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Fairness criteria for allocating indivisible chores: connections and efficiencies

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Abstract

We study several fairness notions in allocating indivisible *chores* (i.e., items with disutilities) to agents who have additive and submodular cost functions. The fairness criteria we are concerned with are envy-free up to any item, envy-free up to one item, maximin share (MMS), and pairwise maximin share (PMMS), which are proposed as relaxations of envy-freeness in the setting of additive cost functions. For allocations under each fairness criterion, we establish their approximation guarantee for other fairness criteria. Under the additive setting, our results show strong connections between these fairness criteria and, at the same time, reveal intrinsic differences between goods allocation and chores allocation. However, such strong relationships cannot be inherited by the submodular setting, under which PMMS and MMS are no longer relaxations of envy-freeness and, even worse, few non-trivial guarantees exist. We also investigate efficiency loss under these fairness constraints and establish their prices of fairness.

Keywords Fair division · Indivisible chores · Price of fairness

1 Introduction

Fair division is a central matter of concern in economics, multiagent systems, and artificial intelligence [6, 16, 18]. Over the years, there emerges a tremendous demand for fair division when a set of indivisible resources, such as classrooms, tasks, and properties, are divided among a group of agents. This field has attracted the attention of researchers and most results are established when resources are considered as goods that bring positive utility to agents. However, in real-life division problems, the resources to be allocated can

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also be chores which, instead of positive utility, bring non-positive utility or cost to agents. For example, one might need to assign tasks among workers, teaching load among teachers, sharing noxious facilities among communities, and so on. Compared to goods, fairly dividing chores is relatively under-developed. At first glance, dividing chores is similar to dividing goods. However, in general, chores allocation is not covered by goods allocation and results established on goods do not necessarily hold on chores. Existing works have already pointed out this difference in the context of envy-freeness [14, 15, 19] and equitability [30, 31]. As an example, Freeman et al. [30] indicate that, when allocating goods, a *leximin*[1] allocation is Pareto optimal and *equitable up to any item*,[2] however, a leximin solution does not guarantee equitability up to any item in chores allocation.

Among the variety of fairness notions in the literature, *envy-freeness* (EF) is one of the most compelling, which has drawn research attention over the past few decades [17, 27, 29]. In an envy-free allocation, no agent envies another agent. Unfortunately, the existence of an envy-free allocation cannot be guaranteed in general when the items are indivisible. A canonical example is that one needs to assign one chore to two agents and the chore has a positive cost for either agent. Clearly, the agent who receives the chore will envy the other. In addition, deciding the existence of an EF allocation is computationally intractable, even for two agents with identical preference. Given this predicament, recent studies mainly devote to relaxations of envy-freeness. One direct relaxation is known as *envy-free up to one item* (EF1) [20, 36]. In an EF1 allocation, one agent may be jealous of another, but by removing one chore from the bundle of the envious agent, envy can be eliminated. A similar but stricter notion is *envy-free up to any item* (EFX) [22]. In such an allocation, envy can be eliminated by removing any positive-cost chore from the envious agent's bundle. Another fairness notion, *maximin share* (MMS) [3, 20], generalizes the idea of "cut-and-choose" protocol in cake cutting. The maximin share is obtained by minimizing the maximum cost of a bundle of an allocation over all allocations. The last fairness notion we consider is called *pairwise maximin share* (PMMS) [22], which is similar to maximin share but different from MMS in that each agent partitions the combined bundle of himself and any other agent into two bundles and then receives the one with the larger cost.

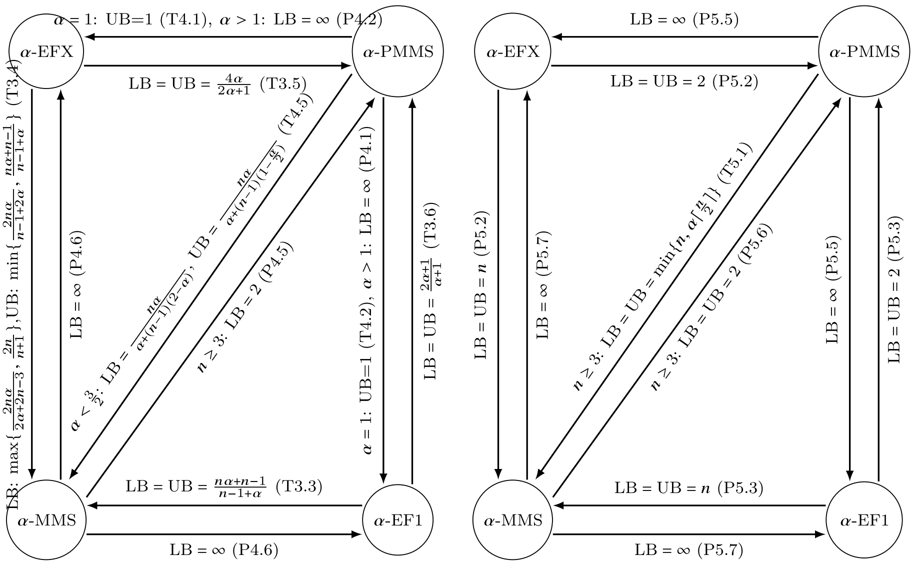
The existing research on envy-freeness and its relaxations concentrates on algorithmic features of fairness criteria, such as their existence and (approximation) algorithms for finding them. Relatively little research studies the connections between these fairness criteria themselves, or the trade-off between these fairness criteria and the system efficiency, known as *the price of fairness*.

When allocating goods, Amanatidis et al. [2] compare the above four relaxations of envy-freeness and provide results on the approximation guarantee of one to another. However, these connections are unclear in allocating chores. On the price of fairness, Bei et al. [11] study allocation of indivisible goods and focus on the notions for which the corresponding fair allocations are guaranteed to exist, such as EF1, maximum Nash welfare,[3] and leximin. Caragiannis et al. [21] study the price of fairness for both chores and goods,

[1] A leximin solution selects the allocation that maximizes the utility of the least well-off agent, subject to maximizing the utility of the second least, and so on.

[2] Equitability requires that any pair of agents are equally happy with their bundles. In equitability up to any item allocations, the violation of equitability can be eliminated by removing any single item from the happier (in goods allocation)/ less happy agent (in chores allocation).

[3] Nash welfare is the product of agents' utilities.



(a) Additive (b) Submodular

Fig. 1 Connections between fairness criteria. *Note* Fig. 1a, b illustrate connections between fairness criteria under additive and submodular cost functions, respectively. LB and UB stand for lower and upper bound, respectively. P*x.y* and T*x.y* point to Proposition *x.y* and Theorem *x.y*, respectively

and focus on the classical fairness notions, namely, EF, proportionality[4] and equitability. To the best of our knowledge, no existing work covers the price of fairness for the aforementioned four relaxations (which we will call *additive relaxations* from time to time in this paper) of envy-freeness in chores allocation.

In this paper, we fill these gaps by investigating the four relaxations of envy-freeness on two aspects. On the one hand, we study the connections between these criteria and, in particular, we consider the following questions: *Does one fairness criterion implies another? To what extent can one criterion guarantee for another?* On the other hand, we study the trade-off between fairness and *efficiency* (or *social cost* defined as the sum of costs of the individual agents). Specifically, for each fairness criterion, we investigate its *price of fairness*, which is defined as the supremum ratio of the minimum social cost of a fair allocation to the minimum social cost of any allocation.

1.1 Main results

On the connections between fairness criteria, we summarize our main results in Fig. 1 on the approximation guarantee of one fairness criterion for another. Figures 1a and 1b show connections under additive and submodular settings, respectively. As shown in Fig. 1a below, when agents have additive cost functions, there exist evidently significant connections

[4] An allocation of goods (resp. chores) is proportional if the value (resp. cost) of every agent's bundle is at least (resp. at most) one n-th fraction of his value (resp. cost) for all items.

Table 1 Prices of fairness

	EFX	EF1	PMMS	$\frac{3}{2}$-PMMS	2-MMS	
$n = 2$	2 (T6.4)	$\frac{5}{4}$ (T6.1)	2 (T6.4)	$\frac{7}{6}$ (T6.3)	1 (L2.2)	additive
	[3, 4) (T7.1)	[2, 4) (T7.2)	3 (T7.3)	$[\frac{4}{3}, \frac{8}{3})$ (T7.4)		submodular
$n \geq 3$			∞ (T6.5)		$[\frac{n+3}{6}, n)$ (T6.7)	additive
					$[\frac{n+3}{6}, \frac{n^2}{2})$ (T7.6)	submodular

Note Interval $[a, b)$ means that the lower bound is equal to a and upper bound is equal to b. T$x.y$ and L$x.y$ point to Theorem $x.y$ and Lemma $x.y$, respectively. The price of MMS with respect to submodular cost functions is open

between these fairness notions. While some of our results show similarity to those in goods allocation [2], others also reveal their difference. Figure 1b provides the corresponding results under the submodular setting, which then show a sharp contrast to results under the additive setting. More specifically, except that PMMS can have a bounded approximation guarantee on MMS, no non-trivial guarantee exists between any other pair of fairness notions.

After comparing each pair of fairness notions, we compare the efficiency of fair allocations with the optimal one. To quantify the efficiency loss, we apply the idea of the price of fairness and our results are summarized in Table 1. The terminology "α-XYZ" below refers to an α approximation for fairness notion XYZ. The formal definitions will be given in Sect. 2.

As detailed later in the paper, most of the results summarized in Fig. 1 and Table 1 are tight.

1.2 Related works

The fair division problem has been studied for both indivisible goods [13, 22, 36] and indivisible chores [5, 7, 31]. Among various fairness notions, a prominent one is EF proposed by Foley [29]. But an EF allocation may not exist and even worse, checking the existence of an EF allocation is NP-complete [6]. For the relaxations of envy-freeness, the notion of EF1 originates from Lipton et al. [36] and is formally defined by Budish [20]. Lipton et al. [36] provide an efficient algorithm for EF1 allocations of goods when agents have monotone valuations functions. When allocating chores, Aziz et al. [4] show that, in the additive setting, EF1 is achievable by allocating chores in a round-robin fashion. Another fairness notion that has been a subject of much attention in the last few years is MMS, proposed by Budish [20]. However, existence of an MMS allocation is not guaranteed either for goods [35] or for chores [7], even with additive valuation functions. Consequently, more efforts are on approximation of MMS in the additive setting, with Amanatidis et al. [3], Ghodsi et al. [33], Garg and Taki [32] on goods and Aziz et al. [7], Huang and Lu [34] on chores. Some other studies consider approximating MMS when agents have (a subclass of) submodular valuation functions. Barman and Krishnamurthy [9] consider the submodular setting and show that 0.21-approximation of MMS can be efficiently computed by the round-robin algorithm. Barman and Verma [10] show that an MMS allocation is guaranteed to exist and can be computed efficiently if agents have submodular valuation functions with binary margin.

The notions of EFX and PMMS are introduced by Caragiannis et al. [22]. They consider goods allocation and establish that a PMMS allocation is also EFX when the valuation functions are additive. Beyond the simple case of $n = 2$, the existence of an EFX allocation has not been settled in general. However, significant progress has been made for some special cases. When $n = 3$, the existence of an EFX allocation of goods is proved by Chaudhury et al. [24]. Based on a modified version of leximin solutions, Plaut and Roughgarden [37] show that an EFX allocation is guaranteed to exist when all agents have identical valuations. The work most related to ours is by Amanatidis et al. [2], which is on goods allocation under additive setting, and provides connections between the above four relaxations of envy-freeness.

As for the price of fairness, Caragiannis et al. [21] show that, in the case of *divisible* goods, the price of proportionality is $\Theta(\sqrt{n})$ and the price of equitability is $\Theta(n)$. Bertsimas et al. [12] extend the study to other fairness notions, *maximin*[5] fairness and proportional fairness, and they provide a tight bound on the price of fairness for a broad family of problems. Bei et al. [11] focus on indivisible goods and concentrate on the fairness notions that are guaranteed to exist. They present an asymptotically tight upper bound of $\Theta(n)$ on the price of maximum Nash welfare [26], maximum egalitarian welfare [18] and leximin. They also consider the price of EF1 but leave a gap between the upper bound $O(n)$ and lower bound $\Omega(\sqrt{n})$. This gap is later closed by Barman et al. [8] with the results that, for both EF1 and $\frac{1}{2}$-MMS, the price of fairness is $O(\sqrt{n})$. All the work reviewed above on the price of fairness is on the additive setting. On the other hand, the price of fairness has been studied in other multi-agent systems, such as machine scheduling [1] and kidney exchange [28].

2 Preliminaries

In the problem of a fair division of indivisible chores, we have a set $N = \{1, 2, \dots, n\}$ of agents and a set $E = \{e_1, \dots, e_m\}$ of indivisible chores. As chores are the items with non-positive values, each agent $i \in N$ is associated with a cost function $c_i : 2^E \to R_{\geq 0}$, which maps any subset of E into a non-negative real number. Throughout this paper, we assume $c_i(\emptyset) = 0$ and c_i is monotone, that is, $c_i(S) \leq c_i(T)$ for any $S \subseteq T \subseteq E$. We say a (set) function $c(\cdot)$ is:

- *Additive*, if $c(S) = \sum_{e \in S} c(e)$ for any $S \subseteq E$.
- *Submodular*,[6] if for any $S, T \subseteq E$, $c(S \cup T) + c(S \cap T) \leq c(S) + c(T)$.
- *Subadditive*, if for any $S, T \subseteq E$, $c(S \cup T) \leq c(S) + c(T)$.

Clearly, additivity implies submodularity, which in turn implies subadditivity. For simplicity, instead of $c_i(\{e_j\})$, we use $c_i(e_j)$ to represent the cost of chore e_j for agent i.

An *allocation* $\mathbf{A} := (A_1, \dots, A_n)$ is an n-partition of E among agents in N, i.e., $A_i \cap A_j = \emptyset$ for any $i \neq j$ and $\cup_{i \in N} A_i = E$. Each subset $S \subseteq E$ also refers to a *bundle* of

[5] It maximizes the lowest utility level among all the agents.

[6] An equivalent definition is as follows: $c(\cdot)$ is submodular if for any $S \subseteq T \subseteq E$ and $e \in E \setminus T$, $c(T \cup \{e\}) - c(T) \leq c(S \cup \{e\}) - c(S)$.

chores. For any bundle S and $k \in \mathbb{N}^+$, we denote by $\Pi_k(S)$ the set of all k-partition of S, and $|S|$ the number of chores in S.

2.1 Fairness criteria

We study envy-freeness and its (additive) relaxations and are concerned with both exact and approximate versions of these fairness notions.

Definition 2.1 For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-EF if for any $i,j \in N, c_i(A_i) \leq \alpha \cdot c_i(A_j)$. In particular, 1-EF is simply called EF.

Definition 2.2 For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-EF1 if for any $i,j \in N$, there exists $e \in A_i$ such that $c_i(A_i \backslash \{e\}) \leq \alpha \cdot c_i(A_j)$. In particular, 1-EF1 is simply called EF1.

Definition 2.3 For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-EFX if for any $i,j \in N, c_i(A_i \backslash \{e\}) \leq \alpha \cdot c_i(A_j)$ for any $e \in A_i$ with $c_i(e) > 0$. In particular, 1-EFX is simply called EFX.

Clearly, EFX[7] is stricter than EF1. Next, we formally introduce the notion of maximin share. For any $k \in [n] = \{1, \ldots, n\}$ and bundle $S \subseteq E$, the maximin share of agent i on S among k agents is

$$\mathrm{MMS}_i(k, S) = \min_{A \in \Pi_k(S)} \max_{j \in [k]} c_i(A_j).$$

We are interested in the allocation in which each agent receives cost no more than his maximin share.

Definition 2.4 For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-MMS if for any $i \in N, c_i(A_i) \leq \alpha \cdot \mathrm{MMS}_i(n, E)$. In particular, 1-MMS is called MMS.

Definition 2.5 For any $\alpha \geq 1$, an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is α-PMMS if for any $i,j \in N$,

$$c_i(A_i) \leq \alpha \cdot \min_{\mathbf{B} \in \Pi_2(A_i \cup A_j)} \max \left\{ c_i(B_1), c_i(B_2) \right\}.$$

In particular, 1-PMMS is called PMMS.

Note that the right-hand side of the above inequality is equivalent to $\alpha \cdot \mathrm{MMS}_i(2, A_i \cup A_j)$.

Example 2.1 Let us consider an example with three agents and a set $E = \{e_1, \ldots, e_7\}$ of seven chores. Agents have additive cost functions, displayed in the table below.

[7] Note Plaut and Roughgarden [37] consider a stronger version of EFX by dropping the condition $c_i(e) > 0$. In this paper, all results about EFX, except Theorems 4.1 and 4.4, still hold under the stronger version.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
Agent 1	2	3	3	0	4	2	1
Agent 2	3	1	3	2	5	0	5
Agent 3	1	5	10	2	3	1	3

It is not hard to verify that $\text{MMS}_1(3, E) = 5, \text{MMS}_2(3, E) = 7$, $\text{MMS}_3(3, E) = 10$. For instance, agent 2 can partition E into three bundles: $\{e_1, e_3\}$, $\{e_2, e_7\}, \{e_4, e_5, e_6\}$, so that the maximum cost of any single bundle for her is 7. Moreover, there is no other partitions that can guarantee a better worst-case cost.

We now examine allocation $\mathbf{A}$ with $A_1 = \{e_1, e_4, e_7\}, A_2 = \{e_2, e_3, e_6\}, A_3 = \{e_5\}$. We can verify that $c_i(A_i) \leq c_i(A_j)$ for any $i, j \in [3]$ and thus allocation $\mathbf{A}$ is EF that is then also EFX, EF1, MMS and PMMS. For another allocation $\mathbf{B}$ with $B_1 = \{e_1, e_5, e_7\}, B_2 = \{e_2, e_4, e_6\}, B_3 = \{e_3\}$, agent 1 would still envy agent 2 even if chore e_7 is eliminated from her bundle, and hence, allocation $\mathbf{B}$ is neither exact EF nor EFX. One can verify that $\mathbf{B}$ is indeed $\frac{7}{3}$-EF and 2-EFX. Moreover, allocation $\mathbf{B}$ is EF1 because agent 1 would not envy others if chore e_5 is eliminated from her bundle and agent 3 would not envy others if chore e_3 is eliminated from her bundle. As for the approximation guarantee on the notions of MMS and PMMS, it is not hard to verify that allocation $\mathbf{B}$ is $\frac{7}{5}$-MMS and $\frac{7}{5}$-PMMS.

2.2 Price of fairness

Let $I = \langle N, E, (c_i)_{i \in N} \rangle$ be an instance of the problem for allocating indivisible chores and let $\mathcal{I}$ be the set of all such instances. The *social cost* of an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is defined as $\text{SC}(\mathbf{A}) = \sum_{i \in N} c_i(A_i)$. The optimal social cost for an instance I, denoted by $\text{OPT}(I)$, is the minimum social cost over all allocations for this instance. Following previous work [11, 21], when study the price of fairness, we assume that agents cost functions are normalized to one, i.e., $c_i(E) = 1$ for all $i \in N$.

