)$
if *ψ is of the form $\neg[\mathrm{B}]_j\xi$* **then**
  if *there is an B_j-Predecessor t of s such that $\neg[\mathrm{B}]_j\xi \in L(t)$ and $L^{\neg[\mathrm{B}]_j}(t,\xi) = L^{\neg[\mathrm{B}]_j}(s,\xi)$* **then**
    if *$\xi = [\mathrm{B}]^+_G\zeta$ with $j \in G$ and s is a $\neg[\mathrm{B}]_j\xi$-Successor of t* **then**
      For every descendant m of t on the path from t to s set $B(m) := B(m) \cup \{n\}$, where n is an B_j-successor of t on the path from t to s;
  else if *$\xi = [\mathrm{B}]^+_G\zeta$ with $j \in G$ and there is a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\zeta$-Predecessor t of s such that $L(s)/[\mathrm{B}]_j \cup \{\zeta\} = L(t)/[\mathrm{B}]_k \cup \{\zeta\}$* **then**
    if $[\mathrm{B}]_j\zeta \in L(s)$ **then**
      if $\zeta \notin L(s)$ **then**
        Create an B_j-Successor v of s with $L(v) = L^{\neg[\mathrm{B}]_j}(s,\xi)$;
      else if `DecideSatisfiabilityAux(`$L(s)$, G, ζ, j`) = unsat` **then**
        Mark s `unsat`;
  else Create an B_j-Successor v of s with $L(v) = L^{\neg[\mathrm{B}]_j}(s,\xi)$;
  ;
else if *ψ is of the form $[\mathrm{B}]_j\xi$ and there are no formulas of the form $\neg[\mathrm{B}]_j\zeta \in L(s)$* **then**
  If there is no B_j-Predecessor t of s such that $[\mathrm{B}]_j\xi \in L(t)$ and $L^{[\mathrm{B}]_j}(t) = L^{[\mathrm{B}]_j}(s)$ and $L^{[\mathrm{B}]_j}(s) \nsubseteq L(s)$, then create an B_j-successor u of s with $L(u) = L^{[\mathrm{B}]_j}(s)$;

6.2 Complexity of $\mathcal{L}^{\mathbf{T}}_{\mathbf{R_1}}$

The algorithm for checking TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ requires a different approach since, as Proposition 1 shows, a model for such formulas may contain an exponentially long path. Therefore we modify Algorithm 1, designed for checking the TEAMLOG satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ in polynomial space. The difference lies in using a new procedure for B-successors creation, specifically for the formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ with $j \in G$. Since the satisfying sequence for such a formula may have exponential length with respect to the size of the set of formulas, the algorithm using a polynomial space cannot attempt to construct such a sequence storing it fully in the memory, as it was done in the case of Algorithm 1. For this reason, the new algorithm constructs a pre-tableau just like Algorithm 1, creating G- and I-successors in the same way, but stopping creation of B-successors for formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ when certain condition is satisfied. In such case Function 9 is used for checking if the label of the $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor is satisfiable. If it is decided by Function 9 that the label of the $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor is not satisfiable, then the state is marked `unsat`. Otherwise, the decision on how the state should be marked depends on the other successors and the same procedure of marking nodes, as the one used in

Algorithm 1, is applied. The new algorithm is referred two as Algorithm 2, and its formulation differs from the formulation of Algorithm 1 by the procedure of B-successors creation being replaced with a new one, called Procedure 8.

In the algorithm we are referring to the following sets, defined for a given set of formulas $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ and $G \subseteq \mathcal{A}$:

$$\begin{aligned}\Phi/[\mathrm{B}]_G^+ &= \{\psi : [\mathrm{B}]_H^+\psi \in \Phi \text{ and } G \subseteq H\},\\ \Phi \sqcap [\mathrm{B}]_G^+ &= \{[\mathrm{B}]_H^+\psi : [\mathrm{B}]_H^+\psi \in \Phi \text{ and } G \subseteq H\},\\ \Phi^{[\mathrm{B}]_G^+} &= (\Phi/[\mathrm{B}]_G^+) \cup (\Phi \sqcap [\mathrm{B}]_G^+).\end{aligned}$$

Given a formula ψ, $G \subseteq \mathcal{A}$, $j \in G$ and a set of formulas Φ such that $\{[\mathrm{B}]_j\psi, \psi\} \subseteq \Phi$ as an input, Function 9 decides whether the set $\Phi^{\neg[\mathrm{B}]_j}([\mathrm{B}]_G^+\psi)$ is satisfiable or not. To describe the idea of this algorithm, let Ψ_1 and Ψ_2 be sets of formulas. Given $k \in \mathcal{A}$, we say that Ψ_1 and Ψ_2 are *connected with* k if $\Psi_1 \sqcap k = \Psi_2 \sqcap k$. Moreover, given a set $H \subseteq \mathcal{A}$, we say that Ψ_1 and Ψ_2 are *H-connected* if they are connected with some $k \in H$. Let Γ be a set of formulas and let $\mathcal{S}(\Gamma)$ be the set of all minimal sets of formulas containing Γ as a subset that are [B]-expanded tableaux. Given $H \subseteq \mathcal{A}$, let $\mathcal{G}_H(\Gamma) = (V, E)$ be an undirected graph such that V consists of all elements $\Psi \in \mathcal{S}(\Gamma)$ such that Algorithm 2 returns `sat` on input $\bigwedge\Psi$ and for all $(\Psi_1, \Psi_2) \in V \times V$, $(\Psi_1, \Psi_2) \in E$ iff they are H-connected. A *path* in $\mathcal{G}_H(\Gamma)$ is a sequence $\Gamma_0, \ldots, \Gamma_n$ of elements of V such that for all $1 \le i \le n$, Γ_{i-1} and Γ_i are H-connected. The *length of path* $\Gamma_0, \ldots, \Gamma_n$ is n. Given a path $\Gamma_0, \ldots, \Gamma_n$ of length $n \ge 1$ in $\mathcal{G}_H(\Gamma)$ we call a sequence $j_1, \ldots, j_n$ of elements from H such that for each $1 \le i \le n$, Γ_{i-1} and Γ_i are connected with j_i, a *sequence associated with path* $\Gamma_0, \ldots, \Gamma_n$. If $n = 0$, then the sequence associated with the path is the empty sequence ε. Given two sets of formulas Ψ_0 and Ψ_1, we say that Ψ_1 is *reachable from* Ψ_0 in $\mathcal{G}_H(\Gamma)$ (in n steps) iff there exists a path $\Gamma_0, \ldots, \Gamma_n$ in $\mathcal{G}_H(\Gamma)$ such that $\Psi_0 = \Gamma_0$ and $\Psi_1 = \Gamma_n$.

To decide the satisfiability of $\bigwedge\Phi^{\neg[\mathrm{B}]_j}([\mathrm{B}]_G^+\psi)$, Function 9 checks whether there exist two sets of formulas $\{\Psi_0, \Psi_1\} \subseteq \mathcal{S}((\Phi/[\mathrm{B}]_H^+) \cup \{\psi\})$, with $H = \mathrm{ag}\Big(\Big(\Phi \sqcap [\mathrm{B}]_{\{j\}}^+\Big) \cup \{\neg[\mathrm{B}]_G^+\psi\}, [\mathrm{B}]^+\Big)$, such that

- $(\Phi \sqcap j) \setminus ((\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{B}]_j)) \subseteq \Psi_0$ and
- either there exists $k \in H$ such that Algorithm 2 returns `sat` on the input $\bigwedge\Big(\Psi_1^{[\mathrm{B}]_k} \cup (\Phi/[\mathrm{B}]_H^+) \cup \{\sim\psi\}\Big)$ and Ψ_1 is reachable from Ψ_0 in $\mathcal{G}_H((\Phi/[\mathrm{B}]_H^+) \cup \{\psi\})$ with path $\Gamma_0, \ldots, \Gamma_n$ such that if $n = 0$, then $j \ne k$, and if $n \ge 1$, then there exists $j_n \in H \setminus \{k\}$ such that Γ_{n-1} and Γ_n are connected with j_n.
- or Ψ_1 is reachable from Ψ_0 in $\mathcal{G}_H((\Phi/[\mathrm{B}]_H^+) \cup \{\psi\})$ and there exists $k \in G \setminus H$ such that either Algorithm 2 returns `sat` on the input $\bigwedge\Big(\Psi_1^{[\mathrm{B}]_k} \cup \Phi^{[\mathrm{B}]_{H\cup\{k\}}^+} \cup \{\sim\psi\}\Big)$ or Algorithm 2 returns `sat` on $\bigwedge\Big(\Psi_1^{[\mathrm{B}]_k} \cup \Phi^{[\mathrm{B}]_{H\cup\{k\}}^+} \cup \{\psi, \neg[\mathrm{B}]_G^+\psi, [\mathrm{B}]_k\psi, \neg[\mathrm{B}]_k[\mathrm{B}]_G^+\psi\}\Big)$

To check reachability, Function 10 is used. Given sets of formulas Φ_1, Ψ and Φ_2, sets $H \subseteq \mathcal{A}$ and $F \subseteq H$, $p \in H$ and $K \ge 0$, Function 10 checks if there exists a set of formulas $\Gamma \in \mathcal{S}(\Phi_1)$ such that Algorithm 2 returns `sat` on input $\bigwedge\Gamma$ and Φ_2 is reachable from Γ in $\mathcal{G}_H(\Phi_1)$ in at most $2^K - 1$ steps with a path $\Gamma_0, \ldots, \Gamma_n$ such that if $n = 0$, then $p \notin F$ and if $n \ge 1$, then there exists $j_n \in H \setminus F$ such that Γ_{n-1} and Γ_n are connected with j_n. The set F with which the algorithm is called will always be either $\varnothing$ or a singleton. It is used to forbid, in certain situations, one of the possible connections between the last two sets in the constructed sequence.

Function 9: DecideSatisfiabilityAux

Input: A formula ψ, $G \subseteq \mathscr{A}$, $j \in G$ and a set of formulas Φ with $\{[\mathrm{B}]_j\psi, \psi\} \subseteq \Phi$
Output: A decision whether $\Phi^{\neg[\mathrm{B}]_j}([\mathrm{B}]^+_G\psi)$ is satisfiable or not

Let $H = \mathrm{ag}\Big(\Big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$;
Construct a pre-tableau consisting of single node `root`, with $L(\mathtt{root}) = \big(\Phi/[\mathrm{B}]^+_H\big) \cup \{\psi\}$ and all successor relations being empty;
Let Z be the set of all leaves of the pre-tableau with labelling sets that are not trivially inconsistent;
repeat
  if *there is* $n \in Z$ *such that* n *is not a state and* $\xi \in L(n)$ *is a witness to that* **then**
    `FormState(`n,ξ`);`
until *no change occurred*;
foreach $n \in Z$ *such* n *is a state* **do**
  if `DecideSatisfiability(`$\bigwedge L(n)$`) = sat` **then**
    foreach $k \in H$ **do**
      if `Reachable(`$(\Phi/[\mathrm{B}]^+_H) \cup \{\psi\}$, $(\Phi \sqcap j) \setminus ((\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{B}]_j))$, $L(n)$, H, $\{k\}$, j, $|L(n)|$`)` **then**
        if `DecideSatisfiability(`$\bigwedge\big(L^{[\mathrm{B}]_k}(n) \cup \Phi/[\mathrm{B}]^+_H \cup \{\sim\psi\}\big)$`) = sat` **then**
          return `sat`;
    if `Reachable(`$(\Phi/[\mathrm{B}]^+_H) \cup \{\psi\}$, $(\Phi \sqcap j) \setminus ((\Phi \sqcap [\mathrm{B}]_j) \cup (\Phi \sqcap \neg[\mathrm{B}]_j))$, $L(n)$, H, $\varnothing$, j, $|L(n)|$`)` **then**
      foreach $k \in G \setminus H$ **do**
        if `DecideSatisfiability(`$\bigwedge\Big(L^{[\mathrm{B}]_k}(n) \cup \Phi^{[\mathrm{B}]^+_{H\cup\{k\}}} \cup \{\sim\psi\}\Big)$`) = sat` **then**
          return `sat`;
      foreach $k \in G \setminus H$ **do**
        if `DecideSatisfiability(`$\bigwedge\big(L^{[\mathrm{B}]_k}(n) \cup \Phi^{[\mathrm{B}]^+_{H\cup\{k\}}} \cup \{\psi, \neg[\mathrm{B}]^+_G\psi, [\mathrm{B}]_k\psi,$ $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi\}\big)$`) = sat` **then**
          return `sat`;
return `unsat`;

The idea of the algorithm is based on the idea of Savitch's algorithm for checking reachability in graph that uses quadratic logarithmic space with respect to $|V|$ (c.f. Papadimitriu 1994). Notice that all the sets in $\mathcal{S}((\Psi/[\mathrm{B}]^+_H) \cup \{\psi\})$ have the same number of elements and if $\Gamma \in \mathcal{S}((\Psi/[\mathrm{B}]^+_H) \cup \{\psi\})$, then $|\mathcal{S}((\Psi/[\mathrm{B}]^+_H) \cup \{\psi\})| \leq 2^{|\Gamma|}$. Thus to check reachability in $\mathcal{G}_H((\Phi/[\mathrm{B}]^+_H) \cup \{\psi\})$ it is enough to check whether there is reachability in at most $2^{|\Gamma|} - 1$ steps, where $\Gamma \in \mathcal{S}((\Psi/[\mathrm{B}]^+_H) \cup \{\psi\})$.

The procedure of marking nodes remains as in Algorithm 1, however the notion of unresolved formula used by it is different in the case of formulas of the form $\neg[\mathrm{B}]^+_G\psi$. The modification is related to the fact that in the case of any $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-Successor state s and any formula $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi \in L(s)$ with $k \neq j$, either the $\neg[\mathrm{B}]_k[\mathrm{B}]^+_k\psi$-successor of s is created or Function 9 is used to check the satisfiability of $L^{\neg[\mathrm{B}]_k}(s, [\mathrm{B}]^+_G\psi)$. Hence the only situation in which such a formula can be unresolved is when $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ and all the other formulas from $\neg\widetilde{\mathrm{Cl}}([\mathrm{B}]^+_G\psi)$ appear positively in $L(s)$. Unresolved formula of the form $\neg[\mathrm{B}]^+_G\psi$ is defined as follows.

Definition 17 (Unresolved formula) Let n be a node in a pre-tableau and let $\neg[\mathrm{B}]^+_G\psi \in L(n)$. A formula $\neg[\mathrm{B}]^+_G\psi$ is *unresolved in* n if one of the following holds:

- n is an internal node and a $\neg[\mathrm{B}]_j[\mathrm{B}]_G^+\psi$-descendant with $j \in G$, none of its successors is marked `sat`, there exists a successor of n marked `undec` and $B(n) \neq \emptyset$,
- n is a state and a $\neg[\mathrm{B}]_j[\mathrm{B}]_G^+\psi$-Successor with $j \in G$, $B(n) \neq \emptyset$, $[\mathrm{B}]_k[\mathrm{B}]_G^+\psi \in L(n)$, for all $k \in G \setminus \{j\}$, and $[\mathrm{B}]_k\psi \in L(n)$, for all $k \in G$.

Function 10: Reachable

Input: Three sets of formulas Φ_1, Ψ and Φ_2, $H \subseteq \mathscr{A}$, $F \subseteq H$, $p \in H$, and $K \geq 0$

Output: A decision whether there exists a satisfiable set of formulas $\Gamma \in \mathscr{S}(\Phi_1)$ such that $\Psi \subseteq \Gamma$ and Φ_2 is reachable from Γ in $\mathscr{G}_H(\Phi_1)$ in at most $2^K - 1$ steps with a path $\Gamma_0, \ldots, \Gamma_n$ such that if $n = 0$, then $p \notin F$ and if $n \geq 1$, then there exists $j_n \in H \setminus F$ such that Γ_{n-1} and Γ_n are connected with j_n. The algorithm is always used with F being either $\varnothing$ or containing exactly one element.

```
if p ∉ F then
    Construct a pre-tableau consisting of single node root with L(root) := Φ1 ∪ Ψ and all successor
      relations being empty;
    Let Z denote the set of all leaves of the pre-tableau with labelling sets that are not trivially inconsistent;
    repeat
        if there is n ∈ Z such that n is not a state and ξ ∈ L(n) is a witness to that then
            FormState(n,ξ);
    until all nodes of Z are states;
    foreach n ∈ Z such that n is a state do
        if L(n) = Φ2 then
            return
        true;
if K = 0 then
    return false;
else
    Construct a pre-tableau consisting of single node root with L(root) := Φ1 and all successor relations
      being empty;
    Let Z be the set of all leaves of the pre-tableau with labelling sets that are not trivially inconsistent;
    repeat
        if there is n ∈ Z such that n is not a state and ξ ∈ L(n) is a witness to that then
            FormState(n,ξ);
    until all nodes of Z are states;
    foreach n ∈ Z do
        if DecideSatisfiability(⋀L(n)) = sat then
            if Reachable(Φ1,Ψ,L(n),H,∅,p,K−1) then
                foreach j ∈ H do
                    if Reachable(Φ1,L(n)⊓j,Φ2,H,F,j,K−1) then
                        return true;
    return false;
```

We show first that for any input $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ Algorithm 2 terminates. Notice that Lemma 6 stating the bounds on the length of the sequence of I-successors in the pre-tableau holds for Algorithm 2 as well, as it uses the same procedure of I-successors creation as Algorithm 2. The procedure of B-successors creation is changed in Algorithm 2 and the following lemma, stating the bounds on the length of a sequence of B-successors, can be shown.

Lemma 8 *The maximal length of sequence of* B*-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm* 2 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *is* $\leq \mathcal{O}\left(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\right)$.

The following lemma states bounds on the state height of a pre-tableau constructed by Algorithm 2 for an input formula satisfying modal context restriction $\mathbf{R}_1$.

Lemma 9 *State height of the pre-tableau constructed by Algorithm 2 for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *is* $\leq \mathcal{O}\Big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\Big)$ *and its height is* $\leq \mathcal{O}\Big(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\Big)$.

The two lemmas above imply that Algorithm 2 terminates for any input satisfying modal context restriction $\mathbf{R}_1$, as stated below.

Proposition 6 *For any input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *Algorithm 2 terminates.*

Algorithm 2 is sound and valid, as stated below.

Proposition 7 *A formula* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *is satisfiable iff Algorithm 2 returns* `sat` *on the input* φ.

The following theorem states lower and upper bounds on the complexity of the TeamLog satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$. To show the theorem we have to show that Algorithm 2 can be executed in space polynomial with respect to the size of the input formula.

Theorem 4 *The* TeamLog *satisfiability problem for formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *is PSPACE complete.*

The tableau constructed in proof of Proposition 7 can have exponential depth with respect to the input formula. For that reason an algorithm similar to that used in proof of Theorem 3 for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ with modal depth bounded by a constant that works in polynomial time cannot be used in the case $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ with modal depth of formulas bounded by a constant. In fact finding such an algorithm may be very difficult, as the satisfiability problem for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ with modal depth bounded by 2 is PSPACE hard, as stated below.

Theorem 5 *The problem of checking* TeamLog *satisfiability of formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *with modal depth bounded by* 2 *is PSPACE complete.*

6.3 Restriction $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1(\mathbf{c})}$

To check TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1(\mathbf{c})}$ Algorithm 1 can be used. We will show that the algorithm terminates on any input from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1(\mathbf{c})}$ and that its state height is bounded by a polynomial depending on modal depth of the input formula. We start with a lemma stating bounds on the length of a sequence of B-Successors with unchanged modal depth of labels in the pre-tableau.

Lemma 10 *The maximal length of sequence of* B*-Successors with unchanged modal depth of labels in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ *is* $\leq \mathcal{O}\Big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\Big)$.

Notice that the bounds given in Lemma 10 are of the same order as in the case of restriction $\mathbf{R}_2$. However, the constant factor in the case of restriction $\mathbf{R}_{1(c)}$ is different. This factor depends exponentially on c, as all the maximal consistent subsets of the sets of literals of the form $[O]_j$ with $O \in \{\mathrm{G}, \mathrm{I}\}$ and $j \in G$ in direct scope of operators $[\mathrm{B}]^+_G$ in the labels of sets may need to be enumerated until the branch expansion stops.

Bounds on the state height of the pre-tableau constructed by Algorithm 1 for an input formula satisfying modal context restriction $\mathbf{R}_{1(c)}$ are stated in the lemma below.

Lemma 11 *State height of the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ *is* $\leq \mathcal{O}\Big(\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\Big)$ *and its height is* $\leq \mathcal{O}\Big(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1}\Big)$.

Since the size of a pre-tableau constructed by the algorithm for an input formula from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ is bounded so Algorithm 1 terminates on any input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$.

Proposition 8 *For any input formula* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ *Algorithm 1 terminates.*

Since Algorithm 1 terminates on any input from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ so, by Proposition 4, it is also sound and valid for checking the TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$, as stated in the proposition below.

Proposition 9 *A formula* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ *is satisfiable iff Algorithm 1 returns* `sat` *on the input* φ.

Moreover, since the state height of the pre-tableau constructed by Algorithm 1 for an input formula $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ is bounded by a polynomial depending on $\mathrm{dep}(\varphi)$ and c, so the size of the pre-tableau constructed for φ in proof of Lemma 11 on the basis of this pre-tableau has the size which is bounded by a polynomial depending on $|\varphi|$ with the degree depending on $\mathrm{dep}(\varphi)$. Thus we have the following theorem, analogous to Theorem 3 for $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_2}}$. Proof of the theorem is analogous to proof of Theorem 3.

Theorem 6 *For any fixed k, if modal depth of formulas from* $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(c)}}$ *is bounded by k, then the* TeamLog *satisfiability for them is NPTIME complete.*

7 Conclusions

In this paper we presented a family of language restrictions for modal logics for multiagent systems that can be used to provide NPTIME solvability of the satisfiability problem. The family of restrictions, called modal context restrictions, generalizes modal depth restriction. We applied two restrictions of this kind to one of existing multiagent formalisms called TeamLog and studied the complexity of the satisfiability problem of the restricted language of the formalism.

Like in the case of other modal formalisms for multiagent systems with fixpoint modalities, such as common beliefs or mutual intentions, the satisfiability problem of TeamLog is EXPTIME complete even if the modal depth of formulas is bounded by 2. For this reason a language restriction which would be less forbidding than modal depth restriction would be needed to reduce the complexity of the problem. In the paper we introduced three restrictions called $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_1(c)$. In the case of the least restrictive one of them, called $\mathbf{R}_1$, the problem remains PSPACE hard even if modal depth of formulas is bounded by 2. In the case of the most restrictive one, called $\mathbf{R}_2$, combining it with restricting modal depth of formulas by a constant results in NPTIME completeness of the satisfiability problem. Since restriction $\mathbf{R}_2$ is too strong, at least in situations when aspects of multiagent systems combining informational and motivational attitudes are specified, like for example collective intentions and collective commitments, we proposed a refinement of restriction $\mathbf{R}_1$ called $\mathbf{R}_1(c)$. Combining this restriction with restricting modal depth of formula results in NPTIME solvability of the satisfiability problem.

The restrictions of the language studied in this paper do not lead to tractable fragments of the formalisms considered. However, we were able to find NPTIME complete fragments, even in the case of full TeamLog, which originally has EXPTIME complete satisfiability problem. Two possible approaches could be undertaken to address this issue: reducing the satisfiability of the NPTIME complete fragments to some other NPTIME complete problems for which well performing, heuristics based algorithms exist, or studying further restrictions of the language that could lead to PTIME solvable satisfiability problem. The first of these approaches was successfully used by Kacprzak, Lomuscio and Penczek in Kacprzak et al. (2004a, 2004b), where model checking of temporal modal logic is studied. The authors reduce this problem to the problem of satisfiability of propositional calculus (SAT) and use existing SAT-solvers for it. Applying a similar approach to the NPTIME complete fragments of TeamLog could be a promising direction for further research. For the second approach, different language restrictions that were already studied in the literature could be considered. Firstly, it would be interesting to investigate the Horn fragment of TeamLog. In Nguyen (2000) Linh Nguyen studied Horn fragments of various basic multimodal logics and he found out that when modal depth of formulas is bounded by a constant, then the satisfiability problem is PTIME complete in the case of several of standard multimodal logics.