The *price of fairness* is the supremum ratio over all instances between the social cost of the "best" fair allocation and the optimal social cost, where "best" refers to the one with the minimum cost. Since we consider several fairness criteria, let F be any given fairness criterion and define by $F(I)$ as the set (possibly empty) of all allocations for instance I that satisfy fairness criterion F.

Definition 2.6 For any given fairness property F, the price of fairness with respect to F is defined as

$$\text{PoF} = \sup_{I \in \mathcal{I}} \min_{\mathbf{A} \in F(I)} \frac{\text{SC}(\mathbf{A})}{\text{OPT}(I)}.$$

Note that in the case where it is unclear whether $F(I) \neq \emptyset$ for any instances, we only consider those instances I with $F(I) \neq \emptyset$.

2.3 Some simple observations

We begin with some initial results, which reveal some intrinsic difference in allocating goods and allocating chores as far as approximation guarantee is concerned. Our proof of any result in this paper either immediately follows the statement of the result, or can be found in the Appendix clearly indicated. First, we state a simple lemma concerning lower bounds of the maximin share.

Lemma 2.1 *When agents have subadditive cost functions, for any $i \in N$ and $S \subseteq E$, we have*

$$\mathrm{MMS}_i(k,S) \geq \frac{1}{k} c_i(S), \forall k \in [n]; \qquad \mathrm{MMS}_i(k,S) \geq c_i(e), \forall e \in S, \forall k \in [n].$$

Proof Let $\mathbf{T} = (T_1, \dots, T_k)$ be the k-partition of S defining $\mathrm{MMS}_i(k,S)$; that is $\max_{T_j} c_i(T_j) = \mathrm{MMS}_i(k,S)$. We start with the lower bound $\frac{1}{k} c_i(S)$. Without loss of generality, assume $c_i(T_1) \geq c_i(T_2) \geq \cdots \geq c_i(T_k)$ and as a result, we have $c_i(T_1) = \mathrm{MMS}_i(k,S)$. Then, the following holds

$$kc_i(T_1) \geq \sum_{j=1}^{k} c_i(T_j) \geq c_i\left(\bigcup_{j=1}^{k} T_j\right) = c_i(S),$$

where the second transition is due to subadditivity. Due to $c_i(T_1) = \mathrm{MMS}_i(k,S)$, we have $\mathrm{MMS}_i(k,S) \geq \frac{1}{k} c_i(S)$. As for the lower bound $c_i(e)$, for any given chore $e \in S$, there must exist a bundle $T_{j'}$ containing e. Due to the monotonicity of cost function, we have $c_i(T_{j'}) \geq c_i(e)$, which combines $\mathrm{MMS}_i(k,S) = c_1(T_1) \geq c_1(T_{j'})$, implying $\mathrm{MMS}_i(k,S) \geq c_i(e)$. □

Based on the lower bounds in Lemma 2.1, we provide a trivial approximation guarantee for PMMS and MMS.

Lemma 2.2 *When agents have subadditive cost functions, any allocation is 2-PMMS and n-MMS.*

Proof Let $\mathbf{A} = (A_1, \dots, A_n)$ be an arbitrary allocation without any specified properties. We first show it's already an n-MMS allocation. By Lemma 2.1, for each agent i, we have $c_i(E) \leq n \cdot \mathrm{MMS}_i(n,E)$. Then, due to the monotonicity of the cost function, $c_i(A_i) \leq c_i(E) \leq n \cdot \mathrm{MMS}_i(n,E)$ holds.

Next, by a similar argument, we prove the result about 2-PMMS. By Lemma 2.1, $c_i(A_i \cup A_j) \leq 2\mathrm{MMS}_i(2, A_i \cup A_j)$ holds for any $i, j \in N$. Again, due to the monotonicity of the cost function, we have $c_i(A_i) \leq c_i(A_i \cup A_j)$ that implies $c_i(A_i) \leq 2\mathrm{MMS}_i(2, A_i \cup A_j)$. Therefore, allocation $\mathbf{A}$ is also 2-PMMS, completing the proof. □

As can be seen from the proof of Lemma 2.2, in allocating chores, if one assigns all chores to one agent, then the allocation still has a bounded approximation for PMMS and MMS. However, when allocating goods, if an agent receives nothing but his maximin share is positive, then clearly the corresponding allocation has an infinite approximation guarantee for PMMS and MMS.

3 Guarantess from envy-based relaxations

Let us start with EF. According to the definitions, for any $\alpha \geq 1$, α-EF is stronger than α-EFX and α-EF1. The following theorems present the approximation guarantee of α-EF for MMS and PMMS.

Theorem 3.1 *When agents have additive cost functions, for any $\alpha \geq 1$, an α-EF allocation is also $\frac{n\alpha}{n-1+\alpha}$-MMS, and this result is tight.*

Proof We first prove the upper bound and focus on agent i. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-EF allocation, then according to its definition, $c_i(A_i) \leq \alpha \cdot c_i(A_j)$ holds for any $j \in N$. By summing up j over $N \backslash \{i\}$, we have $(n-1)c_i(A_i) \leq \alpha \cdot \sum_{j \in N \backslash \{i\}} c_i(A_j)$ and as a result, $(n-1+\alpha)c_i(A_i) \leq \alpha \cdot \sum_{j \in N} c_i(A_j) = \alpha \cdot c_i(E)$ where the last transition owing to the additivity of cost functions. On the other hand, from Lemma 2.1, it holds that $\mathrm{MMS}_i(n, E) \geq \frac{1}{n} c_i(E)$, implying the ratio

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(n, E)} \leq \frac{n\alpha}{n-1+\alpha}.$$

Regarding tightness, consider the following instance with n agents and n^2 chores denoted as $\{e_1, \dots, e_{n^2}\}$. Agents have an identical cost profile and for every $i \in [n]$, $c_i(e_j) = \alpha$ for $1 \leq j \leq n$ and $c_i(e_j) = 1$ for $n+1 \leq j \leq n^2$. Consider allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_i = \{e_{(i-1)n+1}, \dots, e_{in}\}$ for any $i \in N$. It is not hard to verify that allocation $\mathbf{B}$ is α-EF. As for $\mathrm{MMS}_1(n, E)$, since in total we have n chores with each cost α and $(n-1)n$ chores with each cost 1, then in the partition defining $\mathrm{MMS}_1(n, E)$, each bundle contains exactly one chore with cost α and $n-1$ chores with cost 1. Consequently, we have $\mathrm{MMS}_1(n, E) = n-1+\alpha$ and the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(n,E)} = \frac{n\alpha}{n-1+\alpha}$. □

Theorem 3.2 *When agents have additive cost functions, for any $\alpha \geq 1$, an α-EF allocation is also $\frac{2\alpha}{1+\alpha}$-PMMS, and this result is tight.*

Proof We first prove the upper bound. Let $\mathbf{A} = (A_1, A_2, \dots, A_n)$ be an α-EF allocation, then according to the definition, for any $i, j \in N$, $c_i(A_i) \leq \alpha \cdot c_i(A_j)$ holds. By additivity, we have $c_i(A_i \cup A_j) = c_i(A_i) + c_i(A_j) \geq (1 + \frac{1}{\alpha}) \cdot c_i(A_i)$, and consequently, $c_i(A_i) \leq \frac{\alpha}{\alpha+1} \cdot c_i(A_i \cup A_j)$ holds. On the other hand, from Lemma 2.1, we know $c_i(A_i \cup A_j) \leq 2 \cdot \mathrm{MMS}_i(2, A_i \cup A_j)$, and therefore the following holds

$$c_i(A_i) \leq \frac{2\alpha}{\alpha+1} \cdot \mathrm{MMS}_i(2, A_i \cup A_j).$$

As for tightness, consider an instance with n agents and $2n$ chores denoted as $\{e_1, e_2, \dots, e_{2n}\}$. Agents have identical cost profile and for every $i \in [n]$, $c_i(e_1) = c_i(e_2) = \alpha$ and $c_i(e_j) = 1$ for $3 \leq j \leq 2n$. Now, consider an allocation $\mathbf{B} = (B_1, \dots, B_n)$ where $B_i = \{e_{2i-1}, e_{2i}\}$ for any $i \in N$. It is not hard to verify that allocation $\mathbf{B}$ is α-EF and except for agent 1, no one else will violate the condition of PMMS. For any $j \geq 2$, one can calculate $\mathrm{MMS}_1(2, B_1 \cup B_j) = 1 + \alpha$, yielding the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(2, B_1 \cup B_j)} = \frac{2\alpha}{1+\alpha}$, as required. □

Theorem 3.2 indicates that the approximation guarantee of α-EF for PMMS is independent of the number of agents. However, according to Theorem 3.1, its approximation guarantee for MMS is affected by the number of agents. Moreover, this guarantee ratio converges to α as n goes to infinity.

We remark that none of EFX, EF1, PMMS and MMS has a bounded guarantee for EF. We show this by a simple example. Consider an instance of two agents and one chore, and the chore has a positive cost for both agents. Assigning the chore to an arbitrary agent results in an allocation that satisfies EFX, EF1, PMMS and MMS, simultaneously. However, since one agent has a positive cost on his own bundle and zero cost on other agents' bundle, such an allocation has an infinite approximation guarantee for EF.

Next, we consider approximation of EFX and EF1.

Proposition 3.1 *When agents have additive cost functions, an α-EFX allocation is α-EF1 for any $\alpha \geq 1$. On the other hand, an EF1 allocation is not β-EFX for any $\beta \geq 1$.*

Proof We first show the positive part. Let $\mathbf{A} = (A_1, A_2, \dots, A_n)$ be an α-EFX allocation, then according to its definition, $\forall i, j \in N, \forall e \in A_i$ with $c_i(e) > 0$, $c_i(A_i \setminus \{e\}) \leq \alpha \cdot c_i(A_j)$ holds. This implies $\mathbf{A}$ is also α-EF1.

For the impossibility result, consider an instance with n agents and $2n$ chores denoted as $\{e_1, e_2, \dots, e_{2n}\}$. Agents have identical cost profile. The cost function of agent 1 is: $c_1(e_1) = p, c_1(e_j) = 1, \forall j \geq 2$ where $p \gg 1$. Now consider an allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_i = \{e_{2i-1}, e_{2i}\}, \forall i \in N$. It is not hard to see allocation $\mathbf{B}$ is EF1 and except for agent 1, no one else will envy the bundle of others. Thus, we only concern agent 1 when calculate the approximation guarantee for EFX. By removing chore e_2 from bundle B_1, $\frac{c_1(B_1 \setminus \{e_2\})}{c_1(B_j)} = \frac{p}{2}$ holds for any $j \in N \setminus \{1\}$, and the ratio $\frac{p}{2} \to \infty$ as $p \to \infty$. □

Next, we consider the approximation guarantee of EF1 for MMS. In allocating goods, Amanatidis et al. [2] present a tight result that an α-EF1 allocation is $O(n)$-MMS. In contrast, in allocating chores, α-EF1 can have a much better guarantee for MMS.

Theorem 3.3 *When agents have additive cost functions, for any $\alpha \geq 1$ and $n \geq 2$, an α-EF1 allocation is also $\frac{n\alpha+n-1}{n-1+\alpha}$-MMS, and this result is tight.*

Proof We first prove the upper bound. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-EF1 allocation and the approximation guarantee for MMS is determined by agent i. We can further assume $c_i(A_i) > 0$; otherwise agent i meets the condition of MMS and we are done. Let $\bar{e}$ be the chore with largest cost for agent i in bundle A_i, i.e., $\bar{e} \in \arg\max_{e \in A_i} c_i(e)$.

By the definition of α-EF1, for any $j \in N \setminus \{i\}$, $c_i(A_i \setminus \{\bar{e}\}) \leq \alpha \cdot c_i(A_j)$ holds. Then, by summing up j over $N \setminus \{i\}$ and adding a term $\alpha c_i(A_i)$ on both sides, the following holds,

$$\alpha \cdot \sum_{j \in N} c_i(A_j) \geq (n - 1 + \alpha)c_i(A_i) - (n - 1)c_i(\bar{e}). \tag{1}$$

From Lemma 2.1, we have $\mathrm{MMS}_i(n, E) \geq \max\{\frac{1}{n}c_i(E), c_i(\bar{e})\}$, and by additivity, it holds that

$$n\alpha \mathrm{MMS}_i(n, E) \geq (n - 1 + \alpha)c_i(A_i) - (n - 1)\mathrm{MMS}_i(n, E). \tag{2}$$

Inequality (2) is equivalent to $\frac{c_i(A_i)}{\mathrm{MMS}_i(n,M)} \leq \frac{n\alpha+n-1}{n-1+\alpha}$, as required.

As for tightness, consider the following instance with n agents and a set $E = \{e_1, \dots, e_{n^2-n+1}\}$ of $n^2 - n + 1$ chores. Agents have an identical cost profile and for every $i \in [n]$, $c_i(e_1) = \alpha + n - 1$, $c_i(e_j) = \alpha$ for any $2 \le j \le n$ and $c_i(e_j) = 1$ for $j \ge n+1$. Now, consider an allocation $\mathbf{B} = \{B_1, \dots, B_n\}$ with $B_1 = \{e_1, \dots, e_n\}$ and $B_j = \{e_{n+(n-1)(j-2)+1}, \dots, e_{n+(n-1)(j-1)}\}$ for any $j \ge 2$. Then, we have $c_i(B_j) = n - 1$ for any $i \in [n]$ and $j \ge 2$. Accordingly, except for agent 1, no one else will violate the condition of α-EF1 and MMS. As for agent 1, since $c_1(B_1 \setminus \{e_1\}) = (n-1)\alpha = \alpha c_1(B_j), \forall j \ge 2$, then we can claim that allocation $\mathbf{B}$ is α-EF1. To calculate $\mathrm{MMS}_1(n, E)$, consider an allocation $\mathbf{T} = (T_1, \dots, T_n)$ with $T_1 = \{e_1\}$ and $T_j = \{B_j \cup \{e_j\}\}$ for any $2 \le j \le n$. It is not hard to verify that $c_1(T_j) = \alpha + n - 1$ for any $j \in N$. Therefore, we have $\mathrm{MMS}_1(n, E) = \alpha + n - 1$ implying the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(n,E)} = \frac{n\alpha+n-1}{n-1+\alpha}$, completing the proof. □

We now study α-EFX in terms of its approximation guarantee for MMS and provide upper and lower bounds for general $\alpha \ge 1$ or $n \ge 2$.

Theorem 3.4 *When agents have additive cost functions, for any $\alpha \ge 1$ and $n \ge 2$, an α-EFX allocation is $\min\left\{\frac{2n\alpha}{n-1+2\alpha}, \frac{n\alpha+n-1}{n-1+\alpha}\right\}$-MMS, while it is not β-MMS for any $\beta < \max\left\{\frac{2n\alpha}{2\alpha+2n-3}, \frac{2n}{n+1}\right\}$.*

Proof We first prove the upper bound. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-EFX allocation with $\alpha \ge 1$ and the approximation guarantee for MMS is determined by agent i. The upper bound $\frac{n\alpha+n-1}{n-1+\alpha}$ directly follows from Theorems 3.1 and 3.3. In what follows, we prove the upper bound $\frac{2n\alpha}{n-1+2\alpha}$. We assume $c_i(A_i) > 0$; otherwise agent i meets the condition of MMS and we are done. We further assume that every chore in A_i has positive cost for agent i since zero-cost chore does not affect the approximation guarantee for EFX or MMS. Let e^* be the chore in bundle A_i having the minimum cost for agent i, i.e., $e^* \in \arg\min_{e \in A_i} c_i(e)$. Next, we divide the proof into two cases.

Case 1: $|A_i| = 1$. Then e^* is the unique element in A_i, and thus $c_i(A_i) = c_i(e^*)$. By Lemma 2.1, $c_i(e^*) \le \mathrm{MMS}_i(n, E)$ holds, and thus, $c_i(A_i) \le \mathrm{MMS}_i(n, E)$.

Case 2: $|A_i| \ge 2$. By the definition of α-EFX, for any $j \in N \setminus \{i\}$, $c_i(A_i \setminus \{e^*\}) \le \alpha \cdot c_i(A_j)$. Since $e^* \in \arg\min_{e \in A_i} c_i(e)$ and $|A_i| \ge 2$, we have $c_i(e^*) \le \frac{1}{2} c_i(A_i)$. Then, the following holds,

$$\alpha \cdot c_i(A_j) \ge c_i(A_i) - c_i(e^*) \ge \frac{1}{2} c_i(A_i), \qquad \forall j \in N \setminus \{i\}. \tag{3}$$

By summing up j over $N \setminus \{i\}$ and adding a term $\alpha c_i(A_i)$ on both sides of inequality (3), the following holds

$$\alpha \cdot c_i(E) = \alpha \cdot \sum_{j \in N \setminus \{i\}} c_i(A_j) + \alpha \cdot c_i(A_i) \ge \frac{n-1+2\alpha}{2} c_i(A_i). \tag{4}$$

On the other hand, from Lemma 2.1, we know $\mathrm{MMS}_i(n, E) \ge \frac{1}{n} c_i(E)$, which combines inequality (4) yielding the ratio

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(n, M)} \le \frac{2n\alpha}{n-1+2\alpha}.$$

Regarding the lower bound $\frac{2n}{n+1}$, consider an instance with n agents and a set $E = \{e_1, e_2, \dots, e_{2n}\}$ of $2n$ chores. Agents have identical cost profile and $c_i(e_j) = \lceil \frac{j}{2} \rceil$ for any i, j. It is not hard to verify that for any $i \in [n]$, $\text{MMS}_i(n, E) = n + 1$. Then, consider the allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_1 = \left\{e_{2n-1}, e_{2n}\right\}$ and $B_i = \{e_{i-1}, e_{2n-i}\}$ for any $i \geq 2$. Accordingly, we have $c_i(B_j) = n$ for any $i \in [n]$ and $j \geq 2$. Thus, except for agent 1, no one else will violate the condition of MMS and EFX. As for agent 1, since $c_1(B_1 \backslash \{e_{2n}\}) = c_1(B_1 \backslash \{e_{2n-1}\}) = n$, envy can be eliminated by removing any single chore. Hence, the allocation $\mathbf{B}$ is EFX and its approximation guarantee for MMS equals to $\frac{c_1(B_1)}{\text{MMS}_1(n,E)} = \frac{2n}{n+1}$, as required.

Next, for lower bound $\frac{2n\alpha}{2\alpha+2n-3}$, let us consider an instance with n agents and a set $E = \{e_1, \dots, e_{2n^2-2n}\}$ of $2n^2 - 2n$ chores. We focus on agent 1 with cost function $c_1(e_j) = 2\alpha$ for $1 \leq j \leq n$ and $c_1(e_j) = 1$ for $j \geq n + 1$. Consider the allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_1 = \{e_1, \dots, e_n\}, B_2 = \{e_{n+1}, \dots, e_{3n-2}\}$ and $B_j = \{e_{3n-1+(j-3)(2n-1)}, \dots, e_{3n-2+(j-2)(2n-1)}\}$ for any $j \geq 3$. Accordingly, bundle B_2 contains $2n - 2$ chores and B_j contains $2n - 1$ chores for any $j \geq 3$. For any agent $i \geq 2$, her cost functions is $c_i(e) = 0$ for $e \in B_i$ and $c_i(e) = 1$ for $e \in E \backslash B_i$. Consequently, except for agent 1, no one else violate the condition of MMS and α-EFX. As for agent 1, his cost on B_2 is the smallest over all bundles and $c_1(B_1 \backslash \{e_1\}) = 2\alpha(n - 1) = \alpha c_1(B_2)$, as a result, the allocation $\mathbf{B}$ is α-EFX. For $\text{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\text{MMS}_1(n, E) = 2\alpha + 2n - 3$ implying the ratio $\frac{c_1(B_1)}{\text{MMS}_1(n,E)} = \frac{2n\alpha}{2\alpha+2n-3}$, completing the proof. $\square$

The performance bound in Theorem 3.4 is almost tight since $\frac{n\alpha+n-1}{n-1+\alpha} - \frac{2n\alpha}{2\alpha+2n-3} < \frac{n-1}{n-1+\alpha} < 1$. In addition, we highlight that the upper and lower bounds provided in Theorem 3.4 are tight in two interesting cases: (i) $\alpha = 1$ and (ii) $n = 2$.

On the approximation of EFX and EF1 for PMMS, we have the following theorems.

Theorem 3.5 *When agents have additive cost functions, for any $\alpha \geq 1$, an α-EFX allocation is also $\frac{4\alpha}{2\alpha+1}$-PMMS, and this guarantee is tight.*

Proof We first prove the upper bound. Let $\mathbf{A} = (A_1, A_2, \dots, A_n)$ be an α-EFX allocation and the approximation guarantee for PMMS is determined by agent i. We can assume $c_i(A_i) > 0$; otherwise agent i meets the condition of PMMS and we are done. Let e^* be the chore in A_i having the minimum cost for agent i, i.e., $e^* \in \arg\min_{e \in A_i} c_i(e)$. Then, we divide the proof into two cases.

Case 1: $|A_i| = 1$. Then chore e^* is the unique element in A_i, and thus $c_i(e^*) = c_i(A_i)$. By Lemma 2.1, $c_i(e^*) \leq \text{MMS}_i(2, A_i \cup A_j)$ holds for any $j \in N \backslash \{i\}$. As a result, we have $c_i(A_i) \leq \text{MMS}_i(2, A_i \cup A_j), \forall j \in N \backslash \{i\}$.

Case 2: $|A_i| \geq 2$. Since $e^* \in \arg\min_{e \in A_i} c_i(e)$ and $|A_i| \geq 2$, we have $c_i(e^*) \leq \frac{1}{2} c_i(A_i)$, and equivalently, $c_i(A_i \backslash \{e^*\}) = c_i(A_i) - c_i(e^*) \geq \frac{1}{2} c_i(A_i)$. Then, based on the definition of α-EFX allocation, for any $j \in N \backslash \{i\}$, the following holds

$$\alpha \cdot c_i(A_j) \geq c_i(A_i \setminus \{e^*\}) \geq \frac{1}{2} \cdot c_i(A_i). \tag{5}$$

Combining Lemma 2.1 and Inequality (5), for any $j \in N \backslash \{i\}$, we have

$$\text{MMS}_i(2, A_i \cup A_j) \geq \frac{1}{2}(c_i(A_i) + c_i(A_j)) \geq \frac{2\alpha + 1}{4\alpha} c_i(A_i).$$

Therefore, for any $j \in N \backslash \{i\}$, $c_i(A_i) \leq \frac{4\alpha}{2\alpha+1} \cdot \text{MMS}_i(2, A_i \cup A_j)$ holds, as required.

As for the tightness, consider an instance with n agents and a set $E = \{e_1, \dots, e_{2n}\}$ of $2n$ chores. Agents have identical cost profile and for every $i \in [n]$, $c_i(e_1) = c_i(e_2) = 2\alpha$ and $c_i(e_j) = 1$ for $3 \le j \le 2n$. Consider the allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_i = \{e_{2i-1}, e_{2i}\}, \forall i \in N$. It is not hard to verify that, except for agent 1, no one else would violate the condition of EFX and PMMS. For agent 1, by removing any single chore from his bundle, the remaining cost is α times of the cost on others' bundle. Thus, allocation $\mathbf{B}$ is α-EFX. Notice that for any $j \ge 2$, bundle $B_1 \cup B_j$ contains exactly two chores with cost 2α and two chores with cost 1, then $\mathrm{MMS}_1(2, B_1 \cup B_j) = 2\alpha + 1$, implying for any $j \ne 1$, $\frac{c_1(B_1)}{\mathrm{MMS}_1(2, B_1 \cup B_j)} = \frac{4\alpha}{2\alpha+1}$. □

Theorem 3.6 *When agents have additive cost functions, for any $\alpha \ge 1$, an α-EF1 allocation is also $\frac{2\alpha+1}{\alpha+1}$-PMMS, and this guarantee is tight.*

Proof We first prove the upper bound part. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-EF1 allocation and the approximation guarantee for PMMS is determined by agent i. We can assume $c_i(A_i) > 0$; otherwise agent i meets the condition of PMMS and we are done. To study PMMS, we fix another agent $j \in N \backslash \{i\}$, and let $e^* \in A_i$ be the chore such that $c_i(A_i \backslash \{e^*\}) \le \alpha \cdot c_i(A_j)$. We divide our proof into two cases.

Case 1: $c_i(e^*) > c_i(A_i \cup A_j \setminus \{e^*\})$. Consider $\left\{\{e^*\}, A_i \cup A_j \backslash \{e^*\}\right\}$, a 2-partition of $A_i \cup A_j$. Since $c_i(e^*) > c_i(A_i \cup A_j \backslash \{e^*\})$, we can claim that this partition defining $\mathrm{MMS}_i(2, A_i \cup A_j)$, and accordingly, $\mathrm{MMS}_i(2, A_i \cup A_j) = c_i(e^*)$ holds. From Lemma 2.1 and the definition of α-EF1, the following holds

$$c_i(e^*) \ge \frac{1}{2}(c_i(A_i) + c_i(A_j)) \ge \frac{1}{2} c_i(A_i) + \frac{1}{2\alpha} \cdot c_i(A_i \setminus \{e^*\}). \tag{6}$$

Then, based on (6) and the fact $\mathrm{MMS}_i(2, A_i \cup A_j) = c_i(e^*)$, we have

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(2, A_i \cup A_j)} \le \frac{2\alpha + 1}{\alpha + 1}.$$

Case 2: $c_i(e^*) \le c_i(A_i \cup A_j \backslash \{e^*\})$. By the definition of α-EF1, we have $c_i(A_i \backslash \{e^*\}) \le \alpha \cdot c_i(A_j)$. As a consequence,

$$c_i(A_i) = c_i(e^*) + c_i(A_i \setminus \{e^*\}) \le 2c_i(A_i \setminus \{e^*\}) + c_i(A_j) \le (2\alpha + 1) \cdot c_i(A_j), \tag{7}$$

where the first inequality transition is due to $c_i(e^*) \le c_i(A_i \cup A_j \backslash \{e^*\})$. Using Inequality (7) and additivity of cost function, we have $c_i(A_i) \le \frac{2\alpha+1}{2\alpha+2} \cdot c_i(A_i \cup A_j)$. By Lemma 2.1, we have $\mathrm{MMS}_i(2, A_i \cup A_j) \ge \frac{1}{2} c_i(A_i \cup A_j)$ and then, the following holds,

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(2, A_i \cup A_j)} \le \frac{2\alpha + 1}{\alpha + 1}.$$

As for tightness, consider the following instance of n agents and a set $E = \{e_1, \dots, e_{n+1}\}$ of $n + 1$ chores. Agents have an identical cost profile and for every $i \in [n]$, $c_i(e_1) = \alpha + 1, c_i(e_2) = \alpha$ and $c_i(e_j) = 1$ for $j \ge 3$. Then, consider the allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_1 = \{e_1, e_2\}$ and $B_j = \{e_{j+1}\}, \forall j \ge 2$. It is not hard to verify that allocation $\mathbf{B}$ satisfying α-EF1, and moreover, the guarantee for PMMS is determined by agent 1. Notice that for any $j \ge 2$, the combined bundle $B_1 \cup B_j$ contains three chores with cost $\alpha + 1, \alpha, 1$, respectively. Thus, for any $j \geq 2$, we have $\mathrm{MMS}_1(2, B_1 \cup B_j) = \alpha + 1$, implying the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(2, B_1 \cup B_j)} = \frac{2\alpha+1}{\alpha+1}$. □

In addition to the approximation guarantee for PMMS, Theorem 3.6 also has a direct implication in approximating PMMS algorithmically. It is known that an EF1 allocation can be found efficiently by allocating chores in a *round-robin* fashion — each of the agent $1, \dots, n$ picks her most preferred item in that order, and repeat until all chores are assigned [4]. Therefore, Theorem 3.6 with $\alpha = 1$ leads to the following corollary, which is current the best algorithmic result for PMMS (in chores allocation), to the best of our knowledge.

Corollary 3.1 *When agents have additive cost functions, the round-robin algorithm outputs a $\frac{3}{2}$-PMMS allocation in polynomial time.*

We remark that the established guarantees from envy-based fairness criteria, together with the result in [2], can reveal a sharp contrast between the settings of indivisible goods and chores. In particular, as shown by Theorem 3.3, in allocation of chores, an EF1 allocation is also 2-MMS, while Amanatidis et al. [2] show that in allocation of goods, an EF1 allocation cannot achieve better than n-MMS. In addition, these guarantees can be implemented to translate an approximation algorithm of one fairness notion to another. Theorem 3.3 indicates that a polynomial-time algorithm of EF1, such as round-robin, is also an algorithm of 2-MMS. The 2-MMS approximation matches the result in [7], and recently this approximation ratio has been pushed to 11/9 by Huang and Lu [34]. As for the guarantees from α-EFX, the results at first glance are not as useful as the guarantees from α-EF1, because few results regarding EFX have been established with respect to chores. Note that with respect to goods, the guarantees from α-EFX lead to 4/7-GMMS[8] allocations [25]. We believe that an efficient algorithm of α-EFX with respect to chores would help develop efficient algorithms for other fairness criteria, such as PMMS.

4 Guarantees from share-based relaxations

Note that PMMS implies EFX in goods allocation according to Caragiannis et al. [22]. This implication also holds in allocating chores as stated in our theorem below.

Theorem 4.1 *When agents have additive cost functions, a PMMS allocation is also EFX.*

Proof Let $\mathbf{A} = (A_1, \dots, A_n)$ be a PMMS allocation. For the sake of contradiction, assume $\mathbf{A}$ is not EFX and agent i violates the condition of EFX, which implies $c_i(A_i) > 0$.

As agent i violates the condition of EFX, there must exist an agent $j \in N$ and $e^* \in A_i$ with $c_i(e^*) > 0$ such that $c_i(A_i \setminus \{e^*\}) > c_i(A_j)$. Note chore e^* is well-defined owing to $c_i(A_i) > 0$. Now, consider the 2-partition $\left\{A_i \setminus \{e^*\}, A_j \cup \{e^*\}\right\} \in \Pi_2(A_i \cup A_j)$. By $c_i(A_i \setminus \{e^*\}) > c_i(A_j)$, the following holds:

[8] An allocation is called group-wise maximin share (GMMS) if for every subgroup of agents of size k, each member of the subgroup receives her 1-out-of-k maximin share restricted to the items received by this subgroup.

$$c_i(A_i) > \max\left\{c_i(A_i \setminus \{e^*\}), c_i(A_j \cup \{e^*\})\right\}$$
$$\geq \min_{\mathbf{B} \in \Pi_2(A_i \cup A_j)} \max\left\{c_i(B_1), c_i(B_2)\right\} \geq c_i(A_i), \tag{8}$$

where the last transition is by the definition of PMMS. Inequality (8) is a contradiction, and therefore, **A** must be an EFX allocation. □

Since EFX implies EF1, Theorem 4.1 directly leads to the following result.

Theorem 4.2 *When agents have additive cost functions, a PMMS allocation is also EF*1.

For approximate version of PMMS, when allocating goods it is shown in Amanatidis et al. [2] that for any α, α-PMMS can imply $\frac{\alpha}{2-\alpha}$-EF1. However, in the case of chores, our results indicate that α-PMMS has no bounded guarantee for EF1.

Proposition 4.1 *When agents have additive cost functions, an α-PMMS allocation with $1 < \alpha \leq 2$ is not necessarily β-EF*1 *for any $\beta \geq 1$.*

Proof It suffices to show an α-PMMS allocation with $\alpha \in (1, 2)$ can not have a bounded guarantee for the notion of EF1. Consider an instance with n agents and $n + 1$ chores e_1 ..., e_{n+1}. Agents have identical cost profile and for any i, we let $c_i(e_1) = \frac{1}{\alpha-1}$, $c_i(e_2) = 1$ and $c_i(e_j) = \epsilon$ for $3 \leq j \leq n + 1$ where ϵ takes arbitrarily small positive value. Then, consider an allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_1 = \{e_1, e_2\}$ and $B_j = \{e_{j+1}\}$ for $2 \leq j \leq n$. Consequently, except for agent 1, other agents violate neither EF1 nor α-PMMS. As for agent 1, notice that $\frac{1}{\alpha-1} > 1 + \epsilon$ and thus, for any $j \geq 2$, the combined bundle $B_1 \cup B_j$ admits $\mathrm{MMS}_1(2, B_1 \cup B_j) = \frac{1}{\alpha-1}$ implying $\frac{c_1(B_1)}{\mathrm{MMS}_1(2, B_1 \cup B_j)} = \alpha$. Thus, allocation **B** is α-PMMS. For the guarantee on EF1, as $c_1(B_j) = \epsilon$ for any $j \geq 2$, then removing the chore with the largest cost from B_2 still yields the ratio $\frac{c_1(B_1 \setminus \{e_1\})}{c_1(B_j)} = \frac{1}{\epsilon} \to \infty$ as $\epsilon \to 0$. □

Since for any $\alpha \geq 1$, α-EFX is stricter than α-EF1, the impossibility result on EF1 in Proposition 4.1 is also true for EFX.

Proposition 4.2 *When agents have additive cost functions, an α-PMMS allocation with $1 < \alpha \leq 2$ is not necessarily a β-EFX allocation for any $\beta \geq 1$.*

We now study the approximation guarantee of PMMS for MMS. Since these two notions coincide when there are only two agents, we consider the situation where $n \geq 3$. We first provide a tight bound for $n = 3$ and then give an almost tight bound for general n.

Theorem 4.3 *When agents have additive cost functions, for $n = 3$, a PMMS allocation is also $\frac{4}{3}$-MMS, and moreover, this bound is tight.*

Proof See Appendix A.1. □

For general n, we use the connections between PMMS, EFX and MMS to find the approximation guarantee of PMMS for MMS. According to Theorem 4.1, a PMMS allocation is also EFX, and by Theorem 3.4, EFX implies $\frac{2n}{n+1}$-MMS. As a result, we can claim that PMMS also implies $\frac{2n}{n+1}$-MMS. With the following theorem we show that this guarantee is almost tight.

Theorem 4.4 *When agents have additive cost functions, for $n \geq 4$, a PMMS allocation is $\frac{2n}{n+1}$-MMS but not necessarily β-MMS for any $\beta < \frac{2n+2}{n+3}$.*

Proof The positive part directly follows from Theorems 4.1 and 3.4. As for the lower bound, consider an instance with n (odd) agents and a set $E = \{e_1, \dots, e_{2n}\}$ of $2n$ chores. We focus on agent 1 and his cost function is $c_1(e_j) = \frac{n+1}{2}$ for $1 \leq j \leq n$ and $c_1(e_j) = 1$ for $n+1 \leq j \leq 2n$. Consider the allocation $\mathbf{B} = (B_1, \dots, B_n)$ with $B_1 = \{e_1, e_2\}$, $B_n = \{e_{n+1}, \dots, e_{2n}\}$ and $B_j = \{e_{j+1}\}$ for any $j = 2, \dots, n-1$. For agents $i \geq 2$, her cost function is $c_i(e) = 0$ for any $e \in B_i$ and $c_i(e) = 1$ for any $e \in E \setminus B_i$, and thus agent i has zero cost under allocation $\mathbf{B}$. As a result, except for agent 1, other agents violate neither MMS nor PMMS. For agent 1, we have $c_1(B_1) \leq \mathrm{MMS}_1(2, B_1 \cup B_j)$ holds for any $j \geq 2$, which implies allocation $\mathbf{B}$ is PMMS. For $\mathrm{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\mathrm{MMS}_1(n, E) = \frac{n+3}{2}$ yielding the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(n,E)} = \frac{2n+2}{n+3}$. ☐

Next, we investigate the approximation guarantee of approximate PMMS for MMS. Let us start with an example of six chores $E = \{e_1, \dots, e_6\}$ and three agents. We focus on agent 1 and the cost function of agent 1 is $c_1(e_j) = 1$ for $j = 1, 2, 3$ and $c_1(e_j) = 0$ for $j = 4, 5, 6$, thus clearly, $\mathrm{MMS}_1(3, E) = 1$. Consider an allocation $\mathbf{A} = (A_1, A_2, A_3)$ with $A_1 = \{e_1, e_2, e_3\}$. It is not hard to verify that allocation $\mathbf{A}$ is a $\frac{3}{2}$-PMMS allocation and also a 3-MMS allocation. Combining the result in Lemma 2.2, we observe that allocation $\mathbf{A}$ only has a trivial guarantee on the notion of MMS. Motivated by this example, we focus on α-PMMS allocations with $\alpha < \frac{3}{2}$.

Theorem 4.5 *When agents have additive cost functions, for $n \geq 3$ and $1 < \alpha < \frac{3}{2}$, an α-PMMS allocation is $\frac{n\alpha}{\alpha+(n-1)(1-\frac{\alpha}{2})}$-MMS, but not necessarily $(\frac{n\alpha}{\alpha+(n-1)(2-\alpha)} - \epsilon)$-MMS for any $\epsilon > 0$.*

Before we can prove the above theorem, we need the following two lemmas.