Another possibility would be to look at restrictions of propositional operators used in formulas, following the approach of Bauland et al. (2006) and Bauland et al. (2009) and to CTL* and CTL in Meier et al. (2008).

Appendix

Proof of Proposition 1 We will use propositional variables $q_1, \ldots, q_N$ to enumerate worlds of the model. Each world v receives two numbers of length N in binary representation, that are encoded by the valuation of the formulas $[\mathrm{I}]_1 q_j$ and $[\mathrm{I}]_2 q_j$. Bits of the first number, $M_1(v)$, are encoded by the valuations of formulas $[\mathrm{I}]_1 q_j$, with $[\mathrm{I}]_1 q_1$ corresponding to the least significant bit and $[\mathrm{I}]_1 q_j$ being satisfied in $(\mathcal{M}, v)$ encoding the value 1 and $\neg[\mathrm{I}]_1 q_j$ being satisfied in $(\mathcal{M}, v)$ encoding the value 0 of bit j of $M_1(v)$. Value of $M_2(v)$ is encoded in analogous way with formulas $[\mathrm{I}]_2 q_j$.

Let

$$\varphi = \mathsf{INIT} \wedge [\mathrm{B}]^+_{\{1,2\}}\left(\mathsf{INC}_0 \wedge \bigwedge_{j=1}^{N-1} \mathsf{INC}_1(j)\right) \wedge \neg[\mathrm{B}]^+_{\{1,2\}}\left(\bigvee_{j=1}^{N} \neg[\mathrm{I}]_1 q_j\right),$$

and

$$\mathsf{INIT} = \left(\bigwedge_{j=1}^{N} \neg[\mathrm{I}]_1 q_j\right) \wedge \left([\mathrm{I}]_2 q_1 \wedge \bigwedge_{j\notin}^{N} \neg[\mathrm{I}]_2 q_j\right) \tag{1}$$

$$\mathsf{INC}_0 = \neg[\mathrm{I}]_1 q_1 \rightarrow \left(\bigwedge_{j\notin}^{N} ([\mathrm{I}]_1 q_j \leftrightarrow [\mathrm{I}]_2 q_j)\right) \tag{2}$$

$$\begin{aligned}\mathsf{INC}_1(i) = &\left(\neg[\mathrm{I}]_1 q_{i+1} \wedge \bigwedge_{j=1}^{i} [\mathrm{I}]_1 q_j\right) \rightarrow \left(\bigwedge_{j=i+2}^{N} ([\mathrm{I}]_1 q_j \leftrightarrow [\mathrm{I}]_2 q_j) \wedge\right.\\ &\left.\left(\left(\neg[\mathrm{I}]_2 q_{i+1} \wedge \bigwedge_{j=1}^{i} [\mathrm{I}]_2 q_j\right) \vee \left([\mathrm{I}]_2 q_{i+1} \wedge \bigwedge_{j=1}^{i} \neg[\mathrm{I}]_2 q_j\right)\right)\right)\end{aligned} \tag{3}$$

Notice that $|\varphi| = \mathcal{O}(N^2)$.[16] Take any $(\mathcal{M}, w)$ such that $(\mathcal{M}, w) \vDash \varphi$. The formula INIT enforces that the value of M_1 at the initial world w is 0, that is $M_1(w) = (0, \ldots, 0)_2$, and the value of M_2 at the initial world w is 1, that is $M_2(w) = (0, \ldots, 0, 1)_2$. The formulas INC_0 and $\mathsf{INC}_1(i)$, for $1 \leq i < N$, enforce that at each world $v \in \mathrm{B}^+_{\{1,2\}}(w)$, $M_1(v) \leq M_2(v) \leq M_1(v) + 1$. More precisely, the formula INC_0 enforces that in the case of the least significant bit of $M_1(v)$ being 0, while the formula $\mathsf{INC}_1(i)$ enforced that in the case of the $i+1$ bit being 0 and the bits from i to 1 being 1.

Mixed axioms **BI4** and **BI5** guarantee that for any world $v \in \mathrm{B}^+_{\{1,2\}}(w)$ and any world

[16] Formula φ could be also constructed with use of operators $[\mathrm{G}]_1$ and $[\mathrm{G}]_2$ instead of $[\mathrm{I}]_1$ and $[\mathrm{I}]_2$.

$u \in \mathrm{B}_1(v)$, $M_1(v) = M_1(u)$ and for any world $u \in \mathrm{B}_2(v)$, $M_2(v) = M_2(u)$. Thus if there exists a world $u \in \mathrm{B}^+_{\{1,2\}}(w)$ such that $M_1(u) = (1,\ldots,1)_2$, then for each $0 < x \leq 2^N - 1$ there must exist a world $v \in \mathrm{B}^+_{\{1,2\}}(w)$ such that $M_1(v) = x$ and $u \in \mathrm{B}^+_{\{1,2\}}(v)$. Hence if φ is satisfied in $(\mathcal{M}, w)$, then $(\mathcal{M}, w)$ must contain exponentially long, with respect to $|\varphi|$, sequence of pairwise different worlds.

To see that $|\varphi|$ is satisfiable, take the following model $\mathcal{M} = (W, \{O_j : j \in \{1,2\},\ O \in \{\mathrm{B},\mathrm{G},\mathrm{I}\}, Val\})$, where

- $W = \{s_1,\ldots,s_K\} \cup \{t_0,\ldots,t_K\} \cup \{v_1,\ldots,v_N\} \cup \{u_1,\ldots,u_N\}$, where $K = 2^N - 1$,
- $\mathrm{B}_1(s_k) = \{s_k\}$ and $\mathrm{B}_2(t_k) = \{t_k\}$, for $1 \leq k \leq K$,
- $\mathrm{B}_1(t_k) = \{s_{k+1}\}$ and $\mathrm{B}_2(s_k) = \{t_k\}$, for $1 \leq k \leq K-1$,
- $\mathrm{B}_1(t_K) = \{t_K\}$, $\mathrm{B}_1(v_k) = \mathrm{B}_2(v_k) = \{v_k\}$, $\mathrm{B}_1(u_k) = \mathrm{B}_2(u_k) = \{u_k\}$, for $1 \leq k \leq N$,
- $\mathrm{I}_1(s_1) = \{w_1(0),\ldots,w_N(0)\}$ and $\mathrm{I}_1(t_k) = \mathrm{I}_1(s_{k+1}) = \{w_1(k),\ldots,w_N(k)\}$, for all $1 \leq k \leq K$,
- $\mathrm{I}_2(s_k) = \mathrm{I}_2(t_k) = \{w_1(k),\ldots,w_N(k)\}$, for all $1 \leq k \leq K$,
- $\mathrm{I}_1(v_k) = \mathrm{B}_2(v_k) = \{v_k\}$, $\mathrm{I}_1(u_k) = \mathrm{I}_2(u_k) = \{u_k\}$, for $1 \leq k \leq N$,
- $\mathrm{G}_j = \emptyset$, for $j \in \{1,2\}$,
- $Val(q_j, v_j) = 1$, $Val(q_j, u_j) = 0$, for all $1 \leq j \leq N$, and $Val(q, v) = 0$ on all the remaining arguments, and

$$w_i(k) = \begin{cases} v_i & \text{if the value of } i\text{-th bit (counting from the least significant bit) in binary} \\ & \text{representation of } k \text{ is } 1 \\ u_i & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathcal{M}$ is a TeamLog model and that $(\mathcal{M}, v_0) \vDash \varphi$. $\square$

Proof (Proposition 2) Most of the proof is very similar to the proof given in Halpern and Moses (1992) for $\mathrm{S5}_n$ tableaux (notice that conditions **TB *O*4** and **TB *O*5** are similar to conditions **T4** and **T5**). In what follows we will focus on fixpoint modalities. For the detailed proof see the dissertation Dziubiński (2011).

For the left to right direction we have to show how to construct a TeamLog tableaux for φ on the basis of a model for φ. Let $(\mathcal{M}, w) \vDash \varphi$ where $\mathcal{M} = \left(W, \{O_j : [O]_j \in \Omega^{\mathrm{ind}}\}, Val\right)$ is a TeamLog model and $w \in W$. Consider a model graph $\mathcal{T} = \left(W, \{O_j : [O]_j \in \Omega^{\mathrm{ind}}\}, L\right)$, where $\psi \in L(v)$ iff $(\mathcal{M}, v) \vDash \psi$. Since $\varphi \in L(w)$, so it is enough to show that $\mathcal{T}$ is a TeamLog tableau. Since we focus on fixpoint modalities, we will show that for all $v \in W$, $L(v)$ satisfies conditions **Cl** and **T1** and **T2** for the formulas of the form $[O]^+_G \xi$ and $\neg[O]^+_G \xi$. For condition **Cl** let $v \in W$ and assume that $[O]^+_G \xi \in L(v)$ with $O \in \{\mathrm{B},\mathrm{I}\}$. Then $(\mathcal{M}, v) \vDash [O]^+_G \xi$. Take any $u \in O_j(v)$, for some $j \in G$. Then $u \in O^+_G(v)$, and so $(\mathcal{M}, u) \vDash \xi$. Moreover for any $t \in O^+_G(u)$ it also holds that $t \in O^+_G(v)$, by transitivity of O^+_G. Hence $(\mathcal{M}, u) \vDash [O]^+_G \xi$. Thus $(\mathcal{M}, v) \vDash [O]_j \xi$ and $(\mathcal{M}, v) \vDash [O]_j [O]^+_G \xi$, and so $[O]_j \xi \in L(v)$ and $[O]_j [O]^+_G \xi \in L(v)$. This shows that condition **Cl** is satisfied. For condition **T1**, let $v \in W$ and assume that $[O]^+_G \xi \in L(v)$ with $O \in \{\mathrm{B},\mathrm{I}\}$. Then $(\mathcal{M}, v) \vDash [O]^+_G \xi$ and for any $u \in O^+_G(v)$, $(\mathcal{M}, u) \vDash \xi$ and, consequently, $\xi \in L(u)$. Hence condition **T1** is satisfied. For condition **T2**, let $v \in W$ and assume that $\neg[O]^+_G \xi \in L(v)$ with $O \in \{\mathrm{B},\mathrm{I}\}$. Then

$(\mathcal{M}, v) \nvDash [O]^+_G \xi$ and there must be $u \in O^+_G(v)$ such that $(\mathcal{M}, u) \in \xi$. Thus $(\mathcal{M}, u) \vDash {\sim}\xi$ and ${\sim}\xi \in L(u)$. Hence condition **T2** is satisfied.

For the right to left implication we need to show how to construct a model for φ on the basis of a TeamLog tableau for φ. Let $\mathcal{T} = \left(W, \{O_j : [O]_j \in \Omega^{\text{ind}}\}, L\right)$ be a TeamLog tableau for φ, so that $\varphi \in L(w)$ for some $w \in W$. Consider a Kripke model $\mathcal{M} = \left(W, \{O'_j : [O]_j \in \Omega^{\text{ind}}\}, Val\right)$, where

$$Val(p, v) = \begin{cases} 1, & \text{if } p \in L(v) \\ 0, & \text{if } p \notin L(v) \end{cases}$$

Before defining accessibility relations B'_j, G'_j and I'_j, we will introduce a notion that will be useful here. We will say that states v and u are B_j-connected with a sequence of states $s_0, \ldots, s_m$ if $v = s_0$, $u = s_m$, $m > 0$ and for any $0 < j \leq m$, either $s_j \in \mathrm{B}_j(s_{j-1})$ or $s_{j-1} \in \mathrm{B}_j(s_j)$. We will say that states v and u are B_j-*connected* if there is a sequence of states $s_0, \ldots, s_m$ such that v and u are B_j-connected with it.

Relation B'_j is defined as follows. Let $\bar{\mathrm{B}}_j = \mathrm{B}_j \cup \{(v, v) \in W \times W : \mathrm{B}_j(v) = \varnothing\}$. Then $(v, u) \in \mathrm{B}'_j$ iff $(v, u) \in \bar{\mathrm{B}}_j$ or there exists s such that v and s are B_j-connected and $u \in \mathrm{B}_j(s)$ (notice that it means, in the case of $(v, u) \in \mathrm{B}_j$ or in the latter case, that v and u are B_j-connected). Relation G'_j is defined as follows. A pair of states $(v, u) \in \mathrm{G}'_j$ iff $(v, u) \in \mathrm{G}_j$ or there exists s such that v and s are B_j-connected and $u \in \mathrm{G}_j(s)$. Relation I'_j is defined as follows. Let $\bar{\mathrm{I}}_j = \mathrm{I}_j \cup \{(v, v) \in W \times W : \mathrm{I}_j(v) = \varnothing\}$. Then $(v, u) \in \mathrm{I}'_j$ iff $(v, u) \in \bar{\mathrm{I}}_j \cup \mathrm{G}'_j$ or there exists s such that v and s are B_j-connected and $u \in \mathrm{I}_j(s)$.

Showing that $\mathcal{M}$ is a TeamLog model (i.e. that all the required properties of accessibility relations are satisfied) is analogous to the similar proof in Halpern and Moses (1992) for $\mathrm{S5}_n$ tableaux and we omit it here. See Dziubiński (2011) for the detailed proof. Second thing that needs to be shown is that for any $\psi \in \mathcal{L}^{\mathrm{T}}$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \vDash \psi$. This is done using induction on the length of formulas. Again, most of the steps is analogous to the similar proof in Halpern and Moses (1992) for $\mathrm{S5}_n$ the details can be found in Dziubiński (2011). We restrict attention to the formulas of the form $[O]^+_G \xi$ and $\neg [O]^+_G \xi$.

Assume that $\psi = [O]^+_G \xi$, with $O \in \{\mathrm{B}, \mathrm{I}\}$, and $[O]^+_G \xi \in L(v)$. Take any $u \in O'^+_G(v)$. By condition **Cl**, $\{[O]_j \xi, [O]_j [O]^+_G \xi\} \subseteq L(v)$, for any $j \in G$. Moreover, by simple induction on the length of sequences over G, it can be shown that that for any $u \in O'^+_G(v)$ it holds that $\{\xi, [O]_j \xi, [O]_j [O]^+_G \xi\} \subseteq L(u)$. Thus, by the induction hypothesis, $(\mathcal{M}, u) \vDash \xi$ and so $(\mathcal{M}, v) \vDash [O]^+_G \xi$. Now let $\psi = \neg [O]^+_G \xi$, with $O \in \{\mathrm{B}, \mathrm{I}\}$, and $\neg [O]^+_G \xi \in L(v)$. By condition **TC** there exists $u \in O^+_G(v)$ such that ${\sim}\xi \in L(u)$. Thus, by the induction hypothesis and the fact that $O_j \subseteq O'_j$, for all $j \in \mathcal{A}$, it holds that $u \in O'^+_G(v)$ and $(\mathcal{M}, u) \vDash {\sim}\xi$. Hence $(\mathcal{M}, u) \nvDash \xi$ and $(\mathcal{M}, v) \vDash \neg [O]^+_G \xi$.

We have shown that for any $\psi \in \mathcal{L}^{\mathrm{T}}$ and $v \in W$, $\psi \in L(v)$ implies $(\mathcal{M}, v) \vDash \psi$, and, in particular, $\varphi \in L(w)$ implies $(\mathcal{M}, w) \vDash \varphi$, that is φ is satisfiable. □

Proofs associated with termination property of Algorithm 1

Proof (of Lemma 1) Notice that for all $i \in N$, $F_\Phi^{(i)} \subseteq \mathrm{Gr}(\Phi)$ and $\mathrm{dep}\big(F^{(i)}\big) \leq \mathrm{dep}(\mathrm{Gr}(\Phi))$. We will show first that for all $i \in \mathbb{N}$, $F_\Phi^{(2i)} \subseteq F_\Phi^{(2(i+1))}$ and

$$F_\Phi^{(2i)} \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right) = \varnothing.$$

We will use induction on i. For $i = 0$ the claim is obvious, as $F_\Phi^{(0)} = \varnothing$. Let $i \geq 1$. Since, by the induction hypothesis, $F_\Phi^{(2(i-1))} \subseteq F_\Phi^{(2i)}$, so

$$\neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2(i-1))}/[O]^+\right) \subseteq \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i)}/[O]^+\right)$$

and so $F_\Phi^{(2i+1)} \subseteq F_\Phi^{(2i-1)}$. Since, by the definition of $F_\Phi^{(2i)}$,

$$F_\Phi^{(2i)} \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i-1)}/[O]^+\right) = \varnothing$$

and $F_\Phi^{(2i+1)} \subseteq F_\Phi^{(2i-1)}$, so

$$F_\Phi^{(2i)} \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right) = \varnothing.$$

This, together with the definition of $F_\Phi^{(2(i+1))}$, implies that $F_\Phi^{(2(i))} \subseteq F_\Phi^{(2(i+1))}$.

Since $\mathrm{Gr}(\Phi)$ is finite and, for all $i \in \mathbb{N}$, $F_\Phi^{(i)} \subseteq \mathrm{Gr}(\Phi)$ and, as we have shown above, $F_\Phi^{(2(i))} \subseteq F_\Phi^{(2(i+1))}$, so there exists $n \in \mathbb{N}$ such that for all $i > n$, $F_\Phi^{(2i)} = F_\Phi^{2(i+1)}$.

Secondly, for $i \in \mathbb{N}$,

$$F_\Phi^{(2i+3)} \setminus F_\Phi^{(2i+2)} = F_\Phi^{(2i+3)} \setminus \left(\mathrm{Gr}(\Phi) \setminus \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right)\right)$$

and since $F_\Phi^{(2i+3)} \subseteq \mathrm{Gr}(\Phi)$, so

$$F_\Phi^{(2i+3)} \setminus F_\Phi^{(2i+2)} = F_\Phi^{(2i+3)} \cap \neg\mathrm{PT}\left(\bigcup_{O\in\{\mathrm{B},\mathrm{I}\}} \neg F_\Phi^{(2i+1)}/[O]^+\right).$$

Thus if $F_\Phi^{(2i+1)} \setminus F_\Phi^{(2i)} \neq \varnothing$, then

$$\mathrm{dep}\Big(F_\Phi^{(2i+1)} \setminus F_\Phi^{(2i)}\Big) > \mathrm{dep}\Big(F_\Phi^{(2i+3)} \setminus F_\Phi^{(2i+2)}\Big)$$

and since, for all $i \in \mathbb{N}$, $\mathrm{dep}\big(F^{(i)}\big) \leq \mathrm{dep}(\mathrm{Gr}(\Phi))$, so there exists $n \in \mathbb{N}$ such that for all $i > n$, $F_\Phi^{(2i+1)} \setminus F_\Phi^{(2i)} = \varnothing$. Hence there exists $n \in \mathbb{N}$, such that $F^{(i)} = F^{(i+1)}$, for all $i > n$, and so $F^{(\infty)}$ exists. $\square$

Proof (Lemma 4) For points 1 and 2 notice first that if $\neg[O]_j\xi \in L(s)$, then $\neg[O]_j\xi \in L^{[\mathrm{B}]_j}(s)$ and $\neg[O]_j\xi \in L(t)$, as t is a B_j-Successor of s. Similarly, if $[O]_j\xi \in L(s)$, then $[O]_j\xi \in L(t)$, as t is a B_j-Successor of s. This shows the left to right implications of the two points. For the right to left implication we will show first that $[O]_j\xi \in \neg L(t)$ implies $[O]_j\xi \in \neg L(s)$. Assume that there is a formula $[O]_j\xi \in \neg L(t)$. Then one of the following cases holds:

(i) $[O]_j\xi \in \neg L(s)$,
(ii) there is a formula $[\mathrm{B}]_j\psi \in \neg L(s)$ such that $[O]_j\xi \in \neg\mathrm{BT}(\psi)$.

If case (i) holds, then the claim holds. If case (ii) holds, then $[O]_j\psi \in \neg L(s)$, as s is a state and $L(s)$ is a [B]-expanded tableau.

Now, $\neg[O]_j\xi \in L(t)$ implies $[O]_j\xi \in \neg L(s)$ and it must hold that $\neg[O]_j\xi \in L(s)$, as otherwise it would be $[O]_j\xi \in L(s)$ and $[O]_j\xi \in L(t)$, which would contradict the assumption that t is a state and $L(t)$ cannot be trivially inconsistent. If $[O]_j\xi \in L(t)$, then it must be that $[O]_j\xi \in L(s)$ by similar arguments. Hence points 1 and 2 hold. Points 3 and 4 are straightforward implication of points 1 and 2 and the definitions of $L^{[O]_j}(\cdot)$, for $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$. □

Proof (Lemma 2) For point (i) take any formula of the form $[\mathrm{I}]_G^+\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]_G^+\psi) \geq D$ and suppose that $[\mathrm{I}]_G^+\psi \notin \neg\widehat{\mathrm{Gr}}(\Phi)$. Then either there is a formula $[\mathrm{I}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{B}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{I}]_G^+\psi \in \neg\mathrm{PT}(\xi)$. The first case is impossible as $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$ for all $d \geq D+1$ and so it must hold that $j2H$ which would violate modal context restriction $\mathbf{R}_1$. The second case is impossible as well since $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\right)$, for all $d \geq D+1$. Hence it must be that $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$.

For point (ii) take any formula of the form $[\mathrm{I}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{I}]_j\psi) \geq D+1$. Since $\mathrm{dep}(\mathrm{Ind}(\Phi)) \leq D$ so $[\mathrm{I}]_j\psi \notin \neg\mathrm{Ind}(\Phi)$. Thus either there is a formula $[\mathrm{I}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{B}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{I}]_j\psi \in \neg\mathrm{BT}(\xi)$, or $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. The first case is impossible as $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$ for all $d \geq D+1$ and so it must hold that $j \in G$ which would violate modal context restriction $\mathbf{R}_1$.[17] The second case is impossible as well, as $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\right)$, for all $d \geq D+1$. Hence it must be that $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. Thus either there is a formula $[\mathrm{I}]_G^+\psi \in \neg\Phi$ such that $j \in G$ or ψ is of the form $[\mathrm{I}]_G^+\xi$ with $j \in G$ and $\psi \in \neg\Phi$. Hence, by point (i), it holds that $[\mathrm{I}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(\Phi)\right)$. □

Proof (Lemma 3) For point (i) take any formula of the form $[\mathrm{B}]_G^+\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{B}]_G^+\psi) \geq D$ and suppose that $[\mathrm{B}]_G^+\psi \notin \neg\widehat{\mathrm{Gr}}(\Phi)$. Thus either there is a formula $[\mathrm{B}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{B}]_G^+\psi \in \neg\mathrm{PT}(\xi)$. The first case is impossible as $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\right)$ for all $d \geq D+1$ and so it must hold that $j \in H$ which would

[17] Notice that for this argument to hold it is necessary to forbid sequences $S_{\mathrm{IB}}(G)$ in modal context of formulas.

violate modal context restriction $\mathbf{R}_1$. The second case is impossible as well since $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$, for all $d \geq D + 1$. Hence it must be that $[\mathrm{B}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$.

For point (ii) take any formula of the form $[\mathrm{B}]_j\psi \in \neg\Phi$ with $\mathrm{dep}([\mathrm{I}]_j\psi) \geq D + 1$. Since $\mathrm{dep}(\mathrm{Ind}(\Phi)) \leq D$ so $[\mathrm{B}]_j\psi \notin \neg\mathrm{Ind}(\Phi)$. Thus either there is a formula $[\mathrm{B}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{B}]_j\psi \in \neg\mathrm{BT}(\xi)$, or $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. The first case is impossible as $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\right)$ for all $d \geq D + 1$ and so it must hold that $j \in G$ which would violate modal context restriction $\mathbf{R}_1$. The second case is impossible as well, as $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$, for all $d \geq D + 1$. Hence it must be that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. Thus either there is a formula $[\mathrm{B}]_G^+\psi \in \neg\Phi$ such that $j \in G$ or ψ is of the form $[\mathrm{B}]_G^+\xi$ with $j \in G$ and $\psi \in \neg\Phi$. Hence, by point (i), it holds that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(\Phi)\right)$.

Before showing point (iii) we will show that if $\Phi \subseteq \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ and there is a formula of the form $[\mathrm{I}]_G^+\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]_G^+\psi) \geq D$, then $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ (*).

So take any formula of the form $[\mathrm{I}]_G^+\psi \in \neg\Phi$ with $j \in G$ and $\mathrm{dep}([\mathrm{I}]_G^+\psi) \geq D$ and suppose that $[\mathrm{I}]_G^+\psi \notin \neg\widehat{\mathrm{Gr}}(\Phi)$. Then either there is a formula $[\mathrm{B}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[\mathrm{I}]_G^+\psi \in \neg\mathrm{PT}(\xi)$. The first case is impossible as $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\right)$ for all $d \geq D + 1$ and so it must hold that $j \in H$ which would violate modal context restriction $\mathbf{R}_2$. The second case is impossible as well since $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$, for all $d \geq D + 1$. Hence it must be that $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(\Phi)$.

For point (iii), take any formula of the form $[O]_j\psi \in \neg\Phi$ with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $\mathrm{dep}([O]_j\psi) \geq D + 1$. Since $\mathrm{dep}(\mathrm{Ind}(\Phi) \sqcap j) \leq D$ so $[O]_j\psi \notin \neg\mathrm{Ind}(\Phi)$. Thus either there is a formula $[\mathrm{B}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ or a formula $[\mathrm{I}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(\Phi)$ with $[O]_j\psi \in \neg\mathrm{BT}(\xi)$, or $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$.

The first case is impossible as $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{B}]^+, d\right)$ for all $d \geq D + 1$ and so it must hold that $j \in G$ which would violate modal context restriction $\mathbf{R}_2$. The second case is impossible as well, since $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$, for all $d \geq D + 1$. Hence it must be that $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}(\Phi)$. Thus $O \in \{\mathrm{B}, \mathrm{I}\}$ and either there is a formula $[O]_G^+\psi \in \neg\Phi$ such that $j \in G$ or ψ is of the form $[O]_G^+\xi$ with $j \in G$ and $\psi \in \neg\Phi$. Hence, by point (i) and by (*), it holds that $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(\Phi)\right)$. Since $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(\Phi), [\mathrm{I}]^+, d\right)$, for all $d \geq D$, so it must be that $O = \mathrm{B}$. $\square$

Proof (Corollary 1) Take any $\psi \in L(t)$ with $\mathrm{dep}(\psi) = d \geq D$. If $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ or $\psi \in L(s) \sqcap j$, then the claim holds. Suppose otherwise. Notice that if ψ was added to $L(t)$ during $[\mathrm{B}]^+$-expanded tableau formation, then ψ must be of the form $[\mathrm{B}]_k\xi$ or $\neg[\mathrm{B}]_k\xi$. Moreover, it must be that $k \neq j$ as, by Lemma 4, it holds that $L(s) \sqcap j = L(t) \sqcap j$. Hence if neither $\psi \in \widetilde{\mathrm{Cl}}(L(t))$ nor $\psi \in L(s) \sqcap j$, then there must be a formula $[\mathrm{B}]_j\xi \in \neg L(s)$ such that $\psi \in \neg\mathrm{BT}(\xi)$. By point (ii) or Lemma 3 it holds that $[\mathrm{B}]_j\xi \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$. Hence either there is a formula $[\mathrm{B}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $j \in G$ or ψ is of the form $[\mathrm{B}]_G^+\eta$ with

$j \in G$, $[\mathrm{B}]^+_G\eta \in \neg\widehat{\mathrm{Gr}}(L(s))$ and $[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \widetilde{\mathrm{Cl}}(L(s))$, or ψ is of the form $\neg[\mathrm{B}]^+_G\eta$ with $j \in G$, $\neg[\mathrm{B}]^+_G\eta \in \widehat{\mathrm{Gr}}(L(s))$ and $[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \neg\widetilde{\mathrm{Cl}}(L(s))$. In the last case it must hold that $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \widetilde{\mathrm{Cl}}(L(s))$ as otherwise it would be $[\mathrm{B}]_j[\mathrm{B}]^+_G\eta \in \widetilde{\mathrm{Cl}}(L(s))$ and $[\mathrm{B}]^+_G\eta \in L(t)$, which would contradict the assumption that t is a state and $L(t)$ cannot be trivially inconsistent. Moreover, in this case it must be that t is a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\eta$-Successor of s. To see why assume the opposite. Then there must be a formula $[\mathrm{B}]_j\xi \in L(s)$ such that $[\mathrm{B}]^+_G\eta \in \neg\mathrm{BT}(\xi)$. As we already observed, it cannot be that $\xi = [\mathrm{B}]^+_G\eta$. This, together with the assumption that $[\mathrm{B}]^+_G\eta \in \neg\mathrm{BT}(\xi)$, implies that ξ cannot be of the form $[\mathrm{B}]^+_H\zeta$. Now, by point (ii) or Lemma 3, it must be that $[\mathrm{B}]_j\xi \in \neg\widetilde{\mathrm{Cl}}(\widehat{\mathrm{Gr}}(L(s)))$. Since ξ cannot be of the form $[\mathrm{B}]^+_H\zeta$, so there must be formula $[\mathrm{B}]^+_H\xi \in \widehat{\mathrm{Gr}}(L(s))$ with $j \in H$. But this is impossible, as it violates modal context restriction $\mathbf{R}_1$. □

Proof (Lemma 5) The structure of the proof is as follows. First we prove four claims that are crucial for the result to hold, then we assess how the length of a sequence of B-Successors with unchanged modal depth of labels can be bounded. The general idea is as follows. We show that the sets $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(\cdot)), [O]^+, d\right)$ for subsequent states in the sequence must gradually increase at subsequent levels d, starting from the topmost level. This relies on the following observations. If subsequent states s and t, such that t is a $[\mathrm{B}]_j$-Successor of state s, satisfy properties **PB1**, **PI1** and **PB2** from above some level $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$, then it must be that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\right)$ at levels $d \geq D$ (Claim 1). Moreover, in this case the sets $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{B}]^+, d\right)$ and $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(\cdot), [\mathrm{I}]^+, d\right)$ can only increase between s and t at the level D and $dep(\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j))$ must either be below D or $\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j)$ must be empty. Also, there can be at most one formula of the form $\neg[\mathrm{B}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(t))$ at level D or above, and if there is one, then t must be a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-Successor of s (Claim 2). Notice that since properties **PB1**, **PI1** and **PB2** are always satisfied at levels above $\mathrm{dep}(L(s_0))$ for all the sets in the sequence (where s_0 is the first state in the sequence), so either they will have to be satisfied at levels above $\mathrm{dep}(L(s_0)) - 1$ starting from the second state in the sequence or the sets $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(\cdot)), [O]^+, d\right)$ will have to increase at this level. This observation applies to lower levels of this set for subsequent states in the sequence and leads to recurrence Equations (4) and (7) which are used to assess the maximal length of the sequence. The basis of these recurrence equations is the length of a sequence where $\mathrm{Ind}(L(\cdot)) = \varnothing$ and properties **PB1**, **PI1** and **PB2** are satisfied at all levels. In such a case the only differences in the labels the successor nodes of states get come from formulas in $\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(\cdot))\right)$ (Claims 3 and 4). By construction of the algorithm, if the labels the successor nodes of states are equal, no new successor can be added to the sequence. Since there can be at most one formula of the form $\neg[\mathrm{B}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(\cdot))$ starting from the second state of the sequence, so these differentiating formulas come from a very restricted set and we show that within a constant number of steps repetition of the label of the successor node in the sequence must occur. The detailed analysis of the bounds on the sequence is given after the claims. Proof of Claim 3 uses the assumption that the input formula satisfies restriction $\mathbf{R}_2$. For all the remaining claims modal context restriction $\mathbf{R}_1$ is a sufficient assumption.

Claim 1 *Let t be an* $[\mathrm{B}]_j$*-Successor of state s in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *and let* $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$. *If s and t satisfy properties* ***PI*1**, ***PB*1** *and* ***PB*2**, *for all* $d > D$, *then the following hold for all* $d \geq D$ *and* $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *with* $\mathrm{dep}(\psi) \geq D$

(i) $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\right)$,
(ii) $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\right)$,
(iii) if $[\mathrm{B}]^+_G \xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ and $\psi \in \neg\mathrm{BT}(\xi)$, then $\psi \notin \mathrm{Ind}(L(t))$ and $\psi \notin \widehat{\mathrm{Gr}}(L(t))$.

Claim 2 *Let t be an* $[\mathrm{B}]_j$*-Successor of state s in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *and let* $D \geq \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$ *be such that s and t satisfy properties* ***PI*1**, ***PB*1** *and* ***PB*2**, *for all* $d > D$. *Then the following hold*:

(i) $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\right) \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\right)$ and $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\right) \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\right)$.
(ii) Either $\mathrm{dep}(\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j)) < D$ or $\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j) = \varnothing$.
(iii) There can be at most one formula of the form $\neg[\mathrm{B}]^+_G \psi \in \widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]^+_G \psi) \geq D$. Moreover, if there is such a formula, then $\neg[\mathrm{B}]^+_G \psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G \psi \in L(s)$ and t is a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G \psi$-Successor of s.

Claim 3 *Let t be an* $[\mathrm{B}]_j$*-Successor of s in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$. *If s and t satisfy properties* ***PB*1**, ***PI*1** *and* ***PB*2**, *for all* $d \geq 0$, *and* $\mathrm{Ind}(L(s)) \sqcap \varphi = \varnothing$, *then for all* $k \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ *and* $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ *it holds that*

(i) $\psi \in L(s)/[\mathrm{B}]_j$ implies $\psi \in L(t)/[\mathrm{B}]_k$ or $\neg[\mathrm{B}]_k \psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$, and
(ii) $\psi \in L(t)/[\mathrm{B}]_k$ implies $\psi \in L(s)/[\mathrm{B}]_j$ or $\neg[\mathrm{B}]_j \psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$.

Claim 4 *Let t be an* $[\mathrm{B}]_j$*-Successor of s in the pre-tableau constructed by Algorithm* 1 *for an input* $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$. *If s and t satisfy properties* ***PB*1**, ***PI*1** *and* ***PB*2**, *for all* $d \geq 0$, *and* $\mathrm{Ind}(L(s)) \sqcap j = \varnothing$, *then for all* $k \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ *and* $\psi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_2}$ *it holds that*

(i) $[O]_j \psi \in \neg L(s)$ implies $O = \mathrm{B}$ and $[O]_k \psi \in \neg L(t)$, and
(ii) $[O]_k \psi \in \neg L(t)$ implies $O = \mathrm{B}$ and $[O]_j \psi \in \neg L(s)$.

Consider a sequence of states $s_0, \ldots, s_m$ such that for any $0 < k \leq m$, s_k is an B_{j_k}-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) \sqcap j_k = \varnothing$ and the sequence satisfies properties **PB1**, **PI1** and **PB2** for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \varnothing$ then, by Claim 3, the length of such sequence must be ≤ 2. This is because for any

$j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(s1), [\mathrm{B}]^+\right)$ it holds that $L(s_1)/[\mathrm{B}]_j \subseteq L(s_0)/[\mathrm{B}]_{j_1} \subseteq L(s_1)$, $L(s_1) \sqcap j \subseteq L(s_1)$ and there are no formulas of the form $\neg[\mathrm{B}]_j\psi \in L(s_1)$. Hence $L^{[\mathrm{B}]_j}(s_1) \subseteq L(s_1)$ and no $[\mathrm{B}]_j$-Successor of s_1 can be created. On the other hand, if $j \notin \mathrm{ag}\left(\widehat{\mathrm{Gr}}(s1), [\mathrm{B}]^+\right)$, then for any $[\mathrm{B}]_j$-Successor s_2 of s_1, property **PB1** will not be satisfied for s_1 and s_2. If there is more than one formula of the form $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such sequence must be ≤ 2 as if it was larger then, by point (iii) of Claim 2, property **PB2** would have to be violated.

Lastly, if $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$, then the length of the sequence must be $\leq 2|G| + 1$. To see why, assume the opposite, that is $m > 2|G| + 1$. Notice that, by point (iii) of Claim 2, for all $0 < k \leq m$ it must be that $\widehat{\mathrm{Gr}}(L(s_k)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$ and s_k must be a $\neg[\mathrm{B}]_{j_k}[\mathrm{B}]^+_G\psi$-Successor of s_{k-1} with $j_k \in G$. By Claim 3, for any two states s_{i-1} and s_{l-1} in the sequence, with $i, l \leq m$, it holds that $L(s_{i-1})/[\mathrm{B}]_{j_i} \subseteq L(s_{l-1})/[\mathrm{B}]_{j_l} \cup \{\psi\}$ and $L(s_{l-1})/[\mathrm{B}]_{j_l} \cup \{\psi\} \subseteq L(s_{i-1})/[\mathrm{B}]_{j_i}$. Moreover, Claim 4 together with Claim 3 implies that for any two subsequent states s_{k-1} and s_k, with $k < m$, $L(s_{k-1}) \sqcap j_k \subseteq L(s_k) \sqcap j_{k+1} \cup \{[\mathrm{B}]_{j_{k+1}}\psi\}$ and $L(s_k) \sqcap j_{k+1} \subseteq L(s_{k-1}) \sqcap jk \cup \{[\mathrm{B}]_{j_k}\psi\}$. To see why, consider the first inclusion and take any $\xi \in L(s_{k-1}) \sqcap j_k$. By point (i) of Claim 4 it must be that ξ is either of the form $[\mathrm{B}]_{j_k}\zeta$ or $\neg[\mathrm{B}]_{j_k}\zeta$ and $[\mathrm{B}]_{j_{k+1}}\zeta \in \neg L(s_k)$. Suppose that the first case holds. Then it must be that $\zeta \in L(s_{k-1})/[\mathrm{B}]_{j_k}$ and, by the fact that $L(s_{k-1})/[\mathrm{B}]_{j_k} \subseteq L(s_k)/[\mathrm{B}]_{j_{k+1}} \cup \{\psi\}$, either $[\mathrm{B}]_{j_{k+1}}\zeta \in L(s_k) \sqcap j_{k+1}$ or $\zeta = \psi$. Suppose now that the second case holds. If $\neg[\mathrm{B}]_{j_{k+1}}\zeta \neq L(s_k)$, then $[\mathrm{B}]_{j_{k+1}}\zeta \in L(s_k)$ and, consequently, $\zeta \in L(s_k)/[\mathrm{B}]_{j_{k+1}}$. Since $L(s_k)/[\mathrm{B}]_{j_{k+1}} \subseteq L(s_{k-1})/[\mathrm{B}]_{j_k} \cup \{\psi\}$, so either $[\mathrm{B}]_{j_k}\zeta \in L(s_{k-1})$ or $\zeta = \psi$. The first case is impossible, as $\neg[\mathrm{B}]_{j_k}\zeta \in L(s_{k-1})$, and so it must be that $\zeta = \psi$ and $[\mathrm{B}]_{j_{k+1}}\psi \in L(s_k)$. Since for any two subsequent states s_{k-1} and s_k, with $k < m$, $L(s_{k-1}) \sqcap j_k \subseteq L(s_k) \sqcap j_{k+1} \cup \{[\mathrm{B}]_{j_{k+1}}\psi\}$ and $L(s_k) \sqcap j_{k+1} \subseteq L(s_{k-1}) \sqcap j_k \cup \{[\mathrm{B}]_{j_k}\psi\}$, so the analogous fact holds for any two states in the sequence (possibly excluding the last state).

If $m > 2|G| + 1$, then there must exist $0 < k_1 < k_2 < k_3 \leq m$ such that $j_{k_1} = j_{k_2} = j_{k_3}$. By what we have shown above the sets $L^{[\mathrm{B}]_{j_{k_1}}}(s_{k_1-1})$, $L^{[\mathrm{B}]_{j_{k_2}}}(s_{k_2-1})$ and $L^{[\mathrm{B}]_{j_{k_3}}}(s_{k_3-1})$ may differ by at most two formulas, ψ and $[\mathrm{B}]_j\psi$. Moreover, each of these sets either contains both these formulas or does not contain ψ and contains $\neg[\mathrm{B}]_j\psi$. Thus at least two of the sets must be equal. But then the $\neg[\mathrm{B}]_{j_i}[\mathrm{B}]^+_G\psi$-successor of one of the states s_{k_1-1}, s_{k_2-1} or s_{k_3-1} with $i = 1, 2$ or 3, respectively, cannot be created, which contradicts the assumption that all s_{k_1}, s_{k_2} and s_{k_3} are in the sequence. Hence the length of the sequence must be $\leq 2|G| + 1$.

Let $G \subseteq \mathcal{A}$, $D \geq 0$ and let T^G_D denote the maximal length of a sequence of B_G-Successors in the pre-tableau constructed by Algorithm 1 such that

1. properties **PB1** and **PI1** are satisfied for the sequence for all $d \geq D$,
2. property **PB2** is satisfied for the sequence for all $d > D$,
3. for each state s in the sequence $\mathrm{dep}(\mathrm{Ind}(L(s))) \leq D$ and
4. there is exactly one formula of the form $\neg[\mathrm{B}]^+_H\psi \in \widehat{\mathrm{Gr}}(L(s))$ with $\mathrm{dep}([\mathrm{B}]^+_H\psi) > D$.

Then $T_D^G \leq \overline{T}_D^{|G|}$, where

$$\overline{T}_m^n = \begin{cases} 2n+1, & \text{if } m=0 \\ 4+\sum_{i=1}^{i} \overline{T}_{m-1}^i, & \text{if } m>0 \end{cases} \tag{4}$$

To show that this inequality holds, we will use induction on D. The fact that $T_0^G \leq 2|G|+1$ follows from what we have shown above. The fact that $T_D^G \leq 4+\sum_{i=1}^{|G|} \overline{T}_{D-1}^i$, for $D>0$, follows from Claims 1 and 2. To see why, notice that by point (iii) of Claim 2, starting from the second state in the sequence under consideration, property **PB2** is satisfied for the remaining subsequence, for all $d \geq D$. Thus, by point (ii) of Claim 2, any subsequence of the sequence under consideration with $\text{dep}(\text{Ind}(L(s)))$ remaining unchanged for its every state s, can have length at most 3. Hence starting from the third state in the sequence $\text{dep}(\text{Ind}(L(s))) \leq D-1$.

To assess the length of the remaining part of the sequence, we divide it into parts marked by the first appearance of a new element $j \in G$ in the set $\text{ag}\left(\widehat{\text{Gr}}(L(\cdot)), [\text{B}]^+, D-1\right)$. By doing this we divide the sequence into $|G|$ parts, $P_1,\ldots,P_{|G|}$. A part P_i is a subsequence $s_j,\ldots,s_{k-1}$ such that some i'th element of G appeared for the first time in $\text{ag}\left(\widehat{\text{Gr}}(L(s_j)), [\text{B}]^+, D-1\right)$ and some $i+1$'th element of G appeared for the first time in $\text{ag}\left(\widehat{\text{Gr}}(L(s_k)), [\text{B}]^+, D-1\right)$. Notice that it may be that for some state s in the sequence more than one element of G appears for the first time in $\text{ag}\left(\widehat{\text{Gr}}(L(s)), [\text{B}]^+, D-1\right)$. In such cases we can assume that the length of some of the sequences P_i is 0.

Let $G^{(1)},\ldots,G^{(|G|)}$ be the sequence of subsets of G such that $G^{(i)}$ is the set of all j that appear in the sets $\text{ag}\left(\widehat{\text{Gr}}(L(s)), [\text{B}]^+, D-1\right)$ for s in the sequence P_i. By point (ii) of Claim 2 and point (i) of Claim 2, $\omega \in \text{ag}\left(\widehat{\text{Gr}}(L(s)), [\text{I}]^+, D-1\right)$ for every state s in the sequence. Hence the condition **PI1** is satisfied for all $d \geq D-1$ and every state of the sequence. Moreover, by point (i) of Claim 2, starting from the first occurrence of some $j \in G$ in $\text{ag}\left(\widehat{\text{Gr}}(L(s)), [\text{B}]^+, D-1\right)$ we have $j \in \text{ag}\left(\widehat{\text{Gr}}(L(t)), [\text{B}]^+, D-1\right)$ for all the remaining states t of the sequence. Hence the condition **PB1** is satisfied for all $d \geq D-1$ on every part P_i of the sequence. Thus the length of a each part P_i is $\leq T_{D-1}^{G^{(i)}}$. Hence, by the induction hypothesis, the length of the sequence consisting of the parts $P_1,\ldots,P_{|G|}$ with points 1– 4 being satisfied is $\leq \sum_{i=1}^{|G|} \overline{T}_{D-1}^i$.

To solve 4 we use the following fact (proved in the Appendix).

Fact 1 Let X_m^n be defined as follows, for $m \geq 0$ and $n \geq 1$:

$$X_m^n = \begin{cases} 2n+1, & \text{if } m=0 \\ B+\sum_{i=1}^{n} X_{m-1}^i, & \text{if } m>0. \end{cases} \tag{5}$$

Then, for $n, m \geq 1$,

$$X_m^n = B\binom{n+m-1}{m-1} + (n+2)\binom{n+m-1}{m}. \tag{6}$$

By Fact 1 from (4) we get (for $D > 0$)

$$T_D^G \leq 4\frac{(|G|+D-1)!}{|G|!(D-1)!} + (|G|+2)\frac{(|G|+D-1)!}{(|G|-1)!D!} = \mathcal{O}\left(D^{|G|}\right).$$

Let now S_D^G denote the maximal length of a sequence of B_G-Successors in the pre-tableau constructed by Algorithm 1 such that points 1– 3 are satisfied for it. Then $S_D^G \leq \overline{S}_D^{|G|}$, where

$$\overline{S}_m^n = \begin{cases} 2n+1, & \text{if } m = 0 \\ 4 + \overline{T}_{m-1}^n + \sum_{i=1}^n \overline{S}_{m-1}^i, & \text{if } m > 0. \end{cases} \tag{7}$$

Explanation for this equation is similar to that of Eq. (4). The only new thing is $\overline{T}_{D-1}^{|G|}$ in the case of $D > 0$. It comes from the fact the after two states in the sequence, by point (iii) of Claim 2 there can be at most one formula of the form $\neg[\mathrm{B}]_H^+\xi \in \widehat{\mathrm{Gr}}(L(s))_{D-1}$. If there is no such a formula in $\widehat{\mathrm{Gr}}(L(s))_{D-1}$, then, by point (iii) of Claim 2 property **PB2** will be satisfied for the remaining part of the sequence for all $d \geq D$, and there can be at most $\sum_{i=1}^{|G|} \overline{S}_{D-1}^i$ states in this remaining part. However, if there is exactly one such formula in $\widehat{\mathrm{Gr}}(L(s))_{D-1}$, then the maximal length of the subsequence in which it remains is bounded by T_{D-1}^G. After that, there can be no formula of the form $\neg[\mathrm{B}]_H^+\xi \in \widehat{\mathrm{Gr}}(L(s))_{D-1}$ and there can be at most $\sum_{i=1}^{|G|} \overline{S}_{D-1}^i$ states in the remaining part of the sequence.