Lemma 4.3 *For any $i \in N$ and $S \subseteq E$, suppose $\mathrm{MMS}_i(2, S)$ is defined by a 2-partition $\mathbf{T} = (T_1, T_2)$ with $c_i(T_1) = \mathrm{MMS}_i(2, S)$. If the number of chores in T_1 is at least two, then $\frac{c_i(S)}{\mathrm{MMS}_i(2,S)} \geq \frac{3}{2}$.*

Proof For the sake of contradiction, we assume $\frac{c_i(S)}{\mathrm{MMS}_i(2,S)} < \frac{3}{2}$. Since $c_i(T_1) = \mathrm{MMS}_i(2, S)$, we have $c_i(T_1) > \frac{2}{3}c_i(S)$, and accordingly, $c_i(T_2) < \frac{1}{3}c_i(S)$ due to additivity. Thus, $c_i(T_1) - c_i(T_2) > \frac{1}{3}c_i(S)$ holds, and we claim that each single chore in T_1 has cost strictly larger than $\frac{1}{3}c_i(S)$ for agent i; otherwise, by moving the chore with the smallest cost in T_1 to T_2, one can find a 2-partition in which the cost of larger bundle is smaller than $c_i(T_1)$, a contradiction. Based on our claim, we have $|T_1| = 2$. Notice that for any $e \in T_1$, $c_i(e) > c_i(T_2)$ holds. As a result, moving one chore from T_1 to T_2 results in a 2-partition, in which the cost of larger bundle is strictly smaller than $c_i(T_1)$, contradicting the construction of allocation $\mathbf{T}$. □

Lemma 4.4 *For any* $i \in N$ *and* $S_1, S_2 \subseteq E$, *if* $\mathrm{MMS}_i(2, S_1 \cup S_2) > \mathrm{MMS}_i(2, S_1)$, *then* $\mathrm{MMS}_i(2, S_1 \cup S_2) \le \frac{1}{2}c_i(S_1) + c_i(S_2)$.

Proof Suppose $\mathrm{MMS}_i(2, S_1)$ is defined by partition (T_1, T_2) and we have $\mathrm{MMS}_i(2, S_1) = c_i(T_1)$. We distinguish two cases according to the value of $c_i(T_1)$. If $c_i(T_1) = \frac{1}{2}c_i(S_1)$, then consider $(T_1 \cup S_2, T_2)$, a 2-partition of $S_1 \cup S_2$. Clearly, $\mathrm{MMS}_i(2, S_1 \cup S_2) \le c_i(T_1 \cup S_2) = \frac{1}{2}c_i(S_1) + c_i(S_2)$. If $c_i(T_1) > \frac{1}{2}c_i(S_1)$, since $\mathrm{MMS}_i(2, S_1 \cup S_2) > \mathrm{MMS}_i(2, S_1)$, we can claim that $c_i(T_1) - c_i(T_2) < c_i(S_2)$; otherwise, considering partition $\{T_1, T_2 \cup S_2\}$, we have $\mathrm{MMS}_i(2, S_1 \cup S_2) \le c_i(T_1) = \mathrm{MMS}_i(2, S_1)$, a contradiction. Now let us consider $\{T_2 \cup S_2, T_1\}$, another 2-partition of $S_1 \cup S_2$. According to our claim, we have $c_i(T_2 \cup S_2) > c_i(T_1)$, and thus, $\mathrm{MMS}_i(2, S_1 \cup S_2) \le c_i(T_2 \cup S_2) < \frac{1}{2}c_i(S_1) + c_i(S_2)$, where the last inequality is due to $c_i(T_2) = c_i(S_1) - c_i(T_1) < \frac{1}{2}c_i(S_1)$. □

Proof of Theorem 4.5 We first prove the upper bound. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an α-PMMS allocation and we focus our analysis on agent i. Let $\alpha^{(i)} = \max_{j \ne i} \frac{c_i(A_i)}{\mathrm{MMS}_i(2, A_i \cup A_j)}$ and $j^{(i)}$ be the index such that $\mathrm{MMS}_i(2, A_i \cup A_{j^{(i)}}) \le \mathrm{MMS}_i(2, A_i \cup A_j)$ for any $j \in N \backslash \{i\}$ (tie breaks arbitrarily). By such a construction, clearly, $\alpha = \max_{i \in N} \alpha^{(i)}$ and $c_i(A_i) = \alpha^{(i)} \cdot \mathrm{MMS}_i(2, A_i \cup A_{j^{(i)}})$. Then, we split our proof into two different cases.

Case 1: $\exists j \ne i$ such that $\mathrm{MMS}_i(2, A_i \cup A_j) = \mathrm{MMS}_i(2, A_i)$. Then $\alpha^{(i)} = \frac{c_i(A_i)}{\mathrm{MMS}_i(2, A_i)}$ holds. Suppose $\mathrm{MMS}_i(2, A_i)$ is defined by the 2-partition (T_1, T_2) with $c_i(T_1) = \mathrm{MMS}_i(2, A_i)$. If $|T_1| \ge 2$, by Lemma 4.3, we have $\alpha^{(i)} = \frac{c_i(A_i)}{\mathrm{MMS}_i(2, A_i)} \ge \frac{3}{2}$, contradicting to $\alpha^{(i)} \le \alpha < \frac{3}{2}$. As a result, we can further assume $|T_1| = 1$. Then, by Lemma 2.1, we have $\mathrm{MMS}_i(n, E) \ge c_i(T_1)$ and accordingly, $\frac{c_i(A_i)}{\mathrm{MMS}_i(n, E)} \le \frac{c_i(A_i)}{c_i(T_1)} = \alpha^{(i)} \le \alpha$. For $1 < \alpha < \frac{3}{2}$ and $n \ge 3$, it is not hard to verify that $\alpha \le \frac{n\alpha}{\alpha + (n-1)(1 - \frac{\alpha}{2})}$, completing the proof for this case.

Case 2: $\forall j \ne i$, $\mathrm{MMS}_i(2, A_i \cup A_j) > \mathrm{MMS}_i(2, A_i)$ holds. According to Lemma 4.4, for any $j \ne i$, the following holds

$$\mathrm{MMS}_i(2, A_i \cup A_j) \le \frac{1}{2}c_i(A_i) + c_i(A_j). \tag{9}$$

Due to the construction of $\alpha^{(i)}$, for any $j \ne i$, we have $c_i(A_i) \le \alpha^{(i)} \cdot \mathrm{MMS}_i(2, A_i \cup A_j)$. Combining Inequality (9), we have $c_i(A_j) \ge \frac{2 - \alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)$ for any $j \ne i$. Thus, the following holds,

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(n, E)} \le \frac{n c_i(A_i)}{c_i(E)} \le \frac{n c_i(A_i)}{c_i(A_i) + (n-1)\frac{2 - \alpha^{(i)}}{2\alpha^{(i)}} c_i(A_i)}. \tag{10}$$

The last expression in (10) is monotonically increasing in $\alpha^{(i)}$, and accordingly, we have

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(n, E)} \le \frac{n\alpha}{\alpha + (n-1)(1 - \frac{\alpha}{2})}.$$

As for the lower bound, consider an instance of n (even) agents and a set $E = \{e_1, \dots, e_{n^2}\}$ of n^2 chores. Agents have identical cost functions and for any i, we let $c_i(e_j) = \alpha$ for

$1 \le j \le n$ and $c_i(e_j) = 2 - \alpha$ for $n + 1 \le j \le n^2$. Consider the allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_i = \{e_{(i-1)n+1}, \ldots, e_{ni}\}$ for any $i \in [n]$. Since $\alpha > 1$, it is not hard to verify that, except for agent 1, no one else violates the condition of PMMS, and accordingly, the approximation guarantee for PMMS is determined by agent 1. For agent 1, since n is even, $\mathrm{MMS}_1(2, B_1 \cup B_j) = n$ holds for any $j \ge 2$, and due to $c_1(B_1) = n\alpha$, we can claim that the allocation $\mathbf{B}$ is α-PMMS. Moreover, it is not hard to verify that $\mathrm{MMS}_1(n, E) = \alpha + (n-1)(2-\alpha)$ and so $\frac{c_1(B_1)}{\mathrm{MMS}_1(n,E)} = \frac{n\alpha}{\alpha+(n-1)(2-\alpha)}$, completing the proof. □

The motivating example right before Theorem 4.5, unfortunately, only works for the case of $n = 3$. When n becomes larger, an α-PMMS allocation with $\alpha \ge \frac{3}{2}$ can still provide a non-trivial approximation guarantee on the notion of MMS.

We remain to consider the approximation guarantee of MMS for other fairness criteria. Notice that all of EFX, EF1 and PMMS can have non-trivial guarantee for MMS (i.e., better than n-MMS). However, the converse is not true and even the exact MMS does not provide any substantial guarantee for the other three criteria.

Proposition 4.5 *When agents have additive cost functions, for any $n \ge 3$, an MMS allocation is not necessarily β-PMMS for any $1 \le \beta < 2$.*

Proof Consider an instance with n agents and $p + 2n - 1$ chores denoted as $\{e_1, \ldots, e_{2n+p-1}\}$ where $p \in \mathbb{N}^+$ and $p \gg 1$. We focus on agent 1 and his cost function is: $c_1(e_j) = 1$ for any $1 \le j \le n + p$ and $c_1(e_j) = p$ for any $j \ge n + p + 1$. Consider allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_1 = \left\{e_1, \ldots, e_{p+1}\right\}$, $B_i = \left\{e_{p+i}\right\}, \forall i = 2, \ldots, n-2$, $B_{n-1} = \{e_{n+p-1}, e_{n+p}\}$ and $B_n = \{e_{n+p+1}, \ldots, e_{2n+p-1}\}$. For any agent $i \ge 2$, her cost function is $c_i(e) = 0$ for any $e \in B_i$ and $c_i(e) = 1$ for any $e \notin B_i$. Consequently, except for agent 1, other agents violate neither MMS nor PMMS, and accordingly the approximation guarantee for PMMS and MMS is determined by agent 1. For $\mathrm{MMS}_1(n, E)$, it happens that E can be evenly divided into n bundles of the same cost (for agent 1), so we have $\mathrm{MMS}_1(n, E) = p + 1$. Accordingly, $c_1(B_1) = \mathrm{MMS}_1(n, E)$ holds and thus, allocation $\mathbf{B}$ is MMS. As for the approximation guarantee on PMMS, consider the combined bundle $B_1 \cup B_2$ and it is not hard to verify that $\mathrm{MMS}_1(2, B_1 \cup B_2) = \lceil \frac{p+2}{2} \rceil$ implying $\frac{c_1(B_1)}{\mathrm{MMS}_1(2,B_1 \cup B_2)} = \frac{p+1}{\lceil \frac{p+2}{2} \rceil} \to 2$ as $p \to \infty$. □

Proposition 4.6 *When agents have additive cost functions, an MMS allocation is not necessarily β-EF1 or β-EFX for any $\beta \ge 1$.*

Proof By Proposition 3.1, the notion β-EFX is stricter than β-EF1, and thus, we only need to show the unbounded guarantee on EF1. Again, we consider the instance given in the proof of Proposition 4.5. As stated in that proof, $\mathbf{B}$ is an MMS allocation, and except for agent 1, no one else will violate the condition of PMMS. Note that PMMS is stricter than EF1, then no one else will violate the condition of EF1. As for agent 1, each chore in B_1 has the same cost for him, so we can remove any single chore in B_1 and check its performance in terms of EF1. When comparing to bundle B_2, we have $\frac{c_1(B_1 \setminus \{e_1\})}{c_1(B_2)} = p \to \infty$ as $p \to \infty$. □

The guarantees presented in this section again indicate that goods and chores are not mirror images of one another. To take an example, according to Proposition 4.1, α-PMMS allocations of chores with $\alpha > 1$ have no bounded guarantee on EF1, whereas in allocations of goods, α-PMMS can guarantee $\frac{\alpha}{2-\alpha}$-MMS [2].

5 Guarantees beyond additive setting

The results in previous sections demonstrate the strong connections between the four (additive) relaxations of envy-freeness in the setting of additive cost functions. Under this umbrella, what would also be interesting is that whether there still exists certain connections when agents' cost profile is no longer additive. In this section, we also study the connections between fairness criteria and, instead of additive cost functions, we assume that agents have submodular cost functions, which have also been widely concerned with in the fair division literature [23, 33].

As a starting point, we consider EF, the strongest notion in the setting of additive, and see whether it can still provide guarantee on other fairness notions. According to the definitions, the notion of EF is, clearly, still stricter than EFX and EF1 if cost functions are monotone. Then, we study the approximation guarantee of EF on MMS and PMMS. As shown by our results below, in contrast to the results under additive setting, PMMS and MMS are no longer the relaxations of EF, and even worse, the notion of EF does not provide any substantial guarantee on PMMS and MMS.

Proposition 5.1 *When agents have submodular cost functions, an EF allocation is not necessarily β_1-MMS or β_2-PMMS for any $1 \le \beta_1 < n, 1 \le \beta_2 < 2$.*

Proof It suffices to show that there exists an EF allocation with approximation guarantee n and 2 for MMS and PMMS, respectively. Consider an instance with n (even) agents and a set E of chores with $|E| = n^2$. Chores are placed in the form of $n \times n$ matrix $E = \left[e_{ij}\right]_{n\times n}$. All agents have an identical cost function $c(S) = \sum_{i=1}^{n} \min\left\{\left|E_i \cap S\right|, 1\right\}$ for any $S \subseteq E$, where E_i is the set of all elements in the i-th row of matrix E, i.e., $E_i = \{e_{i1}, \dots, e_{in}\}$. Since capped cardinality function $\left|E_i \cap S\right|$ of $S \subseteq E$ is monotone and submodular for any fix i $(1 \le i \le n)$, it follows that $c(\cdot)$ is also monotone and submodular.[9]

Next, we prove that this instance permits an EF allocation, with which the approximation guarantee for MMS and PMMS is n and 2, respectively. Consider an allocation $\mathbf{B} = (B_1, \dots, B_n)$ where for any j, bundle B_j contains all elements in the j-th column of matrix E, i.e., $B_j = (e_{1j}, e_{2j}, \dots, e_{nj})$. One can compute that $c(B_j) = \sum_{i=1}^{n} \min\left(\left|E_i \cap B_j\right|, 1\right) = n$ holds for any $j \in [n]$, which implies that allocation $\mathbf{B}$ is EF. Next, we check the approximation guarantee of $\mathbf{B}$ on MMS. With a slight abuse of notation, we let $\mathbf{E}$ be the allocation defined by n-partition $E_1, \dots, E_n$, i.e., $\mathbf{E} = (E_1, \dots, E_n)$. It is not hard to see that for any $i \in N, c(E_i) = 1$. Then we claim that allocation $\mathbf{E}$ defines MMS for all agents; otherwise, there exists another allocation in which each bundle has cost strictly smaller than 1, and this never happens because $c(e) = 1$ for any $e \in E$ and $c(\cdot)$ is monotone. Therefore, for any $i \in N, \mathrm{MMS}_i(n, E) = 1$, which implies $\frac{c(B_i)}{\mathrm{MMS}_i(n,E)} = n$, as required.

[9] More generally, if $f(\cdot)$ is submodular, then $g(f(\cdot))$ is also submodular for any $g(\cdot)$ that is non-decreasing and concave. Furthermore, conical combination (with sum as a special case) of submodular functions is also submodular.

Next, we argue that the allocation $\mathbf{B}$ is 2-PMMS. Fix $i, j \in N$ and $j \neq i$. Notice that the combined bundle $B_i \cup B_j$ contains two columns of chores, so we can consider another allocation $\mathbf{B}' = (B'_i, B'_j)$ with $B'_i = \left\{ e_{1i}, \dots, e_{\frac{n}{2}i}, e_{1j}, \dots, e_{\frac{n}{2}j} \right\}$ and $B'_j = \{ e_{\frac{n}{2}+1i}, \dots, e_{ni}, e_{\frac{n}{2}+1j}, \dots, e_{nj} \}$. The idea of $\mathbf{B}'$ is to split each column into two parts with equal size and one part staring from the first row to $\frac{n}{2}$-th row while the other one containing the rest half. By the definition of cost function $c(\cdot)$, we know $c(B'_i) = c(B'_j) = \frac{n}{2}$ implying $\mathrm{MMS}_i(2, B_i \cup B_j) \leq \max\{c(B'_i), c(B'_j)\} = \frac{1}{2}c_i(B_i)$. Therefore, $\mathbf{B}$ is a 2-PMMS allocation. □

In the aspect of worst-case analysis, combining Lemma 2.2 and Proposition 5.1, EF can only have a trivial guarantee (n and 2, respectively) on MMS and PMMS, which is a sharp contrast to the results in additive setting where EF is strictly stronger than these two notions. As we mentioned above, EF is stricter than EFX and EF1, then we can directly argue that neither EFX nor EF1 can have better guarantees than trivial ones, namely, 2-PMMS and n-MMS.

Proposition 5.2 *When agents have submodular cost functions, an EFX allocation is not necessarily β_1-MMS or β_2-PMMS for any* $1 \leq \beta_1 < n, 1 \leq \beta_2 < 2$.

Proposition 5.3 *When agents have submodular cost functions, an EF*1 *allocation is not necessarily β_1-MMS or β_2-PMMS for any* $1 \leq \beta_1 < n, 1 \leq \beta_2 < 2$.

As for the connections between EFX and EF1, the statement of Proposition 3.1 is still true in the case of submodular.

Proposition 5.4 *When agents have submodular cost functions, an α-EFX allocation is also α-EF*1 *for any* $\alpha \geq 1$. *On the other hand, an EF*1 *allocation is not necessarily a β-EFX for any* $\beta \geq 1$.

Proof The positive part follows directly from definitions of EFX and EF1. As for the impossibility result, the instance in the proof of Proposition 3.1 is established in the case of additive. Since an additive function is also submodular, we also have such an impossibility result here. □

Next, we study the notion of PMMS in terms of its approximation guarantee on EFX and EF1. Recall the results of Theorems 4.1 and 4.2, a PMMS allocation is stricter than EFX and EF1 in the additive setting. However, in the case of submodular, this relationship does not hold any more, and even worse, PMMS provides non-trivial guarantee on neither EFX nor EF1.

Proposition 5.5 *When agents have submodular cost functions, a PMMS allocation is not necessarily a β-EF*1 *or β-EFX allocation for any* $\beta \geq 1$.

Proof By Proposition 5.4, for any $\beta \geq 1$, β-EFX is stronger than β-EF1, and thus it suffices to show the approximation guarantee for EF1 is unbounded. In what follows, we provide an instance that has a PMMS allocation with only trivial guarantee on EF1.

Consider an instance with two agents and a set $E = \{e_1, e_2, e_3\}$ of chores. Agents have identical cost function $c(S) = \min\{|S|, 2\}$. Since $|S|$ is monotone and submodular, it follows that $c(\cdot)$ is also monotone and submodular (see Footnote 9).