By Fact 1 from (7) we get (for $D > 0$)

$$S_D^G \leq \left(4 + T_{D-1}^G\right)\frac{(|G|+D-1)!}{|G|!(D-1)!} + (|G|+2)\frac{(|G|+D-1)!}{(|G|-1)!D!} = \mathcal{O}\left(D^{2|G|}\right).$$

Thus the maximal length of a sequence of B-Successors with the same modal depth of labels is $\leq S_{\mathrm{dep}(\varphi)+1}^{\mathcal{A}} = \mathcal{O}\left(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\right)$. □

Proof (Lemma 6) The structure of the proof of Lemma 6 is similar to that of proof of Lemma 5. Claims analogous to Claims 1–3 are used and an analogue of Claim 4 is not required. Sets $\mathrm{Ind}(L(\cdot)) \sqcap j$ are replaced by sets $\mathrm{Ind}(L(\cdot))$. The details of the proof can be found in Dziubiński (2011). □

Proof (Claim 1) Take any $d \geq D$. Notice that if $d > \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$, then points (i) and (ii) hold for it. Also if $d \geq \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$, then, since $\mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right) = \mathrm{dep}(Gr(L(s)))$ and $\mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(t))\right) = \mathrm{dep}(Gr(L(t)))$, so point (iii) holds for it as well.

For $d \leq \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$ we will use induction, starting with maximal value of d. So suppose that $d = \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. As we observed above, point (iii) holds for d and we need to show points (i) and (ii) only. For point (i), assume that $j \notin \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\right)$. Then there must be a formula $[\mathrm{B}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]_G^+\psi) = d$ and $j \notin G$. Hence either there is a formula $[\mathrm{B}]_j\xi \in \neg L(s)$, such that $[\mathrm{B}]_G^+\psi \in \neg\mathrm{P}(\xi)$ or there is a formula $[\mathrm{B}]_j\xi \in \neg L(t)$ such that $[\mathrm{B}]_G^+\psi \in \neg\mathrm{P}(\xi)$ and ξ was added during [B]-expanded tableau formation. The first case is impossible, as $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$ and $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$. The second case is impossible as well, for either $[\mathrm{B}]_j\xi \in \neg L(s)$ which, as we shown above, is not possible, or there is a formula $[\mathrm{B}]_j\zeta \in \neg L(s)$ such that $[\mathrm{B}]_j\xi \in \neg\mathrm{BT}(\zeta)$. This case is impossible by analogous arguments to those used for the previous case. Thus it must be that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\right)$. For point (ii), assume that $\omega \notin \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\right)$. Then there must be a formula $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}\left([\mathrm{I}]_G^+\psi\right) = d$. Hence there must be a formula $[\mathrm{B}]_j\xi \in \neg L(s)$ such that $[\mathrm{I}]_G^+\psi \in \neg\mathrm{P}(\xi)$. Again this is impossible, as $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$ and $\mathrm{dep}([\mathrm{B}]_j\xi) > \mathrm{dep}(\mathrm{Ind}(L(s)) \sqcap j)$. Thus it must be that $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\right)$.

For the induction step, suppose that $d < \mathrm{dep}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. For point (iii) notice that if $[\mathrm{B}]_G^+\xi \in \neg L(s)$ and $\zeta \in \neg\mathrm{BT}(\xi)$ and $\mathrm{dep}(\psi) \geq D$, then $\mathrm{dep}\left([\mathrm{B}]_G^+\xi\right) \geq d + 1$. Moreover since, by point (i), $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d + 1\right)$ so, by property **PB1**, it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d + 1\right)$ and so $j \in G$. Thus if $[\mathrm{B}]_G^+\xi \in \widehat{\mathrm{Gr}}(L(s))$, then $[\mathrm{B}]_G^+\xi \in L(t)$ and if $\neg[\mathrm{B}]_G^+\xi \in \widehat{\mathrm{Gr}}(L(s))$, then $[\mathrm{B}]_G^+\xi \in \neg L(t)$, by condition **PB2**. Since $j \in G$ and $\mathrm{dep}\left([\mathrm{B}]_G^+\xi\right) > d$ so, by point (i) of Lemma 3, $[\mathrm{B}]_G^+\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$. Hence it must be that $\psi \notin \mathrm{Ind}(L(t))$ and $\psi \notin \widehat{\mathrm{Gr}}(L(t))$.

For point (i) assume that $j \notin \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\right)$. Then there must be a formula $[\mathrm{B}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$, with $\mathrm{dep}([\mathrm{B}]_G^+\psi) = d$ and $j \notin G$. By the induction hypothesis it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d' + 1\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d' + 1\right)$, for all $d' \geq d$. Moreover, by properties **PI1** and **PB1** it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d' + 1\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d' + 1\right)$, for all $d' \geq d$. Thus, by Corollary 1, there exists a formula $[\mathrm{B}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]_G^+\psi \in \neg\mathrm{BT}(\xi)$ (notice that since $j \notin G$, so neither point (iv) nor point (v) of Corollary 1 can apply here). Then, by point (iii) it holds that $[\mathrm{B}]_G^+\psi \notin \neg\widehat{\mathrm{Gr}}(L(t))$, which contradicts our assumptions. Hence it must be that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, d\right)$.

For point (ii) assume that $\omega \notin \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\right)$. Then there must be a formula $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{I}]_G^+\psi) = d$. By arguments similar to those used above, it can be shown that there must be a formula $[\mathrm{B}]_H^+\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$ such that $[\mathrm{I}]_G^+\psi \in \neg\mathrm{BT}(\xi)$, which contradicts the assumption that $[\mathrm{I}]_G^+\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$. Hence it must be that $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, d\right)$. $\square$

Proof (Claim 2)

Point (i)

For the fact that $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\right) \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\right)$ assume that the opposite holds. Then there must exist a formula $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$ such that $\mathrm{dep}([\mathrm{B}]^+_G\psi) = D$ and $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\right) \not\subseteq G$. Notice that by point (i) of Claim 1 it holds that $j \in G$.

By points (i) and (ii) of Claim 1 and properties **PB1** and **PI1** it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\right)$, for all $d > D$. Thus, by Corollary 1, either there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{BT}(\xi)$, or $[\mathrm{B}]^+_G\psi \in \neg L(s)$.

The first case is impossible, as it implies that $j \in H$ and so it violates modal context restriction $\mathbf{R}_1$. Thus it must be that the second case holds and, by the fact that $j \in G$ and by point (i) of Lemma 3 it must be that $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. But then it must hold that $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\right) \subseteq G$, which contradicts our assumptions. Hence this case is impossible as well and it must be that $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, D\right) \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+, D\right)$.

For the fact that $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\right) \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\right)$ notice that, by point (ii) of Claim 1, it holds that $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\right)$ and so $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\right) = \mathcal{A} \cup \{\omega\}$. Hence it holds that $\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, D\right) \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+, D\right)$.

Point (ii)

Assume that the opposite holds. Then there exists a formula $\psi \in \mathrm{Ind}(L(t))n(L(t) \sqcap j)$ with $\mathrm{dep}(\psi) \geq D$. By points (i) and (ii) of Claim 1 and properties **PB1** and **PI1** it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+, d\right)$, for all $d > D$. Thus, by Corollary 1, there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ with $\psi \in \neg\mathrm{BT}(\xi)$. This is impossible as, by point (iii) of Claim 1, it implies that $\psi \notin \mathrm{Ind}(L(t))$ which contradicts our assumptions. Hence it must be that either $\mathrm{dep}(\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j)) < D$ or $\mathrm{Ind}(L(t)) \setminus (L(t) \sqcap j) = \varnothing$.

Point (iii)

Take any formula $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$ with $\mathrm{dep}([\mathrm{B}]^+_G\psi) \geq D$. By point (i) of Claim 1 it must be that $j \in G$. By points (i) and (ii) of Claim 1 and properties **PB1** and **PI1**, it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+, d\right)$, for all $d > D$. Thus, by Corollary 1, either there exists a formula $[\mathrm{B}]^+_H\xi \in \neg\widehat{\mathrm{Gr}}(L(s))$ such that $[\mathrm{B}]^+_G\psi \in \neg\mathrm{PT}(\xi)$, or $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ and t is a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-Successor of t. The first case is impossible as $j \in H$ and it would violate modal context restriction $\mathbf{R}_1$. Thus it must be that the second case holds. This implies, in particular, that there can be at most one formula of the form $\neg[\mathrm{B}]^+_G\psi$ in $\widehat{\mathrm{Gr}}(L(t)) \sqcap \neg[\mathrm{B}]^+$ with $\mathrm{dep}\left(\neg[\mathrm{B}]^+_G\psi\right) \geq D$. $\square$

Proof (Claim 3) Let $k2\mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$. If $j = k$, then, by Lemma 4, it holds that $L^{[\mathrm{B}]_j}(s) = L^{[\mathrm{B}]_k}(t)$ and so the claim holds in this case. Suppose that $j \neq k$.

For point (i), let $\psi \in L(s)/[\mathrm{B}]_j$. Then there exists a formula $[\mathrm{B}]_j\psi \in L(s)$ and, by point (ii) of Lemma 3, $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. Thus either there exists a formula $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$ or ψ is of the form $[\mathrm{B}]^+_G\xi$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$.

Suppose that the first case holds. If $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s))$, then $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, by property **PB2**. Otherwise $[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$, as s is a state and $L(s)$ is a closed propositional tableau. Thus $[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(t))$, as $j \in G$, $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ and point (i) of Lemma 3 applies. Since $[\mathrm{B}]^+_G\psi \in \neg\widehat{\mathrm{Gr}}(L(t))$, $k \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ and $L(t)$ is a [B]-expanded closed propositional tableau, so either $\psi \in L(t)/[\mathrm{B}]_k$ or $\neg[\mathrm{B}]_k\psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$.

Suppose that the second case holds, that is ψ is of the form $[\mathrm{B}]^+_G\xi$ and $\psi \in \neg\widehat{\mathrm{Gr}}(L(s))$. Then, by arguments analogous to those used for the first case, it holds that $[\mathrm{B}]^+_G\xi \in \neg\widehat{\mathrm{Gr}}(L(t))$ and the point holds by the fact that t is a state and $L(t)$ is a [B]-expanded tableau.

For point (ii) we will show first that $[\mathrm{B}]_k\psi \in \neg L(t)$ implies $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. Notice that, by point (ii) of Claim 2, it must be that $\mathrm{Ind}(L(t)) \setminus (L(t)uj) = \varnothing$ and since $j \neq k$, so $\mathrm{Ind}(L(t)) \sqcap k = \varnothing$. Now, suppose that $[\mathrm{B}]_k\psi \in \neg L(t)$. Since $k \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ so, by point (ii) of Lemma 3, it holds that $[\mathrm{B}]_k\psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$. By points (i) and (ii) of Claim 1 it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\right)$. Hence, since t is a state and $L(t)$ is a closed, fully expanded and $[O]^+$-expanded tableau, so it holds that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$. Then, by Lemma 4, it holds that $[\mathrm{B}]_j\psi \in \neg L(s)$. Moreover, by the fact that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t), [\mathrm{B}]^+)\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\right)$ and by properties **PB1** and **PI1** it holds that $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\right)$. Thus, by Lemma 3, it holds that $[\mathrm{B}]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. Now, let $\psi \in L(t)/[\mathrm{B}]_k$. Then there is a formula $[\mathrm{B}]_k\psi \in L(t)$ and, by what we shown above, either $[\mathrm{B}]_j\psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$ and, consequently, $\psi \in L^{[\mathrm{B}]_j}(s)$ or $\neg[\mathrm{B}]_j\psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. □

Proof (Claim 4) Before we start showing points (i) and (ii), notice that, by Claim 1, we have $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\right)$. Moreover, by properties **PB1** and **PI1**, we have also $\{j,k\} \subseteq \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\right)$.

For point (i), let $[O]_j\psi \in L(s)$. Since $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{B}]^+\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(s)), [\mathrm{I}]^+\right)$, so, by point (iii) of Lemma 3, $[O]_j\psi \in \neg L(s)$ implies $O = \mathrm{B}$ and $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(s))\right)$. Using arguments similar to those used in proof of Claim 3 it follows that $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$. Now, since $k \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ so $[O]_k\psi \in \widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$ and so $[O]_k\psi \in \neg L(t)$.

For point (ii), let $[O]_k\psi \in L(t)$. As we observed in proof of Claim 3, by point (ii) of Claim 2, it must be that $\mathrm{Ind}(L(t)) \sqcap k = \emptyset$. Since $k \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$ and $\omega \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{I}]^+\right)$, so, by point (iii) of Lemma 3, $[O]_k\psi \in \neg L(t)$ implies $O = \mathrm{B}$ and $[O]_k\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$. Since $j \in \mathrm{ag}\left(\widehat{\mathrm{Gr}}(L(t)), [\mathrm{B}]^+\right)$, so $[O]_j\psi \in \neg\widetilde{\mathrm{Cl}}\left(\widehat{\mathrm{Gr}}(L(t))\right)$. Thus $[O]_j\psi \in \neg L(t)$ and, by Lemma 4, $[O]_j\psi \in \neg L(s)$. □

Proof (Fact 1) To proof the fact we will use induction over m. Suppose that $m = 1$. Since

$$X_1^n = B + \sum_{i=1}^{n} X_0^i = B + \sum_{i=1}^{n}(2i+1) = B + (n+2)n = B\binom{n}{0} + (n+2)\binom{n}{1},$$

so the claim holds for any $n \geq 1$. For the induction take $m > 1$ and suppose that the claim holds for $m-1$ and any $n \geq 1$. Then

$$\begin{aligned} X_m^n &= B + \sum_{i=1}^{n} X_{m-1}^i = B + \sum_{i=1}^{n}\left[B\binom{i+m-2}{m-2} + (n+2)\binom{i+m-2}{m-1}\right] \\ &= B\sum_{i=0}^{n}\binom{i+m-2}{m-2} + (n+2)\sum_{i=1}^{n}\binom{i+m-2}{m-1} \\ &= B\sum_{j=m-2}^{n+m-2}\binom{j}{m-2} + (n+2)\sum_{j=m-1}^{n+m-2}\binom{j}{m-1} \end{aligned}$$

From the properties of binomial coefficients we know that

$$\sum_{j=k}^{n}\binom{j}{k} = \binom{n+1}{k+1},$$

thus

$$X_m^n = B\binom{n+m-1}{m-1} + (n+2)\binom{n+m-1}{m},$$

for all $n \geq 1$. □

Proofs associated with validity of Algorithm 1

Before proving the validity we need to introduce the following useful notions. Given a formula $\neg[O]_G^+\psi$ and a model $\mathcal{M}$ with a world v in it such that $(\mathcal{M}, v) \vDash \neg[O]_G^+\psi$, we call any sequence of worlds $v_0, \ldots, v_k$ such that $v_0 = v$, for all $0 < l \leq k$ it holds that $vl \in O_{j_l}(v_{l-1})$ and $j_l \in G$, for all $0 < l < k$ it holds that $(\mathcal{M}, v) \vDash \psi$ and $(\mathcal{M}, v_k) \vDash \neg\psi$, a *satisfying sequence* for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$. A satisfying sequence for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$ that has the minimal length is called a *minimal satisfying sequence for* $\neg[O]_G^+\psi$ *in* $(\mathcal{M}, v)$. Given a formula φ we call any pair $(\mathcal{M}, v)$ such that $(\mathcal{M}, v) \vDash \varphi$ a *satisfying pair for* φ. Given a set of formulas Φ with $\neg[O]_G^+\psi \in \Phi$ we say that $(\mathcal{M}, v)$ is a *satisfying pair for* $\bigwedge\Phi$ *with minimal satisfying sequence for* $\neg[O]_G^+\psi$, if $(\mathcal{M}, v) \vDash \bigwedge\Phi$ and a minimal satisfying sequence for $\neg[O]_G^+\psi$ in $(\mathcal{M}, v)$ is minimal over all satisfying pairs for $\bigwedge\Phi$.

Firstly, we will use the following lemma. The proof is straightforward and therefore omitted (for details see Dziubiński 2011).

Lemma 12 *Let n be an internal node in the pre-tableau constructed by Algorithm* 1 *for some input formula $\varphi \in \mathcal{L}^{\mathrm{T}}$. For any Kripke model $\mathcal{M}$ and a world v in it such that $(\mathcal{M}, v) \vDash \bigwedge L(n)$, there exists a successor m of n such that $(\mathcal{M}, v) \vDash \bigwedge L(m)$.*

We will also use the following lemma. We omit the proof here as it is mostly technical and lengthy. The detailed proof can be found in Dziubiński (2011).

Lemma 13 *Let $\Phi \subseteq \mathcal{L}^{\mathrm{T}}$ be a* $[\mathrm{B}]$*-expanded tableau. Then the following hold*:

1. if $\neg[\mathrm{I}]_G^+\varphi \in \Phi$ and $(\mathcal{M}, w)$ is a satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence $v_0, \ldots, v_n$ for $[\mathrm{I}]_G^+\varphi$ such that $n \geq 2$, then $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{I}]_{j_1}}\left([\mathrm{I}]_G^+\varphi\right)$ with minimal satisfying sequence $v_1, \ldots, v_n$ for $\neg[\mathrm{I}]_G^+\varphi$.
2. if $\neg[\mathrm{B}]_G^+\varphi \in \Phi$ and $(\mathcal{M}, w)$ is a satisfying pair for $\bigwedge \Phi$ with minimal satisfying sequence $v_0, \ldots, v_n$ for $[\mathrm{B}]_G^+\varphi$ such that $n \geq 2$, then $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge \Phi^{\neg[\mathrm{B}]_{j_1}}\left([\mathrm{B}]_G^+\varphi\right)$ with minimal satisfying sequence $v_1, \ldots, v_n$ for $\neg[\mathrm{B}]_G^+\varphi$.

One of the main differences in the proof of validity of the algorithm, as compared to analogous proofs for modal logics without iterated modalities, comes from the possibility of existence of nodes marked undec in a pre-tableau constructed by the algorithm. The following lemma is crucial for dealing with nodes that are marked unsat because of an unresolved formula in their label and existence of successors marked `undec`.

Lemma 14 *Let n be a node in the pre-tableau constructed by Algorithm* 1 *for some input formula $\varphi \in \mathcal{L}^{\mathrm{T}}$, with a formula $\neg[O]_G^+\psi \in L(n)$ (where $O \in \{\mathrm{B}, \mathrm{I}\}$) unresolved in n. Suppose also that for any descendant r of n it holds that if r is marked* `unsat`, *then $\bigwedge L(r)$ is not satisfiable. If $\bigwedge L(n)$ is satisfiable, then $B(n) \neq \{n\}$.*

Proof We will show first that if n is a node of the pre-tableau, $\bigwedge L(n)$ is satisfiable, $\neg[O]_G^+\psi$ with $O \in \{\mathrm{B}, \mathrm{I}\}$ is unresolved in n and for each successor r of n, r being marked `unsat` implies that $\bigwedge L(r)$ is not satisfiable, then for any satisfying pair $(\mathcal{M}, v)$ for $\bigwedge L(n)$ with minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]_G^+\psi$, there exists $m \in B(n) \setminus \{n\}$ and $0 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. To show that we will use induction over the maximal distance from n to a descendant leaf of the pre-tableau. Suppose that n is a leaf of the pre-tableau. Since $\bigwedge L(n)$ is satisfiable so $L(n)$ cannot be trivially inconsistent and n must be a state. Moreover n must be a $\neg[O]_j[O]_G^+\psi$-Successor, for some $j \in G$, and must be marked `undec`, as $\neg[O]_G^+\psi$ is unresolved in n. Take any satisfying pair $(\mathcal{M}, v)$ for $\bigwedge L(n)$ with minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]_G^+\psi$. Since n is a state and $\neg[O]_G^+\psi$ is unresolved in it so, for all $j \in G$, $[O]_j\psi \in L(n)$. Thus it must be that $(\mathcal{M}, v_1) \vDash \psi$ and $(\mathcal{M}, v_1) \vDash \neg[O]_G^+\psi$. This, together with the fact that $L(n)$ is a $[O]^+$-expanded tableau, implies $\neg[O]_{j_1}[O]_G^+\psi \in L(n)$. Since n is a leaf of the pre-tableau so creation of a successor for $\neg[O]_{j_1}[O]_G^+\psi$ must be blocked by some node m and since n is a $\neg[O]_j[O]_G^+\psi$-successor, for some $j \in G$, so $m \in B(n)$. By construction of the algorithm it must hold that

$L^{\neg[O]_{j_1}}(n, [O]^+_G\psi) = L(m)$ which, by the fact that $(\mathcal{M}, v_1) \vDash \neg[O]^+_G\psi$, implies $(\mathcal{M}, v_1) \vDash \bigwedge L(m)$. Notice that since $[O]_{j_1}\psi \in L(n)$ so $(\mathcal{M}, v_1) \vDash \psi$ and so it must be that $k > 1$. To see that $m \neq n$, assume the opposite. Then $(\mathcal{M}, v_1)$ is a satisfying pair for $\bigwedge L(n)$ with $v_1, \ldots, v_k$ being a satisfying sequence for $\neg[O]^+_G\psi$. Thus we get a contradiction with the assumption of minimality of $v_0, \ldots, v_k$ and so it must be that $m \neq n$.

For the induction step, suppose that n is not a leaf of the pre-tableau. Take any satisfying pair $(\mathcal{M}, v)$ for $\bigwedge L(n)$ with minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\psi$. Suppose first that n is an internal node. By Lemma 12 there must exist a successor r of n such that $(\mathcal{M}, v) \vDash \bigwedge L(r)$. Since n is marked `undec` and $L(r)$ is satisfiable, so r must be marked `undec` as well and $\neg[O]^+_G\psi$ must be unresolved in r. Moreover, by construction of the algorithm, it must be that $B(r) \subseteq B(n)$. Thus, by the induction hypothesis, there is $m \in B(n)$ and $0 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. Moreover, it must be that $m \neq n$ as otherwise, by similar arguments to those used for the induction basis, we get a contradiction with the assumption of minimality of $v_0, \ldots, v_k$.

Suppose now that n is a state. As we argued for the induction basis, n is marked `undec`, for all $j \in G$ it holds that $[O]_j\psi \in L(n)$ and $\neg[O]_{j_1}[O]^+_G\psi \in L(n)$. By the fact that $\neg[O]^+_G\psi$ is unresolved in n, either a $\neg[O]_{j_1}[O]^+_G\psi$-successor r of n is marked `undec` or its creation is blocked by some ancestor m. In the latter case the claim holds by analogous arguments to those used for the induction basis. Suppose that the first case holds. Then $\neg[O]^+_G\psi$ is unresolved in r. Moreover, since $(\mathcal{M}, v)$ is a satisfying pair for $\bigwedge L(n)$ with a minimal satisfying sequence $v_0, \ldots, v_k$ for $\neg[O]^+_G\psi$ and $[O]_j\psi \in L(n)$, for all $j \in G$, so $k \geq 2$ and, by Lemma 13, $(\mathcal{M}, v_1)$ must be a satisfying pair for $\bigwedge L(r)$ with minimal satisfying sequence $v_1, \ldots, v_k$ for $\neg[O]^+_G\psi$. Hence, by the induction hypothesis, there must exist $m \in B(r) \setminus \{r\}$ and $1 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. By construction of the algorithm $m \in B(n)$ and, by minimality of $v_0, \ldots, v_k$, it must be that $m \neq n$.

Now, suppose that n is a node of the pre-tableau with $\neg[O]^+_G\psi \in L(n)$ and satisfying all the assumptions stated in the lemma. Suppose also that $\bigwedge L(n)$ is satisfiable. Then there exists a satisfying pair for $L(n)$ with minimal satisfying sequence for $\neg[O]^+_G\psi$ and, by what was shown above, there must exist $m \in B(n) \setminus \{n\}$ and $0 < l < k$ such that $(\mathcal{M}, v_l) \vDash \bigwedge L(m)$. Hence it must be $B(n) \neq \{n\}$. □

Now we are ready to prove Proposition 4 stating validity of the algorithm.

Proof (Proposition 4) For the left to right implication suppose that Algorithm 1 terminates on the input $\varphi \in \mathcal{L}^{\mathrm{T}}$. Then it must have constructed a finite pre-tableau for that input. We start by showing, for any node n of the pre-tableau constructed by the algorithm, that if n is marked `unsat`, then $\bigwedge L(n)$ is unsatisfiable. The proof is by induction on the maximal length of paths from a node to one of its descendant leaves. If n is a leaf and it is marked `unsat`, then it is easy to see that $\bigwedge L(n)$ must be unsatisfiable. For the induction step, suppose that n is a node that is marked `unsat` and that has at least one successor. Let n be an internal node. If all successors of n are marked `unsat`, then, by the induction hypothesis, for any successor m of n, $\bigwedge L(m)$ must be unsatisfiable. Suppose that $\bigwedge L(n)$ is satisfiable. By Lemma 12 there exists a successor m of n such that $\bigwedge L(m)$ is satisfiable and we get a contradiction. Thus $\bigwedge L(n)$ must be unsatisfiable in this case. Suppose that there exists a successor of n which is not marked `unsat`. Then it must be that all successors of n are marked either `unsat` or `undec`, there exists $\psi \in L(n)$ such that ψ is unresolved in n and

$B(n) = \{n\}$. Hence, by Lemma 14, $\bigwedge L(n)$ must be unsatisfiable. Let n be a state. If there exists a successor of n which is marked `unsat`, than showing that $\bigwedge L(n)$ is straightforward. Suppose then that none of the successors of n is marked `unsat`. Then there must exist a formula $\neg[O]^+_G\psi \in L(n)$ which is unresolved in n and it must hold that $B(n) = \{n\}$. Hence, by Lemma 14, $\bigwedge L(n)$ must be unsatisfiable.

Observe that root of any pre-tableau constructed by the algorithm is marked either `unsat` or `sat`, as it cannot block creation of a successor for any formula of the form $\neg[O]_j[O]^+_G\psi$ with $j \in G$, and so there cannot be any formula which is unresolved in `root`. Thus if `root` in the pre-tableau is not marked `sat`, then it is marked `unsat` and φ must be unsatisfiable. Hence if φ is satisfiable, then `root` node must be marked `sat` and the algorithm returns `sat`.

For the right to left implication, assume that Algorithm 1 returned `sat` on the input φ. Then it constructed a finite pre-tableau $(N, \mathtt{root}, succ, \{O_j\text{-succ} : j \in \mathcal{A}, O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$ for φ (N is the set of nodes, `root` is the root, *succ* and O_j-succ are the successor relations and L is the labelling function of the pre-tableau). We will show how to construct, on the basis of this pre-tableau, a TeamLog tableau for φ such that the number of states of the tableau is $\leq ((2|\mathcal{A}|+1)|\varphi|)^s$, where s is the state height of the pretableau. Consider a model graph $\mathcal{T} = \left(W, \{O_j : [O]_j \in \Omega^{\mathrm{ind}}\}, L|_W\right)$, where W is constructed as follows. In further part of the proof we will refer to the following set, defined for a given node n:

$$S(n) = \begin{cases} \{n\}, & \text{if } n \text{ is a state} \\ \bigcup_{m \in succ(n)} S(m), & \text{otherwise,} \end{cases}$$

which is the set of states in the subtree of the pre-tableau with the root n that are closest to n. Notice that n is marked `sat` if and only if there exists $s \in S(n)$ such that s is marked `sat`. Moreover, for any $s \in S(n)$ it holds that $L(n) \subseteq L(s)$, as labels of successors created during the steps of propositional tableau formation and [B]-expanded tableau formation extend the labels of their predecessors.

We start with W consisting of a state marked `sat` from $S(\mathtt{root})$. Then, for each state $w \in W$ whose O-Successors were not added to the set yet, we take, for each O-successor node n of w, a state $v \in S(n)$ which is marked `sat` (if there is such) or which is marked `undec` (otherwise). We proceed like that until leaves of the pre-tableau are reached. Since each state of the pre-tableau has at most $(2|A|+1)|\varphi|$ O-successors (as the number of elements in its label is bounded by $(2|A|+1)|\varphi|$), so W has $\leq ((2|A|+1)|\varphi|)^s$ elements. Labelling function L is like in the pre-tableau but restricted to W. Before defining the accessibility relations, we need to define the set of states associated with nodes blocking creation of $\neg[O]_j[O]^+_G\xi$-successors of a given state. Let v be a state with a formula $\neg[O]_j[O]^+_G\xi \in L(v)$ and suppose that n is a node that blocks creation of a $\neg[O]_j[O]^+_G\xi$-successor of v. That is $n \in B(v)$ and there exists a $\neg[O]^+_G\xi$-Ancestor t of v with n being its $\neg[O]_j[O]^+_G\xi$-successor and such that $L^{[O]_j}(v) = L^{[O]_j}(t)$. In such a case we will call any state $u2S(n)$ a *O_j-loop-back state for v*. Given a state $v \in W$, $j \in \mathcal{A}$ and $O \in \{\mathrm{B}, \mathrm{I}\}$ let

- $\mathit{LB}^O_j(v) = \{u \in W : u \text{ is a } O_j\text{-loop-back state for } v\}$.

When constructing the accessibility relations, we will need to properly extend them with loop back connections. The accessibility relations of $\mathcal{T}$ are defined as follows:

- $\mathrm{B}_j = \left(\mathrm{B}_j\text{-Succ} \cap W \times W\right) \cup \{(v,u) \in W \times W:$ there exists $w \in W$ such that $\{v,u\} \subseteq \mathrm{B}_j\text{-Succ}(w)\} \cup \{(v,u) \in W \times W : u \in \mathit{LB}_j^{\mathrm{B}}(v)\}$,
- $\mathrm{G}_j = \left(\mathrm{G}_j\text{-Succ} \cap W \times W\right) \cup \{(v,u) \in W \times W :$ there exists $w \in W$ such that $v \in \mathrm{B}_j\text{-Succ}(w)$ and $u \in \mathrm{G}_j\text{-Succ}(w)\}$,
- $\mathrm{I}_j = \left(\mathrm{I}_j\text{-Succ} \cap W \times W\right) \cup \{(v,u) \in W \times W:$ there exists $w \in W$ such that $v \in \mathrm{B}_j\text{-Succ}(w)$ and $u \in \mathrm{I}_j\text{-Succ}(w) \cup \mathit{LB}_j^{\mathrm{I}}(w)\} \cup \{(v,u) \in W \times W : u \in \mathit{LB}_j^{\mathrm{I}}(v)\}$,

Since there exists $w \in W$ such that $w \in S(\mathtt{root})$ and $j \in L(w)$ so it is enough to show that $\mathcal{T}$ is a TeamLog tableau. Before we show that, notice that the construction above guarantees that for any state $u \in W \setminus \{w\}$ and any $j \in \mathcal{A}$ it holds that $w \notin \mathrm{B}_j(u)$ (this is because w is not a B-Successor of any state). If we show that $\mathcal{T}$ is a TeamLog tableau, then, by Proposition 2, it will follow that φ is satisfiable. Since all elements of W are states, so they must be $[\mathrm{B}]$-expanded tableaux. In showing that the properties of TeamLog tableau, we will focus on those related to fixpoint modalities (the proof for other properties is similar to the case without fixpoint modalities given in Dziubiński et al. 2007; Dziubiński 2011). We will mostly concentrate on condition **TC** which is related to iterated modalities.

Conditions **T1**, **T2**, **T4**, **TG4**, **TI4**, **T5**, **TG5**, **TI5** and **TIG** can be shown by arguments similar to those used in Dziubiński et al. (2007) for TeamLog without fixpoint modalities, as for all $u \in \mathit{LB}_j^O(v)$ it holds that $L^{[O]_j}(v) \subseteq L(u)$ and $L^{\neg[O]_j}\left(v, [O]_G^+\xi\right) \subseteq L(u)$, where $\neg[O]_G^+\xi$ is the formula associated with u in $\mathit{LB}_j^O(v)$. Condition **TD** for $O \in \{\mathrm{B},\mathrm{I}\}$ can also be shown by arguments similar to those for the case of TeamLog without fixpoint modalities, with additional argument that if a $[O]_j[O]_G^+\xi$-successor of some state $v \in W$ is not created, then it holds that $L^{[O]_j}(v) \subseteq L(v)$, so that $[O]_G^+\xi \in L(v)$.

For condition **TC**, suppose that $v \in W$ and suppose that $\neg[O]_G^+\psi \in L(v)$ with $O \in \{\mathrm{B},\mathrm{I}\}$. We will show first that the condition holds for the states that are marked `sat` and are $\neg[O]_j[O]_G^+\psi$-Successors, with some $j \in G$. To show this we will use induction on the maximum distance from the state to its descendant leafs. Suppose that v is a leaf of the pre-tableau. Since v is a state so $L(v)$ is a $[O]^+$-expanded tableau. By condition **CE**, for all $j \in G$ it holds that $[O]_j\psi \in \neg L(v)$. If there was no $j \in G$ such that $\neg[O]_j\psi \in L(v)$ then $\neg[O]_G^+\psi$ would be unresolved in v and v would be marked `undec`. Hence there must exist such $j \in G$, in which case condition **TC** follows from condition **T2**. For the induction step notice that if there is $j \in G$ such that $\neg[O]_j\psi \in L(v)$ then the condition holds, by condition **T2**. Otherwise, by condition **CE** and by the fact that v is marked `sat`, there must exist $j \in G$ such that $\neg[O]_j[O]_G^+\psi \in L(v)$ and $\neg[O]_j[O]_G^+\psi$-successor of v is created and marked `sat`. Hence condition **TC** holds by the induction hypothesis.

Secondly we will show that condition **TC** is satisfied for states that are marked `undec` and are $\neg[O]_j[O]_G^+\psi$-Successors with some $j \in G$. To show this we will use induction on $M(v) = min_{n \in B(v)}\mathrm{sheight}(n)$ (where $\mathrm{sheight}(n)$ is the state height of n), starting from the minimal value. Let $V \subseteq W$ be the set of states such that for each $v \in V$, $M(v)$ is minimal. To show that the condition is satisfied for all $v \in V$ we will use induction on the maximum distance from the state to its descendant leafs. Suppose that v is a leaf. By the fact that v is marked `undec` and by condition **CE**, for all $j \in G$ it holds that $[O]_j\xi \in L(v)$ and there exists $j \in G$ such that $\neg[O]_j[O]_G^+\psi \in L(v)$. Let $m(v) = \mathrm{argmin}_{n \in B(v)}\mathrm{sheight}(n)$ and let $j \in G$ be such that creation of $\neg[O]_j[O]_G^+\psi$-successor of v is blocked by $m(v)$. Then there exists $u \in S(m(v))$ such that $u \in O_j(v)$. By construction of the algorithm, $m(v)$ must be a

$\neg[O]_k[O]^+_G\psi$-successor with $k \in G$. Moreover, by minimality of $M(v)$, $B(m(v)) = \{m(v)\}$ and so $m(v)$ must be marked `sat`, as it would be marked `unsat` otherwise. Hence u must be marked `sat` as well. Moreover it holds that $\neg[O]^+_G\psi \in L(u)$ and, by what we have shown above, condition **TC** is satisfied for it. Hence condition **TC** is satisfied for v as well. If v is not a leaf of the pre-tableau, then, by condition **CE** and by the fact that v is marked `undec`, there must exist $j \in G$ such that $\neg[O]_j[O]^+_G\psi \in L(v)$. Again we take $m(v)$. If there is $j \in G$ such that creation of $\neg[O]_j[O]^+_G\psi$-successor of v is blocked by $m(v)$, then condition **TC** is satisfied by arguments analogous to those used above. Otherwise there must be $j \in G$ such that there is a $\neg[O]_j[O]^+_G\psi$-Successor u of v with $m(v) \in B(u)$ and condition **TC** holds by the induction hypothesis. For the induction step (of the main induction) suppose that $M(v)$ is not minimal. Consider the set of states $V \subseteq W$ with the same value of $M(v)$. Arguments here are analogous to those used for the induction basis. The difference lies in the fact that $B(m(v)) \notin \{m(v)\}$ this time, as $M(v)$ is not minimal. However, this implies that there is $m' \in B(m(v))$ such that $\mathrm{sheight}(m') < \mathrm{sheight}(m(v))$ and the induction hypothesis applies.

Lastly we will show that condition **TC** is satisfied for states that are not $\neg[O]_j[O]^+_G\psi$-Successors with any $j \in G$. By condition **CE** there must exist $j \in G$ such that either $\neg[O]_j\psi \in L(v)$ or $\neg[O]_j[O]^+_G\psi \in L(v)$. In the first case, the condition holds by condition **T2**. Similarly in the second case, if $\neg[O]_j[O]^+_G\psi$-successor of v was created. If it was not, then v must be a B_j-Successor of some state $w \in W$ and there there exists $u \in O_j(w)$ such that u is a $\neg[O]_k[O]^+_G\psi$-Successor with $k \in G$ and $\neg[O]^+_G\psi \in L(u)$. By what we have shown above, condition **TC** is satisfied for u and $\neg[O]^+_G\psi$ and, consequently, it is satisfied for v and $\neg[O]^+_G\psi$ as well. Hence we have shown that $\mathcal{T}$ is a TeamLog tableau for φ and that φ is satisfiable. $\square$

Proofs associated with termination property of Algorithm 2

Proof (Lemma 8) Notice that Claims 1–3 shown in proof of Lemma 5 hold in the case of Algorithm 2 as well and they require modal context restriction $\mathbf{R}_1$ only.

Consider a sequence of states $s_0, \ldots, s_m$ in the pre-tableau such that for any $0 < k \leq m$, s_k is a B_{j_k}-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) \sqcap j_k = \emptyset$ and the sequence satisfies properties **PB1**, **PB1** and **PB2** for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \emptyset$ or there is more than one formula of the form $\neg[\mathrm{B}]^+_G\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such sequence must be ≤ 2, by the same arguments as those used in proof of Lemma 5. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^+ = \{\neg[\mathrm{B}]^+_G\psi\}$, then the length of the sequence must be ≤ 2. This is because, by point (iii) of Claim 2, for all $0 < k \leq m$ it must be that s_k is a $\neg[\mathrm{B}]_{j_k}[\mathrm{B}]^+_G\psi$-Successor of s_{k-1} with $j_k \in G$. Suppose that the length of the sequence is > 2. By Lemma 4 it must be that $j_1 \neq j_2$ and $j_2 \neq j_3$. Claim 3 implies that $\left(L(s_0)/[\mathrm{B}]_{j_1}\right) \cup \{\psi\} = \left(L(s_1)/[\mathrm{B}]_{j_2}\right) \cup \{\psi\}$, $\left(L(s_1)/[\mathrm{B}]_{j_2}\right) \cup \{\psi\} = \left(L(s_2)/[\mathrm{B}]_{j_3}\right) \cup \{\psi\}$ and $\left(L(s_2)/[\mathrm{B}]_{j_3}\right) \cup \{\psi\} = \left(L(s_3)/[\mathrm{B}]_{j_3}\right) \cup \{\psi\}$. Since, for all $1 \leq i \leq 4$, s_{i-1} is a state so $L(s_{i-1})$ is a $[O]^+$-expanded tableau and so $[\mathrm{B}]_{j_i}\psi \in \neg L(s_{i-1})$. Notice that if $\neg[\mathrm{B}]_{j_2}\psi \in L(s_1)$, then $\neg[\mathrm{B}]_{j_2}[\mathrm{B}]^+_G\psi$-Successor of s_1 would not be created and s_2 would not be in the sequence. Hence it must be that $[\mathrm{B}]_{j_2}\psi \in L(s_1)$. But then $\psi \in L(s_2)$ and either $[\mathrm{B}]_{j_3}\psi \in$

$L(s_2)$ or $\neg[\mathrm{B}]_{j_3}\psi \in L(s_2)$. In any of these cases $\neg[\mathrm{B}]_{j_2}[\mathrm{B}]_G^+\psi$-Successor of s_2 would not be created and so s_3 cannot be in the sequence, which contradicts our assumptions. Hence the length of the sequence must be ≤ 2. Using arguments similar to those used in Lemma 5 it can be shown that the maximal length of a sequence of B-Successors with the same modal depth of labels is $\mathcal{O}\left(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\right)$. □

Proof (Lemma 9) The proof is by analogous arguments to those used in proof of Lemma 7, using Lemma 8 instead of Lemma 5. □

Proof (Proposition 6) In the proof we will use the notion of $[\mathrm{B}]^+$-depth of a formula defined below.

Definition 18 ($[\mathrm{B}]^+$-depth) The $[\mathrm{B}]^+$-*depth of a formula* φ, denoted by $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, is defined inductively as follows:

- $\mathrm{dep}_{[\mathrm{B}]^+}(p) = 0$, where $p \in \mathcal{P}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}(\neg\varphi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$,
- $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi_1 \wedge \varphi_2) = \max\{\mathrm{dep}_{[\mathrm{B}]^+}(\varphi_1), \mathrm{dep}_{[\mathrm{B}]^+}(\varphi_2)\}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}\left([O]_j\varphi\right) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, where $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $j \in \mathcal{A}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}\left([\mathrm{I}]_G^+\varphi\right) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, where $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$,
- $\mathrm{dep}_{[\mathrm{B}]^+}\left([\mathrm{B}]_G^+\varphi\right) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi) + 1$, where $G \in \mathrm{P}(\mathcal{A}) \setminus \{\varnothing\}$.

Let Φ be a finite set of formulas, then

$$\mathrm{dep}_{[\mathrm{B}]^+}(\Phi) = \begin{cases} 0, & \text{if } \Phi = \varnothing \\ \max\{\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) : \varphi \in \Phi\}, & \text{otherwise.} \end{cases}$$

We will show that the algorithm terminates for any input $\varphi \in \mathcal{L}_{\mathbf{R_1}}^{\mathrm{T}}$ using induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\left(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\right)$ (in lexicographic order). For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$ (in this case $\mathrm{ag}\left(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\right) = \mathcal{A} \cup \{\omega\}$). Then there are no formulas of the form $[\mathrm{B}]_G^+\psi \in \neg\mathrm{Sub}(\varphi)$ and Algorithm 2 works like Algorithm 1 on the input φ. Thus, by Proposition 3, Algorithm 2 terminates on the input φ. For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$. By Lemma 9 either Algorithm 2 terminates or Function 9 is called. Since any call to Algorithm 2 in Function 9 is made with an input $\xi = \bigwedge \Xi$ with $\Xi \subseteq \neg\mathrm{Sub}(\varphi)$ such that either $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\left(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\right) \subsetneq \mathrm{ag}\left(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\right) \subseteq \mathcal{A}$ so, by the induction hypothesis, each such call terminates. Thus each call to Function 10 in Function 9 terminates and since the number of calls to Algorithm 2 and Function 10 made in Function 9 is finite, so Function 9 terminates. Hence Algorithm 2 must terminate on the input φ, as Function 9 is called there finitely many times (this is because it is called for the states of the pre-tableau constructed by Algorithm 2 and the number of these states is finite, as the number of successors of any node is finite and, by Lemma 9, the depth of the pre-tableau is also finite). $\square$

Proofs associated with validity of Algorithm 2

We first show two auxiliary lemmas that will be used in proof of validity. These lemmas show, essentially, that when the Function 9 is invoked for some state s and its $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor, then the labels of this successor and subsequent $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-successors, with $k \in \mathrm{ag}\Big(\Big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, that could follow, would differ on elements from $\neg\Big((L(s) \sqcap j) \setminus \Big(\Big(L(s) \sqcap [\mathrm{B}]_j\Big) \cup \Big(L(s) \sqcap \neg[\mathrm{B}]_j\Big)\Big)\Big)$ only. More precisely, each of them would include a maximal subset of $\neg\Big((L(s) \sqcap j) \setminus \Big(\Big(L(s) \sqcap [\mathrm{B}]_j\Big) \cup \Big(L(s) \sqcap \neg[\mathrm{B}]_j\Big)\Big)\Big)$. This justifies the use of Function 9 in such cases.

Lemma 15 *Let s be a state in the pre-tableau constructed by Algorithm 2 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$. Let there be a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ with $j \in G$ and such that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of s was not created because of a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Predecessor t of s with $k \neq j$ and such that $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$. Then for any $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$ it holds that $L(s) \sqcap [\mathrm{B}]_l \subseteq \widetilde{\mathrm{Cl}}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$.*

Proof Suppose that s is like stated in the lemma. We will start by showing that $L(s) \sqcap [\mathrm{B}]_j \subseteq \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. To show that we will use induction on the modal depth of a formula, starting from maximal values. Take any formula of the form $[\mathrm{B}]_j\xi \in L(s)$ and suppose its modal depth is maximal in $L(s) \sqcap [\mathrm{B}]_j$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ and $j \neq k$ so either $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ or there is a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $[\mathrm{B}]_j\xi \in \neg\mathrm{BT}(\zeta)$. If the first case holds that the claim is satisfied. Suppose that the second case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. The first case is impossible, as it would contradict the assumption of maximality of $\mathrm{dep}([\mathrm{B}]_j\xi)$. On the other hand, if $\zeta = \psi$, then $[\mathrm{B}]_k\zeta \in \widetilde{\mathrm{Cl}}(\{\neg[\mathrm{B}]^+_G\psi\})$ and since $[\mathrm{B}]_j\xi \in \neg\mathrm{BT}(\zeta)$ so this violates modal context restriction $\mathbf{R}_1$. Hence this case is impossible as well.

For the induction step, suppose that modal depth of $[\mathrm{B}]_j\xi$ is not maximal. Like in the case of the induction basis, $[\mathrm{B}]_j\xi \in L(s)$ implies that either $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ or there is a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $[\mathrm{B}]_j\xi \in \neg\mathrm{BT}(\zeta)$. If the first case holds that the claim is satisfied. Suppose that the second case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. Suppose that the first case holds. Then, by the induction hypothesis, $[\mathrm{B}]_j\zeta \in \mathrm{Cl}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. But then $[\mathrm{B}]_j\xi \in \neg\mathrm{BT}(\zeta)$ means that we have a violation of modal context restriction $\mathbf{R}_1$. Hence this case is not possible. On the other hand, if $\zeta = \psi$, then, as we argued for the induction basis, $[\mathrm{B}]_k\zeta \in \widetilde{\mathrm{Cl}}(\{\neg[\mathrm{B}]^+_G\psi\})$ and we get a violation of modal context restriction $\mathbf{R}_1$ again. Hence this case is impossible as well.

Secondly, we will show that $L(t) \sqcap [\mathrm{B}]^+_{\{k\}} \subseteq L(s) \sqcap [\mathrm{B}]^+_{\{j\}}$. To see this, take any formula of the form $[\mathrm{B}]^+_H\xi \in L(t)$ with $k \in H$. Since t is a state, so $L(t)$ is a closed tableau and so $[\mathrm{B}]_k[\mathrm{B}]^+_H\xi \in L(t)$. Thus $[\mathrm{B}]^+_H\xi \in L(t)/[\mathrm{B}]_k$ and, consequently, $[\mathrm{B}]^+_H\xi \in L(s)$ (as $L(t)/[\mathrm{B}]_k \subseteq L(s)$). Hence we need to show that $j \in H$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]^+_H\xi = \psi$ or there is a formula $[\mathrm{B}]_j[\mathrm{B}]^+_H\xi \in L(s)$. The first case is not possible, as $k \in G$ and it would lead to violation of modal context restriction $\mathbf{R}_1$. Suppose that the second case holds. Then, by what we have shown above, it must be that $[\mathrm{B}]_j[\mathrm{B}]^+_H\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Suppose that $j \notin H$. Then it would have to be that either $[\mathrm{B}]^+_H\xi = \psi$ or there is a formula $[\mathrm{B}]^+_{H'}[\mathrm{B}]^+_H\xi \in L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}$. As we have already shown, the first case would lead to violation of modal context restriction $\mathbf{R}_1$. The second case would lead to violation of modal context restriction $\mathbf{R}_1$ as well, as $k \in H'$ and $k \in H$. Thus it must be that $j \in H$ and so $[\mathrm{B}]^+_H\xi \in L(s) \sqcap [\mathrm{B}]^+_{\{j\}}$.