Next, we prove this instance having a PMMS allocation whose guarantee for EF1 is unbounded. Since in total, we have three chores, and thus in any 2-partition there always exists an agent receiving at least two chores. Thus, we can claim that $\mathrm{MMS}_i(2, E) = 2$ for any $i \in [2]$. Then, consider an allocation $\mathbf{B} = (B_1, B_2)$ with $B_1 = E$ and $B_2 = \emptyset$. Allocation $\mathbf{B}$ is PMMS since, for any $i \in [2]$, $\max\{c(B_1), c(B_2)\} = \mathrm{MMS}_i(2, E) = 2$ holds. However, bundle B_2 is empty and so $c_1(B_2) = c(B_2) = 0$. Then, no matter which chore is removed from bundle B_1, agent 1 still has a positive cost, which implies an unbounded approximation guarantee for the notion of EF1. $\square$

The approximation guarantee of an MMS allocation for EFX, EF1 and PMMS can be directly derived from the results in the additive setting. According to Propositions 4.5 and 4.6, in additive setting MMS does not provide non-trivial guarantee on all other three notions. Since additive functions belong to the class of submodular functions, we directly have the following two results.

Proposition 5.6 *When agents have submodular cost functions, an MMS allocation is not necessarily β-PMMS for any* $1 \leq \beta < 2$.

Proposition 5.7 *When agents have submodular cost functions, an MMS allocation is not necessarily a β-EF*1 *or β-EFX allocation for any* $\beta \geq 1$.

At this stage, what remains is the approximation guarantee of PMMS on MMS. Before presenting the main result, we provide a lemma, which states that the quantity of MMS is monotonically non-decreasing on the set of chores to be assigned.

Lemma 5.8 *Given a monotone function* $c(\cdot)$ *defined on ground set* E, *for any subsets* $S \subseteq T \subseteq E$, *if quantities* $\mathrm{MMS}(2, S)$ *and* $\mathrm{MMS}(2, T)$ *are computed based on function* $c(\cdot)$, *then* $\mathrm{MMS}(2, S) \leq \mathrm{MMS}(2, T)$.

Proof Let $\{T_1, T_2\}$ be the 2-partition of set T and moreover it defines $\mathrm{MMS}(2, T) = c(T_1) \geq c(T_2)$. We then consider $\{T_1 \cap S, T_2 \cap S\}$, which is, clearly, a 2-partition of S due to $S \subseteq T$. According to the definition of MMS, we have

$$\mathrm{MMS}(2, S) \leq \max\{c(T_1 \cap S), c(T_2 \cap S)\} \leq \max\{c(T_1), c(T_2)\} = \mathrm{MMS}(2, T),$$

where the second inequality transition is because $c(\cdot)$ is monotone. $\square$

Theorem 5.1 *When agents have submodular cost functions, for any* $1 \leq \alpha \leq 2$, *an α-PMMS allocation is also* $\min\{n, \alpha\lceil \frac{n}{2} \rceil\}$*-MMS, and this guarantee is tight.*

Proof We first prove the upper bound. According to Lemma 2.2, any allocation is n-MMS and so what remains is to prove the upper bound of $\alpha\lceil \frac{n}{2} \rceil$. Fix agent i with cost function $c_i(\cdot)$. Suppose n-partition $\{T_1, \ldots, T_n\}$ defines $\mathrm{MMS}_i(n, E)$ and w.l.o.g, we assume $c_i(T_1) \geq c_i(T_2) \geq \cdots \geq c_i(T_n)$, i.e., $c_i(T_1) = \mathrm{MMS}_i(n, E)$. Then, we let 2-partition $\{Q_1, Q_2\}$ defines $\mathrm{MMS}_i(2, E)$ and $c_i(Q_1) \geq c_i(Q_2)$, i.e., $c_i(Q_1) = \mathrm{MMS}_i(2, E)$. Let $\mathbf{A}$ be an arbitrary α-PMMS allocation, and accordingly, for any $j \neq i$, we have

$c_i(A_i) \le \alpha \cdot \mathrm{MMS}_i(2, A_i \cup A_j)$. Since $A_i \cup A_j$ is a subset of E, according to Lemma 5.8, we have $\mathrm{MMS}_i(2, A_i \cup A_j) \le \mathrm{MMS}_i(2, E)$. We then construct an upper bound of $\mathrm{MMS}_i(2, E)$ through partition $\{T_1, \dots, T_n\}$.

Let us consider a 2-partition $\{B_1, B_2\}$ of E with $B_1 = \{T_1, T_2, \dots, T_{\lceil \frac{n}{2} \rceil}\}$, $B_2 = \{T_{\lceil \frac{n}{2} \rceil + 1}, \dots, T_n\}$. Then, the following holds:

$$\begin{aligned}\max\{c_i(B_1), c_i(B_2)\} &= \max\left\{c_i\left(\cup_{j=1}^{\lceil \frac{n}{2} \rceil} T_j\right), c_i\left(\cup_{j=\lceil \frac{n}{2} \rceil+1}^{n} T_j\right)\right\} \\ &\le \max\left\{\sum_{j=1}^{\lceil \frac{n}{2} \rceil} c_i(T_j), \sum_{j=\lceil \frac{n}{2} \rceil+1}^{n} c_i(T_j)\right\} \le \lceil \frac{n}{2} \rceil \cdot c_i(T_1),\end{aligned}$$

where the first inequality transition is due to subadditivity of $c_i(\cdot)$ and the second inequality transition is because $c_i(T_1) \ge c_i(T_2) \cdots \ge c_i(T_n)$. Recall $c_i(Q_1) = \mathrm{MMS}_i(2, E) \le \max\{c_i(B_1), c_i(B_2)\}$, and accordingly we have $\mathrm{MMS}_i(2, E) \le \lceil \frac{n}{2} \rceil \cdot c_i(T_1) = \lceil \frac{n}{2} \rceil \cdot \mathrm{MMS}_i(n, E)$. Therefore, for any $j \ne i$, the following holds:

$$\frac{c_i(A_i)}{\mathrm{MMS}_i(n, E)} \le \frac{\alpha \cdot \mathrm{MMS}_i(2, A_i \cup A_j)}{\mathrm{MMS}_i(n, E)} \le \frac{\alpha \cdot \mathrm{MMS}_i(2, E)}{\mathrm{MMS}_i(n, E)} \le \alpha \cdot \lceil \frac{n}{2} \rceil.$$

As for the lower bound, it suffices to show that for any $\alpha \in [1, 2]$, there exists an α-PMMS allocation with approximation guarantee $\alpha\lceil \frac{n}{2} \rceil$ of MMS when $\alpha\lceil \frac{n}{2} \rceil \le n$. Let us consider an instance with n (even) agents and a set E of chores with $|E| = n(n+1)$. Since $\alpha \le 2$ and n is even, clearly we have $\alpha\lceil \frac{n}{2} \rceil \le n$. Chores are placed in $n \times (n+1)$ matrix $E = \left[e_{ij}\right]_{n\times(n+1)}$. For $j \in [n+1]$, denote by P_j the j-th column, i.e., $P_j = \{e_{1j}, e_{2j}, \dots, e_{nj}\}$. We concentrate on allocation $\mathbf{A}$ with $A_1 = P_1 \cup \cdots \cup P_{\lfloor \alpha \frac{n}{2} \rfloor} \cup P_n$, $A_j = \left\{e_{j,\lfloor \alpha \frac{n}{2} \rfloor+1}, \dots, e_{j,n-1}, e_{j,n+1}\right\}$ for any $2 \le j \le n-1$, and $A_n = \left\{e_{n,\lfloor \alpha \frac{n}{2} \rfloor+1}, \dots, e_{n,n-1}, e_{n,n+1}\right\} \cup \left\{e_{1,\lfloor \alpha \frac{n}{2} \rfloor+1}, \dots, e_{1,n-1}, e_{1,n+1}\right\}$. For any $2 \le i \le n$, agent i has additive cost function $c_i(\cdot)$ with $c_i(e) = 0$ for any $e \in A_i$, and $c_i(e) = 1$ for any $e \in E \backslash A_i$. Then, for every $2 \le i \le n$, agent i has an additive, clearly monotone and submodular, cost function, and violates neither PMMS nor MMS due to $c_i(A_i) = 0$. Consequently, the approximation guarantee of $\mathbf{A}$ on both PMMS and MMS are determined by agent 1.

As for the cost function $c_1(\cdot)$ of agent 1, for any $S \subseteq E$, we let

$$c_1(S) = \sum_{j=1}^{n-1} \min\{|S \cap P_j|, 1\} + \delta \cdot \min\{|S \cap P_n|, 1\} + (1-\delta) \cdot \min\{|S \cap P_{n+1}|, 1\},$$

where $\delta = \alpha\frac{n}{2} - \lfloor \alpha\frac{n}{2} \rfloor$. Function $c_1(\cdot)$ is clearly monotone. As in the proof of Proposition 5.1 (see Footnote 9), $c_1(\cdot)$ as a conical combination of submodular functions is also submodular.

We argue $\mathbf{A}$ is an α-PMMS allocation with approximation guarantee $\alpha\frac{n}{2}$ on the notion of MMS. In fact, under allocation $\mathbf{A}$, one can compute $c_1(A_1) = \lfloor \alpha\frac{n}{2} \rfloor + \delta = \alpha\frac{n}{2}$ and $c_1(A_1 \cup A_j) = c_1(E) = n$ for any $2 \le j \le n$. Then, for any $j \ge 2$, due to Lemma 2.1, it holds that $\mathrm{MMS}_1(2, A_1 \cup A_j) \ge \frac{n}{2}$, which then imply $c_1(A_1) \le \alpha \mathrm{MMS}_1(2, A_1 \cup A_j)$. Thus, allocation $\mathbf{A}$ is α-PMMS. As for the quantity of $\mathrm{MMS}_1(n, E)$, consider partition $\{B_i\}_{i\in[n]}$ with

$B_i = P_i$ for $1 \le i \le n-1$ and $B_n = P_n \cup P_{n+1}$. It is not hard to verify $c_1(B_i) = 1$ for any $i \in [n]$. According to Lemma 2.1, we have $\mathrm{MMS}_1(n, E) \ge \frac{1}{n} c_1(E) = 1$. Hence, partition $\{B_i\}_{i \in [n]}$ defines $\mathrm{MMS}_1(n, E) = 1$, and accordingly, the approximation guarantee of $\mathbf{A}$ for MMS is $\alpha \frac{n}{2}$, equivalent to $\alpha \lceil \frac{n}{2} \rceil$ since n is even. □

We remark that all statements in this section are still true if agents have subadditive cost functions. Results in this section show that although PMMS (or MMS) is proposed as relaxation of EF under additive setting, there are few connections between PMMS (or MMS) and EF in the submodular setting. This motivates new submodular fairness notions which is not only a relaxation of EF but also inherit the spirit of PMMS (or MMS).

6 Price of fairness under additive setting

After having compared the fairness criteria between themselves, in this section we study the efficiency of these fairness criteria in terms of the price of fairness with respect to social optimality of an allocation.

6.1 Two agents

We start with the case of two players. Our first result concerns EF1.

Theorem 6.1 *When $n = 2$ and agents have additive cost functions, the price of EF1 is 5/4.*

Proof For the upper bound part, we analyze the allocation returned by algorithm ALG_1, whose detailed description is in Appendix A.2. In this proof, we denote $L(k) = \{e_1, \dots, e_k\}$ and $R(k) = \{e_k, \dots, e_m\}$. We first show that ALG_1 is well-defined and can always output an EF1 allocation. Note that $\mathbf{O}$ is the optimal allocation for the underlying instance due to the order of chores. We consider the possible value of index s. Because of the normalized cost function, trivially, $s < m$ holds. If $s = 0$, ALG_1 outputs the allocation returned by round-robin (line 6) and clearly, it's EF1. If the optimal allocation $\mathbf{O}$ is EF1 (line 9), we are done. For this case, we claim that if $s = m-1$, then $\mathbf{O}$ must be EF1. The reason is that for agent 1, his cost $c_1(O_1) \le c_2(O_2) \le c_1(O_2)$ where the first transition due to line 1 of ALG_1, and thus he does not envy agent 2. For agent 2, since he only receives a single chore in optimal allocation due to $s = m-1$, clearly, he does not violate the condition of EF1, either. Thus, allocation $\mathbf{O}$ is EF1 in the case of $s = m-1$. Next, we study the remaining case (lines 11–13) that can only happen when $1 \le s \le m-2$. We first show that the index f is well-defined. It suffices to show $c_2(R(s+2)) > c_2(L(s))$. For the sake of contradiction, assume $c_2(R(s+2)) \le c_2(L(s))$. This is equivalent to $c_2(O_2 \setminus \{e_{s+1}\}) \le c_2(O_1)$, which means agent 2 satisfying EF1 in allocation $\mathbf{O}$. Due to the assumption (line 1), $c_1(O_1) \le c_2(O_2) \le c_1(O_2)$ holds, and thus, agent 1 is EF under the allocation $\mathbf{O}$. Consequently, the allocation $\mathbf{O}$ is EF1, a contradiction. Then, we prove allocation $\mathbf{A}$ (line 13) is EF1. According to the order of chores, it holds that

$$\frac{c_1(L(f))}{c_2(L(f))} \le \frac{c_1(R(f+2))}{c_2(R(f+2))}.$$

Since $c_2(R(f+2)) > c_2(L(f)) \ge 0$, this implies,

$$\frac{c_1(L(f))}{c_1(R(f+2))} \le \frac{c_2(L(f))}{c_2(R(f+2))}.$$

By the definition of index f, we have $c_2(R(f+2)) > c_2(L(f))$ and therefore $c_1(L(f)) < c_1(R(f+2))$ which is equivalent to $c_1(A_1 \setminus \{e_{f+1}\}) < c_1(A_2)$. Thus, agent 1 is EF1 under allocation $\mathbf{A}$. As for agent 2, if $f = m-2$, then $|A_2| = 1$ and clearly, agent 2 does not violate the condition of EF1. We can further assume $f \le m-3$. Since f is the maximum index satisfying $f \ge s$ and $c_2(R(f+2)) > c_2(L(f))$, it must hold that $c_2(R(f+3)) \le c_2(L(f+1))$, which is equivalent to $c_2(A_2 \setminus \{e_{f+2}\}) \le c_2(A_1)$ and so agent 2 is also EF1 under allocation $\mathbf{A}$.

Next, we show the social cost of the allocation returned by ALG_1 is at most 1.25 times of the optimal social cost. If $s = 0$, both agents have the same cost profile, then any allocations have the optimal social cost and we are done in this case. If allocation $\mathbf{O}$ is EF1, then clearly, we are done. The remaining case is of lines 11–13 of ALG_1. Since $c_1(O_1) \le c_2(O_2) \le c_1(O_2)$, we have $c_1(O_1) \le \frac{1}{2}$. Notice that $\mathbf{O}$ is not EF1, then $c_2(O_2) > \frac{1}{2}$ must hold; otherwise, $c_2(O_2) \le c_2(O_1)$ and allocation $\mathbf{O}$ is EF, a contradiction. Therefore, under the case where allocation $\mathbf{O}$ is not EF1, we must have $c_1(O_1) \le \frac{1}{2}$ and $c_2(O_2) > \frac{1}{2}$. Due to $f+2 \ge s+1$ and the order of chores, it holds that

$$\frac{c_1(R(f+2))}{c_2(R(f+2))} \ge \frac{c_1(O_2)}{c_2(O_2)}.$$

This implies $c_1(R(f+2)) \ge \frac{c_1(O_2)}{c_2(O_2)} c_2(R(f+2))$, and equivalently,

$$c_1(A_1) = c_1(L(f+1)) \le 1 - \frac{c_1(O_2)}{c_2(O_2)} c_2(R(f+2)).$$

Again, by the construction of f, we have

$$c_2(A_2) = c_2(R(f+2)) > c_2(L(f)) \ge c_2(L(s)) = c_2(O_1).$$

Therefore, we derive the following upper bound,

$$\begin{aligned} c_1(A_1) + c_2(A_2) &\le 1 - \left(\frac{c_1(O_2)}{c_2(O_2)} - 1\right) c_2(A_2) \le 1 - \left(\frac{c_1(O_2)}{c_2(O_2)} - 1\right) c_2(O_1) \\ &= 1 - \left(\frac{1 - c_1(O_1)}{c_2(O_2)} - 1)(1 - c_2(O_2)\right), \end{aligned} \tag{11}$$

where the second inequality is due to $\frac{c_1(O_2)}{c_2(O_2)} \ge 1$ and $c_2(A_2) \ge c_2(O_1)$. Based on (11), we have an upper bound on the price of EF1 as follows:

$$\text{Price of EF1} \le \frac{1 - \left(\frac{1 - c_1(O_1)}{c_2(O_2)} - 1\right)(1 - c_2(O_2))}{c_1(O_1) + c_2(O_2)}. \tag{12}$$

Recall $0 \le c_1(O_1) \le \frac{1}{2} < c_2(O_2) \le 1$. The partial derivatives of the fraction in (12) with respect to $c_1(O_1)$ is equal to the following:

$$\frac{1}{(c_1(O_1) + c_2(O_2))^2} \left(\frac{1}{c_2(O_2)} - 2\right).$$

It is not hard to see this derivative has a negative value for any $\frac{1}{2} < c_2(O_2) \leq 1$. Thus, the fraction in (12) takes maximum value only when $c_1(O_1) = 0$ and hence,

$$\text{Price of EF1} \leq \frac{3 - \frac{1}{c_2(O_2)}}{c_2(O_2)} - 1.$$

Similarly, by taking the derivative with respect to $c_2(O_2)$, the maximum value of this expression happens only when $c_2(O_2) = \frac{2}{3}$, then one can easily compute the maximum value of the RHS of (12) is 1.25. Therefore, the price of EF1 ≤ 1.25.

As for the lower bound, consider an instance with a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is $c_1(e_1) = 0$ and $c_1(e_2) = c_1(e_3) = \frac{1}{2}$. For agent 2, his cost is $c_2(e_1) = \frac{1}{3} - 2\epsilon$ and $c_2(e_2) = c_2(e_3) = \frac{1}{3} + \epsilon$ where $\epsilon > 0$ takes arbitrarily small value. An optimal allocation assigns chore e_1 to agent 1 and the rest chores to agent 2, which yields the optimal social cost $\frac{2}{3} + 2\epsilon$. However, this allocation is not EF1 since agent 2 envies agent 1 even removing one chore from his bundle. To achieve EF1, agent 2 can not receive both of chores e_2 and e_3, and so, agent 1 must receive one of chore e_2 and e_3. Therefore, the best EF1 allocation can be assigning chore e_1 and e_2 to agent 1 and chore e_3 to agent 2 resulting in the social cost $\frac{5}{6} + \epsilon$. Thus, the price of EF1 is at least $\frac{\frac{5}{6}+\epsilon}{\frac{2}{3}+2\epsilon} \rightarrow \frac{5}{4}$ as $\epsilon \rightarrow 0$, completing the proof. □

According to Theorems 3.3 and 3.6, EF1 implies 2-MMS and $\frac{3}{2}$-PMMS. The following two theorems confirm an intuition — if one relaxes the fairness condition, then less efficiency will be sacrificed.

Theorem 6.2 *When $n = 2$ and agents have additive cost functions, the price of 2-MMS is 1.*

Proof The proof directly follows from Lemma 2.2. □

Theorem 6.3 *When $n = 2$ and agents have additive cost functions, the price of $\frac{3}{2}$-PMMS is* 7/6.