Now take any formula of the form $[\mathrm{B}]_l\xi \in L(s)$ with $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. We will show that it must be that $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$.

Since $[\mathrm{B}]_l\xi \in L(s)$, so either

1. there exists a formula $[\mathrm{B}]_k\zeta \in L(t)$ with $[\mathrm{B}]_l\xi \in \neg\mathrm{BT}(\zeta)$, or
2. $[\mathrm{B}]_l\xi \in L(t) \sqcap k$, or
3. $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}(L(s))$.

Case 1 Suppose that the first case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. If $[\mathrm{B}]_j\zeta \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\zeta \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Thus either there is a formula $[\mathrm{B}]^+_H\zeta \in L(t)$ with $\{j,k\} \subseteq H$, or there is a formula $[\mathrm{B}]^+_H\eta \in L(t)$ with $\zeta = [\mathrm{B}]^+_H\eta$ and $\{j,k\} \subseteq H$. Since $[\mathrm{B}]_l\xi \in \neg\mathrm{BT}(\zeta)$, so it must be that the first of these cases holds, that is $[\mathrm{B}]^+_H\zeta \in L(t)$ with $\{j,k\} \subseteq H$. Now, since $(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq L(s)$, so $(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq \Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}$. Moreover, since $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, so it must be that $l \in \mathrm{ag}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Thus $l \in H$ and we get a violation of modal context restriction $\mathbf{R}_1$. If $\zeta = \psi$, then we get a violation of modal context restriction $\mathbf{R}$ $\mathbf{R}_1$ again. This is because, by the assumption that $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, it holds that $l \in G$.

Case 2.

Suppose that the second case holds, that is $[\mathrm{B}]_l\xi \in L(t) \sqcap k$. Then it must be that $l = k$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\xi \in L(s)$ or $\xi = \psi$. If $[\mathrm{B}]_j\xi \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ and, since $l = k$ and $L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}} \subseteq L(t) \sqcap [\mathrm{B}]^+_{\{k\}}$, so

$[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. If $\xi = \psi$, then we also have $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$.

Case 3. Suppose that the third case holds, that is $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}(L(s))$. Then either there is a formula $[\mathrm{B}]^+_H\xi \in L(s)$ with $l \in H$ or ξ is of the form $[\mathrm{B}]^+_H\zeta$, with $l \in H$, and $[\mathrm{B}]^+_H\zeta \in L(s)$.

Suppose that the first of these cases, that is $[\mathrm{B}]^+_H\xi \in L(s)$ with $l \in H$. Then there must be a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $[\mathrm{B}]^+_H\xi \in \neg\mathrm{PT}(\zeta)$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\zeta \in L(s)$ or $\zeta = \psi$. If $\zeta = \psi$, then we get a violation of modal context restriction $\mathbf{R}_1$ again. This is because, by the assumption that $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, it holds that $l \in G$. If $[\mathrm{B}]_j\zeta \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\zeta \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Thus either there is a formula $[\mathrm{B}]^+_F\zeta \in L(t)$ with $\{j,k\} \subseteq F$, or there is a formula $[\mathrm{B}]^+_F\eta \in L(t)$ with $\zeta = [\mathrm{B}]^+_F\eta$ and $\{j,k\} \subseteq F$. Consider the first of these cases, that is $[\mathrm{B}]^+_F\zeta \in L(t)$ with $\{j,k\} \subseteq F$. Since $(L(t) \sqcap [\mathrm{B}]_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq L(s)$, so $(L(t) \sqcap [\mathrm{B}]_{\{j,k\}}) \cup \{\neg[\mathrm{B}]^+_G\psi\} \subseteq \Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}$. Moreover, since $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, so it must be that $l \in \mathrm{ag}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Thus $l \in F$ and we get a violation of modal context restriction $\mathbf{R}_1$. Consider the second of these cases. Since $[\mathrm{B}]^+_H\xi \in \neg\mathrm{PT}(\zeta)$, so the only possibility in this case is $F = H$ and $\xi = \eta$, which means that $[\mathrm{B}]^+_H\xi \in L(t)$ and $\{l,j,k\} \subseteq H$. Thus $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ in this case.

Suppose that the second of these cases holds, that is ξ is of the form $[\mathrm{B}]^+_H\zeta$, with $l \in H$, and $[\mathrm{B}]^+_H\zeta \in L(s)$. Then there must be a formula $[\mathrm{B}]_k\eta \in L(t)$ such that $[\mathrm{B}]^+_H\zeta \in \neg\mathrm{PT}(\eta)$. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$, so either $[\mathrm{B}]_j\eta \in L(s)$ or $\eta = \psi$. If $\eta = \psi$, then we get a violation of modal context restriction $\mathbf{R}_1$ again. This is because, by the assumption that $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, it holds that $l \in G$. If $[\mathrm{B}]_j\eta \in L(s)$, then, by what we have shown above, $[\mathrm{B}]_j\eta \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{j,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Thus either there is a formula $[\mathrm{B}]^+_F\eta \in L(t)$ with $\{j,k\} \subseteq F$, or there is a formula $[\mathrm{B}]^+_F\chi \in L(t)$ with $\eta = [\mathrm{B}]^+_F\chi$ and $\{j,k\} \subseteq F$. By arguments analogous to those used above, the first of these cases leads to violation of modal context restriction $\mathbf{R}_1$. Consider the second of these cases. Since $[\mathrm{B}]^+_H\zeta \in \neg\mathrm{PT}(\eta)$, so the only possibility in this case is $F = H$ and $\zeta = \chi$, which means that $[\mathrm{B}]^+_H\zeta \in L(t)$ and $\{l,j,k\} \subseteq H$. Thus $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$ in this case.

We have shown that if $[\mathrm{B}]_l\xi \in L(s)$ with $l \in \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$, then $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. Since $L(t) \sqcap [\mathrm{B}]^+_{\{l,k\}} \subseteq L(t) \sqcap [\mathrm{B}]^+_{\{k\}}$ and, as we have shown above, $L(t) \sqcap [\mathrm{B}]^+_{\{k\}} \subseteq L(s) \sqcap [\mathrm{B}]^+_{\{j\}}$, so $[\mathrm{B}]_l\xi \in \widetilde{\mathrm{Cl}}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}\Big)$. □

Lemma 16 *Let s be a state in the pre-tableau constructed by Algorithm 2 for an input $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$. Let there be a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(s)$ with $j \in G$ and such that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of s was not created because of a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\zeta$-Predecessor t of s with $k \neq j$ and such that $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$. Suppose that $\psi \in L(s)$ and let $H = \mathrm{ag}\Big(\Big(L(s) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Then for any $\Gamma \in \mathcal{S}\big(\big(L(s)/[\mathrm{B}]^+_H\big) \cup \{\psi\}\big)$ it holds that $\neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j))) = \neg(\Gamma \sqcap j)$.*

Proof Suppose that s is like stated in the lemma. Take any $\Gamma \in \mathcal{S}\big(\big(L(s)/[\mathrm{B}]^+_H\big) \cup \{\psi\}\big)$. For the left to right inclusion, take any $\xi \in \neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. Then $\xi \in \neg L(s)$ and either $\xi \in \neg\mathrm{BT}(\psi)$ or there exists a formula $[\mathrm{B}]_k\zeta \in L(t)$ such that $\xi \in \neg\mathrm{BT}(\zeta)$. If the first case holds, then $\xi \in \neg\Gamma$. Suppose that the first case does not hold and that the second case holds. Since $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ so there exists a formula $[\mathrm{B}]_j\zeta \in L(s)$ and, by Lemma 15, $\zeta \in L(s)/[\mathrm{B}]^+_H$. Thus $\xi \in \neg\Gamma$.

For the right to left inclusion, take any $\xi \in \neg(\Gamma \sqcap j)$. Then either $\xi \in \neg\mathrm{BT}(\psi)$ or there exists $\zeta \in L(s)/[\mathrm{B}]^+_H$ such that $\xi \in \neg\mathrm{BT}(\zeta)$. If the first case holds, then $\xi \in \neg(L(s) \sqcap j)$, as $\psi \in L(s)$, and, by modal context restriction $\mathbf{R}_1$, ξ cannot be of the form $[\mathrm{B}]_j\eta$ nor of the form $\neg[\mathrm{B}]_j\eta$. Hence $\xi \in \neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. Suppose that the first case does not hold and that the second case holds. Then there must exist a formula $[\mathrm{B}]^+_H\zeta \in L(s)$ and since $j \in H$ and $L(s)$ is a closed tableau, so it must be that $[\mathrm{B}]_j\zeta \in L(s)$. Then, by the fact that $(L(s)/[\mathrm{B}]_j) \cup \{\psi\} = (L(t)/[\mathrm{B}]_k) \cup \{\psi\}$ and $L(t)/[\mathrm{B}]_k \subseteq L(s)$, it holds that $\zeta \in L(s)$ and, consequently, $\xi \in \neg L(s)$. Hence $\xi \in L(s) \sqcap j$ and since, by modal context restriction $\mathbf{R}_1$, ξ cannot be of the form $[\mathrm{B}]_j\eta$ nor of the form $\neg[\mathrm{B}]_j\eta$ (as $j \in H$), so $\xi \in \neg((L(s) \sqcap j) \setminus ((L(s) \sqcap [\mathrm{B}]_j) \cup (L(s) \sqcap \neg[\mathrm{B}]_j)))$. □

Now we are ready to proof Proposition 7 stating validity of Algorithm 2.

Proof (Proposition 7) For the left to right implication we will use induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order). For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$ (in this case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) = \mathcal{A} \cup \{\omega\}$). Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$ and Algorithm 2 works like Algorithm 1 on the input φ. Thus, by Proposition 4, if φ is satisfiable, then Algorithm 2 returns `sat` on the input φ. For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$. In this case, like in proof of Proposition 4, we will show, for any node n of the pre-tableau constructed by the algorithm for input φ, that if n is marked `unsat`, then $\bigwedge L(n)$ is unsatisfiable. Like in the case of proof of Proposition 4 we will use induction on the maximal length of paths from a node to one of its descendant leaves. Arguments for most of the cases are like in the aforementioned proof, apart from the case of the nodes with a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ in their label, for which a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor was not created and Function 9 was used to check whether the label of such a successor is satisfiable.

So suppose that n is a state of the pre-tableau with a formula of the form $\neg[\mathrm{B}]^+_G\psi \in L(n)$ and suppose that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of n, with $j \in G$, was not created and that n was marked `unsat` because Function 9 returned `unsat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$. Since

Function 9 was used to check the satisfiability of $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, so it must be that $\{[\mathrm{B}]_j\psi, \psi\} \subseteq L(n)$. Suppose that $\bigwedge L(n)$ is satisfiable and let $(\mathcal{M}, u)$ be such that $(\mathcal{M}, u) \models \bigwedge L(n)$. Since $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(n)$ and $(\mathcal{M}, u) \models \bigwedge L(n)$, so there must exist $v \in \mathrm{B}_j$ such that $(\mathcal{M}, v) \models \neg[\mathrm{B}]^+_G\psi$. Moreover, it must be that $(\mathcal{M}, v) \models \bigwedge L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$. Since $(\mathcal{M}, v) \models \neg[\mathrm{B}]^+_G\psi$ so there must exist a satisfying sequence for $\neg[\mathrm{B}]^+_G\psi$ in $(\mathcal{M}, v)$. Let $v_0, \ldots, v_k$ be minimal such sequence. Let $H = \mathrm{ag}\Big(\big(L(n) \sqcap [\mathrm{B}]^+_{\{j\}}\big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$. Notice that since $j \in G$, so $H \neq \emptyset$, as $j \in H$. Let $0 \leq l < k$ be maximal such that for all $1 \leq i \leq l$ it holds that $j_i \in H$. Observe that for all $0 \leq i \leq l$, $(\mathcal{M}, v_i) \models \bigwedge(L(n)/[\mathrm{B}]^+_H)$ and $(\mathcal{M}, v_i) \models \psi$ (in the case of v_0 this follows from the fact that $[\mathrm{B}]_j\psi \in L(n)$). Thus, for all $0 \leq i \leq l$, $(\mathcal{M}, v_i) \models \bigwedge\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ and for each $0 \leq i \leq l$ there exists a set of formulas $\Gamma_i \in \mathcal{S}\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$ such that $(\mathcal{M}, v_i) \models \bigwedge \Gamma_i$. Since $\mathrm{dep}_{[\mathrm{B}]^+}\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big) < \mathrm{dep}_{[\mathrm{B}]^+}(L(n))$ so, for all $0 \leq i \leq l$, $\mathrm{dep}_{[\mathrm{B}]^+}(\Gamma_i) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$. Hence, by the induction hypothesis, Algorithm 2 cannot return `unsat` on the input $\bigwedge \Gamma_i$. Moreover, by transitivity, generalized transitivity, Euclidity and generalized Euclidity of accessibility relations B_{j_i}, for all $0 < i \leq l$ it holds that $\Gamma_{i-1} \sqcap j_i = \Gamma_i \sqcap j_i$. Hence Γ_l is reachable from Γ_0 in $\mathcal{G}_H\big((L(n)/[\mathrm{B}]^+_H) \cup \{\psi\}\big)$. Moreover, by Lemma 16 it holds that $\neg((L(n) \sqcap j) \setminus ((L(n) \sqcap [\mathrm{B}]_j) \cup (L(n) \sqcap \neg[\mathrm{B}]_j))) = \neg(\Gamma_0 \sqcap j)$ and, by transitivity, generalized transitivity, Euclidity and generalized Euclidity of accessibility relation B_j it must be that $(L(n) \sqcap j) \setminus ((L(n) \sqcap [\mathrm{B}]_j) \cup (L(n) \sqcap \neg[\mathrm{B}]_j)) = \Gamma_0 \sqcap j$. Hence $(L(n) \sqcap j) \setminus ((L(n) \sqcap [\mathrm{B}]_j) \cup (L(n) \sqcap \neg[\mathrm{B}]_j)) \subseteq \Gamma_0$.

Now, if $l + 1 = k$, then it must be that $(\mathcal{M}, v_{l+1}) \models \neg\psi$ and so $(\mathcal{M}, v_{l+1}) \models {\sim}\psi$. Moreover, $(\mathcal{M}, v_{l+1}) \models \bigwedge\Big(L(n)/[\mathrm{B}]^+_{H\cup\{j_{l+1}\}}\Big)$ and $(\mathcal{M}, v_{l+1}) \models \bigwedge \Gamma_l^{[\mathrm{B}]_{j_k}}$. If $j_{l+1} \in H$, then $(\mathcal{M}, v_{l+1}) \models \xi$, where $\xi = \bigwedge\Big(\Gamma_l^{[\mathrm{B}]_{j_k}} \cup \big(L(n)/[\mathrm{B}]^+_H\big) \cup \{{\sim}\psi\}\Big)$ and since $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ so, by the induction hypothesis, Algorithm 2 cannot return `unsat` on the input ξ. Notice that if $k = 1$, then it must be that $j_k \neq j$, as $(\mathcal{M}, v_0) \models [\mathrm{B}]_j\psi$. Notice also that if $k \geq 2$, then, by minimality of $v_0, \ldots, v_k$, it must be that $j_k \neq j_{k-1}$. Thus Function 9 must return `sat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, which contradicts our assumptions. If $j_{l+1} \in G \setminus H$, then, by simple induction on i, for all $0 \leq i \leq l+1$ it holds that $(\mathcal{M}, v_i) \models \bigwedge\Big(L(n) \sqcap [\mathrm{B}]^+_{H\cup\{j_{l+1}\}}\Big)$. Hence $(\mathcal{M}, v_{l+1}) \models \xi$, where $\xi = \bigwedge\Big(\Gamma_l^{[\mathrm{B}]_{j_{l+1}}} \cup L^{[\mathrm{B}]^+_{H\cup\{j_{l+1}\}}}(n) \cup \{{\sim}\psi\}\Big)$. Since either $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)) \subseteq H \subsetneq H \cup \{j_{l+1}\} = \mathrm{ag}(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\xi))$ so, by the induction hypothesis, Algorithm 2 cannot return `unsat` on the input ξ. Thus Function 9 must return `sat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, which contradicts our assumptions.

Otherwise, if $l + 1 < k$, then it must be that $j_{l+1} \in G \setminus H$. By minimality of $v_0, \ldots, v_k$, it must be that $(\mathcal{M}, v_l) \models [\mathrm{B}]_{j_{l+1}}\psi$, $(\mathcal{M}, v_{l+1}) \models \psi$ and $(\mathcal{M}, v_{l+1}) \models \neg[\mathrm{B}]^+_G\psi$. Hence, it must be that $(\mathcal{M}, v_l) \models \neg[\mathrm{B}]_{j_{l+1}}[\mathrm{B}]^+_G\psi$ and, by transitivity of $\mathrm{B}_{j_{l+1}}$, $(\mathcal{M}, v_{l+1}) \models [\mathrm{B}]_{j_{l+1}}\psi$ and $(\mathcal{M}, v_{l+1}) \models \neg[\mathrm{B}]_{j_{l+1}}[\mathrm{B}]^+_G\psi$. As we observed above, $(\mathcal{M}, v_i) \models \bigwedge\Big(L(n)/[\mathrm{B}]^+_{H\cup\{j_{l+1}\}}\Big)$ and $(\mathcal{M}, v_i) \models \bigwedge\Big(L(n) \sqcap [\mathrm{B}]^+_{H\cup\{j_{l+1}\}}\Big)$. Moreover $(\mathcal{M}, v_{l+1}) \models \bigwedge \Gamma_l^{[\mathrm{B}]_{j_{l+1}}}$. Thus $(\mathcal{M}, v_{l+1}) \models \xi$, where $\xi = \bigwedge\Big(\Gamma_l^{[\mathrm{B}]_{j_{l+1}}} \cup L^{[\mathrm{B}]^+_{H\cup\{j_{l+1}\}}}(n) \cup \{\psi, \neg[\mathrm{B}]^+_G\psi, [\mathrm{B}]_{j_{l+1}}\psi, \neg[\mathrm{B}]_{j_{l+1}}[\mathrm{B}]^+_G\psi\}\Big)$. Since either

$\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subseteq H \subsetneq H \cup \{j_{l+1}\} = \mathrm{ag}\Big(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\xi)\Big)$ so, by the induction hypothesis, Algorithm 2 cannot return `unsat` on the input ξ. Thus Function 9 must return `sat` on the input $L^{\neg[\mathrm{B}]_j}(n, [\mathrm{B}]^+_G\psi)$, which contradicts our assumptions.

As was pointed out in proof of Proposition 4, `root` of any pre-tableau constructed by the algorithm must be marked either `unsat` or `sat` and if `root` is not marked `sat`, then it must be marked `unsat` and φ must be unsatisfiable. Hence if φ is satisfiable, then `root` node must be marked `sat` and the algorithm must return `sat`.

For the right to left implication we will show that if Algorithm 2 returns `sat` on the input φ, then a TeamLog tableau for φ can be constructed. More precisely, we will show that if Algorithm 2 returns `sat` on the input φ, then a TeamLog tableau for φ can be constructed, which has a state w such that $\varphi \in L(w)$ and for no other state u of this tableau there exists $j \in \mathcal{A}$ such that $w \in \mathrm{B}_j(u)$. To show that, we will again use induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order).

For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$. Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$, Algorithm 2 works like Algorithm 1 on the input φ and TeamLog tableau for φ can be constructed like in proof of Proposition 4. Recall that that construction guarantees that there exists a state w in that tableau such that $\varphi \in L(w)$ and there is no other state u in that tableau and no $j \in \mathcal{A}$ such that $w \in \mathrm{B}_j(u)$.

For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$ and let

$$(N, \mathrm{root}, succ, \{O_j\text{-succ} : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L)$$

be the pre-tableau constructed by Algorithm 2 for φ. Let

$$\mathcal{T} = (W, \{O_j : j \in \mathcal{A},\ O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}\}, L|_W)$$

be a model graph constructed on the basis of this pre-tableau like in proof of Proposition 4. Let $V \subseteq W$ be the set of states such that for each $v \in V$ there is a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ with $j \in G$ and such that a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor of v was not created because of a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\zeta$-Predecessor u of v, with $k \in G$, such that $k \neq j$ and $(L(v)/[\mathrm{B}]_j) \cup \{\psi\} = (L(u)/[\mathrm{B}]_k) \cup \{\psi\}$.

As we remarked above, the construction from Proposition 4 guarantees that there exists $w \in W$ such that $\varphi \in L(w)$ and for all $u \in W \setminus \{w\}$ and $j \in \mathcal{A}$, $w \notin \mathrm{B}_j(u)$.

Notice that conditions **T1**, **T4**, **T5**, **TBG4** , **TBI4**, **TBG5**, **TBI5** and **TIG** are satisfied for all the states of $\mathcal{T}$, as the same argumentation as the one used in proof of Proposition 4 will work here. Similarly it can be shown like in proof of Proposition 4 that conditions **T2** and **TD** are satisfied for all the states of $\mathcal{T}$ that are not in V. Also, in the case of states from V and formulas of the form $[\mathrm{G}]_k\xi$ it can be shown like in proof of Proposition 4 that the condition **T2** holds for them. Similarly with states from V, formulas of the form $[\mathrm{I}]_k\xi$ and conditions **T2** and **TD**. Condition **TC** also holds for all states of $\mathcal{T}$ and formulas of the form $[\mathrm{I}]^+_H\xi$. The problem are conditions **T2** and **TD** for states from V and formulas of the form $[\mathrm{B}]_k\xi$, as well as conditions **TC** for all states of $\mathcal{T}$ and formulas of the form $[\mathrm{B}]^+_H\xi$. To satisfy these conditions, the model graph $\mathcal{T}$ has to be extend at the states from V, so that a TeamLog tableau for φ is created. The extension is by adding B-successors of states from V for the formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ for which the successors were not created for the reasons described above. We will describe the extension for a given state $v \in V$ and a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ for which a successor was not created for the reasons described above, showing that this extension sustains the conditions of TeamLog tableau listed above, while making the unsatisfied conditions **T2**, **TD** and **TC** satisfied.

Take any $v \in V$ and $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ with $j \in G$ for which a $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$-successor was not created for the reasons described above. Now two cases are possible: either $[\mathrm{B}]_j\psi \in L(v)$ or not. Suppose first that $[\mathrm{B}]_j\psi \notin L(v)$. Then, by the fact that v, being a state, is a $[\mathrm{B}]^+$-expanded tableau, it holds that $\neg[\mathrm{B}]_j\psi \in L(v)$. Let $\mathcal{T}$ be extended as follows:

- $\mathrm{B}'_j(v) = \mathrm{B}_j(v) \cup \{v\}$.

Notice that since $[\mathrm{B}]_j\psi \notin L(v)$, so $\psi \notin L(v)/[\mathrm{B}]_j$ and so $L(v)/[\mathrm{B}]_j \subseteq L(u)/[\mathrm{B}]_k \subseteq L(v)$. Since it also holds that $L(v) \sqcap j \subseteq L(v)$ and $\neg[\mathrm{B}]^+_G\psi \in L(v)$, so $L^{\neg[\mathrm{B}]_j}(v, [\mathrm{B}]^+_G\psi) \subseteq L(v)$. Hence the conditions **T1**, **T4**, **T5**, **TBG4** , **TBI4**, **TBG5**, **TBI5** and **TIG** are still satisfied for v after the extension. Also, conditions **T2**, **TD** and **TC** are still satisfied for those states and formulas for which they were satisfied before the extension. Notice also that after this extension it still holds that there exists $w \in W$ such that $\varphi \in L(w)$ and for all $u \in W \setminus \{w\}$ and $j \in \mathcal{A}$, $w \notin \mathrm{B}_j(u)$.

Secondly, suppose that $[\mathrm{B}]_j\psi \in L(v)$. Then Function 9 must have been used to check the satisfiability of $\bigwedge L^{\neg[\mathrm{B}]_j}(v, [\mathrm{B}]^+_G\psi)$. Since v, being in W, must be marked `sat` so Function 9 must have returned `sat` on the input $\bigwedge L^{\neg[\mathrm{B}]_j}(v, [\mathrm{B}]^+_G\psi)$. Thus sets of formulas Ψ_0 and Ψ_1 were found such that $\{\Psi_0, \Psi_1\} \subseteq \mathcal{S}\big(\big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\}\big)$, Ψ_1 is reachable from Ψ_0 in $\mathcal{G}_H\big(\big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\psi\}\big)$ (where $H = \mathrm{ag}\Big(\Big(L(v) \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$) with path $(\Gamma_0, \ldots, \Gamma_n)$ and associated sequence $j_1, \ldots, j_n$ such that Algorithm 2 returned `sat` on each input $\bigwedge \Gamma_i$, with $0 \le i \le n$, and

1. either there exists $k \in H$ such that Algorithm 2 returned `sat` on the input $\bigwedge \Xi$, where $\Xi = \Psi_1^{[\mathrm{B}]_k} \cup \big(L(v)/[\mathrm{B}]^+_H\big) \cup \{\sim\psi\}$, in which case the sequence $k \neq j$, if $n = 0$, and $k \neq j_n$, if $n \ge 1$,
2. or there exists $k \in G \setminus H$ such that Algorithm 2 returned `sat` on the input $\bigwedge \Xi$, where $\Xi = \Psi_1^{[\mathrm{B}]_k} \cup L^{[\mathrm{B}]^+_{H \cup \{k\}}}(v) \cup \{\sim\psi\}$,
3. or there exists $k \in G \setminus H$ such that Algorithm 2 returned `sat` on the input $\bigwedge \Xi$, where $\Xi = \Psi_1^{[\mathrm{B}]^+_k} \cup L^{[\mathrm{B}]^+_{H \cup \{k\}}}(v) \cup \{\psi, \neg[\mathrm{B}]^+_G\psi, [\mathrm{B}]_k\psi, \neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi\}$.

Since Algorithm 2 returned `sat` on the input $\bigwedge \Gamma_i$, for each $0 \le i \le n$, and $\mathrm{dep}_{[\mathrm{B}]^+}(\Gamma_i) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ so, by the induction hypothesis, a sequence $(\mathcal{T}_0, \ldots, \mathcal{T}_n)$ of TeamLog tableaux can be created for each of the subsequent elements of $(\Gamma_0, \ldots, \Gamma_n)$.

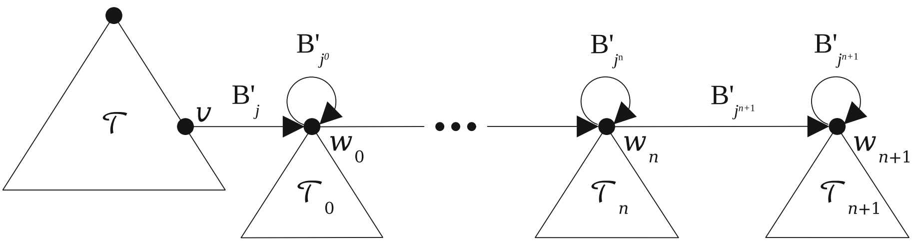

Fig. 3 Extension of model graph $\mathcal{T}$ at $v \in V$ with tableaux $\mathcal{T}_0, \ldots, \mathcal{T}_{n+1}$ constructed by Function 9

Also, in each of the cases 1 – 3 above, by the induction hypothesis, a TeamLog tableau $\mathcal{T}_{n+1}$ can be created for $\bigwedge \Xi$. In the case 1 this is because $\mathrm{dep}_{[\mathrm{B}]^+}(\Xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, in the cases 2 and 3 this is because either $\mathrm{dep}_{[\mathrm{B}]^+}(\Xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\Xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)) \subseteq H \subsetneq H \cup \{k\} = \mathrm{ag}(\mathrm{Sub}(\Xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\Xi))$.

Let $W_0, \ldots, W_{n+1}$ be the sets of states in the subsequent tableaux $\mathcal{T}_0, \ldots, \mathcal{T}_{n+1}$. Also let O_k, with $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $k \in \mathcal{A}$, be the accessibility relations in those tableaux and let L be the labelling function in those tableaux. Let $w_0, \ldots, w_{n+1}$ be the sequence of states in those subsequent tableaux such that $\bigwedge \Gamma_i \in L(w_i)$, for each $0 \le i \le n$, $\bigwedge \Xi \in L(w_{n+1})$, and for each $0 \le i \le n+1$ it holds that for all $u \in W_i \setminus \{w_i\}$ and $j \in \mathcal{A}$, $w_i \notin \mathrm{B}_j(u)$. By the induction hypothesis such sequence of states exists. Let $\mathcal{T}$ be extended as follows (where $j_0 = j$, $j_{n+1} = k$ and $X_j(\neg[\mathrm{B}]^+_G\psi) = \{[\mathrm{B}]_j\psi, \neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi, \neg[\mathrm{B}]^+_G\psi\}$ for $j \in G$) (see Fig. 3 for illustration of this extension):

- $\mathrm{B}'_j(v) = \mathrm{B}_j(v) \cup \{w_0\}$.
- $\mathrm{B}'_{j_i}(w_i) = \mathrm{B}_{j_i}(w_i) \cup \{w_i\}$, for $0 \le i \le n$.
- $\mathrm{B}'_{j_{i+1}}(w_i) = \mathrm{B}_{j_{i+1}}(w_i) \cup \{w_{i+1}\}$, for $0 \le i \le n$.
- $\mathrm{B}'_{j_{i+1}}(w_i) = \mathrm{B}_{j_{i+1}}(w_i) \cup \{w_i, w_{i+1}\}$, for $0 \le i \le n$.
- $L'(w_i) = L(w_i) \cup (L(v) \sqcap [\mathrm{B}]^+_H) \cup \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H) \cup X_{j_i}(\neg[\mathrm{B}]^+_G\psi) \cup X_{j_{i+1}}(\neg[\mathrm{B}]^+_G\psi)$, for $0 \le i \le n-1$.
- $L'(w_n) = L(w_n) \cup (L(v) \sqcap [\mathrm{B}]^+_H) \cup \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H) \cup X_{j_n}(\neg[\mathrm{B}]^+_G\psi)$, if the case 1 or 2 holds.
- $L'(w_n) = L(w_n) \cup (L(v) \sqcap [\mathrm{B}]^+_H) \cup \widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H) \cup X_{j_n}(\neg[\mathrm{B}]^+_G\psi) \cup X_{j_{n+1}}(\neg[\mathrm{B}]^+_G\psi)$, if the case 3 holds.
- $L'(w_{n+1}) = L(w_{n+1}) \cup \left(L(v) \sqcap [\mathrm{B}]^+_{H\cup\{k\}}\right) \cup \widetilde{\mathrm{Cl}}\left(L(v) \sqcap [\mathrm{B}]^+_{H\cup\{k\}}\right)$.

Like in the case of the previous extension, conditions **T1**, **T4**, **T5**, **TBG4** , **TBI4**, **TBG5**, **TBI5** and **TIG** are still satisfied for v after the extension described above. Condition **TIG** is not affected by the extension, as it adds an B_j-successor of state v only. Condition **T1** could be affected in the case of formulas of the form $[\mathrm{B}]_j\xi \in L(v)$ only. Take any such formula. By Lemma 15 it holds that $L(v) \sqcap [\mathrm{B}]_j \subseteq \widetilde{\mathrm{Cl}}\left(\left(L(v) \sqcap [\mathrm{B}]^+_{\{j\}}\right) \cup \{\neg[\mathrm{B}]^+_G\psi\}\right)$. Hence if $[\mathrm{B}]_j\xi \in L(v)$, then either $[\mathrm{B}]_j\xi = [\mathrm{B}]_j\psi$ or there exists a formula $[\mathrm{B}]^+_T\zeta \in L(v)$ such that $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}([\mathrm{B}]^+_T\zeta)$. Thus $\xi \in \left(L(v)/[\mathrm{B}]^+_H\right) \cup \{\psi\} \cup \{[\mathrm{B}]^+_T\zeta\}$ and since $L(w_0) \in \mathcal{S}\left(\left(L(v)/[\mathrm{B}]^+_H\right) \cup \{\psi\}\right)$, so $\xi \in L'(w_0)$ and so condition **T1** is satisfied for v and $[\mathrm{B}]_j\xi$. Like in the case of Condition **T1**, Condition **T4** could be affected in the case of formulas of the form $[\mathrm{B}]_j\xi \in L(v)$ only. Take any such formula. As we argued above either $[\mathrm{B}]_j\xi = [\mathrm{B}]_j\psi$ or there exists a formula $[\mathrm{B}]^+_T\zeta \in L(v)$ such that $[\mathrm{B}]_j\xi \in \widetilde{\mathrm{Cl}}([\mathrm{B}]^+_T\zeta)$. In either case $[\mathrm{B}]_j\xi \in L'(w_0)$ and condition **T4** is satisfied for v and $[\mathrm{B}]_j\xi$. Also Condition **T5** could be affected in the case of formulas of the form $[\mathrm{B}]_j\xi \in L'(w_0)$ only. Take any such formula. By modal context restriction $\mathbf{R}_1$, $[\mathrm{B}]_j\xi \notin \neg\mathrm{BT}(\zeta)$ for any $\zeta \in \left(L(v)/[\mathrm{B}]^+_H\right) \cup \{\psi\}$. Hence $[\mathrm{B}]_j\xi \notin L(w_0)$ and it must be that $[\mathrm{B}]_j\xi \in L'(w_0) \setminus L(w_0)$. Since $L'(w_0) \setminus L(w_0) \subseteq L(v)$, so $[\mathrm{B}]_j\xi \in L(v)$ and condition **T5** is satisfied for v and $[\mathrm{B}]_j\xi$. Conditions **TBG4** and **TBI4** are satisfied because, by construction of the algorithm, it holds that $(L(v) \sqcap j) \setminus ((L(v) \sqcap [\mathrm{B}]_j) \cup (L(v) \sqcap \neg[\mathrm{B}]_j)) \subseteq L(w_0)$. For conditions **TBG5** and **TBI5** notice that since, by construction of the algorithm, $(L(v) \sqcap$

$j) \setminus ((L(v) \sqcap [\mathrm{B}]_j) \cup (L(v) \sqcap \neg[\mathrm{B}]_j)) \subseteq L(w_0)$ so, by Lemma 16, $(L(v) \sqcap j) \setminus ((L(v) \sqcap [\mathrm{B}]_j) \cup (L(v) \sqcap \neg[\mathrm{B}]_j)) = L(w_0) \sqcap j$. Thus these conditions are satisfied for v as well. Notice also that after this extension it still holds that there exists $w \in W$ such that $\varphi \in L(w)$ and for all $u \in W \setminus \{w\}$ and $j \in \mathcal{A}$, $w \notin \mathrm{B}_j(u)$.

All the newly added states satisfy conditions of TeamLog tableau. To see this take any $\mathcal{T}_i$ with $0 \leq i \leq n+1$. Notice first that the extended label $L'(w_i)$ is not trivially inconsistent. This is because $L(w_i) \in \mathcal{S}\big((L(v)/[\mathrm{B}]_H^+) \cup \{\psi\}\big)$ and by modal context restriction $\mathbf{R}_1$ it cannot contain any of the formulas extending it to $L'(w_i)$. Moreover, $L'(w_i)$ is a closed propositional tableau, as $L(w_i)$ and $L'(w_i) \setminus L(w_i)$ are closed propositional tableaux. The labels of all the other states of $\mathcal{T}_i$ remain unchanged. For the remaining conditions of TeamLog tableau, notice first that the only state of $\mathcal{T}_i$ that could be affected by the extension is w_i. This is because, by the induction hypothesis, there is no other state u of $\mathcal{T}_i$ such that $w_i \in O_l(u)$, for any $O \in \{\mathrm{B}, \mathrm{G}, \mathrm{I}\}$ and $l \in \mathcal{A}$. Hence we need to show that the remaining conditions of TeamLog tableau are satisfied for states w_i, for all $0 \leq i \leq n+1$. In showing the conditions, the following observation will be useful: for all $0 \leq i \leq n$ and any formula of the form $[\mathrm{B}]_l\xi \in \neg L'(w_i)$ with $l \in H$ it must be that $[\mathrm{B}]_l\xi \notin \neg L(w_i)$. For take any such formula and suppose that $[\mathrm{B}]_l\xi \in \neg L(w_i)$. Then there must be a formula $\zeta \in \big(L(v)/[\mathrm{B}]_H^+\big) \cup \{\psi\}$ such that $[\mathrm{B}]_l\xi \in \neg\mathrm{BT}(\zeta)$, which is impossible by modal context restriction $\mathbf{R}_1$.

We will consider two cases separately: $i = n+1$ and $0 \leq i \leq n$. Suppose first that $i = n+1$. Notice that $L'(w_{n+1}) \neq L(w_{n+1})$ only in the case 1, when $j_{n+1} \in H$. Thus if $j_{n+1} \notin H$, then the conditions of TeamLog tableau are satisfied for w_{n+1} in $\mathcal{T}_{n+1}$, as it is not affected by the extension. Suppose that $j_{n+1} \in H$. Then the only formulas that could be affected by the extension are formulas from $L'(w_{n+1}) \setminus L(w_{n+1}) = \widetilde{\mathrm{Cl}}\Big(L(v) \sqcap [\mathrm{B}]^+_{H \cup \{k\}}\Big) = \widetilde{\mathrm{Cl}}\big(L(v) \sqcap [\mathrm{B}]_H^+\big)$. Notice that for any formula of the form $[\mathrm{B}]_l\xi \in \neg L'(w_{n+1})$ with $l \in H$ it must be that $[\mathrm{B}]_l\xi \notin \neg L(w_{n+1})$. For take any such formula and suppose that $[\mathrm{B}]_l\xi \in \neg L(w_{n+1})$. Then there must be a formula $\zeta \in L^{[\mathrm{B}]_{j_{n+1}}}(w_n) \cup \{\sim\psi\}$ such that $[\mathrm{B}]_l\xi \in \neg\mathrm{BT}(\zeta)$. This is impossible, as, by the observation above, $L^{[\mathrm{B}]_{j_{n+1}}}(w_n) = \varnothing$ and $[\mathrm{B}]_l\xi \in \neg\mathrm{BT}(\sim\psi)$ would violate modal context restriction $\mathbf{R}_1$. Since there are no formulas of the form $[\mathrm{B}]_l\xi \in L(w_{n+1})$ so, by construction of tableau $\mathcal{T}_{n+1}$, $\mathrm{B}_l(w_{n+1}) = \varnothing$, for all $l \in H$, as no successor of a state can be created for a formula that is not in the label of the state. Thus condition **T1** is satisfied for all formulas from $L'(w_{n+1}) \setminus L(w_{n+1})$. Condition **TD** is satisfied for all the formulas from $L'(w_{n+1}) \setminus L(w_{n+1})$, as $L'(w_{n+1}) \setminus L(w_{n+1}) \subseteq L'(w_{n+1})$. Conditions **T4** and **T5** are satisfied for all the formulas from $L'(w_{n+1}) \setminus L(w_{n+1})$, as $\mathrm{B}_l(w_{n+1}) = \varnothing$, for all $l \in H$. The remaining conditions of TeamLog tableau are not applicable to any formula from $L'(w_{n+1}) \setminus L(w_{n+1})$, and so all the conditions of TeamLog tableau are satisfied for w_{n+1}.

Secondly, suppose that $0 \leq i \leq n$. Condition **T1** is satisfied for w_i and any formula from $L(w_i)$ in $\mathcal{T}_i$ and the only formulas for which it could be affected after the extension are formulas from $L'(w_i) \setminus L(w_i)$ and formulas the form $[\mathrm{B}]_{j_l}\xi \in L'(w_i)$ with $l \in \{i, i+1\}$. As we observed above, for any formula of the form $[\mathrm{B}]_l\xi \in \neg L'(w_i)$ with $l \in H$ it must be that $[\mathrm{B}]_l\xi \notin \neg L(w_i)$. Hence, by construction of tableau $\mathcal{T}_i$, $\mathrm{B}_l(w_i) = \varnothing$, for all $l \in H$, as no successor of a state can be created for a formula that is not in the label of the state. This, together with the fact that $(L'(w_i) \setminus L(w_i))/[\mathrm{B}]_l \subseteq L'(w_i)$, for all $l \in H$, implies that condition **T1** is satisfied for w_i, for all $0 \leq i \leq n-1$. In the case of $i = n$, condition **T1** is satisfied by the fact that $L^{[\mathrm{B}]_k}(w_n) \subseteq L(w_{n+1})$, $\widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]_H^+)/[\mathrm{B}]_k = L^{[\mathrm{B}]^+_{H \cup \{k\}}}(v)$ and, in the case 3, $\psi \in L(w_{n+1})$. Condition **TD** is satisfied for w_i and any formula from $L(w_i)$ in $\mathcal{T}_i$ and the only formulas for which it could be affected after the extension are formulas from $L'(w_i) \setminus L(w_i)$. In the case of formulas of the form $[\mathrm{B}]_{j_{i+1}}\xi \in L'(w_i) \setminus L(w_i)$ the condition is satisfied by the fact that $w_{i+1} \in \mathrm{B}_{j_{i+1}}(w_i)$ and by condition **T1**. In the case of the remaining formulas of the form $[\mathrm{B}]_l\xi \in L'(w_i) \setminus L(w_i)$, it must be that $l \in H$ and since for any $l \in H$ it holds that $\widetilde{\mathrm{Cl}}(L(v) \sqcap [\mathrm{B}]^+_H)/[\mathrm{B}]_l \subseteq L^{[\mathrm{B}]^+_H}(v) \subseteq L(w_i)$ and $\psi \in L(w_i)$ so the condition is satisfied for all $0 \le i \le n-1$. Then only formulas in $L'(w_i)$ for which conditions **T4** and **T5** could be affected after the extension are formulas of the form $[\mathrm{B}]_{j_{i+1}}\xi$. If $0 \le i \le n-1$, then such formulas must be elements of $L'(w_i) \setminus L(w_i)$ and $L'(w_{i+1}) \setminus L(w_{i+1})$, as $j_i \in H$, and since $(L'(w_i) \setminus L(w_i)) \sqcap [\mathrm{B}]_{j_{i+1}} = (L'(w_{i+1}) \setminus L(w_{i+1})) \sqcap [\mathrm{B}]_{j_{i+1}}$ in this case, so the condition is satisfied. Suppose that $i = n$ and $j_{n+1} \in H$. Then $L'(w_n)$ extends $L(w_n)$ according to the case 1 and $L(w_{n+1})$ is like in this case as well. Hence $L'(w_n) \setminus L(w_n) = L'(w_{n+1}) \setminus L(w_{n+1})$ and the conditions are satisfied. Suppose that $i = n$ and $j_{n+1} \notin H$. Then the conditions are satisfied for formulas from $L(w_n)$ and $L^{[\mathrm{B}]_{j_{n+1}}}(w_n)$. For the remaining formulas, take any formula of the form $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_n) \setminus L(w_n)$. If $[\mathrm{B}]_{j_{n+1}}\xi \in \widetilde{\mathrm{Cl}}\big(L(v) \sqcap [\mathrm{B}]^+_H\big)$, then $[\mathrm{B}]_{j_{n+1}}\xi \in \widetilde{\mathrm{Cl}}\Big(L(v) \sqcap [\mathrm{B}]^+_{H \sqcap \{j_{n+1}\}}\Big)$ and, consequently, $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_{n+1})$, as $L(v) \sqcap [\mathrm{B}]^+_{H \sqcap \{j_{n+1}\}} \subseteq L(w_{n+1})$ and $L(w_{n+1})$ is a closed tableau. Otherwise it must be that $[\mathrm{B}]_{j_{n+1}}\xi = [\mathrm{B}]_{j_{n+1}}\psi$ and $L'(w_n)$ is constructed according to the case 3. Then $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_{n+1})$, as $[\mathrm{B}]_{j_{n+1}}\psi \in L(w_{n+1})$ in the case 3. On the other hand, take any formula of the form $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_{n+1}) \setminus L^{[\mathrm{B}]_{j_{n+1}}}(w_n)$. If $[\mathrm{B}]_{j_{n+1}}\xi \in L^{[\mathrm{B}]^+_{H \cup \{j_{n+1}\}}}(v)$, then $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_n)$, as $L^{[\mathrm{B}]^+_{H \cup \{j_{n+1}\}}}(v) \subseteq L'(w_n)$. Otherwise, it must be that $[\mathrm{B}]_{j_{n+1}}\xi = [\mathrm{B}]_{j_{n+1}}\psi$ and $L'(w_n)$ is constructed according to the case 3, in which case $[\mathrm{B}]_{j_{n+1}}\xi \in L'(w_n)$. Hence conditions **T4** and **T5** are satisfied for w_n. Conditions **TBG4**, **TBI4**, **TBG4** and **TBG5** are applicable at w_i to formulas from $L(w_i)$ only and since $L(w_i) \sqcap j_{i+1} = L(w_{i+1}) \sqcap j_{i+1}$, as Γ_i and Γ_{i+1} are connected with j_{i+1}, so the conditions are satisfied. Condition **TIG** holds at w_i as it cannot be affected by the extension. Condition **T2** holds for all the formulas from $L(w_i)$ and the only formulas from $L'(w_i) \setminus L(w_i)$ to which it is applicable are $\neg[\mathrm{B}]_{j_i}[\mathrm{B}]^+_G\psi$ and $\neg[\mathrm{B}]_{j_{i+1}}[\mathrm{B}]^+_G\psi$. If $0 \le i \le n-1$, then the condition is satisfied for both of these formulas as $\neg[\mathrm{B}]^+_G\psi \in L'(w_i)$, $\neg[\mathrm{B}]^+_G\psi \in L'(w_{i+1})$, $w_i \in \mathrm{B}'_{j_i}(w_i)$ and $w_{i+1} \in \mathrm{B}'_{j_{i+1}}(w_i)$. If $i = n$, then the condition is satisfied for $\neg[\mathrm{B}]_{j_i}[\mathrm{B}]^+_G\psi$, as $\neg[\mathrm{B}]^+_G\psi \in L'(w_i)$ and $w_i \in \mathrm{B}'_{j_i}(w_i)$. If $\neg[\mathrm{B}]_{j_{n+1}}[\mathrm{B}]^+_G\psi \in L'(w_n)$, then the case 3 must hold and the condition is satisfied, as in this case $\neg[\mathrm{B}]_{j_{n+1}}[\mathrm{B}]^+_G\psi \in L(w_{n+1})$. Condition **TC** holds for all formulas from $L(w_i)$ and the only formula from $L'(w_i) \setminus L(w_i)$ to which it is applicable is $\neg[\mathrm{B}]^+_G\psi$. Since $w_{n+1} \in {\mathrm{B}'_G}^+(w_i)$ and either $\sim\psi \in L'(w_{n+1})$ or $\neg[\mathrm{B}]^+_G\psi \in L(w_{n+1})$ and since condition **TC** is satisfied for w_{n+1}, so there must exist $u \in {\mathrm{B}'_G}^+(w_{n+1})$ with $\sim\psi \in L(u)$ and $u \in {\mathrm{B}'_G}^+(w_i)$. Thus condition **TC** is satisfied for w_i.