Proof We first prove the upper bound. Given an instance I, let $\mathbf{O} = (O_1, O_2)$ be an optimal allocation of I. If the allocation $\mathbf{O}$ is already $\frac{3}{2}$-PMMS, we are done. For the sake of contradiction, we assume that agent 1 violates the condition of $\frac{3}{2}$-PMMS in allocation $\mathbf{O}$, i.e., $c_1(O_1) > \frac{3}{2}\mathrm{MMS}_1(2, E)$. Suppose $O_1 = \{e_1, \dots, e_h\}$ and the index satisfies the following rule; $\frac{c_1(e_1)}{c_2(e_1)} \geq \frac{c_1(e_2)}{c_2(e_2)} \geq \cdots \geq \frac{c_1(e_h)}{c_2(e_h)}$. In this proof, for simplicity, we write $L(k) := \{e_1, \dots, e_k\}$ for any $1 \leq k \leq h$ and $L(0) = \emptyset$. Then, let s be the index such that $c_1(O_1 \setminus L(s)) \leq \frac{3}{2}\mathrm{MMS}_1(2, E)$ and $c_1(O_1 \setminus L(s-1)) > \frac{3}{2}\mathrm{MMS}_1(2, E)$. In the following, we divide our proof into two cases.

Case 1: $c_1(L(s)) \leq \frac{1}{2}c_1(O_1)$. Consider allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \setminus L(s)$ and $A_2 = O_2 \cup L(s)$. We first show allocation $\mathbf{A}$ is $\frac{3}{2}$-PMMS. For agent 1, due to the construction of index s, he does not violate the condition of $\frac{3}{2}$-PMMS. As for agent 2, we claim that $c_2(A_2) = 1 - c_2(O_1 \setminus L(s-1)) + c_2(e_s) < \frac{1}{4} + c_2(e_s)$ because $c_2(O_1 \setminus L(s-1)) \geq c_1(O_1 \setminus L(s-1)) > \frac{3}{2}\mathrm{MMS}_1(2, E) \geq \frac{3}{4}$ where the first inequality transition is due to the fact that O_1 is the bundle assigned to agent 1 in the optimal allocation. If $c_2(e_s) < \frac{1}{2}$, then clearly, $c_2(A_2) < \frac{3}{4} \le \frac{3}{2}\mathrm{MMS}_2(2, E)$. If $c_2(e_s) \ge \frac{1}{2}$, then $c_2(e_s) = \mathrm{MMS}_1(2, E)$ and accordingly, it is not hard to verify that $c_2(A_2) \le \frac{3}{2}\mathrm{MMS}_1(2, E)$. Thus, **A** is a $\frac{3}{2}$-PMMS allocation.

Next, based on allocation **A**, we derive an upper bound on the price of $\frac{3}{2}$-PMMS. First, by the order of index, $\frac{c_1(L(s))}{c_2(L(s))} \ge \frac{c_1(O_1)}{c_2(O_1)}$ holds, implying $c_2(L(s)) \le \frac{c_2(O_1)}{c_1(O_1)}c_1(L(s))$. Since $A_1 = O_1 \backslash L(s)$ and $A_2 = O_2 \cup L(s)$, we have the following:

$$\begin{aligned}
\text{Price of } \frac{3}{2}\text{-PMMS} &\le 1 + \frac{c_2(L(s)) - c_1(L(s))}{c_1(O_1) + c_2(O_2)} \le 1 + \frac{c_1(L(s))(\frac{c_2(O_1)}{c_1(O_1)} - 1)}{c_1(O_1) + c_2(O_2)} \\
&= 1 + \frac{\frac{c_1(L(s))}{c_1(O_1)}(1 - c_2(O_2) - c_1(O_1))}{c_1(O_1) + c_2(O_2)} \\
&\le 1 + \frac{\frac{1}{2} - \frac{1}{2}(c_1(O_1) + c_2(O_2))}{c_1(O_1) + c_2(O_2)} \le 1 - \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6},
\end{aligned}$$

where the second inequality due to $c_2(L(s)) \le \frac{c_2(O_1)}{c_1(O_1)}c_1(L(s))$; the third inequality due to the condition of Case 1; and the last inequality is because $c_1(O_1) > \frac{3}{2}\mathrm{MMS}_1(2, E) \ge \frac{3}{4}$.

Case 2: $c_1(L(s)) > \frac{1}{2}c_1(O_1)$. We first derive a lower bound on $c_1(e_s)$. Equation $c_1(e_s) = c_1(O_1 \backslash L(s-1)) + c_1(L(s)) - c_1(O_1)$ and the condition of Case 2 imply $c_1(e_s) > c_1(O_1 \backslash L(s-1)) - \frac{1}{2}c_1(O_1)$, and consequently we have $c_1(e_s) > \frac{3}{2}\mathrm{MMS}_1(2, E) - \frac{1}{2}c_1(O_1) \ge \frac{1}{4}$ where the last transition is due to $\mathrm{MMS}_1(2, E) \ge \frac{1}{2}$ and $c_1(O_1) \le 1$. Then, we consider two subcases.

If $0 \le c_2(e_s) - c_1(e_s) \le \frac{1}{8}$, consider an allocation $\mathbf{A} = (A_1, A_2)$ with $A_1 = O_1 \backslash \{e_s\}$ and $A_2 = O_2 \cup \{e_s\}$. We first show the allocation **A** is $\frac{3}{2}$-PMMS. For agent 1, since $c_1(e_s) > \frac{1}{4}$, $c_1(A_1) = c_1(O_1) - c_1(e_s) < \frac{3}{4} \le \frac{3}{2}\mathrm{MMS}_1(2, E)$. As for agent 2, $c_2(A_2) = c_2(O_2) + c_2(e_s) \le 1 - c_1(O_1) + c_2(e_s) < \frac{1}{4} + c_2(e_s)$. If $c_2(e_s) < \frac{1}{2}$, then clearly, $c_2(A_2) \le \frac{3}{4} < \frac{3}{2}\mathrm{MMS}_2(2, E)$ holds. If $c_2(e_s) \ge \frac{1}{2}$, we have $c_2(e_s) = \mathrm{MMS}_2(2, E)$ and accordingly, it is not hard to verify that $c_2(A_2) \le \frac{3}{2}\mathrm{MMS}_2(2, E)$. Thus, the allocation **A** is $\frac{3}{2}$-PMMS. Next, based on the allocation **A**, we derive an upper bound regarding the price of $\frac{3}{2}$-PMMS,

$$\text{Price of } \frac{3}{2}\text{-PMMS} \le \frac{c_1(O_1) - c_1(e_s) + c_2(O_2) + c_2(e_s)}{c_1(O_1) + c_2(O_2)} \le 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6},$$

where the second inequality due to $0 \le c_2(e_s) - c_1(e_s) \le \frac{1}{8}$ and $c_1(O_1) > \frac{3}{4}$.

If $c_2(e_s) - c_1(e_s) > \frac{1}{8}$, consider an allocation $\mathbf{A}' = (A'_1, A'_2)$ with $A'_1 = \{e_s\}$ and $A'_2 = E \backslash \{e_s\}$. We first show that the allocation $\mathbf{A}'$ is $\frac{3}{2}$-PMMS. For agent 1, due to Lemma 2.1, $c_1(e_s) \le \mathrm{MMS}_1(2, E)$ holds. As for agent 2, since $c_2(e_s) \ge c_1(e_s) > \frac{1}{4}$, we have $c_2(A'_2) = c_2(E) - c_2(e_s) < \frac{3}{4} \le \frac{3}{2}\mathrm{MMS}_2(2, E)$. Thus, the allocation $\mathbf{A}'$ is $\frac{3}{2}$-PMMS. In the following, we first derive an upper bound for $c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})$, then based on the bound, we provide the target upper bound for the price of fairness. Since $c_1(O_1) > \frac{3}{4}$ and $c_2(e_s) - c_1(e_s) > \frac{1}{8}$, we have

$c_2(O_1 \backslash \{e_s\}) - c_1(O_1 \backslash \{e_s\}) = c_2(O_1) - c_1(O_1) - (c_2(e_s) - c_1(e_s)) < \frac{1}{8}$, and then, the following holds,

$$\text{Price of } \frac{3}{2}\text{-PMMS} \leq 1 + \frac{c_2(O_1 \setminus \{e_s\}) - c_1(O_1 \setminus \{e_s\})}{c_1(O_1) + c_2(O_2)} \leq 1 + \frac{1}{8} \times \frac{4}{3} = \frac{7}{6},$$

which completes the proof of the upper bound.

Regarding lower bound, consider an instance I with two agents and a set $E = \{e_1, e_2, e_3, e_4\}$ of four chores. The cost function for agent 1 is: $c_1(e_1) = \frac{3}{8}, c_1(e_2) = \frac{3}{8} + \epsilon, c_1(e_3) = \frac{1}{8} - \epsilon, c_1(e_4) = \frac{1}{8}$ where $\epsilon > 0$ takes arbitrarily small value. For agent 2, here cost function is: $c_2(e_1) = c_2(e_2) = \frac{1}{2}, c_2(e_3) = c_2(e_4) = 0$. It is not hard to verify that $\text{MMS}_i(2, E) = \frac{1}{2}$ for any $i = 1, 2$. In the optimal allocation, the assignment is; e_1, e_2 to agent 1 and e_3, e_4 to agent 2, resulting in $\text{OPT}(I) = \frac{3}{4} + \epsilon$. Observe that to satisfy $\frac{3}{2}$-PMMS, agent 1 cannot receive both chores e_1, e_2, and accordingly, the minimum social cost of a $\frac{3}{2}$-PMMS allocation is $\frac{7}{8}$ by assigning e_1 to agent 1 and the rest chores to agent 2. Based on this instance, when $n = 2$, the price of $\frac{3}{2}$-PMMS is at least $\frac{\frac{7}{8}}{\frac{6}{8}+\epsilon} \to \frac{7}{6}$ as $\epsilon \to 0$. □

We remark that if we have an *oracle* for the maximin share, then our constructive proof of Theorem 6.3 can be transformed into an efficient algorithm for finding a 3/2-PMMS allocation whose cost is at most 7/6 times the optimal social cost. Moving to other fairness criteria, we have the following uniform bound.

Theorem 6.4 *When $n = 2$ and agents have additive cost functions, the price of PMMS, MMS, and EFX are all* 2.

Proof We first show results on the upper bound. When $n = 2$, PMMS is identical to MMS and moreover implies EFX according to Theorem 4.1. Thus it suffices to show that the price of PMMS is at most 2. Given an instance I, let allocation $\mathbf{O} = (O_1, O_2)$ be its optimal allocation and w.l.o.g, we assume $c_1(O_1) \leq c_2(O_2)$. If $c_2(O_2) \leq \frac{1}{2}$, then we have $c_1(O_1) \leq 1 - c_1(O_1) = c_1(O_2)$ and $c_2(O_2) \leq 1 - c_2(O_2) = c_2(O_1)$. So allocation $\mathbf{O}$ is an EF and accordingly is PMMS, which yields the price of PMMS equals to one. Thus, we can further assume $c_2(O_2) > \frac{1}{2}$ and hence the optimal social cost is larger than $\frac{1}{2}$.

We next show that there exist a PMMS allocation whose social cost is at most 1. W.l.o.g, we assume $\text{MMS}_1(2, E) \leq \text{MMS}_2(2, E)$ (the other case is symmetric). Let $\mathbf{T} = (T_1, T_2)$ be the allocation defining $\text{MMS}_1(2, E)$ and $c_1(T_1) \leq c_1(T_2) = \text{MMS}_1(2, E)$. If $c_2(T_2) \leq c_2(T_1)$, then allocation $\mathbf{T}$ is EF (also PMMS), and thus it holds that $c_1(T_1) \leq \frac{1}{2}$ and $c_2(T_2) \leq \frac{1}{2}$. Therefore, social cost of allocation $\mathbf{T}$ is no more than one, which implies the price of PMMS is at most two. If $c_2(T_2) > c_2(T_1)$, then consider the allocation $\mathbf{T}' = (T_2, T_1)$. Since $c_1(T_1') = c_1(T_2) = \text{MMS}_1(2, E)$ and $c_2(T_2') = c_2(T_1) < c_2(T_2)$, then $\mathbf{T}'$ is a PMMS allocation. Owing to $\text{MMS}_1(2, E) \leq \text{MMS}_2(2, E)$, we claim that $c_2(T_1) \leq c_1(T_1)$; otherwise, we have $\text{MMS}_1(2, E) = c_1(T_2) > c_2(T_2) > c_2(T_1)$, and equivalently, allocation $\mathbf{T}'$ is a 2-partition where the cost of both bundles for agent 2 is strictly smaller than $\text{MMS}_1(2, E)$, contradicting $\text{MMS}_1(2, E) \leq \text{MMS}_2(2, E)$. By $c_2(T_1) \leq c_1(T_1)$, the social cost of allocation $\mathbf{T}'$ satisfies $c_2(T_1) + c_1(T_2) \leq 1$ and so the price of PMMS is at most two.

Regarding the tightness, consider an instance I with two agents and a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is: $c_1(e_1) = \frac{1}{2}, c_1(e_2) = \frac{1}{2} - \epsilon$ and $c_1(e_3) = \epsilon$ where $\epsilon > 0$ takes arbitrarily small value. For agent 2, his cost is $c_2(e_1) = \frac{1}{2}, c_2(e_2) = \epsilon$ and

$c_2(e_3) = \frac{1}{2} - \epsilon$. An optimal allocation assigns chores e_1, e_2 to agent 2, and e_3 to agent 1, and consequently, the optimal social cost equals to $\frac{1}{2} + 2\epsilon$. We first concern the tightness on the notion of PMMS (or MMS, these two are identical when $n = 2$). In any PMMS allocations, it must be the case that an agent receives chore e_1 and the other one receives chores e_2, e_3, and thus the social cost of PMMS allocations is one. Therefore, the price of PMMS and of MMS is at least $\frac{1}{\frac{1}{2}+\epsilon} \to 2$ as $\epsilon \to 0$. As for EFX, similarly, it must be the case that in any EFX allocations, the agent receiving chore e_1 cannot receive any other chores. Thus, it not hard to verify that the social cost of EFX allocations is also one and the price of EFX is at least $\frac{1}{\frac{1}{2}+\epsilon} \to 2$ as $\epsilon \to 0$. □

6.2 More than two agents

Note that the existence of an MMS allocation is not guaranteed in general [7, 35] and the existence of PMMS or EFX allocation is still open in chores when $n \geq 3$. Nonetheless, we are still interested in the prices of fairness in case such a fair allocation does exist.

Theorem 6.5 *When agent have additive cost functions, for $n \geq 3$, the price of EF1, EFX, PMMS and $\frac{3}{2}$-PMMS are all infinity.*

Proof In this proof, ϵ always takes arbitrarily small positive value. Based on our results on the connections between fairness criteria, we have the relationship: PMMS $\to$ EFX $\to$ EF1 $\to$ $\frac{3}{2}$-PMMS, where $A \to B$ refers to that notion A is stricter than notion B. Therefore, it suffices to give a proof for $\frac{3}{2}$-PMMS.

Consider an instance with n agents and $m \geq 5$ chores. The cost function of agent 1 is $c_1(e_1) = 1 - 4\epsilon$, $c_1(e_j) = 0$ for $j = 2, \dots, m-4$, and $c_1(e_j) = \epsilon$ for $j \geq m-3$. For agent 2, his cost is $c_2(e_1) = 1 - \frac{4}{m}$, $c_2(e_j) = 0$ for $j = 2, \dots, m-4$, and $c_2(e_j) = \frac{1}{m}$ for $j \geq m-3$. The cost function of agent 3 is: $c_3(e_1) = \epsilon$, $c_3(e_j) = \frac{1}{m}$ for $j = 2, \dots, m-1$, and $c_3(e_m) = \frac{1}{m} - \epsilon$. For any $i \geq 4$, the cost function of agent i is $c_i(e_j) = \frac{1}{m}$ for any $j \in [m]$. An optimal allocation assigns $e_{m-3}, e_{m-2}, e_{m-1}, e_m$ to agent 1 and e_1 to agent 3. For each of the remaining chores, it is assigned to the agents having zero cost on it. Accordingly, the optimal social cost is 5ϵ. However, in any optimal allocation $\mathbf{O}$, we have $\text{MMS}_1(2, O_1 \cup O_2) = 2\epsilon$, implying $c_1(O_1) > \frac{3}{2}\text{MMS}_1(2, O_1 \cup O_2)$. Thus, agent 1 violates $\frac{3}{2}$-PMMS. In order to achieve $\frac{3}{2}$-PMMS, at least one of $e_{m-3}, e_{m-2}, e_{m-1}, e_m$ has to be assigned to someone other than agent 1, and so the social cost of a $\frac{3}{2}$-PMMS allocation is at least $\frac{1}{m} + 3\epsilon$, resulting in an unbounded price of $\frac{3}{2}$-PMMS when $\epsilon \to 0$. □

In the context of goods allocation, Barman et al. [8] present an asymptotically tight price of EF1, $O(\sqrt{n})$. However, as shown by Theorem 6.5, when allocating chores, the price of EF1 is infinite, which shows a sharp contrast between goods and chores allocation.

We are now left with MMS fairness. Let us first provide upper and lower bounds on the price of MMS.

Theorem 6.6 *When agents have additive cost functions, for $n \geq 3$, the price of MMS is at most n^2 and at least $\frac{n}{2}$.*

Proof We first prove the upper bound part. For any instance I, if $\mathrm{OPT}(I) \le \frac{1}{n}$, then by Lemma 2.1, any optimal allocations is MMS. Thus, we can further assume $\mathrm{OPT}(I) > \frac{1}{n}$. Notice that the maximum social cost of an allocation is n and thus the upper bound of n^2 is straightforward.

For the lower bound, consider an instance I with n agents and $n+1$ chores $E = \{e_1, \dots, e_{n+1}\}$. For agent $i = 2, \dots, n$, $c_i(e_1) = c_i(e_2) = \frac{1}{2}$ and $c_i(e_j) = 0$ for any $j \ge 3$. As for agent 1, $c_1(e_1) = \frac{1}{n}$, $c_1(e_2) = \epsilon$, $c_1(e_3) = \frac{1}{n} - \epsilon$ and $c_1(e_j) = \frac{1}{n}$ for any $j \ge 4$ where $\epsilon > 0$ takes arbitrarily small value. It is not hard to verify that $\mathrm{MMS}_1(n, E) = \frac{1}{n}$ and $\mathrm{MMS}_i(n, E) = \frac{1}{2}$ for $i \ge 2$. In any optimal allocation $\mathbf{O} = (O_1, \dots, O_n)$, the first two chores are assigned to agent 1 and each of the remaining chores is assigned to agents having cost zero. Thus, we have $\mathrm{OPT}(I) = \frac{1}{n} + \epsilon$. However, in any optimal allocation $\mathbf{O}$, we have $c_1(O_1) > \mathrm{MMS}_1(n, E) = \frac{1}{n}$. In order to achieve MMS, agent 1 can not receive both chores e_1, e_2, and so at least one of them has to be assigned to the agent other than agent 1. As a result, the social cost of an MMS allocation is at least $\frac{1}{2} + \epsilon$, which implies that the price of MMS is at least $\frac{n}{2}$ as $\epsilon \to 0$. $\square$

As mentioned earlier, the existence of MMS allocation is not guaranteed. So we also provide an asymptotically tight price of 2-MMS, whose existence is guaranteed for any instance with additive cost functions.