Let $\mathcal{T}'$ be the tableau extending $\mathcal{T}$ at all states from V in the way described above. In particular the set W' of worlds of $\mathcal{T}'$ extends W with all new worlds added for each state $v \in V$ and each formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ for which the extension described above was made. We will show that the extended model graph $\mathcal{T}'$ is a TeamLog tableau for φ. As we argued above, the extension from $\mathcal{T}$ to $\mathcal{T}'$ sustains all the conditions of TeamLog tableau that were already satisfied for states of $\mathcal{T}$. Also, the conditions of TeamLog tableau are satisfied for all newly added states. Hence what remains to be shown are conditions **T2** and **TD** at states from V as well as condition **TC** for formulas of the form $[\mathrm{B}]^+_G\psi$.

For conditions **T2** and **TD** take any state $v \in V$. The only formulas from $L(v)$ for which condition **T2** is not satisfied in $\mathcal{T}$ are formulas of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ for which successors were not created by the algorithm for the reasons described above. Since for any such formula $\mathrm{B}'_j(v)$ extends $\mathrm{B}_j(v)$ with a new state u such that $\neg[\mathrm{B}]^+_G\psi \in L(u)$, so condition **T2** must be satisfied for that formula and v in $\mathcal{T}'$. For condition **TD**, take any formula of the form $[\mathrm{B}]_j\xi \in L(v)$ such that condition **TD** is not satisfied for it and for v in $\mathcal{T}$. Then there must be a formula of the form $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ for which successors were not created for the reasons described above, in which case $\mathrm{B}'_j(v)$ extends $\mathrm{B}_j(v)$ with a new state u and since condition **T1** is satisfied for v, so condition **TD** is satisfied for v and $[\mathrm{B}]_j\xi$.

What remains to be shown is that condition **TC** is satisfied in $\mathcal{T}'$ for all states from W and formulas of the form $\neg[\mathrm{B}]^+_G\psi$. So take any state $v \in W$ and any formula of the form $\neg[\mathrm{B}]^+_G\psi \in L(v)$. If there is $j \in G$ such that $\neg[\mathrm{B}]_j\psi \in L(v)$, then condition **TC** is satisfied for $\neg[\mathrm{B}]^+_G\psi$ and v by the fact that condition **T2** is satisfied for v. Suppose then that for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$. We will show first that condition **TC** holds for those of such states which are $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors with some $k \in G$. Notice first that no state of $\mathcal{T}$ which is a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor can be marked `undec`. For assume the opposite and suppose that $v \in W$ is such a state. Suppose also that v is a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor of some state u with $k \in G$. Let n be the $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-successor of u on the path from u to v. Then, by construction of the algorithm, it must be that $\neg[\mathrm{B}]_k[\mathrm{B}]^+_k\psi \in L(v)$, a successor could not be created for it and for each descendant m of u on the path from u to v, $n \in B(m)$. Moreover, for all $l \in G \setminus \{k\}$ it must be that $[\mathrm{B}]_l[\mathrm{B}]^+_G\psi \in L(v)$. Since this is true for any $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor of u which is marked `undec` and which is a descendant of n, so it must be that $B(n) = \{n\}$. Moreover, since $v \in W$, so there cannot be any state in $\mathcal{SS}(n)$ which is marked `sat`. But then, by construction of the algorithm, n would have to be marked `unsat` and, consequently, u would have to marked `unsat` as well and it could not be that $u \in W$. This contradicts the assumption that $v \in W$. Hence v must be marked `sat`. Now, to show that condition **TC** holds for $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors $v \in W$ such that for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$, we will use induction on the maximal distance from states to descendant leaves of $\mathcal{T}$. For the induction basis suppose that v is a leaf of $\mathcal{T}$ and a $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successor with $k \in G$. Since v is a state, so $L(v)$ must be a $[\mathrm{B}]^+$-expanded tableau and since for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$, so there must exist $j \in G$ such that $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$. Since v is a leaf of $\mathcal{T}$, so it must be that a successor for $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi$ could not be created, and since v cannot be marked `undec`, so it must be that $v \in V$. Then, by construction of $\mathcal{T}'$, there must exist $u \in \mathrm{B}'(v)$ with $\neg[\mathrm{B}]^+_G\psi \in L(u)$. Moreover, since $\neg[\mathrm{B}]_j\psi \notin L(v)$, so $u \notin W$ and, as we showed above, condition **TC** is satisfied for it and for $\neg[\mathrm{B}]^+_G\psi$. Hence there must exist $t \in \mathrm{B}'^{\,+}_G(u)$ such that $\sim\psi \in L(t)$. Since $t \in \mathrm{B}'^{\,+}_G(v)$, so condition **TC** is satisfied for v and $[\mathrm{B}]^+_G\psi$. For the induction step suppose that v is not a leaf of $\mathcal{T}$. If $v \in V$, then the condition **TC** is satisfied for v and $[\mathrm{B}]^+_G\psi$ by the same arguments as those used for the induction basis. Otherwise, there must exist $j \in G$ such that $\neg[\mathrm{B}]_j[\mathrm{B}]^+_G\psi \in L(v)$ and a successor of v was created for it (recall that v cannot be marked `undec`). By construction of

$\mathcal{T}$, there must exists $u \in \mathrm{B}_j(v)$ such that $\neg[\mathrm{B}]^+_G\psi \in L(u)$ and, by the induction hypothesis, condition **TC** is satisfied for u and $\neg[\mathrm{B}]^+_G\psi$. Hence, by analogous arguments to those used for the induction basis, condition **TC** is satisfied for v and $\neg[\mathrm{B}]^+_G\psi$ as well. Thus we have shown that show that condition **TC** holds for $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors $v \in W$ such that for all $j \in G$, $\neg[\mathrm{B}]_j\psi \notin L(v)$. For the case of states of $\mathcal{T}$ which are not $\neg[\mathrm{B}]_k[\mathrm{B}]^+_G\psi$-Successors with any $k \in G$, arguments analogous to those used in proof of Proposition 4 can be used to show that condition **TC** is satisfied for them and $\neg[\mathrm{B}]^+_G\psi$ as well.

Thus we have shown that if Algorithm 2 returns `sat` on input φ, then a TeamLog tableau for φ can be constructed which implies, by Proposition 2, that φ is satisfiable. □

Proof (Theorem 4) The problem of TeamLog satisfiability of formulas without fixpoint modalities is PSPACE hard. Hence the problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ is PSPACE hard.

To show that the problem is in PSPACE, we will show that Algorithm 2 can be run by a deterministic Turing machine using polynomial space with respect to $|\varphi|$. To show that we will use induction on the pair $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big)$ (in lexicographic order). For the induction basis, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) = 0$ (in this case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) = \mathcal{A} \cup \{\omega\}$). Then there are no formulas of the form $[\mathrm{B}]^+_G\psi \in \neg\mathrm{Sub}(\varphi)$ and Algorithm 2 works like Algorithm 1 on the input φ. Thus, by Theorem 2, it can be run by a deterministic Turing machine using polynomial space with respect to $|\varphi|$. For the induction step, suppose that $\mathrm{dep}_{[\mathrm{B}]^+}(\varphi) > 0$ (in which case $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subseteq \mathcal{A}$). To check the satisfiability of φ a pre-tableau is constructed and the decision with regard to the satisfiability is made on the basis of how the root of this pre-tableau is marked. Since the decision on how each node is marked in this pre-tableau depends on the nodes on the path from this node to the root of the pre-tableau and how descendants of this node are marked, so for deciding how the root node should be marked, the pre-tableau could be traversed in depth first search like manner. By Lemma 9, the depth of the pre-tableau constructed by Algorithm 2 is $\leq \mathcal{O}(|\varphi|\mathrm{dep}(\varphi)^{2|\mathcal{A}|+1})$. To mark some of the leaves of the pre-tableau constructed by Algorithm 2, Function 9 may be called. Each such call requires polynomial space. To see this, let ψ, G, j and Φ be the input for such call. Then $\Phi \subseteq \neg\mathrm{Cl}(\mathrm{Sub}(\varphi))$ and so $|\Phi| \leq (2|\mathcal{A}| + 1)|\varphi| = \mathcal{O}(\Phi)$. The algorithm enumerates the elements of $\mathcal{S}(\Phi/[\mathrm{B}]^+_H)$, where $H = \mathrm{ag}\Big(\Big(\Phi \sqcap [\mathrm{B}]^+_{\{j\}}\Big) \cup \{\neg[\mathrm{B}]^+_G\psi\}, [\mathrm{B}]^+\Big)$ is a subset of G. To enumerate these elements a pre-tableau is constructed with the root labelled with $\Phi/[\mathrm{B}]^+_H$ and then all the states that could be obtained from this set are enumerated. To enumerate these states depth first search method could be used, so that $\leq \mathcal{O}(|\varphi|)$ memory would be needed to remember each path leading from the root to a state. For each such a state it is checked whether its label is satisfiable, which can be done in polynomial space, as $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, where ξ is the conjunction of formulas in the states, and the induction hypothesis applies. Next for each such a state reachability to some other element of a graph $\mathcal{G}\Big(\big(\Phi/[\mathrm{B}]^+_H\big) \cup \{\psi\}\Big)$ is checked, and this requires using Function 10 recursively, with depth of recursion $\leq \mathcal{O}(|\varphi|)$. Function 10 enumerates elements of $\mathcal{S}\Big(\big(\Phi/[\mathrm{B}]^+_H\big) \cup \{\psi\}\Big)$, which requires $\leq \mathcal{O}(|\varphi|)$ memory. Additionally, the satisfiability of the labels of these states is checked and since $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$, where ξ is the conjunction of formulas in the label of a state, so, by the induction hypothesis, each such call requires polynomial space. Notice also that the upper bound on the number of steps of reachability can be stored using $\mathcal{O}(|\varphi|)$ space. Thus Function 10 uses polynomial space with respect to $|\varphi|$. Lastly, after checking reachability for a given state from $\mathcal{S}(\Phi/[\mathrm{B}]^+_H)$, Function 9 checks the satisfiability of some properly constructed sets of formulas obtained from the label of the state. Since for each such a set of formulas $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) < \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ or $\mathrm{dep}_{[\mathrm{B}]^+}(\xi) = \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)$ and $\mathrm{ag}\Big(\mathrm{Sub}(\varphi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\varphi)\Big) \subsetneq \mathrm{ag}\Big(\mathrm{Sub}(\xi), [\mathrm{B}]^+, \mathrm{dep}_{[\mathrm{B}]^+}(\xi)\Big)$, where ξ is the conjunction of the formulas in the set (c.f. proofs of Proposition 6 and Proposition 7), so, by the induction hypothesis, each such call uses at most polynomial space with respect to $|\varphi|$. Hence each call to Function 9 requires polynomial space with respect to $|\varphi|$ Thus the satisfiability of φ can be decided by a deterministic Turing machine using space of polynomial size with respect to $|\varphi|$. Hence the problem of TeamLog satisfiability of formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ is in PSPACE. □

Proof (Theorem 5) The problem is in PSPACE by Theorem 4. To show hardness we will construct a formula $\varphi^I_T \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R}_1}$ with $\mathrm{dep}(\varphi) = 2$ whose models encode the computation of a given polynomial space bounded deterministic Turing machine T on the input I. The constructed formula φ^I_T will be satisfiable if and only if the computation of T on I terminates in accepting state. A deterministic Turing machine is a tuple $T = (Q, \Sigma, \Gamma, \delta, \mathrm{B}, q_0, q_{\mathrm{A}}, q_{\mathrm{R}})$, where

- Q is a finite set of states,
- $q_0 \in Q$ is the starting state,
- $q_{\mathrm{A}} \in Q$ is the accepting state,
- $q_{\mathrm{R}} \in Q$ is the rejecting state,
- Γ is a finite worktape alphabet,
- $\Sigma \subseteq \Gamma$ is a finite input alphabet,
- $\mathrm{B} \in \Gamma \setminus \Sigma$ is the blank symbol,
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{-1, 0, 1\}$ is the transition function.

Let T be a deterministic Turing machine and I be its input and suppose that during its computation on I, T uses $\leq M(|I|)$ cells of the worktape, where $|I|$ denotes the size of I. To define the formula φ^I_T we will first define three formulas that describe the initial configuration of the machine, a valid configuration of the machine and a valid transition of the machine. To define the formulas we will use the following propositional symbols:

- a^x_i, where $x \in \Gamma$ and $1 \leq i \leq M(n)$, to indicate that the symbol in the i'th cell of the worktape is x;
- s^q_i, where $q \in Q$ and $1 \leq i \leq M(n)$, to indicate that the current state is q and the head of the machine is at the i'th cell of the worktape.

A configuration of the machine will be encoded by valuations of formulas $[\mathrm{I}]_k a^x_i$ and $[\mathrm{I}]_k s^q_i$, where $k \in \{1, 2\}$, $x \in \Gamma$, $q \in Q$ and $1 \leq i \leq M(|I|)$. This way two configurations of the machine are encoded at any world v of a TeamLog model, one by valuations of the formulas with operator $[\mathrm{I}]_1$ and another one by the formulas with operator $[\mathrm{I}]_2$ at this world. We will refer to them by $C_1(v)$ and $C_2(v)$, respectively.

Firstly, the initial configuration is when the head is at the first cell of the machine and the input is written in the first $|I|$ cells of the machine, while the remaining $M(|I|) - |I|$ cells are filled in with blanks. The formula describing the initial configuration is called INIT_I. At any given world v of a TeamLog model it holds that if INIT_I is satisfied there, then $C_1(v)$ encodes the initial configuration of the machine T on input I.

$$\mathsf{INIT}_I = [\mathrm{I}]_1 s_1^{q_0} \wedge \bigwedge_{i=1}^{n} [\mathrm{I}]_1 a_i^{I_i} \wedge \bigwedge_{i=n+1}^{M(|I|)} [\mathrm{I}]_1 a_i^{\mathrm{B}}. \tag{8}$$

Secondly, a formula stating that both C_1 and C_2 encode valid configurations is defined. The formula is called CONFIG_T and it is a conjunction of the two formulas below. The first of these formulas states that in each cell from 1 to $M(|I|)$ exactly one symbol from Γ is put. The second of these formulas states that that the machine is in exactly one state and the head is positioned at exactly one cell from 1 to $M(|I|)$. At any given world of a TeamLog model it holds that if CONFIG_T is satisfied there, then $C_1(v)$ and $C_2(v)$ represent valid configurations of T.

$$\bigwedge_{k\in\{1,2\}} \bigwedge_{i=1}^{M(|I|)} \bigvee_{x\in\Gamma} \left([\mathrm{I}]_k a_i^x \wedge \bigwedge_{y\in\Gamma\setminus\{x\}} \neg [\mathrm{I}]_k a_i^y \right) \tag{9}$$

$$\bigwedge_{k\in\{1,2\}} \left(\bigvee_{i=1}^{M(|I|)} \bigvee_{q\in Q} [\mathrm{I}]_k s_i^q \wedge \bigwedge_{i=1}^{M(|I|)} \bigwedge_{q\in Q} \left([\mathrm{I}]_k s_i^q \rightarrow \bigwedge_{r\in Q\setminus\{q\}} \neg [\mathrm{I}]_k s_i^r \right) \right) \tag{10}$$

Thirdly, transitions of the machine are described by a formula TRANS_T, which is a conjunction of the two formulas below. At any given world v of a TeamLog model it holds that if TRANS_T is satisfied there, then either $C_1(v)$ and $C_2(v)$ encode the same configuration of T or $C_2(v)$ encodes the configuration succeeding the configuration encoded by $C_1(v)$ in the run of the machine T on the input I.

$$\begin{aligned} &\bigwedge_{i=1}^{M(|I|)} \bigwedge_{x\in\Gamma} \bigwedge_{q\in Q} \left([\mathrm{I}]_1 a_i^x \wedge [\mathrm{I}]_1 s_i^q \right) \\ &\quad \rightarrow \left(\bigwedge_{j=1, j\neq i}^{M(|I|)} \bigwedge_{z\in\Gamma} \left([\mathrm{I}]_1 a_j^z \leftrightarrow [\mathrm{I}]_2 a_j^z \right) \wedge \left(\left([\mathrm{I}]_2 a_i^x \wedge [\mathrm{I}]_2 s_i^q \right) \vee \left([\mathrm{I}]_2 a_i^{\delta_2(q,x)} \wedge [\mathrm{I}]_2 s_{i+\delta_3(q,x)}^{\delta_1(q,x)} \right) \right) \right) \end{aligned} \tag{11}$$

$$\begin{aligned} &\bigwedge_{i=1}^{M(|I|)} \bigwedge_{x\in\Gamma} \bigwedge_{q\in Q} \left([\mathrm{I}]_2 a_i^{\delta_2(q,x)} \wedge [\mathrm{I}]_2 s_{i+\delta_3(q,x)}^{\delta_1(q,x)} \right) \\ &\quad \rightarrow \left(\bigwedge_{j=1, j\neq i}^{M(|I|)} \bigwedge_{z\in\Gamma} \left([\mathrm{I}]_1 a_j^z \leftrightarrow [\mathrm{I}]_2 a_j^z \right) \wedge \left(\left([\mathrm{I}]_1 a_i^{\delta_2(q,x)} \wedge [\mathrm{I}]_1 s_{i+\delta_3(q,x)}^{\delta_1(q,x)} \right) \vee \left([\mathrm{I}]_1 a_i^x \wedge [\mathrm{I}]_1 s_i^q \right) \right) \right) \end{aligned} \tag{12}$$

Let φ_T^I be defined as follows:

[18] Formula φ_T^I could be also constructed with use of operators $[\mathrm{G}]_1$ and $[\mathrm{G}]_2$ instead of $[\mathrm{I}]_1$ and $[\mathrm{I}]_2$.

$$\varphi_T^I = \mathsf{INIT}_I \wedge [\mathrm{B}]'^{+}_{\{1,2\}}(\mathsf{CONFIG}_T \wedge \mathsf{TRANS}_T) \wedge \neg[\mathrm{B}]^{+}_{\{1,2\}}\left(\neg \bigvee_{i=1}^{M(|I|)} [\mathrm{I}]_1 s_i^{q_A}\right), \qquad (13)$$

where $[\mathrm{B}]'^{+}_{G}\varphi$ is an abbreviation for $\varphi \wedge [\mathrm{B}]^{+}_{G}\varphi$.[18] Notice that the size of φ_T^I is polynomial with respect to |*I*|. To see that the if φ_T^I is satisfiable, then *T* accepts the input *I*, suppose that $(\mathcal{M}, w) \models \varphi_T^I$. Then $C_1(w)$ encodes the initial configuration of *T* on the input *I*, at all worlds $v \in \mathrm{B}_{\{1,2\}}{}^{+}(w)$, $C_1(v)$ and $C_2(v)$ encode valid configurations of *T* and either $C_1(v)$ and $C_2(v)$ encode the same configuration or the configuration encoded by $C_2(v)$ succeeds the configuration encode by $C_1(v)$ in the run of the machine *T* on the input *I*. Moreover, there exists $u \in \mathrm{B}_{\{1,2\}}{}^{+}(w)$ such that $C_1(u)$ encodes a configuration at accepting state of *T*. The computation of *T* on *I* that leads to that configuration can be read from the path leading from *w* to *u*. For any two subsequent states v_1 and v_2 on this path, such that $v_2 \in R^{\mathrm{B}_j}(v_1)$ with $j \in \{1, 2\}$, by generalized transitivity it holds that $C_j(v_1)$ and $C_j(v_2)$ encode the same configuration. Hence the configurations encoded by C_1 at the subsequent worlds on the path represent states of the run of *T* on the input *I* with the possibility that at some worlds not transitions is performed or preceding states of the machine are restored. After removing such states from the sequence, a sequence of configurations C_1 can be obtained that represents the whole run accepting run of *T* on the input *I*. On the other hand suppose *T* accepts the input *I*. Then we could construct a TeamLog model containing a sequence of worlds connected alternately by accessibility relations B_2 and B_1 such that at every second state of this sequence the configurations of the machine encoded by C_1 are the subsequent configuration of the run of *T* on the input *I*. This shows that *T* accepts *I* if and only if φ_T^I is satisfiable. Thus we have shown that the problem of TeamLog satisfiability for formulas from $\mathcal{L}^{\mathrm{T}}_{\mathbf{R_1}}$ with modal depth bounded by 2 is PSPACE hard and since it is also in PSPACE, so it is PSPACE complete. □

Proofs associated with checking TeamLog satisfiability for $\mathcal{L}^{\mathbf{T}}_{\mathbf{R_1(c)}}$

Proof (Lemma 10) Claims 1–1 shown in proof of Lemma 5 hold in the case of $\varphi \in \mathcal{L}^{\mathrm{T}}_{\mathbf{R_1(n)}}$, as they require modal context restriction $\mathbf{R}_1$ only.

Consider a sequence of states $s_0, \ldots, s_m$ in the pre-tableau such that for any $0 < k \leq m$, s_k is an B_{j_k}-Successor of s_{k-1}. Suppose that for any $0 < k \leq m$ it holds that $\mathrm{Ind}(L(s_{k-1})) \sqcap j_k = \varnothing$ and the sequence satisfies properties **PB1**, **PI1** and **PB2** for all $d \geq 0$. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^{+} = \varnothing$ or there is more than one formula of the form $\neg[\mathrm{B}]^{+}_{G}\psi \in \widehat{\mathrm{Gr}}(L(s_0))$, then the length of such a sequence must be ≤ 2, by the same arguments as those used in proof of Lemma 5. If $\widehat{\mathrm{Gr}}(L(s_0)) \sqcap \neg[\mathrm{B}]^{+} = \{\neg[\mathrm{B}]^{+}_{G}\psi\}$, then the length of the sequence must be $\leq 2^{c+1}|\mathcal{A}| + 1$. Arguments here are similar to those used in proof of 5 for analogous case. Suppose that the length of the sequence $m > 2^{c+1}|\mathcal{A}| + 1$. Then there exists $0 < k_1 < \cdots < k_{2^{c+1}+1} \leq m$ such that $j_{k_1} = \cdots = j_{k_{2^{c+1}+1}}$. By Claim 3 the subsequent sets $L^{\neg[\mathrm{B}]_{j_{k_i}}}(s_{k_i-1}, [\mathrm{B}]^{+}_{G}\psi)$ differ on the elements from $Z_{j_{k_1}} \cup \{\psi\}$ only, where $Z_{j_{k_1}} = \neg\left(\left(L(s_{k_1-1}) \sqcap [\mathrm{I}]_{j_{k_1}}\right) \cup \left(L(s_{k_1-1}) \sqcap [\mathrm{G}]_{j_{k_1}}\right) \cup \{[\mathrm{B}]_{j_{k_1}}\psi\}\right)$. Moreover, each of these sets contains a maximal subset of $Z_{j_{k_1}}$ as a subset and contains ψ if and only if it contains

$[\mathrm{B}]_{j_{k_1}}\psi$. Hence there are $2^{\left|Z_{j_{k_1}}\right|/2}$ such different sets and, by modal context restriction $\mathbf{R}_{1(c)}$, $\left|Z_{j_{k_1}}\right| \leq 2(c+1)$. Thus there must exist $1 \leq i' < i \leq 2^{c+1}$ such that $L^{\neg[\mathrm{B}]_{j_{k_{i'}}}}(s_{k_{i'}-1}, [\mathrm{B}]_G^+\psi) = L^{\neg[\mathrm{B}]_{j_{k_i}}}(s_{k_i-1}, [\mathrm{B}]_G^+\psi)$. But then, by construction of the algorithm, the $\neg[\mathrm{B}]_{j_{k_i}}[\mathrm{B}]_G^+\psi$-successor of s_{k_i-1} cannot be created and so s_{k_i} cannot be in the sequence, which contradicts our assumptions.

Using arguments similar to those used in Lemma 5 it can be shown that the maximal length of a sequence of B-Successors with the same modal depth of labels is $\mathcal{O}\left(\mathrm{dep}(\varphi)^{2|\mathcal{A}|}\right)$. Moreover, 2^{c+1} contributes to the factor of $\mathrm{dep}(\varphi)^{2|\mathcal{A}|}$. □

Proof (Lemma 11) Proof is by analogous arguments to those use in proof of Lemma 7, using Lemma 10 instead of Lemma 5. □

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