Theorem 6.7 *When agents have additive cost functions, for $n \ge 3$, the price of 2-MMS is at least $\frac{n+3}{6}$ and at most n, asymptotically tight $\Theta(n)$.*

Proof We first prove the upper bound. By Proposition 3.3, we know that an EF1 allocation is also $\frac{2n-1}{n}$-MMS (also 2-MMS). As we mentioned earlier, round-robin algorithm always output EF1 allocations. Consequently, given any instance I, the allocation returned by round-robin is also 2-MMS. In the following, we incorporate the idea of expectation in probability theory and show that there exists an order of round-robin such that the output allocation has social cost at most 1.

Let σ be a uniformly random permutation of $\{1, \dots, n\}$ and $\mathbf{A}(\sigma) = (A_1(\sigma), \dots, A_n(\sigma))$ be the allocation returned by round-robin based on the order σ. Clearly, each element $A_i(\sigma)$ is a random variable. Since σ is chosen uniformly random, the probability of agent i on j-th position is $\frac{1}{n}$. Fix an agent i, we assume $c_i(e_1) \le c_i(e_2) \le \cdots \le c_i(e_m)$. If agent i is in j-th position of the order, then his cost is at most $c_i(e_j) + c_i(e_{n+j}) + \cdots + c_i(e_{\lfloor \frac{m-j}{n} \rfloor n+j})$. Accordingly, his expected cost is at most $\sum_{j=1}^{n} \frac{1}{n} \sum_{l=0}^{\lfloor \frac{m-j}{n} \rfloor} c_i(e_{ln+j})$. Thus, we have an upper bound of the expected social cost,

$$\mathbb{E}[SC(\mathbf{A}(\sigma))] \le \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{n} \sum_{l=0}^{\lfloor \frac{m-j}{n} \rfloor} c_i(e_{ln+j}) = \frac{1}{n} \sum_{i=1}^{n} c_i(E) = 1.$$

Therefore, there exists an order such that the social cost of the output is at most 1. Notice that for any instance I, if $\mathrm{OPT}(I) \le \frac{1}{n}$, then any optimal allocations are also MMS. Thus, we can further assume $\mathrm{OPT}(I) > \frac{1}{n}$, and accordingly, the price of 2-MMS is at most n.

For the lower bound, consider an instance I with n agents and a set $E = \{e_1, \dots, e_{n+3}\}$ of $n+3$ chores. The cost function of agent 1 is: $c_1(e_1) = c_1(e_2) = \frac{1}{n} - \epsilon$, $c_1(e_3) = 3\epsilon$, $c_1(e_4) = c_1(e_5) = \epsilon$, $c_1(e_6) = \frac{1}{n} - 3\epsilon$ where $\epsilon > 0$ takes arbitrarily small value, and $c_1(e_j) = \frac{1}{n}$ for any $j > 6$ (if exists). For agent $i = 2, \dots, n$, his cost is: $c_i(e_j) = \frac{1}{3}$ for any $j \in [3]$ and $c_i(e_j) = 0$ for $j \geq 4$. It is not hard to verify that $\mathrm{MMS}_1(n, E) = \frac{1}{n}$ and $\mathrm{MMS}_i(n, E) = \frac{1}{3}$ for any $i \geq 2$. In any optimal allocation $\mathbf{O} = (O_1, \dots, O_n)$, the first three chores are assigned to agent 1 and all rest chores are assigned to agents having cost zero on them. Thus, we have $\mathrm{OPT}(I) = \frac{2}{n} + \epsilon$. However, in any optimal allocations $\mathbf{O}$, $\frac{2}{n} + \epsilon = c_1(O_1) > 2\mathrm{MMS}_1(n, E)$ holds, and so agent 1 violates 2-MMS. In order to achieve a 2-MMS allocation, agent 1 can not receive all first three chores, and so at least one of them has to be assigned to the agent other than agent 1. As a result, the social cost of an 2-MMS allocation is at least $\frac{1}{3} + \frac{1}{n} + 2\epsilon$, yielding that the price of 2-MMS is at least $\frac{n}{6} + \frac{1}{2}$. Combing lower and upper bound, the price of 2-MMS is $\Theta(n)$ □

7 Price of fairness beyond additive setting

In this section, we study the price of fairness when agents have submodular cost functions. Notice that for those fairness notions whose price of fairness is unbounded in the additive setting, the efficiency loss would still be unbounded in the submodular setting. As a consequence, for most notions, we only need to study its price of fairness in the case of two agents. Recall that, when studying specific fairness notion, we only consider instances for which allocations satisfying the underlying fairness notion do exist. All results established in this section remain true if agents have subadditive cost functions.

Theorem 7.1 *When $n = 2$ and agents have submodular cost functions, if an EFX allocation exists, the price of EFX is at least 3 and at most 4.*

Proof We first prove the upper bound. For an instance I, let $\mathbf{O} = (O_1, O_2)$ be an optimal allocation, and w.l.o.g., we assume $c_1(O_1) \leq c_2(O_2)$. Since $c_2(\cdot)$ is submodular and also subadditive, then $c_i(O_i) + c_i(O_{3-i}) \geq c_i(E)$ holds for $i \in [2]$. If $c_1(O_1) \leq c_2(O_2) \leq 1/2$, then $c_i(O_{3-i}) \geq c_i(E) - c_i(O_i) \geq 1/2 \geq c_i(O_i)$ holds for $i \in [2]$. Accordingly, allocation $\mathbf{O}$ is already EFX and we are done. Thus, w.l.o.g., we can further assume $c_2(O_2) > 1/2$. Notice that the social cost of an allocation is at most 2, and so the price of EFX is at most 4.

As for the lower bound, let us consider an instance with a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is: $c_1(e_1) = 1/2, c_1(e_2) = 1/2 - \epsilon, c_1(e_3) = \epsilon$ and for any $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e_s)$ where $\epsilon > 0$ takes arbitrarily small value. The cost function of agent 2 is: $c_2(e_1) = 1 - \epsilon, c_2(e_2) = 3\epsilon, c_2(e_3) = 1 - 2\epsilon$ and for any $S \subseteq E, c_2(S) = \min\{\sum_{e \in S} c_2(e), 1\}$. Function $c_1(\cdot)$ is additive and hence clearly monotone and submodular. For function $c_2(\cdot)$, since $\sum_{e \in S} c_2(e)$ is additive (also monotone and submodular) on S, it follows that $c_2(\cdot)$ is also monotone and submodular (see Footnote 9).

For this instance, the optimal allocation $\mathbf{O} = (O_1, O_2)$ is $O_1 = \{e_1, e_3\}$ and $O_2 = \{e_2\}$, yielding social cost $1/2 + 4\epsilon$. But due to $c_1(O_1 \backslash \{e_3\}) = 1/2 > 1/2 - \epsilon = c_1(O_2)$, agent 1 violates EFX in $\mathbf{O}$. In an EFX allocation, agent 2 can not receive the whole E or $\{e_1, e_3\}$ or $\{e_1, e_2\}$. Thus, the EFX allocation with the smallest social cost is $A_1 = \{e_2, e_3\}$ and $A_2 = \{e_1\}$, yielding social cost $3/2 - \epsilon$. As a consequence, the price of EFX is at least $\frac{3/2-\epsilon}{1/2+4\epsilon} \to 3$ as $\epsilon \to 0$. □

Theorem 7.2 *When $n = 2$ and agents have submodular cost functions, if an EF1 allocation exists, the price of EF1 is at least 2 and at most 4.*

Proof For the upper bound part, similar to the proof of Theorem 7.1, we can w.l.o.g assume $c_1(O_1) \le c_2(O_2)$ and $c_2(O_2) > 1/2$; otherwise, $\mathbf{O}$ is already EF1. Notice that the social cost of an allocation is at most 2, and so the price of EF1 is at most 4.

As for the lower bound, let us consider an instance I with a set $E = \{e_1, e_2, e_3\}$ of three chores. The cost function of agent 1 is: $c_1(e_1) = 1/3 + \epsilon, c_1(e_2) = 1/3, c_1(e_3) = 1/3 - \epsilon$ and for any $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e)$ where $\epsilon > 0$ takes arbitrarily small value. The cost function of agent 2 is: $c_2(e_1) = 1 - \epsilon, c_2(e_2) = 1 - \epsilon, c_2(e_3) = \epsilon$ and for any $S \subseteq E, c_2(S) = \min\{\sum_{e \in S} c_2(e), 1\}$. Function $c_1(\cdot)$ is additive and clearly monotone and submodular. For function $c_2(\cdot)$, since $\sum_{e \in S} c_2(e)$ is additive (also monotone and submodular) on S, it follows that $c_2(\cdot)$ is also monotone and submodular (see Footnote 9).

For this instance, the optimal allocation $\mathbf{O} = (O_1, O_2)$ is $O_1 = \{e_1, e_2\}$ and $O_2 = \{e_3\}$, yielding social cost $2/3 + 2\epsilon$. But since $\min_{e \in O_1} c_1(O_1 \setminus \{e\}) = 1/3 > 1/3 - \epsilon = c_1(O_2)$, agent 1 violates EF1 under allocation $\mathbf{O}$. In an EF1 allocation, agent 2 can not receive all chores and can not receive both e_1, e_2, either. Thus, the EF1 allocation with minimal social cost is $\mathbf{A} = (A_1, A_2)$ with $A_1 = \{e_2\}$ and $A_2 = \{e_1, e_3\}$, yielding cost 4/3. As a consequence, the price of EF1 is at least $\frac{4/3}{3/2+2\epsilon} \to 2$ as $\epsilon \to 0$. □

Theorem 7.3 *When $n = 2$ and agents have submodular cost functions, if an PMMS allocation exists, the price of PMMS is 3.*

Proof According to Lemma 2.1, $\mathrm{MMS}_i(2, E) \ge 1/2$ holds for any $i \in [2]$. Given an instance I and allocation $\mathbf{O}$ with minimal social cost, we can assume allocation $\mathbf{O}$ is not MMS and w.l.o.g, agent 2 violates the condition of MMS. Let $\mathbf{A}$ be an MMS allocation. Due to $c_2(A_2) \le \mathrm{MMS}_2(2, E) < c_2(O_2)$, we have

$$\frac{c_1(A_1) + c_2(A_2)}{c_1(O_1) + c_2(O_2)} < \frac{c_1(A_1) + \mathrm{MMS}_2(2, E)}{\mathrm{MMS}_2(2, E)} \le 3,$$

where the last inequality transition is because $c_1(A_1) \le 1$ and $\mathrm{MMS}_2(2, E) \ge 1/2$.

As for the lower bound, let us consider an instance I with a set $E = \{e_1, e_2, e_3\}$ of chores. The cost function of agent 1 is: $c_1(e_1) = 1/2$, $c_1(e_2) = 1/2 - \epsilon$, $c_1(e_3) = \epsilon$ and for $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e)$. The cost function of agent 2 is: $c_2(e_1) = 1 - 2\epsilon$, $c_2(e_2) = 10\epsilon$, $c_2(e_3) = 1 - 3\epsilon$, $c_2(e_1 \cup e_2) = 1$, $c_2(e_1 \cup e_3) = 1$, $c_2(e_2 \cup e_3) = 1 - \epsilon$, $c_2(E) = 1$. Function $c_1(\cdot)$ is additive and hence monotone and submodular. It is not hard to verify $c_2(\cdot)$ is monotone. Suppose $c_2(\cdot)$ is not submodular, and accordingly, there exists $S \subsetneq T \subseteq E$ and $e \in E \setminus T$ such that $c_2(T \cup \{e\}) - c_2(T) > c_2(S \cup \{e\}) - c_2(S)$. Since $c_2(\cdot)$ is monotone, we have $c_2(S \cup \{e\}) - c_2(S) \ge 0$ implying $c_2(T \cup \{e\}) - c_2(T) > 0$. If $|T| = 2$, the only possibility is $T = e_2 \cup e_3$ and adding e_1 to T has margin ϵ. But for any $S \subsetneq T$ the margin of adding e_1 to S is larger than ϵ, a contradiction. If $|T| = 1$, then $c_2(S \cup \{e\}) - c_2(S) = c_2(e)$ that is the largest margin of adding item e to a subset, a contradiction. Thus, function $c_2(\cdot)$ is also submodular.

For this instance, partition $\{\{e_1\}, \{e_2, e_3\}\}$ defines $\mathrm{MMS}_1(2, E) = 1/2$, and $\{\{e_1\}, \{e_2, e_3\}\}$ defines $\mathrm{MMS}_2(2, E) = 1 - \epsilon$. The minimal social cost allocation $\mathbf{O} = (O_1, O_2)$ with $O_1 = \{e_1, e_3\}$ and $O_2 = \{e_2\}$, resulting in minimal social cost $1/2 + 11\epsilon$. But $c_1(O_1) = 1/2 + \epsilon > \mathrm{MMS}_1(2, E)$, and thus $\mathbf{O}$ is not MMS. Observe that in an MMS allocation, agent 2 can only receive either a single chore or $\{e_2, e_3\}$. The MMS allocation with minimal social cost is $\mathbf{A}$ with $A_1 = \{e_2, e_3\}$ and $A_2 = \{e_1\}$ whose social cost is equal to $3/2 - 2\epsilon$. As a consequence, the price of MMS is at least $\frac{3/2-2\epsilon}{1/2+11\epsilon} \to 3$ as $\epsilon \to 0$. □

Theorem 7.4 *When $n = 2$ and agents have submodular cost functions, if a $\frac{3}{2}$-PMMS allocation exists, the price of $\frac{3}{2}$-PMMS is at least 4/3 and at most 8/3.*

Proof We first prove the upper bound. According to Lemma 2.1, $\mathrm{MMS}_i(2, E) \geq 1/2$ holds for any $i \in [2]$. Given an instance I, let $\mathbf{O} = (O_1, O_2)$ be an minimal social cost allocation of I, and w.l.o.g., we assume $c_1(O_1) \leq c_2(O_2)$. Moreover, we can assume $c_2(O_2) > 3/4$; otherwise $\mathbf{O}$ is already a $\frac{3}{2}$-PMMS allocation and we are done. Notice that the social cost of an allocation is at most 2, and so the price of $\frac{3}{2}$-PMMS is at most 8/3.

As for the lower bound, let us consider an instance with a set $E = \{e_1, e_2, e_3, e_4\}$ of four chores. The cost profile of agent 1 is: $c_1(e_1) = 3/8$, $c_1(e_2) = 3/8 + \epsilon$, $c_1(e_3) = 1/8 - \epsilon$, $c_1(e_4) = 1/8$ and for $S \subseteq E, c_1(S) = \sum_{e \in S} c_1(e)$. The cost profile of agent 2 is: $c_2(e_1) = c_2(e_2) = 1 - \epsilon$, $c_2(e_3) = c_2(e_4) = \epsilon$ and for $S \subseteq E, c_2(S) = \min\{\sum_{e \in S} c_2(e), 1\}$ where $\epsilon > 0$ can take arbitrarily small value. Function $c_1(\cdot)$ is additive and hence monotone and submodular. For function $c_2(\cdot)$, since $\sum_{e \in S} c_2(e)$ is additive (also monotone and submodular) on S, it follows that $c_2(\cdot)$ is also monotone and submodular (see Footnote 9).

For the quantity of MMS, partition $\{\{e_1, e_4\}, \{e_2, e_3\}\}$ defines $\mathrm{MMS}_1(2, E) = 1/2$, and any allocation defines $\mathrm{MMS}_2(2, E) = 1$. The minimal social cost allocation $\mathbf{O}$ with $O_1 = \{e_1, e_2\}$ and $O_2 = \{e_3, e_4\}$ whose social cost is equal to $3/4 + 3\epsilon$. But due to $c_1(O_1) = 3/4 + \epsilon > 3/2 \cdot \mathrm{MMS}_1(2, E)$, agent 1 violates $\frac{3}{2}$-PMMS under $\mathbf{O}$. Notice agent 1 can not receive both e_1, e_2, one can check that the $\frac{3}{2}$-PMMS allocation with minimal social cost assigns all chores to agent 2, yielding social cost exactly 1. As a consequence, the price of $\frac{3}{2}$-PMMS is at least $\frac{1}{3/4+3\epsilon} \to \frac{4}{3}$ as $\epsilon \to 0$. □

Theorem 7.5 *When $n = 2$ and agents have submodular cost functions, the price of 2-MMS is 1.*

Proof According to Lemma 2.2, the allocation with minimal social cost must also be 2-MMS, completing the proof. □

Theorem 7.6 *When $n \geq 3$ and agents have submodular cost functions, the price of 2-MMS is at least $\frac{n+3}{6}$ and at most $\frac{n^2}{2}$.*

Proof The lower bound directly follows from the instance constructed in Theorem 6.7. As for the upper bound, given any minimal social cost allocation $\mathbf{O}$, if $\max_{i\in[n]} c_i(O_i) \leq \frac{2}{n}$, then due to $\mathrm{MMS}_i(n, E) \geq \frac{1}{n}$ from Lemma 2.1, we have $c_i(O_i) \leq 2\mathrm{MMS}_i(n, E)$ for any $i \in [n]$. This implies allocation $\mathbf{O}$ is 2-MMS and we are done. Thus, we can assume w.l.o.g. that $\max_{i\in[n]} c_i(O_i) > \frac{2}{n}$. Notice the social cost of an allocation is at most n, and so the price of 2-MMS is at most $\frac{n^2}{2}$. □

8 Conclusions

In this paper, we are concerned with fair allocations of indivisible chores among agents under the setting of both additive and submodular (subadditive) cost functions. First, under the additive setting, we have established pairwise connections between several (additive) relaxations of the envy-free fairness in allocating, which look at how an allocation under one fairness criterion provides an approximation guarantee for fairness under another criterion. Some of our results in that part are in sharp contrast to what is known in allocating indivisible goods, reflecting the difference between goods and chores allocation. We have also extended to the submodular setting and investigated the connections between these fairness criteria. Our results have shown that, under the submodular setting, the interesting connections we have established under the additive setting almost disappear and few non-trivial approximation guarantees exist. Then we have studied the trade-off between fairness and efficiency, for which we have established the price of fairness for all these fairness notions in both additive and submodular settings. We hope our results have provided an almost complete picture for the connections between these chores fairness criteria together with their individual efficiencies relative to social optimum.

Appendix

A.1 Proof of Theorem 4.3

We first prove the upper bound. Let $\mathbf{A} = (A_1, A_2, A_3)$ be a PMMS allocation and we focus on agent 1. For the sake of contradiction, we assume $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3, E)$. We can also assume bundles A_1, A_2, A_3 do not contain chore with zero cost for agent 1 since the existence of such chores do not affect approximation ratio of allocation A on PMMS or MMS. To this end, we let $c_1(A_2) \leq c_1(A_3)$ (the other case is symmetric).

We first show that A_1 must be the bundle yielding the largest cost for agent 1. Otherwise, if $c_1(A_1) \leq c_1(A_2) \leq c_1(A_3)$, then by additivity $c_1(A_1) \leq \frac{1}{3}c_1(E) \leq \mathrm{MMS}_1(3, E)$, contradicting $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3, E)$. Or if $c_1(A_2) < c_1(A_1) \leq c_1(A_3)$, since A_1 and A_2 is a 2-partition of $A_1 \cup A_2$, then $c_1(A_1)$ is at least $\mathrm{MMS}_1(2, A_1 \cup A_2)$. On the other hand, since $\mathbf{A}$ is a PMMS allocation, we know $c_1(A_1) \leq \mathrm{MMS}_1(2, A_1 \cup A_2)$, and thus, $c_1(A_1) = \mathrm{MMS}_1(2, A_1 \cup A_2)$ holds. Based on assumption $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3, E)$ and

Lemma 2.1, we have $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3,E) \geq \frac{4}{9}c_1(E)$, then $c_1(A_3) \geq c_1(A_1) > \frac{4}{9}c_1(E)$ which yields $c_1(A_2) < \frac{1}{9}c_1(E)$ owning to the additivity. As a result, the difference between $c_1(A_1)$ and $c_1(A_2)$ is lower bounded $c_1(A_1) - c_1(A_2) > \frac{1}{3}c_1(E)$. Due to $c_1(A_1) = \mathrm{MMS}_1(2, A_1 \cup A_2)$, we can claim that every single chore in A_1 has cost strictly greater than $\frac{1}{3}c_1(E)$; otherwise, $\exists e \in A_1$ with $c_1(e) \leq \frac{1}{3}c_1(E)$, then reassigning chore e to A_2 yields a 2-partition $\{A_1 \backslash \{e\}, A_2 \cup \{e\}\}$ with $\max\{c_1(A_1 \backslash \{e\}), c_1(A_2 \cup \{e\})\} < c_1(A_1) = \mathrm{MMS}_1(2, A_1 \cup A_2)$, contradicting the definition of maximin share. Since every single chore in A_1 has cost strictly greater than $\frac{1}{3}c_1(E)$, then A_1 can only contain a single chore; otherwise, $c_1(A_3) \geq c_1(A_1) \geq \frac{|A_1|}{3}c_1(E) \geq \frac{2}{3}c_1(E)$, implying $c_1(A_3 \cup A_1) \geq \frac{4}{3}c_1(E)$, a contradiction. However, if $|A_1| = 1$, according to the second point of Lemma 2.1, $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3,E)$ can never hold. Therefore, it must hold that $c_1(A_1) \geq c_1(A_3) \geq c_1(A_2)$, which then implies $c_1(A_1) = \mathrm{MMS}_1(2, A_1 \cup A_3) = \mathrm{MMS}_1(2, A_1 \cup A_2)$ as a consequence of PMMS.

Next, we prove our statement by carefully checking the possibilities of $|A_1|$. According to Lemma 2.1, if $|A_1| = 1$, then $c_1(A_1) \leq \mathrm{MMS}_1(3,E)$. Thus, we can further assume $|A_1| \geq 2$. We first consider the case $|A_1| \geq 3$. Since $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3,E) \geq \frac{4}{9}c_1(E)$, by additivity, we have $c_1(A_2) + c_1(A_3) < \frac{5}{9}c_1(E)$ and moreover, $c_1(A_2) < \frac{5}{18}c_1(E)$ due to $c_1(A_2) \leq c_1(A_3)$. Then the cost difference between bundle A_1 and A_2 satisfies $c_1(A_1) - c_1(A_2) > \frac{1}{6}c_1(E)$. This allow us to claim that every single chore in A_1 has cost strictly greater than $\frac{1}{6}c_1(E)$; otherwise, reassigning a chore with cost no larger than $\frac{1}{6}c_1(E)$ to A_2 yields another 2-partition of $A_1 \cup A_2$ in which the cost of larger bundle is strictly smaller than $\mathrm{MMS}_1(2, A_1 \cup A_2)$, a contradiction. In addition, since $c_1(A_1) = \mathrm{MMS}_1(2, A_1 \cup A_2)$, we claim $c_1(A_2) \geq c_1(A_1 \backslash \{e\}), \forall e \in A_1$; otherwise, $\exists e' \in A_1$ such that $c_1(A_2) < c_1(A_1 \setminus \{e'\})$, then reassigning e' to A_2 yields another 2-partition of $A_1 \cup A_2$ of which both two bundles' cost are strictly smaller than $\mathrm{MMS}_1(2, A_1 \cup A_2)$, a contradiction. Thus, for any $e \in A_1$, we have $c_1(A_2) \geq c_1(A_1 \backslash \{e\}) \geq \frac{1}{6}c_1(E) \cdot |A_1 \backslash \{e\}| \geq \frac{1}{3}c_1(E)$, where the last transition is due to $|A_1| \geq 3$. However, the cost of bundle A_2 is $c_1(A_2) < \frac{5}{18}c_1(E)$, a contradiction.

The remaining work is to rule out the possibility of $|A_1| = 2$. Let $A_1 = \{e_1^1, e_2^1\}$ with $c_1(e_1^1) \leq c_1(e_2^1)$ (the other case is symmetric). Since $c_1(A_1) > \frac{4}{3}\mathrm{MMS}_1(3,E) \geq \frac{4}{9}c_1(E)$, then $c_1(e_2^1) > \frac{2}{9}c_1(E)$. Let $S_2^* \in \arg\max_{S \subseteq A_2}\{c_1(S) : c_1(S) < c_1(e_1^1)\}$ (can be empty set) be the largest subset of A_2 with cost strictly smaller than $c_1(e_1^1)$. Due to $c_1(A_1) = \mathrm{MMS}_1(2, A_1 \cup A_2)$, then swapping S_2^* and e_1^1 would not produce a 2-partition in which the cost of both bundles are strictly smaller than $c_1(A_1)$, and thus $c_1(A_2 \setminus S_2^* \cup \{e_1^1\}) \geq c_1(A_1)$, equivalent to

$$c_1(A_2 \setminus S_2^*) \geq c_1(e_2^1) > \frac{2}{9}c_1(E). \tag{A-1}$$

Then, by $c_1(A_1) - c_1(A_2) > \frac{1}{6}c_1(E)$ and $c_1(A_2 \backslash S_2^*) \geq c_1(e_2^1)$, we have $c_1(e_1^1) - c_1(S_2^*) > \frac{1}{6}c_1(E)$, which allows us to claim that every single chore in $A_2 \setminus S_2^*$ has cost strictly greater than $\frac{1}{6}c_1(E)$; otherwise, we can find another subset of A_2 whose cost is strictly smaller than e_1^1 but larger than $c_1(S_2^*)$, contradicting the definition of S_2^*. As a result, bundle $A_2 \setminus S_2^*$ must contain a single chore; if not, $c_1(A_2) > \frac{1}{6}c_1(E) \cdot |A_2 \backslash S_2^*| \geq \frac{1}{3}c_1(E)$, which implies $c_1(A_1 \cup A_2 \cup A_3) > \frac{10}{9}c_1(E)$ due to $c_1(A_1) > \frac{4}{9}c_1(E)$ and $c_1(A_3) \geq c_1(A_2) > \frac{1}{3}c_1(E)$. Thus, bundle $A_2 \setminus S_2^*$ only contains one chore, denoted by e_1^2. So we can decompose A_2 as $A_2 = \{e_1^2\} \cup S_2^*$ where $c_1(e_1^2) \geq c_1(e_2^1) > \frac{2}{9}c_1(E)$.

Next, we analyse the possible composition of bundle A_3. To have an explicit discussion, we introduce two more notions Δ_1, Δ_2 as follows

$$\begin{aligned} c_1(A_1) &= \frac{4}{9}c_1(E) + \Delta_1, \\ c_1(A_2) &= \frac{2}{9}c_1(E) + c_1(S_2^*) + \Delta_2. \end{aligned} \tag{A-2}$$

Recall $c_1(A_1) > \frac{4}{9}c_1(E)$ and $c_1(e_1^2) \geq c_1(e_2^1) > \frac{2}{9}c_1(E)$, so both $\Delta_1, \Delta_2 > 0$. Similarly, let $S_3^* \in \arg\min_{S \subseteq A_3}\{c_1(S) : c_1(S) < c_1(e_1^1)\}$, then we claim $c_1(A_3 \backslash S_3^*) \geq c_1(e_2^1)$; otherwise, swapping S_3^* and e_1^1 yields a 2-partition of $A_1 \cup A_3$ in which the cost of both bundles are strictly smaller than $c_1(A_1) = \text{MMS}_1(2, A_1 \cup A_3)$, contradicting the definition of maximin share. By additivity of cost functions and Equation (A-2), we have $c_1(A_3) = \frac{3}{9}c_1(E) - c_1(S_2^*) - \Delta_1 - \Delta_2$, and accordingly $c_1(A_1) - c_1(A_3) = \frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2$. This combing $c_1(A_3 \backslash S_3^*) \geq c_1(e_2^1)$ yields

$$c_1(e_1^1) - c_1(S_3^*) \geq \frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2. \tag{A-3}$$

Based on Inequality (A-3), we can claim that every single chore in $A_3 \setminus S_3^*$ has cost at least $\frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2$; otherwise, contradicting the definition of S_3^*. Recall $c_1(A_3) = \frac{3}{9}c_1(E) - c_1(S_2^*) - \Delta_1 - \Delta_2$, then due to the constraint on the cost of single chore in $A_3 \backslash S_3^*$, we have $|A_3 \backslash S_3^*| \leq 2$. Meanwhile, $c_1(A_3 \backslash S_3^*) \geq c_1(e_2^1)$ implying that bundle $A_3 \setminus S_3^*$ can not be empty. In the following, we separate our proof by discussing two possible cases: $|A_3 \backslash S_3^*| = 1$ and $|A_3 \backslash S_3^*| = 2$.

Case 1: $|A_3 \backslash S_3^*| = 1$. Let $A_3 \backslash S_3^* = \{e_1^3\}$. Therefore, the whole set E is composed by four single chores and two subsets S_2^*, S_3^*, i.e., $E = \{e_1^1, e_2^1, e_1^2, S_2^*, e_1^3, S_3^*\}$. Then, we let $\mathbf{T} = (T_1, T_2, T_3)$ be the allocation defining $\text{MMS}_1(3, E)$ and without loss of generality, let $c_1(T_1) = \text{MMS}_1(3, E)$. Next, to find contradictions, we analyse bounds on both $\text{MMS}_1(3, E)$ and $c_1(A_1)$. Since $\min\{c_1(e_1^2), c_1(e_1^3)\} \geq c_1(e_2^1) \geq \frac{1}{2}c_1(A_1)$, we claim that $c_1(A_1) \leq \frac{9}{18}c_1(E)$; otherwise $c_1(A_1) + c_1(e_1^2) + c_1(e_1^3) > c_1(E)$. Notice that E contains three chores with the cost at least $\frac{2}{9}c_1(E)$ each, if any two of them are in the same bundle under $\mathbf{T}$, then $\text{MMS}_1(3, E) > \frac{4}{9}c_1(E)$ and consequently, $\frac{c_1(A_1)}{\text{MMS}_1(3,E)} < \frac{9}{8}$, a contradiction. Or if each of $\{e_2^1, e_1^2, e_1^3\}$ is contained in a distinct bundle, then the bundle also containing chore e_1^1 has cost at least $c_1(A_1)$ as a result of $\min\{c_1(e_1^2), c_1(e_1^3)\} \geq c_1(e_2^1)$ and $A_1 = \{e_1^1, e_2^1\}$. Thus, $\text{MMS}_1(3, E) \geq c_1(A_1)$ holds, contradicting $c_1(A_1) > \frac{4}{3}\text{MMS}_1(3, E)$.

Case 2: $|A_3 \backslash S_3^*| = 2$. Let $A_3 \backslash S_3^* = \{e_1^3, e_2^3\}$ and accordingly, the whole set can be decomposed as $E = \{e_1^1, e_2^1, e_1^2, S_2^*, e_1^3, e_2^3, S_3^*\}$. Note the upper bound $c_1(A_1) \leq \frac{9}{18}c_1(E)$ still holds since $\min\{c_1(A_3 \setminus S_3^*), c_1(e_1^2)\} \geq c_1(e_2^1)$. Then, we analyse the possible lower bound of $\text{MMS}_1(3, E)$. If chores e_2^1, e_1^2 are in the same bundle of $\mathbf{T}$, then $\text{MMS}_1(3, E) > \frac{4}{9}c_1(E)$ holds and so $\frac{c_1(A_1)}{\text{MMS}_1(3,E)} < \frac{9}{8}$, a contradiction. Thus, chores e_2^1, e_1^2 are in different bundles in $\mathbf{T}$.

Then, if both chores e_1^3, e_2^3 are in the bundle containing e_2^1 or e_1^2, then we also have $\mathrm{MMS}_1(3,E) > \frac{4}{9}c_1(E)$ implying $\frac{c_1(A_1)}{\mathrm{MMS}_1(3,E)} < \frac{9}{8}$, a contradiction. Therefore, only two possible cases; that is, both e_1^3, e_2^3 are in the bundle different from that containing e_2^1 or e_1^2; or the bundle having e_2^1 or e_1^2 contains at most one of e_1^3, e_2^3.

Subcase 1: both e_1^3, e_2^3 are in the bundle different from that containing e_2^1 or e_1^2; Recall $c_1(e_1^1) > \frac{3}{18}c_1(E) + c_1(S_2^*)$ and the fact $\min\{c_1(e_2^1), c_1(e_1^2), c_1(e_1^3 \cup e_2^3)\} > \frac{4}{18}c_1(E)$, the bundle also containing e_1^1 has cost strictly greater than $\frac{7}{18}c_1(E)$. Thus, $\mathrm{MMS}_1(3,E) > \frac{7}{18}c_1(E)$, which combines $c_1(A_1) \le \frac{9}{18}c_1(E)$ implying $\frac{c_1(A_1)}{\mathrm{MMS}_1(3,E)} < \frac{9}{7} < \frac{4}{3}$, a contradiction.

Subcase 2: bundle having e_2^1 or e_1^2 contains at most one of e_1^3, e_2^3. Recall $c_1(e_1^2) \ge c_1(e_2^1)$ and $\min\{c_1(e_1^3), c_1(e_2^3)\} \ge \frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2$, thus in allocation $\mathbf{T}$ there always exist a bundle with cost at least $\frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2 + c_1(e_2^1)$ and results in the ratio

$$\frac{c_1(A_1)}{\mathrm{MMS}_1(3,E)} \le \frac{c_1(e_1^1) + c_1(e_2^1)}{\frac{1}{9}c_1(E) + c_1(S_2^*) + 2\Delta_1 + \Delta_2 + c_1(e_2^1)}. \tag{A-4}$$

In order to satisfying our assumption $\frac{c_1(A_1)}{\mathrm{MMS}_1(3,E)} > \frac{4}{3}$, the RHS of Inequality (A-4) must be strictly greater than $\frac{4}{3}$, which implies the following

$$c_1(e_1^1) > \frac{2}{9}c_1(E) + 2c_1(S_1^*) + 4\Delta_1 + 2\Delta_2. \tag{A-5}$$

However, based on the first equation of (A-2) and $c_1(e_1^1) \le c_1(e_2^1)$, we have $c_1(e_1^1) \le \frac{2}{9}c_1(E) + \frac{1}{2}\Delta_1 < \frac{2}{9}c_1(E) + 2c_1(S_1^*) + 4\Delta_1 + 2\Delta_2$ due to $\Delta_1, \Delta_2 > 0$. This contradicts Inequality (A-5). Therefore, $\frac{c_1(A_1)}{\mathrm{MMS}_1(3,E)} > \frac{4}{3}$ can never hold under *Case 2*. Up to here, we complete the proof of the upper bound.

Next, as for tightness, consider an instance with three agents and a set $E = \{e_1, \dots, e_6\}$ of six chores. Agents have identical cost functions. The cost function of agent 1 is as follows: $c_1(e_j) = 2, \forall j = 1,2,3$ and $c_1(e_j) = 1, \forall j = 4,5,6$. It is easy to see that $\mathrm{MMS}_1(3,E) = 3$. Then, consider an allocation $\mathbf{B} = \{B_1, B_2, B_3\}$ with $B_1 = \{e_1, e_2\}, B_2 = \{e_3\}$ and $B_3 = \{e_4, e_5, e_6\}$. It is not hard to verify that allocation $\mathbf{B}$ is PMMS and due to $c_1(B_1) = 4$, we have the ratio $\frac{c_1(B_1)}{\mathrm{MMS}_1(3,E)} = \frac{4}{3}$.

A.2 Algorithm 1

The following efficient algorithm, which we call ALG_1, outputs an EF1 allocation with a cost at most $\frac{5}{4}$ times the optimal social cost under the case of $n = 2$. In the algorithm, we use notations: $L(k) := \{e_1, \dots, e_k\}$ and $R(k) := \{e_k, \dots, e_m\}$ for any $1 \le k \le m$.

Algorithm 1

Input: An instance I with two agents.

Output: an EF1 allocation of instance I.

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Partition $E = E_0 \cup E_1 \cup E_2$ where $E_1 = \{e \in E \mid c_1(e) < c_2(e)\}$ and $E_2 = \{e \in E \mid c_1(e) > c_2(e)\}$ (we assume $c_1(E_1) \leq c_2(E_2)$ and the other case is symmetric).
Order chores such that $\frac{c_1(e_1)}{c_2(e_1)} \leq \frac{c_1(e_2)}{c_2(e_2)} \leq \cdots \leq \frac{c_1(e_m)}{c_2(e_m)}$, tie breaks arbitrarily. For chore $e$ with $c_1(e) = 0$, put it at the front and chore $e$ with $c_2(e) = 0$ at back.
Find index $s$ such that $c_1(e_s) < c_2(e_s)$ and $c_1(e_{s+1}) \geq c_2(e_{s+1})$.
if $s = 0$ then
    Run a round-robin algorithm: let each of the agent $1, \ldots, n$ picks her most preferred item in that order, and repeat until all chores are assigned.
    return the output
else
    Let O be the allocation with $O_1 = L(s)$ and $O_2 = R(s+1)$.
    if allocation O is EF1 then
        return allocation O.
    else
        find the maximum index $f \geq s$ such that $c_2(R(f+2)) > c_2(L(f))$.
        return allocation A with $A_1 = L(f+1)$ and $A_2 = R(f+2)$
    end if
end if
```

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Declarations

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